Please attempt questions 1–11 below. Throughout, try to write out full and convincing proofs. Questions 8–10 may be less familiar and require more thought, and the final part of question 11 is perhaps a bit hard. If you can't do a question, don't panic (and definitely don't look up a solution) – write down your thoughts and hand them in, and then we can discuss them and work towards the solution.

1. By considering $(r+1)^3 - r^3$, derive the formula $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$.

2. Use induction to prove that for every $n \geqslant 1$,

   (a) $1^2 + 3^2 + \cdots + (2n-1)^2 = \frac{1}{3}(4n^3 - n)$;

   (b) $n^3 + 5n$ is divisible by 6;

   (c) $2^{n+2} + 3^{2n+1}$ is divisible by 7.

3. Find non-inductive solutions to question 2,

   (a) for 2(a), by using the formula in question 1;

   (b) for 2(b), by factorising a suitable expression;

   (c) for 2(c), by using modular arithmetic (or, if you haven't met modular arithmetic, using the result that $a-b$ divides $a^k - b^k$ for $a, b, k \in \mathbb{N}$).

4. *Theorem.* All rabbits are the same colour.

   Proof. We will use induction to show that, for each n, any n rabbits are the same colour. The base case, $n = 1$, is easy: any one rabbit is the same colour as itself. Now suppose the result is true for any set of n rabbits, but that we have $n+1$ rabbits. Call them $r_1, \ldots, r_{n+1}$.

   By induction, we know rabbits $r_1, \ldots, r_n$ are the same colour (because there are n of them), and that rabbits $r_2, \ldots, r_{n+1}$ are the same colour (because there are n of them). Therefore, we deduce that rabbits $r_1, \ldots, r_{n+1}$ are the same colour. So, by induction, all rabbits are the same colour. □

   However, not all rabbits are the same colour. So where is the mistake?

5. Let F_n be the n^{th} Fibonacci number, defined by: $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. By observing

$$F_{n+2} = F_{n+1} + F_n, \quad F_{n+3} = 2F_{n+1} + F_n, \quad F_{n+4} = 3F_{n+1} + 2F_n, \quad \ldots,$$

   guess the general formula for F_{n+k} in terms of F_{n+1} and F_n, and verify it by induction.

   Deduce that F_{kn} is a multiple of F_n for all $k, n \in \mathbb{N}$.

   Deduce also that $F_{n+1}^2 + F_n^2$ and $F_{n+2}^2 - F_n^2$ are Fibonacci numbers for all $n \in \mathbb{N}$.

6. (a) Let $m, n \in \mathbb{N}$. Prove carefully that if $\sqrt[m]{n}$ is rational then it is an integer.

   (b) Let $m, n \in \mathbb{N}$ be coprime, with $m > 1$. Prove that if $\log_m n$ is rational then it is 0.

7. Let $\alpha, \beta, r \in \mathbb{R}$, with α, β irrational and r rational. Which, if any, of $r + \alpha$, $r\alpha$, $\alpha + \beta$, $\alpha\beta$, α^r, r^α and α^β must be irrational? Give proofs or counterexamples. *(If you give an example involving irrational numbers, try to use numbers that you are actually able to prove are irrational.)*

8. Let A be a collection of $n + 1$ distinct integers chosen from the set $\{1, \ldots, 2n\}$. Show that A must contain two numbers which are coprime, and also two numbers such that one divides the other.

9. You are asked to drive a lunar rover around the moon (which is just a circle in this question). There are finitely many fuel depots on the way, with the total amount of fuel stored in them enough to get around the moon exactly once. Prove that there exists a depot such that, starting from there, you can drive the whole way around the moon, picking up fuel at each depot as you pass, without running out of fuel between depots.

10. There are six towns, such that between each pair of towns there is either a train or bus service (but not both). Prove that there are three towns that can be visited in a loop, going via no other towns, using only one mode of transport.

    Is the result still true if there are only five towns?

11. Let $n \in \mathbb{N}$. Let N be the number of primes p satisfying $n < p \leqslant 2n$, and let p be such a prime. Show that p divides the binomial coefficient $\binom{2n}{n}$, and deduce that $n^N \leqslant \binom{2n}{n}$.

    Show also that $\binom{2n}{n} \leqslant 4^n$, and hence find a constant a such that $N \leqslant an/\log n$.

    Deduce that $\pi(n) \leqslant cn/\log n$, for some constant c, where $\pi(n)$ is the number of primes less than or equal to n.

Additional questions

These questions are more difficult than the rest of the sheet – attempt them if they interest you, but not at the expense of other work.

12. The region in Question 10 has grown, and there are now eighteen towns. It is still the case that between each pair of towns there is either a train or bus service (but not both). Prove that there are four towns such that all six of their pairwise connections use the same mode of transport. *(You might prefer to try twenty towns first.)*

13. Is there a power of 7 that starts (in base 10) with the digits 2011...?

14. Let R be a rectangle which can be divided into smaller rectangles, each of which has at least one side of integer length. Prove that R has at least one side of integer length.

Please send any corrections or comments to me at glt1000@cam.ac.uk