1. Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$, and suppose that for every $t > 0$ the sequence $f(t)$, $f(2t)$, $f(3t)$, $\ldots$ tends to 0. Prove that $f(x) \to 0$ as $x \to \infty$.

2. (i) Find a sequence $[a_1, b_1], [a_2, b_2], \ldots$ of closed intervals in $\mathbb{R}$ of positive length whose union contains all rationals in $[0, 1]$ and such that $\sum_{i=1}^{\infty}(b_i - a_i) < 1$.

   (ii) Let $[a_1, b_1], [a_2, b_2], \ldots$ be a sequence of closed intervals in $\mathbb{R}$ of positive length whose union contains all irrationals in $[0, 1]$. Can we have $\sum_{i=1}^{\infty}(b_i - a_i) < 1$?

3. Let f be an infinitely-differentiable function from $\mathbb{R}$ to $\mathbb{R}$, such that for every x there is an n with all the derivatives $f^{(n)}(x)$, $f^{(n+1)}(x)$, $f^{(n+2)}(x)$, $\ldots$ being zero. Must f be a polynomial?

4. Show that $\mathbb{R}^2$ cannot be written as the disjoint union of (non-trivial) circles. What about $\mathbb{R}^3$?

5. (i) Does there exist a function from $\mathbb{R}$ to $\mathbb{R}$ which is continuous at precisely the rationals?

   (ii) Does there exist a function from $\mathbb{R}$ to $\mathbb{R}$ which is continuous at precisely the irrationals?

6. We say that a function f from $\mathbb{R}$ to $\mathbb{R}$ *crosses the axis* at a point x if $f(x) = 0$ but for any $\epsilon > 0$ there exist y and z in $(x - \epsilon, x + \epsilon)$ with $f(y) > 0$ and $f(z) < 0$. Can a continuous function cross the axis at uncountably many places?

7. If f is a polynomial of one real variable that is bounded below (on $\mathbb{R}$), explain why f attains its minimum value. If f is a polynomial of two real variables that is bounded below (on $\mathbb{R}^2$), must f attain its minimum value?

8. Show that the set $\mathbb{R}^{\mathbb{R}}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$ bijects with the set $\mathcal{P}(\mathbb{R})$ of all subsets of $\mathbb{R}$, and that the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$ bijects with $\mathbb{R}$. What about the sets of monotonic functions and of integrable functions?

9. A subset of $\mathbb{R}$ is *perfect* if it is closed and has no isolated points. Prove that every non-empty perfect set has cardinality that of $\mathbb{R}$, and that every closed set is the union of a perfect set and a countable set.

10. Construct a function from $\mathbb{R}$ to $\mathbb{R}$ that is infinitely differentiable, and which is identically 1 on $[-1, 1]$ and identically 0 outside $(-2, 2)$.

11. Let f be a differentiable function from $\mathbb{Q}$ to $\mathbb{Q}$ such that $f' = f$. Must f be identically zero?