

## What does $\nabla f$ do?

In lectures, we learn that for a scalar function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , the gradient  $\nabla f$  does two things:

- (1) it points in the direction of greatest increase of  $f$
- (2) it points perpendicularly to curves/surfaces of constant  $f$

But what do these mean, and are they really compatible statements?

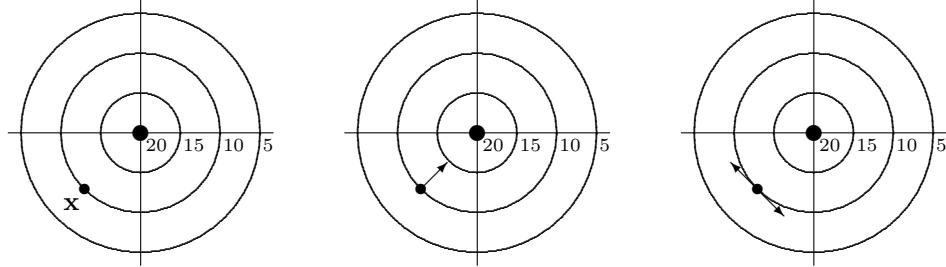
Recall the Taylor expansion for a function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , namely:  $f(x + h) = f(x) + hf'(x) + \dots$

There is a similar expansion for a function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ , which starts:  $f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \mathbf{h} \cdot \nabla f|_{\mathbf{x}} + \dots$

Suppose that we are at the point  $\mathbf{x}$ , and we wish to move in a direction  $\mathbf{h}$  so as to cause the greatest increase in the value of  $f$ . Considering the Taylor expansion, we see that we want  $\mathbf{h} \cdot \nabla f|_{\mathbf{x}}$  to be as large as possible. For two vectors of a given size, the dot product is greatest when those two vectors are parallel. Hence the direction  $\mathbf{h}$  in which we move away from  $\mathbf{x}$  is in the direction given by  $\nabla f$  at  $\mathbf{x}$ . This is property (1) above.

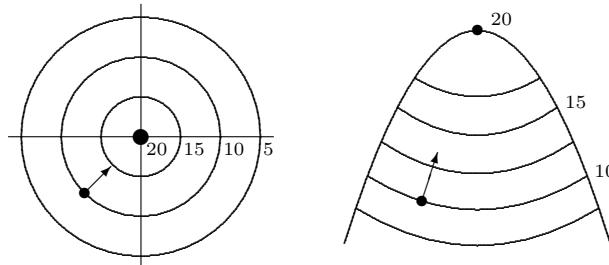
Suppose instead that we don't want to change the value of  $f$ , but instead want to move in some direction while keeping the value of  $f$  constant. Considering the Taylor expansion again, we see that we want  $\mathbf{h} \cdot \nabla f|_{\mathbf{x}}$  to be zero. This is the case when  $\mathbf{h}$  is perpendicular to  $\nabla f|_{\mathbf{x}}$ . Hence the direction  $\mathbf{h}$  in which we move away from  $\mathbf{x}$  is in the direction (any direction, since there will be a line or plane perpendicular to a given vector) given by  $\nabla f$  at  $\mathbf{x}$ . This is property (2) above.

An example. For ease of drawing pictures, let's use a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ , i.e. of the plane. Suppose that we place something hot at the origin, and consider the temperature at different distances away. The function  $f$  will measure the temperature at all points in the plane. Let's plot a few curves of certain temperatures.



Suppose we are at the point  $\mathbf{x}$  shown (first picture). To increase the temperature as quickly as possible, we would move straight towards the origin (second picture). Whereas to move around keeping the temperature constant we move around the circle shown (third picture). The arrow in the second picture is  $\nabla f$ , and the arrows in the third pictures are clearly perpendicular to this.

Now a very slightly different example. Let's say that instead of temperature, the pictures above are the contour lines on a map (the first picture below). So we're looking at the map, while out in the world somewhere we are standing on a hill. The contour values are given by  $f(x, y)$  as before, but we'll now define the height function  $z = f(x, y)$  and obtain a surface (the second picture below).



Suppose we are at the point on the hill marked as a blob in the second picture, and that we want to go up the hill as steeply as possible, i.e. following the arrow shown in the second picture.

We recall that  $\nabla z$  will give us the ‘direction of steepest increase of  $z$ ’, which sounds like what we want. Since  $z$  is a function of two variables  $x, y$ , its gradient will be a vector with two components: it is the arrow shown in the left-hand picture. It says something like ‘to go up as quickly as possible, you must travel north-east’.

So if we’re walking on the hill and follow the compass direction given to us by  $\nabla z$ , then we will go up the hill. That is, we are walking north-east. (I.e.,  $\nabla z$  is the vector lying in the plane in the first picture, and not the vector pointing up the hill in the second picture.)

We also see, as before, that the direction that goes up the hill the most steeply is perpendicular to the paths around the hill, i.e. the contour lines at constant height. This makes sense if we now picture the tangent plane to the hill, at the point where the blob is. Picturing such an inclined plane, it is clear that the steepest and flattest directions are perpendicular.