Let f have Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$, valid for $0 < |z-a| < r$. The important coefficient in Cauchy's Residue Theorem is the *residue* c_{-1}, because that is the only one that survives the integration (essentially because

$$\int_C z^n \, \mathrm{d}z \; = \; \begin{cases} 2\pi\mathrm{i} & \text{if} \quad n = -1 \\ 0 & \text{if} \quad n \neq -1 \end{cases}$$

where C is the circular contour $t \mapsto \mathrm{e}^{\mathrm{i}t}$, $0 \leq t \leq 2\pi$). As remarked previously, Laurent coefficients, and hence residues, are not easy to calculate in general. Fortunately, there are a couple of results that help in a lot of cases.

Proposition 1 *Suppose that h and k are analytic on the disc $|z-a| < r$, with $h(a) \neq 0$, $k(a) = 0$ and $k'(a) \neq 0$. Then $f = h/k$ (which has a pole of order 1) has residue $h(a)/k'(a)$ at a.*

Proof. If f has a pole of order 1 then it can be written

$$f(z) \; = \; \sum_{n=-1}^{\infty} c_n(z-a)^n$$

for $0 < |z-a| < r$. Hence $\lim_{z\to a} f(z)(z-a) = c_{-1}$. On the other hand,

$$\begin{aligned} f(z)(z-a) \; = \; \frac{h(z)(z-a)}{k(z)} \; &= \; h(z)\Big(\frac{z-a}{k(z)-k(a)}\Big) \\ &\to \; h(a)\Big(\frac{1}{k'(a)}\Big) \end{aligned}$$

as $z \to a$. Hence $c^{-1} = h(a)/k'(a)$. □

Example 2 tan has residue -1 at $\frac{\pi}{2}$.

Proposition 3 *Suppose that g is analytic on $|z-a| < r$, with $g(a) \neq 0$. Then*

$$f(z) \; = \; \frac{g(z)}{(z-a)^m}$$

where $m > 0$ (so has a pole of order m) has residue $g^{(m-1)}(a)/(m-1)!$.

Proof. g is analytic so by Taylor's Theorem, $g(z) = \sum_{n=0}^{\infty} d_n(z-a)^n$ for $|z-a| < r$. It follows (by the uniqueness of Laurent coefficients), that f has Laurent expansion

$$f(z) \; = \; \sum_{n=-m}^{\infty} d_{n+m}(z-a)^n$$

for $0 < |z-a| < r$. Hence the residue is $d_{m-1} = g^{(m-1)}(a)/(m-1)!$. □

There will be cases where neither proposition above will apply, for example, if f has an isolated, essential singularity at a. Direct computation of c_{-1} may be necessary. Note that while essential singularities produce extremely wild behaviour, Cauchy's Residue Theorem still applies and residues still exist. For example

$$\mathrm{e}^{\frac{1}{z}} \;=\; 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

has residue 1 at 0, and

$$\int_C \mathrm{e}^{\frac{1}{z}}\,\mathrm{d}z \;=\; 2\pi\mathrm{i}$$

where C is the contour $t \to \mathrm{e}^{\mathrm{i}t}$, $0 \le t \le 2\pi$.