Metric and Topological Spaces

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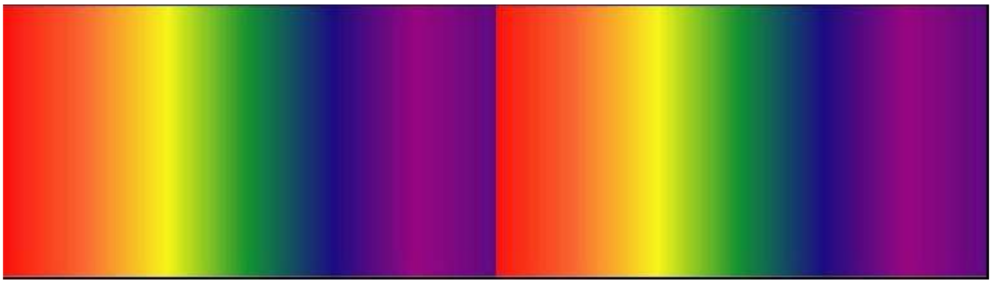
L1: Topological spaces The central idea of topology is that it makes sense to talk about *continuity* without talking about *distance*.



We do this by formalising a notion of "proximity" or "nearness". Specifically, we take a set X and distinguish a certain collection of subsets of X – which we call the *open subsets* – which satisfy certain axioms. The idea is that two points are close if open sets which contain one also contain the other unless the open sets are "small enough". As often happens, the formal definition has so *little* structure, it both captures our original idea, but also lots of much more general things (which makes it even more useful, even if less intuitive).

Definition: A topological space $(X, \mathcal{T}_X)$ is a set X together with a collection $\mathcal{T}_X$ of subsets of X which satisfy:
(i) both $\emptyset$ and X belong to $\mathcal{T}_X$;
(ii) $\mathcal{T}_X$ is closed under taking arbitrary unions;
(iii) $\mathcal{T}_X$ is closed under taking finite intersections.

So for an arbitrary indexing set I and $n \in \mathbb{N}$

$$\{U_i\}_{i \in I} \subset \mathcal{T}_X \;\Rightarrow\; \bigcup_i U_i \in \mathcal{T}_X$$

$$\{V_j\}_{1 \leq j \leq n} \subset \mathcal{T}_X \;\Rightarrow\; \bigcap_j V_j \in \mathcal{T}_X$$

Terminology: we say that the elements of $\mathcal{T}_X$ are the *open* subsets of X. The indexing set I need not be countable. In practise one can check (iii) by checking intersections of two open sets and using induction, but there's no analogous inductive way of checking (ii). The second fundamental definition that goes hand-in-hand with this is the one which allows us to compare topologies on two spaces.

Definition: if $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, a function (of sets) $f : X \to Y$ is *continuous* if $f^{-1}(\mathcal{T}_Y) \subset \mathcal{T}_X$, in other words

$$V \in \mathcal{T}_Y \;\Rightarrow\; f^{-1}(V) \in \mathcal{T}_X$$

So f is cts $\Leftrightarrow$ preimages of open sets are open.

Warning: $f^{-1}(V)$ is the set-theoretic preimage of points $\{x \in X \mid f(x) \in V\}$. We are not assuming f has an inverse function.

Here's a first indication of the power of the formalism:

Lemma: the composition of continuous functions is again continuous.

Proof: Take topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ and functions $f : X \to Y$ and $g : Y \to Z$ which are assumed continuous. If $W \subset Z$ belongs to $\mathcal{T}_Z$, then $V = g^{-1}(W) \in \mathcal{T}_Y$ (since g cts) and $f^{-1}(V) \in \mathcal{T}_X$ (since f cts). Thus $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W)) = f^{-1}(V) \in \mathcal{T}_X$. Since W was arbitrary, preimages of open sets are open, and $g \circ f$ is indeed continuous. $\blacksquare$

This is "cleaner" than the ε, δ proof using the usual idea of continuity in metric spaces... of course, it also recovers (and generalises) that argument, as we'll see next time.

Example: (Indiscrete spaces)
For any X, let $\mathcal{T}_X = \{\emptyset, X\}$. This is the "indiscrete topology" on X.

Example: (Discrete spaces)
For any X, let $\mathcal{T}_X = \mathbb{P}(X)$ be the power set of X, so *every* subset of X is open. This is the "discrete topology" on X.

Warning: if you talk about the "trivial topology" on a set, people won't know which of the two examples above you mean.

Example: (Metric spaces)
If (X, d_X) is a metric space, then let $\mathcal{T}_X$ be those sets $U \subset X$ which are metric-open; i.e. $U \in \mathcal{T}_X$ if for each $x \in U$ there is some $\delta_x > 0$ s.t. the open d_X-ball $B_x(\delta_x) \subset U$. [Check this works!] We'll discuss this at length next time.

Example: consider the set $\mathbb{F}$ of real or complex numbers and declare

$$U \in \mathcal{T}_{\mathbb{F}} \;\Leftrightarrow\quad \mathbb{F}\backslash U \text{ is finite} \quad or \quad U = \emptyset$$

Then certainly $\emptyset \in \mathcal{T}_{\mathbb{F}}$ by definition, and since $\mathbb{F}\backslash\mathbb{F}$ is finite [by convention], $\mathbb{F} \in \mathcal{T}_{\mathbb{F}}$ as well. Suppose $\{U_i\}_{i\in I} \subset \mathcal{T}_{\mathbb{F}}$. Then for each $i \in I$, we know $\mathbb{F}\backslash U_i = V_i$ is finite. Now

$$\mathbb{F}\backslash \bigcup_i U_i = \bigcap_i (\mathbb{F}\backslash U_i) = \bigcap_i V_i$$

is certainly finite, so the second axiom holds. Moreover, if $U_j \neq \emptyset$ for $1 \leq j \leq n$, then

$$\mathbb{F}\backslash \bigcap_{j=1}^{n} U_j = \bigcup_{j=1}^{n} (\mathbb{F}\backslash U_j) = \bigcup_{j=1}^{n} V_j$$

is also finite, so the third axiom holds (note: this axiom is trivial if one of the U_j above is empty). Hence we satisfy all the axioms. $\Box$

Note: in $\mathcal{T}_{\mathbb{F}}$, every two non-empty open sets have non-empty intersection. This is a far cry from the usual Euclidean metric topology. If $\mathbb{F} = \mathbb{C}$ this is called the *Zariski topology*.

The above example shows it's important to have "de Morgan's laws" at your fingertips:

- $X \backslash \bigcup_i A_i = \bigcap_i (X \backslash A_i)$
- $X \backslash \bigcap_i A_i = \bigcup_i (X \backslash A_i)$

The following are also very useful in working out whether a function is continuous or not: note the first is *not* in general an identity.

- $f\left(\bigcap_i A_i\right) \subset \bigcap_i f(A_i)$
- $f\left(\bigcup_i A_i\right) = \bigcup_i f(A_i)$
- $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$
- $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$
- $f^{-1}(Y \backslash B) = X \backslash f^{-1}(B)$ if $f : X \to Y$.

These are all set-theoretic relations, which you can apply to any sets; in topology, they obviously give information when applied to open sets. In all the relations, the indexing set $i \in I$ can be arbitrary, it doesn't have to be finite or countable or...

Definition: suppose $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces and $f : X \to Y$ is a bijective function. If both f and the (well-defined) inverse function f^{-1} are continuous, then f is a *homeomorphism* and the spaces are *homeomorphic*.

Homeomorphic spaces are "topologically indistinguishable". One justification for abstract topology is that there are many interesting pairs of homeomorphic spaces which are not obviously "isometric" for any "natural" metrics. Unfortunately, the neat examples only appear later in the Tripos...

Example: the matrix group $SL(2, \mathbb{R})$ is homeomorphic to a solid torus (inside of a bagel).

Example: The space of cosets of $SL(2, \mathbb{Z})$ inside $SL(2, \mathbb{R})$ is naturally a topological space, homeomorphic to the complement of a trefoil knot in $\mathbb{R}^3$ (this is the knot most commonly found in garden hoses and shoelaces).

L2: Metric spaces Recall that a topological space is a set X with a collection of "open" subsets $\mathcal{T}_X$ s.t. $\emptyset \in \mathcal{T}_X$, $X \in \mathcal{T}_X$ and $\mathcal{T}_X$ is closed under the operations of taking arbitrary unions and finite intersections. A map $f : X \to Y$ is continuous (with respect to topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$ – which we will sometimes suppress from the notation) if $f^{-1}(V)$ is open whenever V is open.

Definition: a metric space (X, d_X) is a set X with a "distance function" $d_X : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the axioms (for any $x, y, z \in X$):
(i) $d(x, y) \geq 0$; $\quad d(x, y) = 0 \Leftrightarrow x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

Example: $(\mathbb{R}^n, d_{eucl})$ with $d(x, y) = \sqrt{\sum (x_i - y_i)^2}$

Example: the *discrete metric* on a set X, with $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$ for all x, y.

Example: $\mathcal{B}[a,b] = \{f : [a,b] \to \mathbb{R} \mid f$ bounded$\}$, with the metric $d(f,g) = sup_{x\in[a,b]}|f(x) - g(x)|$.

Note: (i) the RHS is well-defined, since if $f \leq K_f$ and $g \leq K_g$ on $[a,b]$ then $|f(x) - g(x)| \leq |f(x)| + |g(x)| \leq K_f + K_g$; so we are taking the supremum of a bounded set.
Note: (ii) this really is a metric. For instance, for $f, g, h \in \mathcal{B}$ and $c \in [a,b]$:

$$|f(c) - h(c)| \leq |f(c) - g(c)| + |g(c) - h(c)|$$

The RHS is bounded above by $d(f,g) + d(g,h)$, so we have for each $c \in [a,b]$

$$|f(c) - h(c)| \leq d(f,g) + d(g,h)$$

and now take the sup on the LHS to obtain the triangle inequality. This is called the "sup" or "uniform" metric.

Definition: if $(x_n) \subset X$ satisfy $d(x_n, a) \to 0$ as $n \to \infty$ we say the sequence x_n converges to a, and write $x_n \to a$.
Example: Convergence of a sequence of functions in $(\mathcal{B}[a,b], d_{sup})$ is called "uniform convergence".

It is good to be familiar with many examples of metric spaces:
(i) $\mathbb{R}^n$ has $d_1(x,y) = \sum_i |x_i - y_i|$ [l^1-metric]
(ii) $\mathbb{R}^n$ has $d_\infty(x,y) = max_i\{|x_i - y_i|\}$ [l^∞-metric]

Lemma: if (X,d) is a metric space and $A \subset X$ is a subspace, then $(A, d|_{A\times A})$ defines a metric space structure on A.

Proof: set $d_A(x,y) = d_X(x,y)$ for $x, y \in A$ and check the axioms! ■

Note: this means e.g. surfaces (shapes) in $\mathbb{R}^3$ are metric spaces; but here we're measuring distance "inside the bigger space", so e.g. on the sphere distances between points would *not* be being measured along great circles.

Definition: in a metric space (X,d) the *open ball* $B_\delta(x) = \{a \in X \,|\, d(x,a) < \delta\}$.

Example: $x_n \to x$ iff for each $\delta > 0$, there is N_δ s.t. $x_n \in B_\delta(x)$ for $n > N_\delta$; convergence can be described using open balls.

There are many things we can talk about with metric spaces. Two of the most basic are *boundedness* and *continuity*.

Definition: a function $f : X \to \mathbb{R}$ is bounded if $f(X)$ is a bounded subset of $\mathbb{R}$. If the function $d : X \times X \to \mathbb{R}$ defining the metric is bounded, we say (X, d) is bounded or has finite diameter

$$diam(X) = sup_{\{x,y \in X\}} d(x, y) \ < \infty$$

This seems like a very basic property, but it *isn't* topological, and is just the kind of thing this course is all about ignoring.

Definition: $f : (X, d_X) \to (Y, d_Y)$ is continuous at $x \in X$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_X(x, y) < \delta \ \Rightarrow \ d_Y(f(x), f(y)) < \varepsilon$$

Say f is continuous if it is cts at every $x \in X$.

Note: the defining criterion can be rewritten as: $\forall\, \varepsilon > 0$, $\exists \delta > 0$ s.t. $f^{-1}(B_\varepsilon(f(x)) \supset B_\delta(x)$.

Definition: we say a set $U \subset (X, d)$ is *open* if for each $u \in U$, there is some $\delta_u > 0$ such that $B_{\delta_u}(u) \subset U$. In words, a set is open if it contains an open ball about each of its points.

Lemma: this defines a topology $\mathcal{T}_{X,d}$ on X.

Proof: Obviously $\emptyset$ and X are open, and since the condition just requires existence of something pointwise, arbitrary unions of open sets must be open. We must check that if $U_1, \ldots, U_n$ are open and $\bigcap_j U_j \neq \emptyset$ then $\bigcap_j U_j$ is open. Well, if $u \in \bigcap_j U_j$ then since $u \in U_i$ there is some $\delta_i > 0$ such that $B_{\delta_i}(u) \subset U_i$. Let $\delta = min\{\delta_i : 1 \leq i \leq n\}$ and note that $\delta > 0$. Then $B_\delta(u) \subset \bigcap_j U_j$, so we're done. ■

A topology on a set X which arises in this way for some distance function d is called *metrisable*; later we'll see non-metrisable topologies.

Exercise: on any set, the topology induced by the discrete metric is the discrete topology.

Warning: just as which sets are open in a set depends on which topology you use, so they depend on which metric you use. E.g. $\{0\} \subset \mathbb{R}$ is open in the discrete (metric) topology but not in the usual (Euclidean metric) topology.

Lemma: $f : X \to Y$ is continuous as a map of metric spaces if and only if it is continuous as a map of topological spaces, for the topologies induced by the metrics.

Proof: Recall f is continuous as a map of metric spaces if for every $x \in X$ and $\varepsilon > 0$ there is $\delta > 0$ so that $f^{-1}(B_\varepsilon(f(x)) \supset B_\delta(x)$. If $V \subset Y$ is an open set, $V \in \mathcal{T}_{Y,d_Y}$, and $f(x) \in V$, by definition of "open set" there is some open ball $B_\varepsilon(f(x)) \subset V$. By continuity of f in the metric space sense, this shows there is some $\delta > 0$ so that $B_\delta(x) \subset f^{-1}(V)$. Hence, $f^{-1}(V)$ contains an open ball around each of its points, thus is an open set. The converse is analogous. ■

Note: unions of open balls need not be open balls, so it's more natural to use "open sets".

Examples: any constant function on any topological space is continuous. For if $f : X \to Y$ takes $X \mapsto y \in Y$,then $f^{-1}(V) = X$ if $y \in V$ and $f^{-1}(V) = \emptyset$ if $y \notin V$, so the only possible preimages of open sets are certainly open.

Warning: this shows that the *image* of an open set need NOT be open. For instance $f : \mathbb{R} \to \mathbb{R}$ taking the constant value 0 sends the open set $(-2, 7)$ to the set $\{0\} \subset \mathbb{R}$, which is not open.

On the other hand, if we looked at the (unique) function $f : \mathbb{R} \to \{P\}$ from $\mathbb{R}$ to a one-point space, the image of $(-2, 7)$ would be the set $\{P\}$ which IS open in $\{P\}$. So to talk about open and closed sets, you must first decide what space you're living in. The set $(0, 1] \subset \mathbb{R}$ is not open with the usual metric; but with the induced Euclidean metric on $A = (-7, 1]$ it is.

More interestingly, $x \mapsto tan(x)$ defines a continuous map $(0, \frac{\pi}{2}) \to (0, \infty)$ with a continuous inverse. So "boundedness is not topological"...

L3: Equivalent metrics We saw last time that a metric on a set gives rise to a topology on that set, and that two metric spaces can be "homeomorphic" (so there is a continuous map between them with continuous inverse) even though one is bounded and one is not. So we want to distil topological notions from metric ones.

Definition: metrics d_1 and d_2 on a set X are *equivalent* if they define the same topologies: that is, d_1-open sets are exactly the same as d_2-open sets.

Recall that $\mathcal{T}_{(X,d)}$ denotes the sets which contain an open ball around each of their points, so equivalently:

Corollary: d_1 and d_2 are equivalent if and only if every open d_1-ball contains some open d_2-ball (of possibly smaller radius), and vice-versa.

Example: the Euclidean and discrete metrics on $\mathbb{R}$ are *not* equivalent [why?].

Definition: d_1, d_2 are Lipschitz equivalent if $\exists\, k, \kappa > 0$ s.t. for all $x, y \in X$:

$$k\, d_1(x,y) \;\leq\; d_2(x,y) \;\leq\; \kappa\, d_1(x,y)$$

Note this is "symmetric", i.e. does define an equivalence relation.

Lemma: Lipschitz equivalent metrics are topologically equivalent.

Proof: from the definition,

$$B^{d_2}_{\varepsilon}(x) \subset B^{d_1}_{\varepsilon/k}(x) \qquad\qquad B^{d_1}_{\delta}(x) \subset B^{d_2}_{\delta\kappa}(x)$$

i.e. $d_2(x,y) < \varepsilon \Rightarrow d_1(x,y) < \varepsilon/k$ etc. Hence any set which contains open d_i-balls about its points contains open $d_{j\neq i}$-balls about them. ■

Example: the $l^1 = \sum |x_j - y_j|$, $l^2 = \sqrt{\sum |x_j - y_j|^2}$ and $l^\infty = max_j |x_j - y_j|$ metrics on $\mathbb{R}^n$ are Lipschitz equivalent, indeed (check!):

$$d_1 \geq d_2 \geq d_\infty \geq \frac{1}{\sqrt{n}} d_2 \geq \frac{1}{n} d_1$$

Example: let $\mathcal{C}[0,1]$ be the set of continuous real-valued functions on $[0,1]$. The metrics $l^1 = \int_t |f-g|dt$ and $l^\infty = max_t|f-g|$ are not topologically (hence not Lipschitz) equivalent.

Proof: Let 0 denote the constant function with value 0; we claim $B_1(0; l^\infty)$ is not an l^1-open set. For otherwise it contains some $B_\delta(0; l^1)$. But we can certainly find a function with l^1-norm less than δ but with supremum 2, by taking a suitable height 2 "spike" function centred on a narrow interval around $\frac{1}{2}$, of small total integral, and vanishing elsewhere. ■

Thus there are interestingly different metric topologies on the same set without invoking "pathologies" like the discrete metric. Indeed, in the above example, one of the metrics is complete and the other is not.

Definition: a sequence $(x_n) \subset (X,d)$ in a metric space is a Cauchy sequence if $\forall \varepsilon > 0$ there is $N_\varepsilon > 0$ s.t. $d(x_n, x_m) < \varepsilon$ for $n, m > N_\varepsilon$.

Definition: a metric space is *complete* if every Cauchy sequence in X converges.

Warning: this means: converges to a point of X. e.g. $(0, 1) \subset \mathbb{R}$, with the Euclidean metric, is NOT complete; since the Cauchy sequence $x_n = 1/n$ does NOT converge in $(0, 1)$, even though it does converge in $\mathbb{R}$. The completeness of $\mathbb{R}$ is one of the basic facts we'll assume.

Remark: we've already seen that, with the usual metric topologies (e.g. via a suitably tweaked tangent function), $(0, 1)$ and $\mathbb{R}$ are topologically equivalent; so *completeness is not a topological property*.

Example: $\mathcal{C}[0, 1]$ is complete with the l^∞ metric but not with the l^1-metric. If (f_n) is Cauchy in $(\mathcal{C}[0, 1], l^\infty)$ then CHECK
(i) for each $x \in [0, 1]$, $(f_n(x))$ is Cauchy in $\mathbb{R}$, so converges to some $F(x)$;
(ii) F is continuous as a function of x; and
(iii) $f_n \to F$ in l^∞.
Why doesn't a similar argument apply for l^1 ?

There are some beautiful theorems that apply only to metric, and not topological spaces, and more specifically to complete metric spaces. For instance:

Theorem: Suppose (X, d) is a complete metric space and $f : X \to X$ is a contraction, i.e. $\exists\, \kappa < 1$ s.t. for all x, y, $d(f(x), f(y)) \leq \kappa d(x, y)$. Then f has a unique fixed point.

We'll use an easy (but important!) Lemma:

Lemma: if $x_n \to x$ and $f : X \to Y$ is continuous then $f(x_n) \to f(x)$.

Proof: Since f is continuous, given ε there is $\delta > 0$ s.t. $f^{-1} B_\varepsilon(f(x)) \supset B_\delta(x)$. Hence given $\varepsilon > 0$ there is N_ε s.t. $x_n \in B_\delta(x)$ for $n > N_\varepsilon$ and hence $f(x_n) \in B_\varepsilon(f(x))$. But this, by definition, means $f(x_n) \to f(x)$. ■

Remark: a variation of this argument says a function of metric spaces is continuous *iff* it preserves all limits of sequences in this sense.

Proof of theorem: First note f is continuous! Choose $x_0 \in X$ and set $x_n = f(x_{n-1})$. We claim (x_n) is a Cauchy sequence. Given this, $x_n \to x_\infty$ converges by completeness of X. Then $f(x_n) = x_{n+1} \to f(x_\infty) = x_\infty$, so we have a fixed point. Moreover, if there are 2 fixed points, say x, y, then

$$d(x,y) = d(f(x), f(y)) \leq \kappa\, d(x,y) \Rightarrow d(x,y) = 0$$

so the limit is unique.

To see (x_n) is Cauchy, note that by induction $d(x_r, x_{r-1}) \leq \kappa^{r-1} d(x_1, x_0)$. So if $m > n$

$$d(x_m, x_{n-1}) \leq \sum_{j=0}^{m-n} d(x_{m-j}, x_{m-j-1})$$

$$\leq (\kappa^{m-1} + \kappa^{m-2} + \cdots + \kappa^{n-1})\, d(x_1, x_0).$$

Write this as $\kappa^{n-1}\frac{(1-\kappa^{m-n})}{1-\kappa}\, d(x_1, x_0)$, which is bounded above by $\frac{\kappa^n}{1-\kappa} d(x_1, x_0)$. Now this goes to zero as $n \to \infty$ as $\kappa < 1$, and this implies that the sequence is Cauchy, as required. ■

This "contraction mapping theorem" is both important and beautiful; it implies existence of solutions to ordinary differential equations, underlies the "inverse function theorem" in analysis and geometry... At a more mundane level it gives an obstruction to completeness and hence of *Lipschitz* equivalence of some given metric to one that *is* known to be complete.

Example: the map $f : x \mapsto x + \frac{1}{x}$ on the complete metric space $[1, \infty)$ satsfies

$$\forall\, x \neq y \qquad |f(x) - f(y)| < |x - y|$$

but has no fixed point; so the existence of a "contraction factor" κ strictly less than 1 is critical for the theorem, as well as the proof!

Remark: there is a curious converse. Given $f : X \to X$ a map of a *set* s.t. each iterate $f^n = f \circ \cdots \circ f$ has a unique fixed point, and if $\kappa \in (0, 1)$, there is some complete metric topology on X s.t. f is a contraction in that topology, of factor κ. [We won't prove this, and it doesn't have many applications.]

L4: Closed sets In looking at metric spaces, an important role is played by convergence of sequences; indeed, this can be used to characterise continuity of a function. There is a (weak) analogue of convergence of sequences in a general topological space, but first we need some more language.

Definition: Let $(X, \mathcal{T}_X)$ be a topological space and $V \subset X$ a subset. If the complement $X \backslash V \in \mathcal{T}_X$ is open, we say V is *closed*.

The closed sets satisfy:
(i) $\emptyset$ and X are closed sets;
(ii) arbitrary intersections of closed sets are closed; V_j closed for $j \in J \Rightarrow \bigcap_j V_j$ closed;
(iii) finite unions of closed sets are closed; V_j closed for $1 \leq j \leq n \Rightarrow \bigcap_{j=1}^n V_j$ closed.

These statements are "dual" to the axioms for open sets under taking complements, so follow from de Morgan's laws (check!). A topology is completely determined by saying which sets are closed, since this tells you which are open.

Warning: in general, not every subset of a topological space is either open or closed! For instance in $(\mathbb{R}, \mathcal{T}_{d_{eucl}})$, the set $[1, 2)$ is neither open nor closed.

Example: in the discrete topology, every set is closed (since every set is open).

Warning: just as with open sets, it matters exactly which set you're working in and in which topology to say if something is closed. So $[0, 1)$ is closed in $(-3, 1)$ but not closed in $\mathbb{R}$, with the usual topology; and $(0, 1)$ is closed in $\mathbb{R}$ with the discrete topology but not with the usual topology.

Example: $\bigcup_j [\frac{1}{j}, 1] = (0, 1] \subset \mathbb{R}$, so an infinite union of closed sets need not be closed, just as an infinite intersection of open sets need not be open.

Exercise: $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous iff the preimages of closed sets are closed.

Definition: if $(X, \mathcal{T}_X)$ is a space and $A \subset X$, then x is a limit point of A if every open set $U \ni x$ contains some point of A other than x. [Note: x may or may not be a point of A in this definition.]

Example: in a metric space, x is a limit point of A iff for all $\varepsilon > 0$, $(B_\varepsilon(x) \cap A)\backslash\{x\} \neq \emptyset$.

Example: in $\mathbb{R}$ with the usual topology, every point is a limit point of the rationals $\mathbb{Q}$; so the set of limit points has larger cardinality than the original set.

Example: in the discrete topology (metric), no point is a limit point of any set. For given x and $\delta < 1$, $B_\delta(x) = \{x\}$. In the indiscrete topology, x is a limit point of a nonempty subset A iff $A \neq \{x\}$. For if $A \subset X$, and $x \in X$, then the only open set containing x is X, and $(X \cap A)\backslash\{x\} = A\backslash\{x\}$.

Definition: the *closure* $Cl(A)$ of a set $A \subset X$ is the union of A and all of its limit points.

Note a point $x \in Cl(A)$ iff every open set containing x meets A.

Lemma: This operation satisfies:
(i) $H \subset K \Rightarrow Cl(H) \subset Cl(K)$;
(ii) $Cl(ClH) = Cl(H)$;
(iii) H is closed iff $H = Cl(H)$;
(iv) $Cl(H)$ is closed.

Proof: (i) is obvious from the definition. If $x \in Cl((Cl(H))$ and $U \ni x$ is open then there is some point $y \in Cl(H) \cap U \backslash \{x\}$. So U is an open set containing $y \in Cl(H)$, hence $U \cap H \neq \emptyset$; so every open U containing x meets H, so $x \in Cl(H)$, proving (ii).

For (iii), suppose H is closed. If $x \in X \backslash H$, then $X \backslash H$ is an open set containing x and not meeting H, so $x \notin Cl(H)$. So $Cl(H) \subset H$, so they co-incide. Conversely, if $Cl(H) = H$ and $x \in X \backslash H$ then $\exists$ open $U_x \ni x$ not meeting H, so this open set lies in $X \backslash H$; so $X \backslash H$ is a union of open sets U_x, hence is open. This proves (iii); now (iv) follows from (ii)+(iii). ■

Corollary: $Cl(H)$ is the smallest closed set containing H, i.e. the intersection of all the closed sets containing H.

Proof: if V is closed and $H \subset V$ then $Cl(H) \subset Cl(V) = V$. ■

Definition: if $Cl(A) = X$ we say A is *dense* (or "everywhere dense") in X.

Example: the rationals are dense in the reals; so are the irrationals; but the natural numbers are not dense in $\mathbb{R}$.

Example: $[0, 1) \cup (1, 2]$ is dense in $[0, 2]$.

Similarly, we can talk about the "largest open subset" of A, called the interior $Int(A)$ of A, defined as the union of all the open sets of X contained inside A. If $Int(Cl(A)) = \emptyset$ then A is said to be "nowhere dense". Note: we take the closure before we take the interior; so the rationals are NOT nowhere dense, even though $Int(\mathbb{Q}) = \emptyset$.

By analogy with metric spaces, we might say:

Definition: a sequence $x_n \to x$ if for every open $U \ni x$ there is $N = N_U$ s.t. $x_n \in U$ for $n > N_U$.

In particular, the limit x of a sequence in this sense is a limit point of the set $\{x_n \,|\, n \in \mathbb{N}\}$ (the converse is not true: not all limit points of the set need be limits of the sequence). However, this notion of convergence is rather weak: limits are far from being unique.

Example: in an indiscrete space X, every sequence in X converges to every point of X.

Next time we'll talk about a condition which guarantees uniqueness of limits even in this more general topological setting. Whilst we're collecting definitions, a helpful and related one to open/closed sets is:

Definition: a *neighbourhood* (nhood) of a point $x \in X$ is any set $A \ni x$ which contains some open set U containing x, so $A \supset U \ni x$.

For our examples and illustrations, we often take intervals in $\mathbb{R}$. Any subset of a metric space inherits a metric, hence a topology; in fact this is true without metrics being present.

Definition: if $(X, \mathcal{T}_X)$ is a topological space and $A \subset X$ is a subset, there is an *induced subspace topology* on A, with by definition

$$\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}_X\}.$$

Lemma: $\mathcal{T}_A$ satisfies the axioms to define a topology on the set A.

Proof: $A = A \cap X$ and $\emptyset = \emptyset \cap X$, so these are open. Now $A \cap \bigcup_j U_j = \bigcup_j (A \cap U_j)$, similarly for finite intersections, shows that $\mathcal{T}_A$ inherits the necessary properties from $\mathcal{T}_X$. $\blacksquare$

Example: $[0, 1)$ is open in $[0, \infty)$ since it equals $[0, \infty) \cap (-1, 1)$ and this is of the form $A \cap U$ with $U \in \mathcal{T}_{\mathbb{R}}$. The closure of $(0, 1) \subset (0, \infty)$ is $(0, 1]$, although its closure in $\mathbb{R}$ is $[0, 1]$.

Example: if $A \subset (X, d)$, $\mathcal{T}_A = \mathcal{T}_{A, d|_{A \times A}}$.

L5: Products and quotients We saw that if $(X, \mathcal{T}_X)$ is a topological space and $A \subset X$ is any subset, then we get an induced topology $(A, \mathcal{T}_A = A \cap \mathcal{T}_X)$. There are other ways of building news spaces out of old. It's helpful to introduce the idea of a "basis" for a topology.

Definition: if $(X, \mathcal{T}_X)$ is a topological space, a *basis* for the topology is any subset $\mathcal{B} \subset \mathcal{T}_X$ s.t. every $U \in \mathcal{T}_X$ is a union of elements of $\mathcal{B}$.

$\mathcal{B}$ may not itself be preserved by taking arbitrary unions, but if we "close it up" under this operation by force, we get all of $\mathcal{T}_X$ (and no more than $\mathcal{T}_X$ – why?).

Warning: "being closed under an operation" is a property of a collection of sets, and has nothing to do with "closedness" of a single set in the sense of "having open complement"...

Example: if (X, d_X) is a metric space, the open balls $B_\delta(x)$ (varying over $x \in X$ and $\delta \in \mathbb{R}_+$) form a basis.

Example: a more economical basis for $\mathbb{R}^2$ is $B_q(x)$ where $x \in \mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}^2$ and $q \in \mathbb{Q}_+$; indeed this is a countable basis. [When the topological space you're working on also happens to have the structure of a vector space, don't confuse "basis for a topology" and "linearly independent spanning set".]

Example: in the Zariski topology on $\mathbb{C}$ every (non-full) closed set is a finite union of points, but there is *no* obvious basis for open sets.

Definition: if $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are topological spaces, the *product topology* on $X \times Y$ is the topology with *basis* the open sets of the form $U_X \times U_Y$ with $U_X \in \mathcal{T}_X$ and $U_Y \in \mathcal{T}_Y$.

Thus: every open set in $X \times Y$ is a *union* of sets of the form $U_X \times U_Y$; an arbitrary open set will NOT have this form in general.

Example: the disc $\{x \in \mathbb{R}^2 \mid \|x\|_{eucl} < 1\}$ is open, but not globally $A \times B \subset \mathbb{R} \times \mathbb{R}$.

Note: since we only close a basis up under unions, it must *already* be closed under finite intersections. To see our definition of the product topology makes sense, check

$$(U_X \times U_Y) \cap (V_X \times V_Y) = (U_X \cap V_X) \times (U_Y \times V_Y)$$

Inductively, finite intersections of sets of the form (open in X)$\times$(open in Y) are in the basis.

To reiterate: from the definition, if $w \in W \subset X \times Y$ with W open, there are open $U_X \subset X$ and $U_Y \subset Y$ s.t. $w \in U_X \times U_Y \subset W$.

Lemma: (i) the projection maps $\pi_i : X_1 \times X_2 \to X_i$ are continuous. (ii) $f : Z \to X_1 \times X_2$ is continuous iff the projections $\pi_i \circ f$ are continuous.

Proof: (i) if $U \subset X_1$ is open, $\pi_1^{-1}(U) = U \times X_2$ is a basic open set for $\mathcal{T}_{X_1 \times X_2}$. (ii) One direction is "composition of cts functions". Suppose $\pi_i \circ f$ are both cts and $U_i \in \mathcal{T}_{X_i}$, then $f^{-1}(U_1 \times U_2) = (\pi_1 \circ f)^{-1}(U_1) \cap (\pi_2 \circ f)^{-1}(U_2)$ is open. So inverse images of all basic open sets, and hence all open sets, are open. ■

We've written out a "minimalist" proof: check that all the steps in the above, and proofs like it, really make sense to you!

Exercise: check that the product topology on $\mathbb{R}^n$ induced by the Euclidean metric on $\mathbb{R}$ is the Euclidean metric topology of $\mathbb{R}^n$.

Example: $f : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (xy, sin(x+y))$ is continuous.

Proof: it suffices to check the projections are continuous. Since composition of continuous functions is continuous, we reduce to checking sin, addition and multiplication are continuous, which is straightforward.

Example: the *graph* of a function $f : X \to Y$ is

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}$$

Then $\Gamma_f \cong X$ (they are homeomorphic), via the maps $X \to \Gamma_f,\ x \mapsto (x, f(x))$ and the projection map (to the first factor) $\Gamma_f \to X$. [For we now know these inverse functions *are* cts.]

As well as taking products, we can also take *quotients* of topological spaces; this is a bit more subtle.

Definition: Let $(X, \mathcal{T}_X)$ be a topological space and suppose $p : X \to Y$ is a *surjective* map of sets. The *quotient topology* on Y is

$$\mathcal{T}_{Y,\,quot(p)} = \mathcal{T}_Y = \{V \subset Y \,|\, p^{-1}(V) \in \mathcal{T}_X\}$$

Lemma: this does define a topology.

Proof: Since $\emptyset \in \mathcal{T}_X$ and $X \in \mathcal{T}_X$ and since p is assumed to be onto, we see $\emptyset \in \mathcal{T}_Y$ and $Y \in \mathcal{T}_Y$. If $V_i \in \mathcal{T}_Y$ for $i \in I$, say $p^{-1}V_i = U_i \in \mathcal{T}_X$, then

$$\bigcup_i U_i = \bigcup_i p^{-1}(V_i) = p^{-1}(\bigcup_i V_i)$$

is open (using the LHS and the fact that $\mathcal{T}_X$ is a topology); so the RHS is open, so (by definition of $\mathcal{T}_Y$) $\bigcup_i V_i \in \mathcal{T}_Y$; hence $\mathcal{T}_Y$ is closed under arbitrary unions. Similarly, if V_i, $1 \leq i \leq n$ are in $\mathcal{T}_Y$, then $p^{-1}(\bigcap_j V_j) = \bigcap_j p^{-1}(V_j)$ and this is open in X [check!], so $\mathcal{T}_Y$ is closed under finite intersections, as required. ■

Remarks:
(i) by definition, the quotient map $p : X \to Y$ is continuous. [Think of p as "projection".]
(ii) if $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is any continuous surjective map of topological spaces, then $\mathcal{T}_{Y,\,quot(f)} \supset \mathcal{T}_Y$. To see this, note that if $W \in \mathcal{T}_Y$, then $f^{-1}(W) \in \mathcal{T}_X$ since f is continuous; but this means precisely that $W \in \mathcal{T}_{Y,\,quot(f)}$. Thus, the quotient topology contains "as many open sets as possible" if we want the projection map to be continuous.

The key feature of the quotient topology is:

Lemma: given a triangle of commuting maps of topological spaces

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & Y \\ & T \arrow[ur, "f'"'] & \end{tikzcd}$$

where p induces the quotient topology on T, then f is continuous iff f' is continuous.

Definition: if $p : X \to Y$ is any map of spaces and $\mathcal{T}_Y = \mathcal{T}_{Y,\,quot(p)}$ we call p a *quotient map*.

Note: every map f' from a quotient space induces some map $f = f' \circ p$ from the original space, so to understand whether a map to or from a quotient space is continuous, you can always work "upstairs" with the original space – which is usually easier to understand (e.g. its open sets are more explicit).

Proof: if f' is continuous, $f = f' \circ p$ is obviously continuous (since we already know all quotient maps are continuous, and p is a quotient map). If f is continuous, and $V \in \mathcal{T}_Y$, then $f^{-1}(V) = (f' \circ p)^{-1}(V)$ is open in X. But this is $p^{-1}(f')^{-1}(V)$. Since by assumption T has a quotient topology from p, a set of the shape $p^{-1}(U)$ is open in X if and only if U is open in T, so we deduce $(f')^{-1}(V) \in \mathcal{T}_{T,\,quot(p)}$. Thus, we have shown whenever $V \in \mathcal{T}_Y$, $(f')^{-1}(V) \in \mathcal{T}_T$. Hence f' is continuous. ■

L6: Hausdorff spaces We introduced the idea of quotient spaces, but didn't yet look at any examples. The most important class is where we quotient a space by an equivalence relation, "gluing" parts of the space together.

Definition: an equivalence relation on a topological space $(X, \mathcal{T}_X)$ is a subset $R \subset X \times X$ satisfying
(i) the diagonal $\Delta \subset R$;
(ii) if $(x, y) \in R$ then $(y, x) \in R$ and
(iii) $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$.

The data of an equivalence relation is exactly the same thing as a decomposition of X into disjoint subsets; but we can now give the set $X/\sim$ of equivalence classes a natural topology. For there is a surjective map of sets $X \to X/\sim$ and this induces the quotient topology on the image.

Example: the quotient of the strip

$$\{(x, y) \in \mathbb{R}^2 \,|\, 0 \leq x \leq 1\}$$

by the relation $(0, y) \sim (1, y)$ is an infinite cylinder.

Exercise: describe an orientable surface with g holes as a quotient of a $4g$-gon, by gluing edges in a similar way. [Start with $g = 1$.]

Definition: if $A \subset X$ is a subset, the quotient space obtained by collapsing A to a point and leaving the rest of X alone [i.e. imposing the equivalence relation $x \sim y \Leftrightarrow \{x, y\} \subset A$] is written X/A.

Warning: if $X = G$ is a group and $A = H \leq G$ is a subgroup, $G/H = \{gH : g \in G\}$ denotes the set of cosets...

Example: $D^2/\partial D^2 = S^2$; the quotient of the closed disc by its boundary is a sphere.

But what do we mean by saying one space "is" another? Later we'll have more technology...

Example: the quotient $\mathbb{R}/\mathbb{Z}$ with the topology induced from $(\mathbb{R}, \mathcal{T}_{eucl})$ is homeomorphic to the unit circle $S^1 \subset \mathbb{R}^2$ with the subspace topology.

Proof: Define a map $\mathbb{R} \to S^1$ by taking $t \mapsto (cos\, 2\pi t, sin\, 2\pi t)$. As a map of sets, this is constant on $\mathbb{Z}$, hence descends to a map on the set $\mathbb{R}/\mathbb{Z}$ of equivalence classes; moreover, this latter map is clearly a bijection.

The map $\mathbb{R} \to S^1$ is continuous, since both its projections to the factors are, so by definition of the quotient topology the map $\mathbb{R}/\mathbb{Z} \to S^1$ is continuous. Now a set in $\mathbb{R}/Z$ is open iff its preimage in $\mathbb{R}$ is open. Such an open set is a union of open intervals; indeed the open intervals of length $< \delta \leq \frac{1}{2}$ form a basis for the topology $\mathcal{T}_{eucl}$. The image of such a short open interval under $\mathbb{R} \to S^1$ is obviously open in the subspace topology of S^1, so all open sets in $\mathbb{R}$ map to open sets in S^1. Hence, the map $\mathbb{R}/\mathbb{Z} \to S^1$ is continuous, bijective and *takes open sets to open sets*. This means it's a homeomorphism (exercise!). ■

Quotients are more subtle than products, subspaces etc. For instance, consider the quotient X of two parallel lines $\{x = 0\} \amalg \{x = 1\} \subset \mathbb{R}^2$ by the relation which identifies $(0, y) \sim (1, y)$ whenever $y \neq 0$. The result is one line, but with a single point "doubled up".

Recall that a sequence $(x_n) \subset (X, \mathcal{T}_X)$ converges to $a \in X$ if for every open set $U \ni a$, there is some N_U so that $x_n \in U$ for all $n \geq N_U$. Our line with one "fat point" is obtained from a nice space by imposing a simple equivalence relation, yet:

Example: the sequence $(0, \frac{1}{n}) \subset X$ has two distinct limits, namely $(0, 0)$ and $(1, 0)$.

Proof: in the quotient topology, the open sets about $(0, 0)$ are just sets which pull back to open sets upstairs. Any such contains $(0, \frac{1}{n})$ for all $n \gg 0$. But $(0, \frac{1}{n}) \sim (1, \frac{1}{n})$, so all these points also lie inside any open neighbourhood of $(1, 0) \in X$. Hence the sequence is also converging to this (distinct) point. $\blacksquare$

Often equivalence relations come from geometry (quotients by groups of symmetries), and we want to know what equivalence class a sequence converges to, so this behaviour is bad – not so much that the quotient space is strange, but that this particular good property (uniqueness of limits of sequences, which holds for subsets of $\mathbb{R}^2$) is explicitly not inherited by its Siamese sibling.

Roughly, Hausdorff spaces are the ones which don't have this problem.

Definition: $(X, \mathcal{T}_X)$ is Hausdorff if for every pair of distinct points $x \neq y \in X$ we can find disjoint open sets $U_x \ni x$ and $U_y \ni y$.

So the points can be "housed off" from one another by open sets.

Example: any metric space is Hausdorff. If $x \neq y \in (X, d)$ then the open balls $B_r(x)$ and $B_r(y)$ are disjoint for any $0 < r < d(x, y)/2$.

Example: the indiscrete topology on a set X is Hausdorff iff X has at most one point. The Zariski topology on $\mathbb{C}$ is not Hausdorff since every two non-empty open sets meet non-trivially.

Lemma: if $(X, \mathcal{T}_X)$ is Hausdorff, a sequence $(x_n) \subset X$ has at most one limit.

Proof: suppose $x_n \to a_1$ and $x_n \to a_2$ for distinct points $a_1 \neq a_2$. The Hausdorff hypothesis says there are open sets $U_i \ni a_i$ with $U_1 \cap U_2 = \emptyset$. But by the definition of convergence in a general topological space, there are N_1 and N_2 s.t. $x_n \in U_i$ for $n > N_i$; for $n \geq max\{N_1, N_2\}$ this is absurd. ■

Being Hausdorff is *not* inherited by all quotients, but is by some:

Proposition: Let $(X, \mathcal{T}_X)$ be a *compact* Hausdorff topological space. If $R \subset X \times X$ is a *closed* subspace, the quotient $X/\sim$ is also Hausdorff. In particular, if $A \subset X$ is closed then X/A is Hausdorff.

Let's end with an illustrative (but perhaps quite hard) example.

Example: the group $\mathbb{R}^*$ acts on $\mathbb{R}^2$ via

$$\lambda \cdot (x, y) = (\lambda x, \lambda y)$$

What does the quotient space look like? If $(x, y) \neq 0$, then the orbit is a copy of $\mathbb{R}^*$ – it's exactly the non-zero points of the unique line $L_{(x,y)}$ through the origin in $\mathbb{R}^2$ which contains (x, y). The set of such lines is (visually!) parametrised by the unit circle in $\mathbb{R}^2$, with antipodal points identified. Topologically the quotient space is still homeomorphic to a circle, the "real projective line" $(\mathbb{R}^2 \backslash \{0\})/\mathbb{R}^*$.

But the orbit of $0 \in \mathbb{R}^2$ is just 0; and this orbit is in the closure in $\mathbb{R}^2$ of every other $\mathbb{R}^*$-orbit. This means the quotient $\mathbb{R}^2/\mathbb{R}^*$ is set-theoretically the union of a circle and a disjoint point $\{\star\}$, and if q is a point of the circle then $Cl(\{q\}) = \{q, \star\}$. In the quotient topology, not all points are closed (which means it's not Hausdorff – why?). $\blacksquare$

L7: Compactness The most important notion in topology is probably compactness.

Definition: a topological space $(X, \mathcal{T}_X)$ is *compact* if every open cover of X admits a finite subcover.

i.e. if $\{U_j \mid j \in J\} \subset \mathcal{T}_X$ have the property that $\bigcup_{j \in J} U_j = X$, then for some $\{j_1, \ldots, j_n\} \subset J$ we have $\bigcup_{i=1}^n U_{j_i} = X$.

Example: a discrete space is compact if and only if it is finite. An indiscrete space is always compact.

Example: with the Zariski topology, $(\mathbb{C}\,, \mathcal{T}_{Zariski})$ is compact. For the non-empty open sets are complements of finite sets. Hence, if $\{U_j \mid j \in J\}$ are open, with finite complements V_j, then $\bigcup_j U_j = \mathbb{C} \setminus \bigcap V_j$. Now if the intersection of the finite sets V_j is empty, then for some finite subcollection V_{j_i} the intersection $\bigcap_{i=1}^n V_{j_i}$ is already empty. This implies all open covers have finite subcovers.

Remark: since it is a property of the set $\mathcal{T}_X$ of open sets, the property of being compact is obviously topological, i.e. preserved under homeomorphisms. Actually, we have the stronger:

Lemma: the continuous image of a compact space is compact.

Proof: Suppose $(X, \mathcal{T}_X)$ is compact and $f : X \to Y$ is any continuous map of topological spaces. We claim $f(X) \subset Y$ is compact (with the subspace topology from Y). Let $\{V_j \mid j \in J\}$ be an open cover of $f(X)$. By definition of the subspace topology, $V_j = W_j \cap f(X)$ for some $W_j \in \mathcal{T}_Y$. Since f is continuous, $f^{-1}(V_j) = f^{-1}(f(X) \cap W_j) = f^{-1}W_j$ is open for each j, and this defines an open cover of X. Hence there is a finite subcover $X = \bigcup_{i=1}^n f^{-1}(W_{j_i})$. But then $f(X) = \bigcup_{i=1}^n V_{j_i}$, so our arbitrary open cover of $f(X)$ had a finite subcover. ■

Contrast: being Hausdorff is a topological property, but does not satisfy the analogous Lemma.

Warning: being compact does not say the space X has *some* finite open cover – this is always true, taking the single open set X itself – we insist that *all* open covers have finite subcovers.

Example: a compact metric space is bounded.

Proof: Fix $x \in (X, d)$ (assumed non-empty). The open balls $B_x(n)$ of radius n about x form (as n varies) an open cover of X, since certainly every $y \in X$ satisfies $d(x, y) < n$ for *some* $n(y)$. Compactness gives a finite subcover, so $X = B_x(N)$ for some fixed N, which says for every $y, z \in X$, we have $d(y, z) \leq 2N$ (using the triangle inequality at the last stage). ■

Corollary: if $(X, \mathcal{T}_X)$ is any compact topological space, every continuous real-valued function $f : X \to \mathbb{R}$ is bounded.

Proof: the image $f(X) \subset (\mathbb{R}, \mathcal{T}_{eucl})$ is compact, hence bounded. ■

The following are often helpful. Contrast (ii) below with what happens in an indiscrete space.

Lemma: (i) A closed subset of a compact space is compact. (ii) A compact subset of a Hausdorff space is closed.

Exercise/Corollary: a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. [This is very helpful for understanding quotient spaces, cf. last time!]

Proof:
(i) Let $(X, \mathcal{T}_X)$ be compact and $Z \subset X$ be closed. Let $V_j = Z \cap W_j$ be an open cover of Z, with $W_j \in \mathcal{T}_X$. Then the collection of sets $\{X \backslash Z, W_j \,|\, j \in J\}$ forms an open cover of X, precisely since the W_j cover Z (and noting that $X \backslash Z$ is open exactly since Z is closed). This open cover of X has a finite subcover, say $\{X \backslash Z, W_1, \ldots, W_n\}$. Then $\{V_1, \ldots, V_n\}$ cover Z, so Z is compact.

(ii) Let $(X, \mathcal{T}_X)$ be Hausdorff and $Z \subset X$ be compact. Fix some $x \in X \backslash Z$. We want to show $X \backslash Z$ is open, so it suffices to show it contains an open set U_x containing x; for then we'll have $X \backslash Z = \bigcup_{x \in X \backslash Z} U_x$.

For each $z \in Z$, pick disjoint open sets $V_{zx} \ni z$ and $W_{zx} \ni x$, which is possible by the Hausdorff property. As we vary over z, we get an open cover $Z \subset \bigcup_{z \in Z} V_{zx}$, so this open cover of Z has a finite subcover by compactness of Z. Call this $\{V_{z_1}, \ldots, V_{z_n}\}$. Then consider the neighbourhood $U_x = \bigcap_{j=1}^{n} W_{z_j x}$ of x. This is a finite intersection of opens so open; we claim it lies in $X \backslash Z$. For if $z \in U_x \cap Z$, then $z \in W_{z_j x}$ for every j, but that means $z \notin V_{z_j x}$ for each j (by the original disjointness), but this is a contradiction since this finite collection covers Z. Thus $U_x \subset X \backslash Z$, so $X \backslash Z$ is indeed open. ■

Note: these results are very important – most arguments with compactness reduce to these lemmas!

Corollary: a compact Hausdorff space is *normal*; every two disjoint closed subsets can be separated by disjoint open sets. i.e. if $V, V' \subset X$ are closed and disjoint, $\exists$ open $U \supset V$ and $U' \supset V'$ which are still disjoint.

Proof: Fix $v' \in V'$ and for each $v \in V$ choose disjoint opens $U_{vv'} \ni v$ and $U'_{vv'} \ni v'$ (Hausdorff). $V \subset X$ is closed in a compact space, so compact, so the open cover $\{U_{vv'} \,|\, v \in V\}$ has a finite subcover $\{U_{v_1v'}, \ldots, U_{v_nv'}\}$; let $U_{v'} = \bigcup_{j=1}^{n} U_{v_jv'}$. Set also $U'_{v'} = \bigcap_{j=1}^{n} U'_{v_jv'}$. This gives us for each $v' \in V'$ an open set $U_{v'} \supset V$ and a disjoint open set $U'_{v'} \ni v'$.

Now the $U'_{v'}$ form an open cover of the (compact) V'; take a finite subcover $\{U'_{v'_1}, \ldots, U'_{v'_m}\}$. Now set $U = \bigcap_{k=1}^{m} U_{v'_k}$ (a finite intersection of open sets which each contain V, hence an open set containing V) and set $U' = \bigcup_{k=1}^{m} U'_{v'_k}$ (an open set containing V' and disjoint from U since $U_{v'_k} \cap U'_{v'_k} = \emptyset$ for all k). $\blacksquare$

Aside: Normal spaces have the following amazing property:

Theorem: Let $(X, \mathcal{T}_X)$ be normal and let A and B be disjoint closed sets in X. There is a continuous real-valued function $f : X \to (\mathbb{R}\,, \mathcal{T}_{eucl})$ s.t. $f(A) = 0$ and $f(B) = 1$.

We won't prove this, but it's remarkable since it "creates" the real numbers where the input just concerns these very abstract topological spaces and distinguished collections of sets etc.

Remark: metric spaces are normal, even if they are not compact (exercise). A nasty fact is that products of normal spaces need not be normal (don't worry about an explicit counterexample!); but a good thing to do would be to check that products of Hausdorff spaces are Hausdorff (indeed $X \times Y$ is Hausdorff if and only if both X and Y are).

L8: Compactness continued We introduced compactness, but haven't yet seen many examples.

Proposition: a closed bounded interval $[a, b]$ in $(\mathbb{R}, \mathcal{T}_{eucl})$ is compact.

Proof: take an open cover $\{U_i \,|\, i \in I\}$ of $[a, b]$ and consider the set A of those $x \in [a, b]$ s.t. $[a, x]$ has a finite subcover. We want to show b lies in this set. Now if $a \in U_i$ then $[a, a + \delta) \subset U_i$ for some $\delta > 0$ so $c = supA > a$. Say $c \in U_j$, and suppose for contradiction $c < b$. Well $(c - \varepsilon, c + \varepsilon) \subset U_j$ for some $\varepsilon > 0$, and by definition, $[a, c - \varepsilon/2]$ has a finite subcover by finitely many U_i. Now use this finite subcover, together with U_j itself, to see that $c + \varepsilon/2 \in A$; but this contradicts $c = supA$. Hence $c = b$.

Now $b \in U_k$ say, so $(b - \delta, b] \subset U_k$, and so $[a, b - \delta/2]$ has a finite subcover (definition of A and fact that $b = supA$); now argue as before! ■

Corollary: a continuous real-valued function on a (nonempty) compact space X is bounded and attains its bounds.

Proof: if $a = \inf f(X)$ and $b = \sup f(X)$, which exist by completeness of $\mathbb{R}$, then these belong to $cl(f(X))$ (this is very basic: why?). But $f(X)$ is compact. ■

To get a feel for compact sets in higher-dimensional Euclidean spaces, we need to talk about compactness of products.

Theorem: The product of two compact topological spaces is compact.

Corollary (Heine-Borel theorem): a subspace of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Proof: if A is compact, we know from before it's closed and bounded. If it's closed and bounded, it lies inside some cube $[a, b]^n$; now it's a closed set in a compact space. ■

Proof of Theorem: let X and Y be compact, and take an open cover $U_i, i \in I$ of the product $X \times Y$. We need a finite subcover. Note the (homeomorphic to Y) copies $\{x\} \times Y \subset X \times Y$ are compact.

Fix $x \in X$. If $(x, y) \in U_i$, there is some open $x \in A_y \subset X$ and $y \in B_y \subset Y$ s.t. $(x, y) \in A_y \times B_y \subset U_i$ (definition of product topology); then $\{B_y\}$ form an open cover of $\{x\} \times Y$. Take a finite subcover, indexed by $y_1, \ldots, y_n$, and let $A_x = \bigcap_{j=1}^{n} A_{y_j}$, an open nhood of x. The strips $A_x \times B_{y_j}$ cover $A_x \times Y$, and all these strips belong to some U_i. Thus, for each $x \in X$, there is an open nhood $A_x \ni x$ s.t. $A_x \times Y$ is covered by finitely many U_i.

For each $x \in X$, choose an open nhood A_x with this property, and then take a finite subcover $A_{x_1}, \ldots, A_{x_m}$ of this open cover. Then $X \times Y$ is covered by finitely many strips $A_{x_i} \times Y$, each of which is covered by finitely many U_j, so altogether we have a finite subcover. ■

Remark: this gives us a large collection of compact sets, but we're a long way from describing all of them – indeed, there's no very nice description of the general compact subset of $\mathbb{R}$, even (e.g. there are uncountably many homeomorphism types of such things). Here's a typical "wacky" compact subset of $\mathbb{R}$.

Example: take the unit interval $[0, 1]$, remove the open middle third, remove the open middle thirds of the resulting two intervals, and iterate onwards. Explicitly, at stage n we have a set A_n made up of 2^{n-1} closed intervals, and we set $C = \bigcap_n A_n$, the *Cantor set*. This is closed in a compact space, so compact. It's famously uncountable, fractal... It can be described as the real numbers in $[0, 1]$ which have a base 3 expansion omitting the digit 1.

Theorem: (hard) Every compact metric space is the continuous image of the Cantor set. In particular, a compact metric space has cardinality at most that of $\mathbb{R}$.

Finally, before moving on, we should return to the claims at the end of Lecture 6 about compactness helping in establish the Hausdorff property for quotients. We begin with some helpful lemmas.

Lemma: (i) if $f : X \to Y$ is continuous and Y is Hausdorff, then $A = \{(x_1, x_2) \,|\, f(x_1) = f(x_2)\} \subset X \times X$ is closed. (ii) if f is open and surjective, and A is closed, then Y is Hausdorff.

Proof: (i) Suppose f is continuous and Y is Hausdorff. Then if $(x_1, x_2) \notin A$, the points $f(x_i) \in Y$ are distinct, so can be separated by open sets U_i; and then $f^{-1}(U_1) \times f^{-1}(U_2)$ is an open nhood of (x_1, x_2) not meeting A. So the complement of A is open, and A is closed. (ii) if $f(x_1)$ and $f(x_2)$ are distinct in Y, then $(x_1, x_2) \notin A$; so we can find open nhoods U_i of x_i such that $(U_1 \times U_2) \cap A = \emptyset$. Since f is open, $f(U_i)$ form disjoint open nhoods of $f(x_i)$. ■

Now suppose we are taking the quotient of a compact Hausdorff space X by a closed equivalence relation $R \subset X \times X$; so the quotient map $f : X \to Y = X/\sim$ satisfies the hypothesis, $R = \{(x_1, x_2) \,|\, f(x_1) = f(x_2)\}$ is indeed closed. To see the quotient is Hausdorff it is enough to see the map is an open map, or equivalently, that it is a closed map, so takes closed sets to closed sets. [Note: openness and closedness are only equivalent since we're dealing with a surjective *quotient* map and not just a surjective continuous map: why?]

If $V \subset X$ is closed, we want $f(V)$ closed, so (we're using the quotient topology on Y) we want $f^{-1}(f(V)) = \{x \in X \,|\, \exists\, v \in V \;\; s.t. \; (x, v) \in R\}$ closed. But this is just $\pi_1((X \times V) \cap R)$ with $\pi_1 : X \times X \to X$ the projection. Now $(X \times V) \cap R$ is closed, so compact; so its continuous image under π_1 is compact. But X is Hausdorff, so this image is closed, and we're done. ■

Corollary: the quotient of a compact Hausdorff space by a closed equivalence relation is Hausdorff. Hence, if a compact group G acts continuously on a compact Hausdorff space, the quotient (set of orbits) is Hausdorff. $\blacksquare$

This enables us to take quotients by groups like $S^1 = SO(2)$ (unit complex numbers), by $SO(n)$, by any finite group, and is very useful in geometry. Here's a non-examinable example.

Example: we can describe the "space of complex lines through the origin in a complex vector space" as a quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ (parametrising all possible directions of real lines) by the action of scalar multiplication by S^1 [since there is an S^1-worth of real lines in a given complex line, if we look only at numbers of unit modulus]. The quotient $S^{2n+1}/S^1 = \mathbb{C}\mathbb{P}^n$ is Hausdorff; it's called *complex projective space*.

Challenge: understand why $\mathbb{C}\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is naturally a 2-dimensional sphere.

L9: Sequential compactness Compact subsets of Hausdorff spaces are closed, so they contain all their limit points. This suggests a deep relationship between compactness, and convergence of sequences, which we explore now.

Definition: Let (M, d) be a metric space and $C \subset M$. Then C is *sequentially compact* if every sequence $(x_n) \subset C$ has a subsequence which converges to a point of C.

Note: $(0,1) \subset (\mathbb{R}, d_{eucl})$ is not sequentially compact, since the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ converges in $\mathbb{R}$ but not in $(0,1)$. The closed interval $[0,1] \subset \mathbb{R}$ is sequentially compact, although the sequence $(0,1,0,1,0,\ldots)$ doesn't converge.

Warning: some books distinguish "C is sequentially compact in itself" ($\exists$ some subsequence converging to a point of C) and "C is sequentially compact in M" ($\exists$ some subsequence converging to a point of M); we will not give any special terminology for the latter.

Here are some useful things to remember:

(i) if a sequence converges in a metric space M then it's a Cauchy sequence;
(ii) limits of sequences are unique when they exist (e.g. since metric spaces are Hausdorff);
(iii) if a sequence converges in (M, d) to a point a, and if the metric d' is equivalent to d, then the sequence converges in (M, d') to a as well.

Lemma: if $C \subset (M, d)$ then $x \in Cl(C)$ iff there is a sequence in C converging to x.

Proof: if $x_n \to x$ then for every $\varepsilon > 0$ there is some $N > 0$ s.t. $x_n \in B_\varepsilon(x)$ for $n \geq N$. This implies that $B_\varepsilon(x) \cap C \neq \emptyset$ for all $\varepsilon > 0$ (since $(x_n) \subset C$). But this means x lies in the closure of C, from the definition of closure. Conversely, if $x \in Cl(C)$, then for each n there is some $x_n \in B_{\frac{1}{n}}(x) \cap C$; then $(x_n) \to x$. ■

Corollary: if C is sequentially compact, it's closed. ■

Lemma: Compact $\Rightarrow$ sequentially compact.

Proof: Let $C \subset (M,d)$ be a compact subspace and suppose $(x_n) \subset C$. If the set S of members of the sequence is finite, at least one point $x \in S$ is repeated infinitely often; then there is a constant subsequence $(x = x_{n_j}) \subset (x_n)$ which obviously converges to a point of C. So suppose S has infinitely many members. It is enough to show S has a limit point in C.

If not, for each $x \in C$ there is $\varepsilon(x) > 0$ s.t. $B_{\varepsilon(x)}(x) \cap S = \emptyset$ or $B_{\varepsilon(x)}(x) \cap S = \{x\}$. [Recall that x is a limit point of a set S if every open nhood of x meets S in some point other than x itself.] We have an open cover $B_{\varepsilon(x)}(x)$ of C which has a finite subcover $\{B_{\varepsilon(x_i)}(x_i) \,|\, 1 \leq i \leq n\}$. But each of these open balls contains at most one point of S, hence the union of finitely many contains at most finitely many points of S. But the finite collection covers $C \supset S$, which contradicts infinitude of S. $\blacksquare$

Note: we used above the fact that if a sequence $(x_n) \subset M$ in a metric space has a limit point $x \in M$ (i.e. if x is a limit point of the set S of members of the sequence), then there is a subsequence converging to x. This is intuitively obvious but a tiny bit fiddly:

Without loss of generality, $x_n \neq x$ for each n [why can we assume this?]. Choose $n(1)$ s.t. $x_{n(1)} \in B_1(x)$, and inductively $n(1) < n(2) < \cdots < n(k)$ s.t $x_{n(i)} \in B_{1/i}(x)$. Set

$$\varepsilon = \min\left\{\frac{1}{k+1}, d(x_j, x) \mid 1 \leq j \leq n(k)\right\}$$

Now $x_n \neq x$ for each $n \Rightarrow \varepsilon > 0$. There is some $n(k+1)$ s.t. $x_{n(k+1)} \in B_\varepsilon(x)$, since x is a limit point of the sequence, and choice of ε forces $n(k+1) > n(k)$. Inductively, this constructs a subsequence converging to x, as required. ■

Corollary: a bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. [For $Cl(\{x_n\})$ is compact, hence sequentially compact.] ■

We have that compactness implies sequential compactness, and we would like to establish the reverse as well (characterising compactness in metric spaces entirely in terms of convergence of subsequences). For this, we need to make some connection between the metric structure and open covers; this is by way of "Lebesgue numbers".

Definition: Given $\varepsilon > 0$, an ε-net for a metric space (M, d) is some subset $S \subset M$ such that $M = \bigcup_{x \in S} B_\varepsilon(x)$.

Lemma: a sequentially compact metric space has a finite ε-net for every $\varepsilon > 0$.

Proof: if not, choose $x_1, x_2, \ldots, x_r$ inductively s.t. $d(x_i, x_j) \geq \varepsilon$ for each $i \neq j$. Since $\{x_1, \ldots, x_r\}$ is not an ε-net, we can choose x_{r+1} to continue the induction, and build a sequence with no convergent (no Cauchy) subsequence. This contradicts sequential compactness. $\blacksquare$

Remark: a metric space is "precompact" or "totally bounded" if it has a finite ε-net for every $\varepsilon > 0$. This is *not* the same as being compact. E.g. $(0,1)$ is precompact.

Definition: if (M,d) is a metric space with open cover $M = \bigcup_{j\in J} U_j$, a Lebesgue number for the cover is $\delta > 0$ s.t. for each $x \in M$, there is some $U_{j(x)}$ in the cover s.t. $B_\delta(x) \subset U_{j(x)}$.

Example: the open cover $\{(1/n, 1) \,|\, n \in \mathbb{N}\}$ of $((0,1), d_{eucl})$ has no Lebesgue number. For if $\delta > 0$ and $k > \frac{1}{\delta}$, then $B_\delta(1/k) = (0, \frac{1}{k} + \delta) \not\subset (1/m, 1)$ for any m.

Lemma: every open cover of a sequentially compact metric space has a Lebesgue number.

Proof: Suppose (M,d) is sequentially compact and the open cover $\mathcal{U} = \bigcup_{j\in J} U_j$ has no Lebesgue number. For each n there is some x_n s.t. $B_{1/n}(x_n) \not\subset U_j$ for each j. Let $(x_{n(r)})$ be a convergent subsequence.

The limit $x_{n(r)} \to x$ of the sequence belongs to some $U \in \mathcal{U}$, since the sets form a cover. The set U is open, so there is some $\delta > 0$ s.t. $B_\delta(x) \subset U$. But $B_{\delta/2}(x) \supset x_{n(r)}$ for all $r \geq R$; in particular, if $n(r) > 2/\delta$ then

$$B_{1/n(r)}(x_{n(r)}) \subset B_\delta(x) \subset U$$

But this contradicts the choice of $x_{n(r)}$. ∎

Corollary: sequentially compact ⇒ compact.

Proof: if $M = \bigcup_{j \in J} U_j$ is an open cover of a sequentially compact metric space M, choose a Lebesgue number $\delta > 0$ for the cover. We have a finite δ-net $\{x_1, \ldots, x_n\}$ for M, and there are sets U_{j_i} in the cover s.t. $B_\delta(x_i) \subset U_{j_i}$ (by definition of Lebesgue number). But then

$$M \subset \bigcup_{i=1}^{n} B_\delta(x_i) \subset \bigcup_{i=1}^{n} U_{j_i} \subset M$$

shows that this arbitrary cover indeed has a finite subcover. This proves compactness. ∎

[*Ed:* the definition with open covers is nicer...]

L10: Connectedness In contrast to compactness, connectedness is a very (well, fairly...) "visual" property – does your topological space fall into several pieces or not?

Definition: a topological space $(X, \mathcal{T}_X)$ is *connected* if for every decomposition $X = A \cup B$ into disjoint open subsets $A \in \mathcal{T}_X$ and $B \in \mathcal{T}_X$, either A or B is empty.

Example: $[0,1]$ is connected; $[0,1] \cup [6,7]$ is not. The graph of $\{y = x^2\} \subset \mathbb{R}^2$ is connected; the graph of $\{y = 1/x\} \subset \mathbb{R}^2$ is not.

Example: $\mathbb{Q} \subset (\mathbb{R}, \mathcal{T}_{eucl})$ is not connected. For the sets $(-\infty, \sqrt{2}) \cap \mathbb{Q}$ and $(\sqrt{2}, \infty) \cap \mathbb{Q}$ are open sets in the subspace topology, they are disjoint, and they cover $\mathbb{Q}$.

Example: $(\mathbb{C}, \mathcal{T}_{Zariski})$ is connected, since every two non-empty open sets have non-empty intersection.

Warning: $[0,1] = [0,1/2] \cup [1/2,1]$ and $[0,1] = [0,1/2) \cup [1/2,1]$ are both illegal decompositions!

Lemma: X is connected if and only if every continuous function from X to the two-point discrete space is constant.

Proof: Suppose $f : X \to \{0,1\}$ is continuous, where we give the RHS the discrete topology. If f is onto, then $f^{-1}(0) \cup f^{-1}(1)$ is a decomposition of X into disjoint non-empty open sets, so X is not connected. Thus, connected $\Rightarrow$ such maps are constant. Conversely, if $X = A \cup B$ is a partition of X into disjoint open sets, then the map $f : X \to \{0,1\}$ taking $f|_A \equiv 0$ and $f|_B \equiv 1$ is continuous. ■

In practise, you can take either of the two characterisations of connectedness as a definition, depending on which is more useful to you at the time. Obviously, connectedness is a topological property. Indeed, more is true.

Lemma: The continuous image of a connected space is connected.

Proof: if $f : X \to Y$ and $f(X) = A \cup B$ is a partition into disjoint open sets, then write $A = f(X) \cap U$ and $B = f(X) \cap V$ for open sets U, V in Y. Now $f^{-1}(U) \cup f^{-1}(V) = X$ partitions X, so by connectedness one of these is empty: say $f^{-1}(U) = \emptyset$. But then $A = \emptyset$, so $f(X)$ is connected. ■

Example: if a subspace $I \subset \mathbb{R}$ is connected it is an interval.

[An interval is any set I such that if $x, y, z \subset \mathbb{R}$ and $x < z < y$, then $x, y \in I \Rightarrow z \in I$. So this includes $(a, b), [a, b), (a, b], [a, b], \{a\}$, where open ends may be infinite.]

Proof: if I is not an interval, say $x < z < y$ with $x, y \in I$ and $z \notin I$, then $(-\infty, z) \cap I$ and $(z, \infty) \cap I$ form a partition into disjoint open subsets. ■

Remark: note that it's much harder to characterise the compact subsets of $\mathbb{R}$, e.g. there were fractals like Cantor sets; so connectedness is amazingly well behaved in this sense. And of course we have the converse to the above:

Proposition: intervals in $\mathbb{R}$ are connected.

Proof: Suppose $I = A \cup B$ is a partition into disjoint nonempty opens. Suppose $a \in A$, $b \in B$ and $a < b$ (or exchange labels). Note $[a, b] \subset I$ and let $A' = A \cap [a, b]$ and $B' = B \cap [a, b]$, which are closed in $[a, b]$, hence are compact. Thus $A' \times B' \subset \mathbb{R}^2$ is compact. Hence the Euclidean distance function $(x, y) \mapsto d(x, y)$ achieves its lower bound on $A' \times B'$ at some (a', b'). Since $A' \cap B' = \emptyset$, $d(a', b') > 0$. Now consider $c = (a' + b')/2 \in [a, b]$. Then

$$|c - b'| = \frac{1}{2}|a' - b'| < |a' - b'|$$

so $(c, b') \notin A' \times B' \Rightarrow c \notin A'$. But similarly $c \notin B'$, which contradicts $[a, b] \subset A' \cup B'$. ■

Corollary: for any connected topological space X and real-valued function $f : X \to \mathbb{R}$, $f(X)$ is an interval.

The intermediate value theorem for functions on $\mathbb{R}$ or $[a, b]$ obviously follows. In general, unions of connected spaces aren't connected, but the "obvious" obstruction is in fact the only one.

Lemma: if $\{X_j \,|\, j \in J\}$ are connected subspaces of a fixed topological space $(Z, \mathcal{T}_Z)$, and $X_i \cap X_j \neq \emptyset$ for each i, j then $X = \bigcup_j X_j$ is connected.

Proof: let $f : X \to \{0, 1\}$ be continuous, the RHS being discrete. For each j, $f|_{X_j}$ is constant; say $f|_{X_{j_0}} \equiv 0$. But then for each j, $X_j \cap X_{j_0} \neq \emptyset$ and $f|_{X_j}$ constant $\Rightarrow f|_{X_j} \equiv 0$ as well, so finally $f \equiv 0$ on X, as required. ■

Remark: it's not enough to say that for each i there is some $j \neq i$ s.t. $X_i \cap X_j \neq \emptyset$ – why?

Lemma: X and Y are connected iff $X \times Y$ is connected.

Proof: if $X \times Y$ is connected, then the continuous images (by projections to the factors) X and Y are certainly connected. Conversely, if X and Y are connected, and $f : X \times Y \to \{0, 1\}$ is continuous, then $f|_{X \times \{y\}}$ is constant for each $y \in Y$. But all these subspaces meet the single, connected space $\{x_0\} \times Y$, for any fixed $x_0 \in X$. Now argue precisely as in the previous Lemma. ■

Exercise: use an analogous argument to say that if $n > 1$, $\mathbb{R}^n \backslash \{0\}$ is connected, and deduce S^{n-1} is connected.

Connectedness gives a natural equivalence relation on a space X: $x \sim y \Leftrightarrow x$ and y belong to a common connected subspace of X. The equivalence classes are called the *components* (or connected components) of X. The number of connected components is a topological invariant of the space.

Remark: from the definition, note that the connected components are the maximal connected subspaces of X (maximal with respect to the inclusion relation amongst subsets). The closure of a connected space is connected [example sheet 2], so components are closed. In general, they are not open:

Definition: if the connected components are points, the space is said to be *totally disconnected*.

Lemma: Discrete $\Rightarrow$ totally disconnected, but not the converse.

Proof: if X is discrete, then for each $x \in X$ there is a decomposition $X = \{x\} \cup X \backslash \{x\}$ into disjoint open sets, which shows maximal connected subspaces are points. However, $\mathbb{Q} \subset (\mathbb{R}, \mathcal{T}_{eucl})$ is totally disconnected, but singleton sets $\{q\}$ are not open in $\mathbb{Q}$. ■

Exercise: the Cantor set is totally disconnected [as are many naturally occuring "fractal dusts"].

L11: Path-connectedness There is a variant of the idea of connectedness which is even more intuitive.

Definition: a space $(X, \mathcal{T}_X)$ is path-connected if for every $a, b \in X$ there is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$. [Here $[0,1]$ has its Euclidean metric topology.]

We think of the image of such a map γ as a path in X from a to b.

Lemma: being joined by some path defines an equivalence relation on the points of X.

Proof: if $\gamma(t)$ is a path from a to b, then $\gamma(1-t)$ is a path from b to a; and the constant map $t \mapsto \gamma(t) \equiv a$ defines a path from a to a. For transitivity, if γ is a path from a to b and τ is a path from b to c then

$$\gamma \star \tau : t \mapsto \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \tau(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

defines a path from a to c. $\blacksquare$

Remark: note the map $\gamma \star \tau$ is indeed continuous, since its restriction to the closed sets $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ are both continuous. Now check:

Exercise: for any space $X = \bigcup_{j=1}^{n} V_j$ written as a finite union of closed sets, $f : X \to Y$ is continuous iff $f|_{V_j}$ is continuous for each j.

Analogous to the results for connectedness, one can check the following:
(i) the continuous image of a path-connected space is path-connected;
(ii) the product of path-connected spaces is path-connected.

However, not all analogues carry over. Recall $H \subset X$ is connected $\Rightarrow Cl(H) \subset X$ is connected, hence connected components are always closed.

Lemma: Let $X \subset \mathbb{R}^2$ be the union of the graph of $sin\,\frac{1}{x}$, for $x > 0$, and the interval $[-1, 1]$ on the y-axis. This is connected but not path-connected.

The space in the Lemma is called the "topologist's sin curve":

Proof: Recall the graph of a function is homeomorphic to its domain, hence

$$G = \{(x, y) \,|\, y = sin\,(1/x), \quad x > 0\}$$

is homeomorphic to $\mathbb{R}$ and so connected. Hence, to show X is connected, it's enough to show the points on the y-axis between $(0, -1)$ and $(0, 1)$ lie in $Cl(G)$. Fix $\varepsilon > 0$; if $p = (0, y)$, and $1/(2\pi n) < \varepsilon$, then since $sin((4n + 1)\pi/2) = 1$ and $sin((4n + 3)\pi)/2 = -1$, the intermediate value theorem says $sin(1/x)$ takes on the value $y \in (-1, 1)$ at a point x_0 in the interval $[\frac{2}{(4n+3)\pi}, \frac{2}{(4n+1)\pi}]$. But $|(x_0, y) - (0, y)| < \varepsilon$, so the ε-ball about $(0, y)$ meets G. So $cl(G) = X$.

To see that the space X is *not* path-connected, suppose for contradiction we have a path $\gamma : [0,1] \to X$ from $(0,0)$ to $(1, sin(1))$. Let

$$c = sup\ \{t \mid \gamma(t) \in A = \{0\} \times [-1,1] \subset X\}$$

Now A is closed in X, so $\gamma(c) \in A$, so $c < 1$. Since $\gamma(t) \in G$ for $t > c$, there is $\delta > 0$ s.t. $im(\gamma|_{[c,c+\delta]}) \subset B_{1/3}(\gamma(c))$.

Now consider the composite $\bar{\gamma} : [0,1] \to \mathbb{R}$ given by $\bar{\gamma} = p \circ \gamma$, where $p : \mathbb{R}^2 \to \mathbb{R}$ is first projection (to the x-axis). We have $\bar{\gamma}(c) = 0$ but $\bar{\gamma}(t) > 0$ for all $t > c$. The image $\bar{\gamma}|_{[c,c+\delta]}$ is connected, hence an interval, so contains $[0, \alpha]$ for some positive α. But, arguing as before, the function $sin\,(1/x)$ takes every value in $[-1,1]$ on the interval $(0, \alpha]$; so the image under γ of $[c, c+\delta]$ contains a point with y-co-ordinate 1 and a point with y-co-ordinate -1. But then $im(\gamma|_{[c,c+\delta]}) \not\subset B_{1/3}(\gamma(c))$, which is our contradiction. ■

Note that the topologist's sin curve has a path-component which is *not* a closed subset.

This kind of pathology is very much the exception in that for nice spaces, connectedness and path-connectedness are equivalent.

Lemma: Path-connected $\Rightarrow$ connected.

Proof: Suppose $X \neq \emptyset$ is path-connected but we have written $X = A \cup B$ as a union of disjoint non-empty open subsets. Then for $a \in A$ and $b \in B$, we have a path $\gamma : [0, 1] \to X$ from a to b, and therefore $[0, 1] = \gamma^{-1}(A) \cup \gamma^{-1}(B)$ is a partition of the interval into disjoint non-empty open sets, which is absurd. ■

Lemma: a connected open subset of $\mathbb{R}^n$ is path-connected.

Proof: Let $U \subset \mathbb{R}^n$ be non-empty, open and connected; fix $w \in U$. We consider the subset

$$Z = \{u \in U \mid \exists \text{ a path from } u \text{ to } w\}$$

It is enough to show this is both open and closed; for in a connected space, the only subset which is open, closed and non-empty, is the whole set. [Why?]

To see Z is open, note if $u \in Z$, by openness there is a ball $B_\delta(u) \subset U$. Let $u' \in B_\delta(u)$ be arbitrary. If $\gamma : [0,1] \to U$ is a path from w to u, then the composite $\gamma \star \tau$ of γ with a radial path $\tau \subset B_\delta(u)$ from u to u' is a path from w to u', hence Z contains the whole ball $B_\delta(u)$; so it contains an open nhood of each of its points, and is open.

But exactly the same argument as above shows that the complement $U \backslash Z$ is open; if $v \in U \backslash Z$ and there is a path from w to some point in the δ-ball about v, by concatenation there would be a path from w to v itself, contradiction. Hence Z has open complement and is therefore closed. So Z is full (all of U) or empty, and we know $w \in Z$, so $Z = U$. ■

Corollary: for *manifolds* (topological spaces locally homeomorphic to $\mathbb{R}^n$, i.e. such that every point has an open nhood homeomorphic to $\mathbb{R}^n$), connectedness is the same as path-connectedness.

This is the start of the beautiful subject of *algebraic topology*. One studies a space $(X, \mathcal{T}_X)$ by studying the set of all paths $\gamma : [0,1] \to X$ up to continuous deformation or "homotopy". More precisely, there is a natural topology on the set of continuous maps $S^1 \to X$ – called the loop space $\mathcal{L}X$ – and one uses the number of connected components of this new space as a subtle invariant of the original space X. An important point is that if you "base" the loops somewhere, considering maps

$$S^1 \to X \quad \text{taking} \quad 1 \mapsto x_0$$

taking a chosen point of S^1 to a fixed point of X, then the *set* of connected components of that based subspace $\Omega X \subset \mathcal{L}X$ is naturally a *group*. The operation is induced by the concatenation $\star$ of paths we've already met [draw a picture and you'll see why *based* loops are important!]. But this means we can apply tools from algebra and group theory to questions in geometry and topology.

L12: Topological dynamics To finish the course, we'll use our results on connectedness to prove one or two intriguing things about self-maps of the real line $(\mathbb{R}, \mathcal{T}_{eucl})$. Recall we proved that a contraction of $\mathbb{R}$ always has a fixed point. One can also study "periodic points", that is fixed points of iterates of a map.

Theorem (Sarkovskii): If $f : \mathbb{R} \to \mathbb{R}$ is continuous and has a periodic point of period 3, then f has periodic points of every period.

Remark: We can order the natural numbers

$$\begin{aligned}3 \triangleright 5 \triangleright 7 \triangleright 9 \triangleright \cdots \\ \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \cdots \\ \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \cdots \\ \cdots \triangleright 2^4 \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1\end{aligned}$$

So we take all the odd numbers > 1, then these times increasing powers of 2, finally the powers of 2 in decreasing order. Then if f has a periodic point of period k and $k \triangleright l$ then f has a periodic point of period l.

Before the proof, here are two easy facts which follow since the continuous image of an interval under f is an interval.

(i) if $I \subset J$ are nested closed intervals and $f(I) \supset J$ then f has a fixed point in I. This follows from applying the intermediate value theorem to $x - f(x)$ on I.
(ii) if $A_0 \supset A_1 \supset \cdots \supset A_n$ are nested closed intervals and $f(A_i) \supset A_{i+1}$ for $0 \leq i \leq n-1$, then there is some subinterval $J_0 \subset A_0$ s.t. $f(J_0) = A_1$ is onto A_1. Now there is also an interval in A_1 which maps onto A_2, and lifting this back to J_0, we find $J_1 \subset J_0$ s.t. $f(J_1) \subset A_1$ and $f^2(J_1) = A_2$. Iterating this, it follows that there is $x \in A_0$ s.t. $f^i(x) \in A_i$ for each i.

Proof of Theorem: Suppose we have $a, b, c \in \mathbb{R}$ with $f(a) = b$, $f(b) = c$ and $f(c) = a$. Suppose $a < b < c$ (the other case, $f(a) = c$, is completely similar). Let $I_0 = [a, b]$ and $I_1 = [b, c]$; then $f(I_0) \supset I_1$ and $f(I_1) \supset I_0 \cup I_1$.

We claim f has a fixed point in $[b, c]$; this is an application of (i) above. Similarly, f^2 has a fixed point in $[a, b]$.

Exercise: there is such a fixed point of f^2 which is not a fixed point of f, hence is a periodic point of period 2 (cf. graph).

Thus, for $n > 3$ we need a periodic point of "prime" period n (i.e. a point fixed by the n-th iterate but not by any $(k < n)$-th iterate). Set $A_0 = I_1$; $f(I_1) \supset I_1 \Rightarrow \exists\, A_1 \subset A_0$ s.t. $f(A_1) = A_0 = I_1$ (by (ii) above). Iterate to find $A_2 \subset A_1$ with $f^2(A_2) = A_0$, and $A_{n-2} \subset A_{n-3}$ s.t. $f(A_{n-2}) = A_{n-3}$. If $x \in A_{n-2}$ then by (ii) above, $\{f(x), f^2(x), \ldots, f^{n-2}(x)\} \subset A_0$ and $f^{n-2}(A_{n-2}) = A_0 = I_1$.

Now $f(I_1) \supset I_0$ implies there is $A_{n-1} \subset A_{n-2}$ s.t. $f^{n-1}(A_{n-1}) = I_0$. But $f(I_0) \supset I_1$ implies $f^n(A_{n-1}) \supset I_1$ so $f^n(A_{n-1})$ covers A_{n-1}. Then by (i), f^n must have a fixed point $p \in A_{n-1} \subset I_1 = [b, c]$.

We have constructed a periodic point of period n; we claim this is its prime period. Note $f^{n-1}(p) \in I_0 = [a, b]$; we can suppose it does not lie on the boundary of this interval (otherwise $n = 2$ or $n = 3$). But then if for some $j < n$ we have $f^j(p) = p$, then since the set $\{p, f(p), \ldots, f^{j-1}(p)\} \subset I_1$, it is impossible that $f^{n-1}(p)$ lies in (a, b). Hence we have a point of prime period n. ■

Corollary: if $f : \mathbb{R} \to \mathbb{R}$ has finitely many periodic points, every periodic point has period a power of 2. [Check: this case does occur!] If f has any periodic point of period not a power of 2, it has infinitely many periodic points.

Example: there is a map with a point of period 5 and no point of period 3. E.g. consider the piecewise linear map $f : [1,5] \to [1,5]$ s.t.

$$f : 1 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5 \mapsto 1$$

We make f linear between the integers, having graph as shown below:

Then f^3 maps the intervals

$$[1,2] \mapsto [2,5]; \ \ [2,3] \mapsto [3,5]; \ \ [4,5] \mapsto [1,4]$$

so f^3 has no fixed point in any of these intervals. $f^3([3,4]) = [1,5] \supset [3,4] \Rightarrow f^3$ has a fixed point in $[3,4]$; but we claim this point is unique, hence must be the fixed point of f and not a period 3 point. This is clear since f^3 is monotonically decreasing on $[3,4]$ (check!). This completes the example. $\blacksquare$

A famous family of self-maps of $\mathbb{R}$ are the quadratic maps $x \mapsto \mu x(1-x)$; we'll consider $\mu = 3.839$ (after Smale...). One can check by hand there's a periodic orbit $\{a_1, a_2, a_3\}$ of period 3, to a few decimal places:

$$0.14988.. \mapsto 0.48917.. \mapsto 0.95929.. \mapsto .14988..$$

This is "attracting" − a small interval around the periodic point is mapped into itself by f^3. A general theorem says that there are *no other attracting periodic points*; hence all other periodic points, guaranteed by Sarkovskii, are computationally invisible. Where are they, and what does f do near them?

The answer is given by "symbolic dynamics", and is best expressed in terms of superficially unrelated topological spaces. Consider all infinite sequences $\mathbf{s} = (s_0 s_1 s_2 ...)$ of 0's and 1's with a metric topology $d(\mathbf{s}, \mathbf{t}) = \sum_{j=0}^{\infty} \frac{|s_j - t_j|}{2^j}$. There is a subspace Σ of sequences in which no two 0's are adjacent. There is a natural function $\sigma : \Sigma \to \Sigma$ taking $(s_0 s_1 s_2 \ldots) \mapsto (s_1 s_2 \ldots)$, called the *shift*.

Lemma: the shift $\sigma : \Sigma \to \Sigma$ is continuous.

Proof: let $\varepsilon > 0$ and suppose $1/2^n < \varepsilon$; set $\delta = 1/2^{n+1}$. Then if $d(\mathbf{s}, \mathbf{t}) < \delta$ it follows that $s_i = t_i$ for $i \leq n+1$; hence $\sigma(\mathbf{s})_i = \sigma(\mathbf{t})_i$ for $i \leq n$, and that immediately implies $d(\sigma(\mathbf{s}), \sigma(\mathbf{t})) < \varepsilon$. This proves (uniform) continuity. ■

Theorem: there is a union of two disjoint closed subintervals $I \cup J \subset [0, 1]$, with $I \subset (a_1, a_2)$ and $J \subset (a_2, a_3)$, which contains all periodic points of $f(x) = 3.839x(1-x)$ other than $\{0, a_1, a_2, a_3\}$.

(i) The set $\Lambda = \{x \,|\, f^n(x) \in I \cup J \ \forall n \in \mathbb{N}\}$ is homeomorphic to Σ;
(ii) under this homeomorphism $f \Leftrightarrow \sigma$.

In particular, the structure of periodic orbit phenomena, even for simple quadratic maps of $\mathbb{R}$, is sometimes best understood using rather abstract topological spaces of sequences and their continuous mappings.