

Hyperbolic functions

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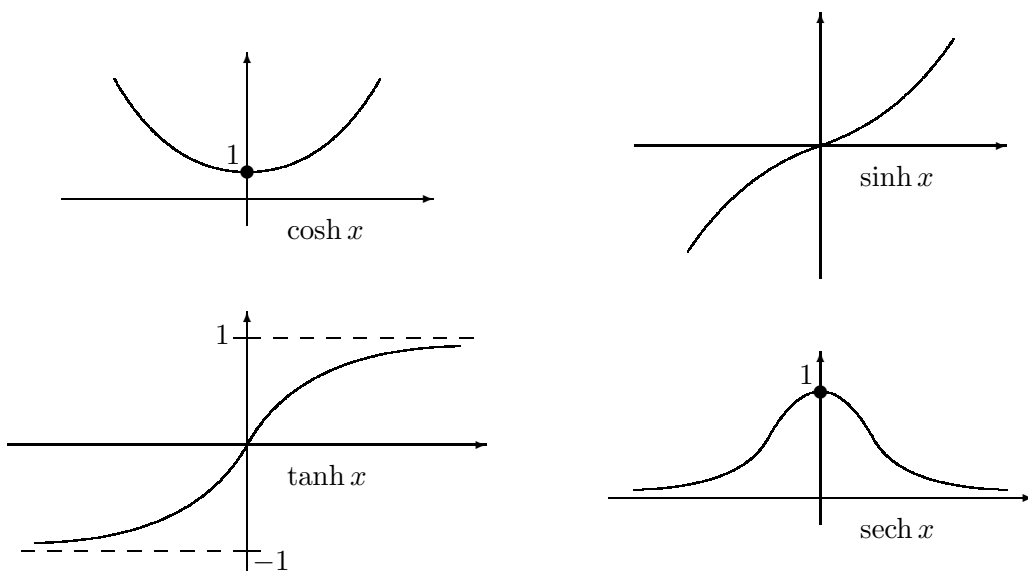
Several paths may be followed that each culminate in the appearance of hyperbolic functions. I am going to define the functions first.

$$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = \frac{e^x - e^{-x}}{2}; \quad \tanh x = \frac{\sinh x}{\cosh x};$$
$$\operatorname{sech} x = \frac{1}{\cosh x}; \quad \operatorname{cosech} x = \frac{1}{\sinh x}; \quad \operatorname{coth} x = \frac{1}{\tanh x} \equiv \frac{\cosh x}{\sinh x}$$

In terms of pronunciation one subscribes to the ‘sinch, cosh, tanch, setch’ school or joins the ‘shine, cosh, than, shec’ brigade. Or you can be a rebel and say ‘sesh’.

Origins

A heavy rope whose ends are held level hangs in a cosh curve. This is called a catenary.



Identities

Note that $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, and so $\cos(ix) = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^x}{2} \equiv \cosh x$

Similarly, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, and $\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{-i(e^{-x} - e^x)}{2} \equiv i \sinh x$.

Thus trig identities can be directly related to hyperbolic identities, except that whenever $\sin^2 x$ appears it is replaced by $-\sinh^2 x$. For the same reason ($i^2 = -1$), $\sin x \sin y$ converts to $-\sinh x \sinh y$, for example in $\cosh(x + y)$. This is Osborn’s rule.

Differentiation properties

Differentiation can be carried out from first principles.

$$\text{eg } \frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} \equiv \cosh x.$$

You can use your knowledge of trig to predict the functions you expect but not their signs.

$$\text{eg } \frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} \equiv +\sinh x.$$

It is also possible to proceed via the trig functions of ix using the chain rule.

$$\text{eg } \sin ix = i \sinh x \Rightarrow i \cos ix = i \frac{d}{dx} \sinh x, \text{ so } \frac{d}{dx} \sinh x = \cosh x.$$

Note that Osborn's rule does not apply to calculus. Note also that there is no periodicity in hyperbolic functions along the real axis.

The solution to the equation $y'' = k^2 y$ is

$$y = Ae^{kx} + Be^{-kx} \equiv C \sinh kx + D \cosh kx \equiv E \sinh k(1-x) + F \cosh k(1-x) \equiv \dots$$

Inverses, logarithmic forms ...

Let $y = \operatorname{arcosh} x$, so that $\cosh y = x$. Note the terminology for inverse hyperbolic functions, compared with, say, arcsine.

$$\text{Then } \frac{1}{2}(e^y + e^{-y}) = x \Rightarrow e^{2y} - 2xe^y + 1 = 0 \Rightarrow e^y = x \pm \sqrt{x^2 - 1} \Rightarrow y = \log(x \pm \sqrt{x^2 - 1}).$$

Because $x - \sqrt{x^2 - 1} = \frac{1}{x + \sqrt{x^2 - 1}}$, this gives $y = \pm \log(x + \sqrt{x^2 - 1})$. To create a one-to-one map we define y to be positive, so take the positive root.

Using a similar derivation, we can show that $\operatorname{arsinh} x = \log(x + \sqrt{x^2 + 1})$.

Note: the logarithmic base that you are most likely to encounter from now on is base e . The terminology 'ln' is mostly redundant, though you may still meet it. I will only ever use log to mean \log_e . If the base is not specified, assume it is e .

... and derivatives

$$\text{If we instead let } y = \operatorname{arsinh} \frac{x}{a}, \text{ then } x = a \sinh y \Rightarrow \frac{dx}{dy} = a \cosh y = a \sqrt{\sinh^2 y + 1} = \sqrt{x^2 + a^2}.$$

$$\text{So } \frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}}.$$

$$\text{Similarly, } \frac{d}{dx} \operatorname{arcosh} \frac{x}{a} = \frac{1}{\sqrt{x^2 - a^2}} \text{ and } \frac{d}{dx} \operatorname{artanh} \frac{x}{a} = \frac{a}{a^2 - x^2}.$$

These formulae can be used for integrating any functions involving the sum of difference of two squares that aren't amenable to normal trig substitutions.

Examples include $1/\sqrt{x^2 + 4}$ and $\sqrt{x^2 - 1}$.

$$\text{Note that, for example, you can write } \int \sqrt{x^2 + 2x - 1} dx = \int \sqrt{(x+1)^2 - (\sqrt{2})^2} dx.$$

We can approach $\int \frac{x+1}{\sqrt{x^2-4}} dx$ in a similar way.

If you wish for more detail on any of this you should consult an A-level book such as Bostock & Chandler's Further Pure Mathematics. Please let me have any comments or corrections.