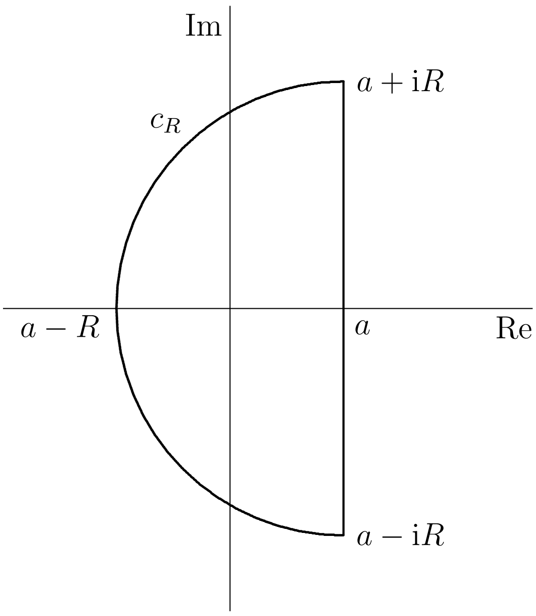

Lemma 1 *Let f be continuous for large $|z|$, and assume that $f(z) \to 0$ as $z \to \infty$. Then, provided $t > 0$, we have*

$$\lim_{R\to\infty} \int_{c_R} f(z)\mathrm{e}^{tz}\,\mathrm{d}z \;=\; 0$$

where c_R denotes the semicircular contour $\theta \mapsto a + R\mathrm{e}^{\mathrm{i}\theta}$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.

Note that the proof of this result is essentially that of Jordan's Lemma rotated through $\frac{\pi}{2}$.

Proof. Let $\varepsilon > 0$. Because $f(z) \to 0$ as $z \to \infty$, there exists $K > 0$ such that $|z| \ge K$ implies $|f(z)| \le \varepsilon$. This means that $|f(z)| \le \varepsilon$ whenever $|z - a| \ge K + |a|$, because $|z| + |a| \ge |z - a|$. For $R \ge K + |a|$, we have

$$\begin{aligned}
\left|\int_{c_R} f(z)\mathrm{e}^{tz}\,\mathrm{d}z\right| &\le \varepsilon \int_{c_R} |\mathrm{e}^{tz}|\,|\mathrm{d}z| && |z-a| = R \ge K + |a| \\
&= \varepsilon \int_{c_R} \mathrm{e}^{\mathrm{Re}(tz)}\,|\mathrm{d}z| \\
&= \varepsilon \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{ta}\mathrm{e}^{tR\cos\theta} R\,\mathrm{d}\theta && z = a + R\mathrm{e}^{\mathrm{i}\theta} \\
&= 2R\mathrm{e}^{ta}\varepsilon \int_{\frac{\pi}{2}}^{\pi} \mathrm{e}^{tR\cos\theta}\,\mathrm{d}\theta && \text{by symmetry} \\
&= 2R\mathrm{e}^{ta}\varepsilon \int_{0}^{\frac{\pi}{2}} \mathrm{e}^{-tR\sin\theta}\,\mathrm{d}\theta \\
&\le 2R\mathrm{e}^{ta}\varepsilon \int_{0}^{\frac{\pi}{2}} \mathrm{e}^{-2tR\theta/\pi}\,\mathrm{d}\theta && \text{Jordan's inequality and } t > 0 \\
&= 2R\mathrm{e}^{ta}\varepsilon \left[\frac{-\pi}{2tR}\mathrm{e}^{-2tR\theta/\pi}\right]_0^{\frac{\pi}{2}} \\
&= \frac{\mathrm{e}^{ta}\pi\varepsilon}{t}(1 - \mathrm{e}^{-tR}) \\
&\le \frac{\mathrm{e}^{ta}\pi\varepsilon}{t}.
\end{aligned}$$

Since $t > 0$ is fixed, this final quantity can be made as small as desired. This completes the proof. $\square$