Mich. 2023 **NUMBERS & SETS – ADDITIONAL QUESTIONS** G. Taylor

1. Let A be the sum of the digits of 4444^{4444}, and let B be the sum of the digits of A. What is the sum of the digits of B?

2. Is there a power of 7 that starts with the digits 2023. . . ?

3. Show that $a^3 + b^5 = 7^{7^{7^7}}$ has no solution with $a, b \in \mathbb{Z}$.

4. Let x, y and z be positive integers satisfying $x^2 + y^2 + 1 = xyz$. Prove that $z = 3$.

5. Let p be a prime other than 7. Show that every integer is a sum of two cubes mod p.

6. Let R be a rectangle which can be divided into smaller rectangles, each of which has at least one side of integer length. Prove that R has at least one side of integer length.

7. Does there exist a cycle in $\mathbb{Z}^3$ (i.e., a path consisting of line segments joining neighbouring points which have integer coordinates, ending up where it started) such that none of the projections in the x, y and z directions contains a cycle?

8. Each integer point in the plane is coloured either red or blue. Must there exist a square (with sides parallel to the axes) with all four corners the same colour?

9. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that takes every value on every interval – in other words, such that for every $a < b$ and every c there is some x with $a < x < b$ and $f(x) = c$.

10. Let $(x_n)_{n=1}^{\infty}$ be a real sequence with $x_n \to 0$. Prove that we may choose $(\varepsilon_n)_{n=1}^{\infty}$, with each $\varepsilon_n = \pm 1$, such that $\sum_{n=1}^{\infty} \varepsilon_n x_n$ is convergent. If $(y_n)_{n=1}^{\infty}$ is another real sequence tending to 0, can we choose the ε_n so that $\sum_{n=1}^{\infty} \varepsilon_n y_n$ is convergent as well?

11. Let S be a (possibly infinite) set of odd positive integers. Prove that there exists a real sequence $(x_n)_{n=1}^{\infty}$ such that, for each odd positive integer k, the series $\sum_{n=1}^{\infty} x_n^k$ converges when k belongs to S and diverges when k does not belong to S.

12. Let $x_1, x_2, \ldots$ be reals such that $\sum_{n=1}^{\infty} |x_n|$ is convergent. Show that if $\sum_{n=1}^{\infty} x_{kn} = 0$ for every $k \in \mathbb{N}$ then $x_n = 0$ for all n. What if we drop the restriction that $\sum_{n=1}^{\infty} |x_n|$ is convergent?

13. When players participate in a *tournament*, each pair play a game, with one or other player winning (there are no draws). Construct a tournament in which, for any two players, there is a player who beats both of them. Is it true that for any k there is a tournament in which, for any k players, there is a player who beats all of them?

14. There is an infinite sequence of boxes, each containing either a red or blue ball. Finitely many students each aim to guess the contents of some box. Each may examine the contents of any proper subset of the boxes, in any order, and (without sharing the gained information with the others) must then make a guess at the contents of a box they didn't examine. How many students can guess correctly?

15. Among a group of n dons, any two have exactly one mutual friend. Show that some don is friends with all the others.

16. Each of n elderly dons knows a piece of gossip not known to any of the others. They communicate by telephone, and in each call the two dons concerned reveal to each other all the information they know so far. What is the smallest number of calls that can be made in such a way that, at the end, all the dons know all the gossip?

17. Let $n, k \in \mathbb{N}$. Suppose that n is a k^{th} power (mod p) for all primes p. Must n be a k^{th} power?

18. Let n be a fixed positive integer. Show that, for sufficiently large prime p (i.e. for all but finitely many primes p), the equation $x^n + y^n = z^n$ has a solution in $\mathbb{Z}_p$ with $x, y, z \neq 0$.