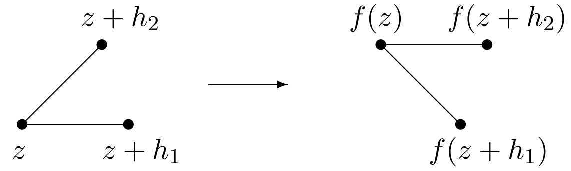
Angles at points where $f'(z) = 0$

Suppose we have a point z somewhere in $\mathbb{C}$, and two tiny displacements from it, at $z+h_1$ and $z+h_2$. If the map is conformal, then the angle between those (as measured from z) is the same before and after the map.



The angle between $z+h_1$ and $z+h_2$ (as measured from z) can be found by pretending that z is the origin (by subtracting z), and taking the difference of the resulting arguments:

$$\arg\big((z+h_2)-z\big) - \arg\big((z+h_1)-z\big) \;=\; \arg(h_2) - \arg(h_1)$$

Now, using a Taylor Series, we have

$$f(z+h) = f(z) + hf'(z) + \tfrac{1}{2}h^2 f''(z) + \ldots$$

so that, for any h,

$$\arg\big(f(z+h) - f(z)\big) \;\approx\; \arg\big(hf'(z)\big) \;=\; \arg(h) + \arg\big(f'(z)\big),$$

providing $f'(z)$ is non-zero. If $f'(z) = 0$, then $hf'(z) = 0$ and its argument is undefined.

So, if $f'(z)$ is non-zero, the angle between $f(z+h_1)$ and $f(z+h_2)$ (as measured from $f(z)$) is

$$\arg\big(f(z+h_2) - f(z)\big) - \arg\big(f(z+h_1) - f(z)\big) \;=\; \arg(h_2) - \arg(h_1),$$

since the $\arg\big(f'(z)\big)$ terms cancel off. Thus conformal maps preserve angles - where the derivative is non-zero.

However, suppose that $f'(z) = 0$, but that $f''(z)$ is non-zero. Then we have to go to the next term in the Taylor Series:

$$f(z+h) = f(z) + \tfrac{1}{2}h^2 f''(z) + \ldots$$

so that

$$\begin{aligned}
\arg\big(f(z+h) - f(z)\big) &\approx \arg\big(\tfrac{1}{2}h^2 f''(z)\big) \\
&= \arg(h^2) + \arg\big(\tfrac{1}{2}f''(z)\big) \\
&= 2\arg(h) + \arg\big(\tfrac{1}{2}f''(z)\big)
\end{aligned}$$

So this time, when we subtract, the $\arg\left(\frac{1}{2}f''(z)\right)$ terms cancel off, and we find

$$\arg\big(f(z+h_2)-f(z)\big)-\arg\big(f(z+h_1)-f(z)\big) \;=\; 2\big(\arg(h_2)-\arg(h_1)\big),$$

so the angle has doubled.

What if $f''(z)=0$ as well? Then we go to the next term in the Taylor Series:

$$f(z+h)=f(z)+\tfrac{1}{6}h^3f'''(z)+\ldots$$

so that

$$\begin{aligned}\arg\big(f(z+h)-f(z)\big) &\approx \arg\left(\tfrac{1}{6}h^3f'''(z)\right)\\ &= \arg(h^3)+\arg\left(\tfrac{1}{6}f'''(z)\right)\\ &= 3\arg(h)+\arg\left(\tfrac{1}{6}f'''(z)\right)\end{aligned}$$

So this time, when we subtract, the $\arg\left(\frac{1}{6}f'''(z)\right)$ terms cancel off, and we find

$$\arg\big(f(z+h_2)-f(z)\big)-\arg\big(f(z+h_1)-f(z)\big) \;=\; 3\big(\arg(h_2)-\arg(h_1)\big),$$

so the angle has trebled.

More generally still, if the first non-zero derivative to appear in the Taylor Series is the n^{th} derivative, then the $\arg\left(\frac{1}{n!}f^{(n)}(z)\right)$ terms cancel, and the argument increases by a factor of n.