Part II

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Logic and Set Theory

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Paper 1, Section II
16H Logic and Set Theory

(a) State and prove the Compactness Theorem for first-order logic. State and prove the Upward Lowenheim-Skolem Theorem. State the Downward Lowenheim-Skolem Theorem, and explain briefly why it is true.

(b) For a language L and an L-structure A, an *automorphism* of A is a bijection from A to itself that preserves all the functions and relations of L. An L-structure is *rigid* if it has no automorphism apart from the identity map.

(i) If a theory T in a language L has arbitrarily large finite non-rigid models, show that T also has an infinite non-rigid model.

(ii) If a theory T in a language L has arbitrarily large finite models, and every finite model of T is rigid, does it follow that every infinite model of T is rigid? Justify your answer.

[*You may assume the Completeness Theorem for first-order logic.*]

Paper 2, Section II
16H Logic and Set Theory

(a) Let α be a non-zero ordinal. Show that there is a greatest ordinal β such that $\omega^\beta \leqslant \alpha$. Deduce that there exist a non-zero natural number n and an ordinal $\gamma < \omega^\beta$ such that $\alpha = \omega^\beta.n + \gamma$.

(b) An ordinal δ is called *additively closed* if whenever $\beta < \delta$ and $\gamma < \delta$ then also $\beta + \gamma < \delta$. Show that a non-zero ordinal is additively closed if and only if it is of the form ω^α for some α.

(c) An ordinal δ is called *multiplicatively closed* if whenever $\beta < \delta$ and $\gamma < \delta$ then also $\beta\gamma < \delta$. Show that an ordinal greater than 2 is multiplicatively closed if and only if it is of the form $\omega^{(\omega^\alpha)}$ for some α.

[*You may assume standard properties of ordinal arithmetic.*]

Paper 3, Section II

16H Logic and Set Theory

In this question we work in a fixed model V of ZFC.

(a) Prove that every set has a transitive closure. [If you apply the Axiom of Replacement to a function-class F, you must explain clearly why F is indeed a function-class.]

(b) State the Axiom of Foundation and the Principle of ϵ-Induction, and show that they are equivalent (in the presence of the other axioms of ZFC).

(c) We say that a set x is *reasonable* if every member of $TC(\{x\})$ is countable. Which of the following are true and which are false? Justify your answers.

(i) A set is reasonable if and only if $TC(\{x\})$ is countable.

(ii) The reasonable sets are all members of V_α, for some α.

(iii) The reasonable sets form a model of ZFC.

[In (c) you may assume any results from the course.]

Paper 4, Section II

16H Logic and Set Theory

In this question we work in ZF, not ZFC. As usual, for cardinals κ and λ we write $\kappa \leqslant \lambda$ if there is an injection from K to L, where K and L are sets of cardinalities κ and λ, respectively.

(a) Show that the assertion that $\leqslant$ is a total ordering on cardinals (in other words, that for any κ and λ we have $\kappa \leqslant \lambda$ or $\lambda \leqslant \kappa$) is equivalent to the Axiom of Choice.

(b) Show that the Axiom of Choice implies that, for any infinite cardinal κ, we have $\kappa^2 = \kappa$.

(c) Suppose that κ and λ are non-zero cardinals such that $\kappa\lambda \leqslant \kappa + \lambda$. Prove that there must exist either an injection or a surjection from K to L.

(d) Show that the assertion that for any infinite cardinal κ we have $\kappa^2 = \kappa$ is equivalent to the Axiom of Choice. [*Hint: for a given set X, you may wish to consider the disjoint union of X with $\gamma(X)$.*]

[*You may assume Hartogs' Lemma, and you may use the equivalence of the Axiom of Choice with any of its equivalents from the course.*]

Paper 1, Section II
16F Logic and Set Theory

State and prove the *Knaster–Tarski fixed-point theorem.*

A subset S of a poset X is called a *down-set* if whenever $x, y \in X$ satisfy $x \in S$ and $y \leqslant x$ then also $y \in S$. Show that the set P of down-sets of X, ordered by inclusion, is a complete poset.

Now let X and Y be totally ordered sets.

(i) Give an example to show that we may have X isomorphic to a down-set in Y, and Y isomorphic to a down-set in X, and yet X is not isomorphic to Y. [*Hint: Consider suitable subsets of the reals.*]

(ii) Show that if X is isomorphic to a down-set in Y, and Y is isomorphic to the complement of a down-set in X, then X is isomorphic to Y.

Paper 2, Section II
16F Logic and Set Theory

(a) Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent.

(b) Which of the following assertions about ordinals α, β and γ are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$.

(ii) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

(iii) If α is a limit ordinal then $\alpha\omega = \omega\alpha$.

(iv) If $\alpha \geqslant \omega_1$ and $\beta < \omega_1$ then $\beta + \alpha = \alpha$.

(v) If $\alpha + \alpha + \beta$ and $\beta + \alpha + \alpha$ are equal then they are both equal to $\alpha + \beta + \alpha$.

Paper 3, Section II
16F   Logic and Set Theory

State and prove the *Compactness Theorem* for first-order predicate logic. State and prove the *Upward Löwenheim–Skolem Theorem.*

[*You may assume the Completeness Theorem for first-order predicate logic.*]

For each of the following theories, is the theory axiomatisable (in the language of posets, extended by some set of constants if necessary) or not? Justify your answers.

   (i) The theory of posets having only finitely many maximal elements.

  (ii) The theory of posets having uncountably many maximal elements.

 (iii) The theory of posets having infinitely many maximal elements or infinitely many minimal elements (or both).

 (iv) The theory of posets having infinitely many maximal elements or infinitely many minimal elements, but not both.

  (v) The theory of the total orders that are isomorphic to a subset of the reals.

Paper 4, Section II
16F   Logic and Set Theory

(a) Define the *von Neumann hierarchy* of sets V_α. Show that each V_α is transitive, and explain why $V_\alpha \subset V_\beta$ whenever $\alpha \leqslant \beta$. Prove that every set x is a member of some V_α.

(b) What does it mean to say that a relation r on a set x is *well-founded* and *extensional*? State *Mostowski's Collapsing Theorem.* Give an example of a set x whose rank is greater than ω but for which the Mostowski collapse of x (equipped with the relation $\in$) is equal to ω.

Which of the following statements are always true and which can be false? Give proofs or counterexamples as appropriate.

   (i) If a relation r on a set x is isomorphic to the relation $\in$ on some transitive set y then r is well-founded and extensional.

  (ii) If a relation r on a set x is isomorphic to the relation $\in$ on some (not necessarily transitive) set y then r is well-founded.

 (iii) If a relation r on a set x is isomorphic to the relation $\in$ on some (not necessarily transitive) set y then r is extensional.

Paper 1, Section II
16G Logic and Set Theory

Let S and T be sets of propositional formulae.

(a) What does it mean to say that S is *deductively closed*? What does it mean to say that S is *consistent*? Explain briefly why if S is inconsistent then some finite subset of S is inconsistent.

(b) We write $S \vdash T$ to mean $S \vdash t$ for all $t \in T$. If $S \vdash T$ and $T \vdash S$ we say S and T are *equivalent*. If S is equivalent to a finite set F of formulae we say that S is *finitary*. Show that if S is finitary then there is a finite set $R \subset S$ with $R \vdash S$.

(c) Now let T_0, T_1, T_2, ... be deductively closed sets of formulae with

$$T_0 \subsetneqq T_1 \subsetneqq T_2 \subsetneqq \cdots .$$

Show that each T_i is consistent.

Let $T = \bigcup_{i=0}^{\infty} T_i$. Show that T is consistent and deductively closed, but that it is not finitary.

Paper 2, Section II
16G Logic and Set Theory

Write down the inductive definition of ordinal exponentiation. Show that $\omega^\alpha \geqslant \alpha$ for every ordinal α. Deduce that, for every ordinal α, there is a least ordinal α^* with $\omega^{\alpha^*} > \alpha$. Show that, if $\alpha \neq 0$, then α^* must be a successor ordinal.

Now let α be a non-zero ordinal. Show that there exist ordinals β and γ, where $\gamma < \alpha$, and a positive integer n such that $\alpha = \omega^\beta n + \gamma$. Hence, or otherwise, show that α can be written in the form

$$\alpha = \omega^{\beta_1} n_1 + \omega^{\beta_2} n_2 + \cdots + \omega^{\beta_k} n_k ,$$

where k, n_1, n_2, ..., n_k are positive integers and $\beta_1 > \beta_2 > \cdots > \beta_k$ are ordinals. [We call this the *Cantor normal form* of α, and you may henceforth assume that it is unique.]

Given ordinals δ_1, δ_2 and positive integers m_1, m_2 find the Cantor normal form of $\omega^{\delta_1} m_1 + \omega^{\delta_2} m_2$. Hence, or otherwise, given non-zero ordinals α and α', find the Cantor normal form of $\alpha + \alpha'$ in terms of the Cantor normal forms

$$\alpha = \omega^{\beta_1} n_1 + \omega^{\beta_2} n_2 + \cdots + \omega^{\beta_k} n_k$$

and

$$\alpha' = \omega^{\beta'_1} n'_1 + \omega^{\beta'_2} n'_2 + \cdots + \omega^{\beta'_{k'}} n'_{k'}$$

of α and α'.

Paper 3, Section II
16G Logic and Set Theory

(a) Let κ and λ be cardinals. What does it mean to say that $\kappa < \lambda$? Explain briefly why, assuming the Axiom of Choice, every infinite cardinal is of the form $\aleph_\alpha$ for some ordinal α, and that for every ordinal α we have $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$.

(b) Henceforth, you should not assume the Axiom of Choice.

Show that, for any set x, there is an injection from x to its power set $\mathcal{P}x$, but there is no bijection from x to $\mathcal{P}x$. Deduce that if κ is a cardinal then $\kappa < 2^\kappa$.

Let x and y be sets, and suppose that there exists a surjection $f\colon x \to y$. Show that there exists an injection $g\colon \mathcal{P}y \to \mathcal{P}x$.

Let α be an ordinal. Prove that $\aleph_\alpha \aleph_\alpha = \aleph_\alpha$.

By considering $\mathcal{P}(\omega_\alpha \times \omega_\alpha)$ as the set of relations on ω_α, or otherwise, show that there exists a surjection $f\colon \mathcal{P}(\omega_\alpha \times \omega_\alpha) \to \omega_{\alpha+1}$. Deduce that $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$.

Paper 4, Section II
16G Logic and Set Theory

Write down the Axiom of Foundation.

What is the *transitive closure* of a set x? Prove carefully that every set x has a transitive closure. State and prove the principle of $\in$-induction.

Let $(V, \in)$ be a model of ZF. Let $F\colon V \to V$ be a function class such that for all $x, y \in V$ we have $F(x) \in F(y)$ if and only if $x \in y$. Show, by $\in$-induction or otherwise, that F is the identity.

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Paper 1, Section II

16H Logic and Set Theory

[Throughout this question, assume the axiom of choice.]

Let κ, λ and μ be cardinals. Define $\kappa + \lambda$, $\kappa\lambda$ and κ^λ. What does it mean to say $\kappa \leqslant \lambda$? Show that $(\kappa^\lambda)^\mu = \kappa^{\lambda\mu}$. Show also that $2^\kappa > \kappa$.

Assume now that κ and λ are infinite. Show that $\kappa\kappa = \kappa$. Deduce that $\kappa + \lambda = \kappa\lambda = \max\{\kappa, \lambda\}$. Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) $\kappa^\lambda = 2^\lambda$;

(ii) $\kappa \leqslant \lambda \implies \kappa^\lambda = 2^\lambda$;

(iii) $\kappa^\lambda = \lambda^\kappa$.

Paper 2, Section II

16H Logic and Set Theory

(a) This part of the question is concerned with propositional logic.

Let P be a set of primitive propositions. Let $S \subset L(P)$ be a consistent, deductively closed set such that for every $t \in L(P)$ either $t \in S$ or $\neg t \in S$. Show that S has a model.

(b) This part of the question is concerned with predicate logic.

(i) State Gödel's completeness theorem for first-order logic. Deduce the compactness theorem, which you should state precisely.

(ii) Let X be an infinite set. For each $x \in X$, let L_x be a subset of X. Suppose that for any finite $Y \subseteq X$ there exists a function $f_Y : Y \to \{1, \dots, 100\}$ such that for all $x \in Y$ and all $y \in Y \cap L_x$, $f_Y(x) \neq f_Y(y)$. Show that there exists a function $F : X \to \{1, \dots, 100\}$ such that for all $x \in X$ and all $y \in L_x$, $F(x) \neq F(y)$.

Paper 3, Section II

16H Logic and Set Theory

Let $(V, \in)$ be a model of ZF. Give the definition of a *class* and a *function class* in V. Use the concept of function class to give a short, informal statement of the Axiom of Replacement.

Let $z_0 = \omega$ and, for each $n \in \omega$, let $z_{n+1} = \mathcal{P}z_n$. Show that $y = \{z_n | n \in \omega\}$ is a set.

We say that a set x is small if there is an injection from x to z_n for some $n \in \omega$. Let **HS** be the class of sets x such that every member of $\text{TC}(\{x\})$ is small, where $\text{TC}(\{x\})$ is the transitive closure of $\{x\}$. Show that $n \in$ **HS** for all $n \in \omega$ and deduce that $\omega \in$ **HS**. Show further that $z_n \in$ **HS** for all $n \in \omega$. Deduce that $y \in$ **HS**.

Is $(\mathbf{HS}, \in)$ a model of ZF? Justify your answer.

[*Recall that $0 = \emptyset$ and that $n + 1 = n \cup \{n\}$ for all $n \in \omega$.*]

Paper 4, Section II

16H Logic and Set Theory

(a) State Zorn's lemma.

[Throughout the remainder of this question, assume Zorn's lemma.]

(b) Let P be a poset in which every non-empty chain has an upper bound and let $x \in P$. By considering the poset $P_x = \{y \in P \mid x \leqslant y\}$, show that P has a maximal element σ with $x \leqslant \sigma$.

(c) A filter is a non-empty subset $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ satisfying the following three conditions:

- if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;

- if $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$;

- $\emptyset \notin \mathcal{F}$.

An ultrafilter is a filter $\mathcal{U}$ such that for all $A \subset \mathbb{N}$ we have either $A \in \mathcal{U}$ or $A^c \in \mathcal{U}$, where $A^c = \mathbb{N}\backslash A$.

(i) For each $n \in \mathbb{N}$, show that $\mathcal{U}_n = \{A \subset \mathbb{N} \mid n \in A\}$ is an ultrafilter.

(ii) Show that $\mathcal{F} = \{A \subset \mathbb{N} \mid A^c \text{ is finite}\}$ is a filter but not an ultrafilter, and that for all $n \in \mathbb{N}$ we have $\mathcal{F} \not\subset \mathcal{U}_n$.

(iii) Does there exist an ultrafilter $\mathcal{U}$ such that $\mathcal{U} \neq \mathcal{U}_n$ for any $n \in \mathbb{N}$? Justify your answer.

Paper 4, Section II

16I   Logic and Set Theory

Define the cardinals $\aleph_\alpha$, and explain briefly why every infinite set has cardinality an $\aleph_\alpha$.

Show that if κ is an infinite cardinal then $\kappa^2 = \kappa$.

Let $X_1, X_2, \ldots, X_n$ be infinite sets. Show that $X_1 \cup X_2 \cup \cdots \cup X_n$ must have the same cardinality as X_i for some i.

Let $X_1, X_2, \ldots$ be infinite sets, no two of the same cardinality. Is it possible that $X_1 \cup X_2 \cup \ldots$ has the same cardinality as some X_i? Justify your answer.

Paper 3, Section II

16I   Logic and Set Theory

Define the *von Neumann hierarchy* of sets V_α. Show that each V_α is transitive, and explain why $V_\alpha \subset V_\beta$ whenever $\alpha \leqslant \beta$. Prove that every set x is a member of some V_α.

Which of the following are true and which are false? Give proofs or counterexamples as appropriate. [You may assume standard properties of rank.]

(i) If the rank of a set x is a (non-zero) limit then x is infinite.

(ii) If the rank of a set x is countable then x is countable.

(iii) If every finite subset of a set x has rank at most α then x has rank at most α.

(iv) For every ordinal α there exists a set of rank α.

Paper 2, Section II

16I   Logic and Set Theory

Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent.

Which of the following assertions about ordinals α, β and γ are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(ii) If α and β are uncountable then $\alpha + \beta = \beta + \alpha$.

(iii) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

(iv) If α and β are infinite and $\alpha + \beta = \beta + \alpha$ then $\alpha\beta = \beta\alpha$.

Paper 1, Section II

16I   Logic and Set Theory

State the *completeness theorem* for propositional logic. Explain briefly how the proof of this theorem changes from the usual proof in the case when the set of primitive propositions may be uncountable.

State the *compactness theorem* and the *decidability theorem*, and deduce them from the completeness theorem.

A poset $(X, <)$ is called *two-dimensional* if there exist total orders $<_1$ and $<_2$ on X such that $x < y$ if and only if $x <_1 y$ and $x <_2 y$. By applying the compactness theorem for propositional logic, show that if every finite subset of a poset is two-dimensional then so is the poset itself.

[Hint: Take primitive propositions $p_{x,y}$ and $q_{x,y}$, for each distinct $x, y \in X$, with the intended interpretation that $p_{x,y}$ is true if and only if $x <_1 y$ and $q_{x,y}$ is true if and only if $x <_2 y$.]

Paper 4, Section II

16G Logic and Set Theory

State and prove the ϵ-Recursion Theorem. [You may assume the Principle of ϵ-Induction.]

What does it mean to say that a relation r on a set x is *well-founded* and *extensional*? State and prove Mostowski's Collapsing Theorem. [You may use any recursion theorem from the course, provided you state it precisely.]

For which sets x is it the case that every well-founded extensional relation on x is isomorphic to the relation ϵ on some transitive subset of V_ω?

Paper 3, Section II

16G Logic and Set Theory

State and prove the Compactness Theorem for first-order predicate logic. State and prove the Upward Löwenheim–Skolem Theorem.

[You may assume the Completeness Theorem for first-order predicate logic.]

For each of the following theories, either give axioms (in the specified language) for the theory or prove that the theory is not axiomatisable.

(i) The theory of finite groups (in the language of groups).

(ii) The theory of groups in which every non-identity element has infinite order (in the language of groups).

(iii) The theory of total orders (in the language of posets).

(iv) The theory of well-orderings (in the language of posets).

If a theory is axiomatisable by a set S of sentences, and also by a finite set T of sentences, does it follow that the theory is axiomatisable by some finite subset of S? Justify your answer.

Paper 2, Section II

16G Logic and Set Theory

State and prove the Knaster–Tarski Fixed-Point Theorem. Deduce the Schröder–Bernstein Theorem.

Show that the poset P of all countable subsets of $\mathbb{R}$ (ordered by inclusion) is not complete.

Find an order-preserving function $f : P \to P$ that does not have a fixed point. [*Hint: Start by well-ordering the reals.*]

Paper 1, Section II

16G   Logic and Set Theory

Give the inductive definition of *ordinal exponentiation*. Use it to show that $\alpha^\beta \leqslant \alpha^\gamma$ whenever $\beta \leqslant \gamma$ (for $\alpha \geqslant 1$), and also that $\alpha^\beta < \alpha^\gamma$ whenever $\beta < \gamma$ (for $\alpha \geqslant 2$).

Give an example of ordinals α and β with $\omega < \alpha < \beta$ such that $\alpha^\omega = \beta^\omega$.

Show that $\alpha^{\beta+\gamma} = \alpha^\beta \alpha^\gamma$, for any ordinals α, β, γ, and give an example to show that we need not have $(\alpha\beta)^\gamma = \alpha^\gamma \beta^\gamma$.

For which ordinals α do we have $\alpha^{\omega_1} \geqslant \omega_1$? And for which do we have $\alpha^{\omega_1} \geqslant \omega_2$? Justify your answers.

[*You may assume any standard results not concerning ordinal exponentiation.*]

Paper 3, Section II
14H Logic and Set Theory

State and prove Zorn's Lemma. [You may assume Hartogs' Lemma.] Indicate clearly where in your proof you have made use of the Axiom of Choice.

Show that $\mathbb{R}$ has a basis as a vector space over $\mathbb{Q}$.

Let V be a vector space over $\mathbb{Q}$. Show that all bases of V have the same cardinality.

[*Hint: How does the cardinality of V relate to the cardinality of a given basis?*]

Paper 2, Section II
14H Logic and Set Theory

Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent.

Which of the following are always true for ordinals α, β and γ and which can be false? Give proofs or counterexamples as appropriate.

 (i) $\alpha + \beta = \beta + \alpha$

 (ii) $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$

 (iii) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

 (iv) If $\alpha\beta = \beta\alpha$ then $\alpha^2\beta^2 = \beta^2\alpha^2$

 (v) If $\alpha^2\beta^2 = \beta^2\alpha^2$ then $\alpha\beta = \beta\alpha$

[In parts (iv) and (v) you may assume without proof that ordinal multiplication is associative.]

Paper 4, Section II
15H Logic and Set Theory

Prove that every set has a transitive closure. [If you apply the Axiom of Replacement to a function-class F, you must explain clearly why F is indeed a function-class.]

State the Axiom of Foundation and the Principle of ϵ-Induction, and show that they are equivalent (in the presence of the other axioms of ZFC).

State the ϵ-Recursion Theorem.

Sets C_α are defined for each ordinal α by recursion, as follows: $C_0 = \emptyset$, $C_{\alpha+1}$ is the set of all countable subsets of C_α, and $C_\lambda = \cup_{\alpha<\lambda} C_\alpha$ for λ a non-zero limit. Does there exist an α with $C_{\alpha+1} = C_\alpha$? Justify your answer.

Paper 1, Section II
15H Logic and Set Theory

State the Completeness Theorem for Propositional Logic.

[You do *not* need to give definitions of the various terms involved.]

State the Compactness Theorem and the Decidability Theorem, and deduce them from the Completeness Theorem.

A set S of propositions is called *finitary* if there exists a finite set T of propositions such that $\{t : S \vdash t\} = \{t : T \vdash t\}$. Give examples to show that an infinite set of propositions may or may not be finitary.

Now let A and B be sets of propositions such that every valuation is a model of exactly one of A and B. Show that there exist finite subsets A' of A and B' of B with $A' \cup B' \models \bot$, and deduce that A and B are finitary.

Paper 2, Section II

14F Logic and Set Theory

Define the *von Neumann hierarchy* of sets V_α, and show that each V_α is a transitive set. Explain what is meant by saying that a binary relation on a set is *well-founded* and *extensional*. State Mostowski's Theorem.

Let r be the binary relation on ω defined by: $\langle m, n \rangle \in r$ if and only if 2^m appears in the base-2 expansion of n (i.e., the unique expression for n as a sum of distinct powers of 2). Show that r is well-founded and extensional. To which transitive set is (ω, r) isomorphic? Justify your answer.

Paper 3, Section II

14F Logic and Set Theory

State the Completeness Theorem for the first-order predicate calculus, and deduce the Compactness Theorem.

Let $\mathbb{T}$ be a first-order theory over a signature Σ whose axioms all have the form $(\forall \vec{x})\phi$ where $\vec{x}$ is a (possibly empty) string of variables and ϕ is quantifier-free. Show that every substructure of a $\mathbb{T}$-model is a $\mathbb{T}$-model, and deduce that if $\mathbb{T}$ is consistent then it has a model in which every element is the interpretation of a closed term of $\mathcal{L}(\Sigma)$. [You may assume the result that if B is a substructure of A and ϕ is a quantifier-free formula with n free variables, then $[\![\phi]\!]_B = [\![\phi]\!]_A \cap B^n$.]

Now suppose $\mathbb{T} \vdash (\exists x)\psi$ where ψ is a quantifier-free formula with one free variable x. Show that there is a finite list $(t_1, t_2, \ldots, t_n)$ of closed terms of $\mathcal{L}(\Sigma)$ such that

$$\mathbb{T} \vdash (\psi[t_1/x] \vee \psi[t_2/x] \vee \cdots \vee \psi[t_n/x]).$$

Paper 1, Section II

15F Logic and Set Theory

Which of the following statements are true? Justify your answers.

(a) Every ordinal is of the form $\alpha + n$, where α is a limit ordinal and $n \in \omega$.

(b) Every ordinal is of the form $\omega^\alpha . m + n$, where α is an ordinal and $m, n \in \omega$.

(c) If $\alpha = \omega . \alpha$, then $\alpha = \omega^\omega . \beta$ for some β.

(d) If $\alpha = \omega^\alpha$, then α is uncountable.

(e) If $\alpha > 1$ and $\alpha = \alpha^\omega$, then α is uncountable.

[*Standard laws of ordinal arithmetic may be assumed, but if you use the Division Algorithm you should prove it.*]

Paper 4, Section II

15F   Logic and Set Theory

(a) State Zorn's Lemma, and use it to prove that every nontrivial distributive lattice L admits a lattice homomorphism $L \to \{0, 1\}$.

(b) Let S be a propositional theory in a given language $\mathcal{L}$. Sketch the way in which the equivalence classes of formulae of $\mathcal{L}$, modulo S-provable equivalence, may be made into a Boolean algebra. [Detailed proofs are not required, but you should define the equivalence relation explicitly.]

(c) Hence show how the Completeness Theorem for propositional logic may be deduced from the result of part (a).

Paper 4, Section II

13I   Logic and Set Theory

State the Axiom of Foundation and the Principle of ϵ-Induction, and show that they are equivalent (in the presence of the other axioms of ZF). [You may assume the existence of transitive closures.]

Explain briefly how the Principle of ϵ-Induction implies that every set is a member of some V_α.

Find the ranks of the following sets:

(i) $\{\omega + 1, \omega + 2, \omega + 3\}$,

(ii) the Cartesian product $\omega \times \omega$,

(iii) the set of all functions from ω to ω^2.

[You may assume standard properties of rank.]

Paper 3, Section II

13I   Logic and Set Theory

(i) State and prove Zorn's Lemma. [You may assume Hartogs' Lemma.] Where in your proof have you made use of the Axiom of Choice?

(ii) Let $<$ be a partial ordering on a set X. Prove carefully that $<$ may be extended to a total ordering of X.

What does it mean to say that $<$ is *well-founded*?

If $<$ has an extension that is a well-ordering, must $<$ be well-founded? If $<$ is well-founded, must every total ordering extending it be a well-ordering? Justify your answers.

Paper 2, Section II

13I Logic and Set Theory

(a) Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent. Give the inductive definitions of ordinal multiplication and ordinal exponentiation.

(b) Answer, with brief justification, the following:

(i) For ordinals α, β and γ with $\alpha < \beta$, must we have $\alpha + \gamma < \beta + \gamma$? Must we have $\gamma + \alpha < \gamma + \beta$?

(ii) For ordinals α and β with $\alpha < \beta$, must we have $\alpha^\omega < \beta^\omega$?

(iii) Is there an ordinal $\alpha > 1$ such that $\alpha^\omega = \alpha$?

(iv) Show that $\omega^{\omega_1} = \omega_1$. Is ω_1 the least ordinal α such that $\omega^\alpha = \alpha$?

[You may use standard facts about ordinal arithmetic.]

Paper 1, Section II

13I Logic and Set Theory

State and prove the Completeness Theorem for Propositional Logic.

[You do *not* need to give definitions of the various terms involved. You may assume the Deduction Theorem, provided that you state it precisely.]

State the Compactness Theorem and the Decidability Theorem, and deduce them from the Completeness Theorem.

Let S consist of the propositions $p_{n+1} \Rightarrow p_n$ for $n = 1, 2, 3, \ldots$. Does S prove p_1? Justify your answer. [Here $p_1, p_2, p_3, \ldots$ are primitive propositions.]

Paper 4, Section II

16I   Logic and Set Theory

Explain what is meant by a *chain-complete* poset. State the Bourbaki–Witt fixed-point theorem.

We call a poset $(P, \leqslant)$ *Bourbakian* if every order-preserving map $f: P \to P$ has a least fixed point $\mu(f)$. Suppose P is Bourbakian, and let $f, g: P \rightrightarrows P$ be order-preserving maps with $f(x) \leqslant g(x)$ for all $x \in P$; show that $\mu(f) \leqslant \mu(g)$. [*Hint: Consider the function $h: P \to P$ defined by $h(x) = f(x)$ if $x \leqslant \mu(g)$, $h(x) = \mu(g)$ otherwise.*]

Suppose P is Bourbakian and $f: \alpha \to P$ is an order-preserving map from an ordinal to P. Show that there is an order-preserving map $g: P \to P$ whose fixed points are exactly the upper bounds of the set $\{f(\beta) \mid \beta < \alpha\}$, and deduce that this set has a least upper bound.

Let C be a chain with no greatest member. Using the Axiom of Choice and Hartogs' Lemma, show that there is an order-preserving map $f: \alpha \to C$, for some ordinal α, whose image has no upper bound in C. Deduce that any Bourbakian poset is chain-complete.

Paper 3, Section II

16I   Logic and Set Theory

Explain what is meant by a *structure* for a first-order signature Σ, and describe briefly how first-order terms and formulae in the language over Σ are interpreted in a structure. Suppose that A and B are Σ-structures, and that ϕ is a conjunction of atomic formulae over Σ: show that an n-tuple $((a_1, b_1), \ldots, (a_n, b_n))$ belongs to the interpretation $[\![\phi]\!]_{A \times B}$ of ϕ in $A \times B$ if and only if $(a_1, \ldots, a_n) \in [\![\phi]\!]_A$ and $(b_1, \ldots, b_n) \in [\![\phi]\!]_B$.

A first-order theory $\mathbb{T}$ is called *regular* if its axioms all have the form

$$(\forall \overrightarrow{x})(\phi \Rightarrow (\exists \overrightarrow{y})\psi),$$

where $\overrightarrow{x}$ and $\overrightarrow{y}$ are (possibly empty) strings of variables and ϕ and ψ are conjunctions of atomic formulae (possibly the empty conjunction $\top$). Show that if A and B are models of a regular theory $\mathbb{T}$, then so is $A \times B$.

Now suppose that $\mathbb{T}$ is a regular theory, and that a sentence of the form

$$(\forall \overrightarrow{x})(\phi \Rightarrow (\psi_1 \vee \psi_2 \vee \cdots \vee \psi_n))$$

is derivable from the axioms of $\mathbb{T}$, where ϕ and the ψ_i are conjunctions of atomic formulae. Show that the sentence $(\forall \overrightarrow{x})(\phi \Rightarrow \psi_i)$ is derivable for some i. [*Hint: Suppose not, and use the Completeness Theorem to obtain a suitable family of $\mathbb{T}$-models $A_1, \ldots, A_n$.*]

Paper 2, Section II

16I   Logic and Set Theory

Write down the recursive definitions of ordinal addition, multiplication and exponentiation. Show that, for any nonzero ordinal α, there exist unique ordinals β, γ and n such that $\alpha = \omega^\beta . n + \gamma$, $\gamma < \omega^\beta$ and $0 < n < \omega$.

Hence or otherwise show that α (that is, the set of ordinals less than α) is closed under addition if and only if $\alpha = \omega^\beta$ for some β. Show also that an infinite ordinal α is closed under multiplication if and only if $\alpha = \omega^{(\omega^\gamma)}$ for some γ.

[You may assume the standard laws of ordinal arithmetic, and the fact that $\alpha \leqslant \omega^\alpha$ for all α.]

Paper 1, Section II

16I   Logic and Set Theory

Explain what is meant by saying that a binary relation $r \subseteq a \times a$ is *well-founded*. Show that r is well-founded if and only if, for any set b and any function $f \colon \mathcal{P}b \to b$, there exists a unique function $g \colon a \to b$ satisfying

$$g(x) = f(\{g(y) \mid \langle y, x \rangle \in r\})$$

for all $x \in a$. [*Hint: For 'if', it suffices to take $b = \{0,1\}$, with $f \colon \mathcal{P}b \to b$ defined by $f(b') = 1 \Leftrightarrow 1 \in b'$.*]

Paper 2, Section II
16G Logic and Set Theory

Explain what is meant by a chain-complete poset. State the Bourbaki–Witt fixed-point theorem for such posets.

A poset P is called *directed* if every finite subset of P (including the empty subset) has an upper bound in P; P is called *directed-complete* if every subset of P which is directed (in the induced ordering) has a least upper bound in P. Show that the set of all chains in an arbitrary poset P, ordered by inclusion, is directed-complete.

Given a poset P, let $[P \to P]$ denote the set of all order-preserving maps $P \to P$, ordered pointwise (i.e. $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all x). Show that $[P \to P]$ is directed-complete if P is.

Now suppose P is directed-complete, and that $f : P \to P$ is order-preserving and inflationary. Show that there is a unique smallest set $C \subseteq [P \to P]$ satisfying

   (a) $f \in C$;

   (b) C is closed under composition (i.e. $g, h \in C \Rightarrow g \circ h \in C$); and

   (c) C is closed under joins of directed subsets.

Show that

   (i) all maps in C are inflationary;

   (ii) C is directed;

   (iii) if $g = \bigvee C$, then all values of g are fixed points of f;

   (iv) for every $x \in P$, there exists $y \in P$ with $x \leqslant y = f(y)$.

Paper 3, Section II

16G Logic and Set Theory

Explain carefully what is meant by *syntactic entailment* and *semantic entailment* in the propositional calculus. State the Completeness Theorem for the propositional calculus, and deduce the Compactness Theorem.

Suppose P, Q and R are pairwise disjoint sets of primitive propositions, and let $\phi \in \mathcal{L}(P \cup Q)$ and $\psi \in \mathcal{L}(Q \cup R)$ be propositional formulae such that $(\phi \Rightarrow \psi)$ is a theorem of the propositional calculus. Consider the set

$$X = \{\chi \in \mathcal{L}(Q) \mid \phi \vdash \chi\} \ .$$

Show that $X \cup \{\neg\psi\}$ is inconsistent, and deduce that there exists $\chi \in \mathcal{L}(Q)$ such that both $(\phi \Rightarrow \chi)$ and $(\chi \Rightarrow \psi)$ are theorems. [*Hint: assuming $X \cup \{\neg\psi\}$ is consistent, take a suitable valuation v of $Q \cup R$ and show that*

$$\{\phi\} \cup \{q \in Q \mid v(q) = 1\} \cup \{\neg q \mid q \in Q, v(q) = 0\}$$

is inconsistent. The Deduction Theorem may be assumed without proof.]

Paper 4, Section II

16G Logic and Set Theory

State the Axiom of Foundation and the Principle of $\in$-Induction, and show that they are equivalent in the presence of the other axioms of ZF set theory. [You may assume the existence of transitive closures.]

Given a model $(V, \in)$ for all the axioms of ZF except Foundation, show how to define a transitive class R which, with the restriction of the given relation $\in$, is a model of ZF.

Given a model $(V, \in)$ of ZF, indicate briefly how one may modify the relation $\in$ so that the resulting structure $(V, \in')$ fails to satisfy Foundation, but satisfies all the other axioms of ZF. [You need not verify that all the other axioms hold in $(V, \in')$.]

Paper 1, Section II
16G Logic and Set Theory

Write down the recursive definitions of ordinal addition, multiplication and exponentiation.

Given that $F \colon \mathbf{On} \to \mathbf{On}$ is a strictly increasing function-class (i.e. $\alpha < \beta$ implies $F(\alpha) < F(\beta)$), show that $\alpha \leqslant F(\alpha)$ for all α.

Show that every ordinal α has a unique representation in the form

$$\alpha = \omega^{\alpha_1}.a_1 + \omega^{\alpha_2}.a_2 + \cdots + \omega^{\alpha_n}.a_n \ ,$$

where $n \in \omega$, $\alpha \geqslant \alpha_1 > \alpha_2 > \cdots > \alpha_n$, and $a_1, a_2, \ldots, a_n \in \omega \setminus \{0\}$.

Under what conditions can an ordinal α be represented in the form

$$\omega^{\beta_1}.b_1 + \omega^{\beta_2}.b_2 + \cdots + \omega^{\beta_m}.b_m \ ,$$

where $\beta_1 < \beta_2 < \cdots < \beta_m$ and $b_1, b_2, \ldots, b_m \in \omega \setminus \{0\}$? Justify your answer.

[The laws of ordinal arithmetic (associative, distributive, etc.) may be assumed without proof.]

Paper 2, Section II

16H Logic and Set Theory

Explain what is meant by a substructure of a Σ-structure A, where Σ is a first-order signature (possibly including both predicate symbols and function symbols). Show that if B is a substructure of A, and ϕ is a first-order formula over Σ with n free variables, then $[\phi]_B = [\phi]_A \cap B^n$ if ϕ is quantifier-free. Show also that $[\phi]_B \subseteq [\phi]_A \cap B^n$ if ϕ is an existential formula (that is, one of the form $(\exists x_1, \ldots, x_m)\psi$ where ψ is quantifier-free), and $[\phi]_B \supseteq [\phi]_A \cap B^n$ if ϕ is a universal formula. Give examples to show that the two latter inclusions can be strict.

Show also that

(a) if T is a first-order theory whose axioms are all universal sentences, then any substructure of a T-model is a T-model;

(b) if T is a first-order theory such that every first-order formula ϕ is T-provably equivalent to a universal formula (that is, $T \vdash (\phi \Leftrightarrow \psi)$ for some universal ψ), and B is a sub-T-model of a T-model A, then $[\phi]_B = [\phi]_A \cap B^n$ for every first-order formula ϕ with n free variables.

Paper 4, Section II

16H Logic and Set Theory

State and prove Hartogs' lemma. [You may assume the result that any well-ordered set is isomorphic to a unique ordinal.]

Let a and b be sets such that there is a bijection $a \sqcup b \to a \times b$. Show, without assuming the Axiom of Choice, that there is either a surjection $b \to a$ or an injection $b \to a$. By setting $b = \gamma(a)$ (the Hartogs ordinal of a) and considering $(a \sqcup b) \times (a \sqcup b)$, show that the assertion 'For all infinite cardinals m, we have $m^2 = m$' implies the Axiom of Choice. [You may assume the Cantor–Bernstein theorem.]

Paper 3, Section II
16H Logic and Set Theory

Write down **either** the synthetic **or** the recursive definitions of ordinal addition and multiplication. Using your definitions, give proofs or counterexamples for the following statements:

(i) For all α, β and γ, we have $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$.

(ii) For all α, β and γ, we have $(\alpha + \beta).\gamma = \alpha.\gamma + \beta.\gamma$.

(iii) For all α and β with $\beta > 0$, there exist γ and δ with $\delta < \beta$ and $\alpha = \beta.\gamma + \delta$.

(iv) For all α and β with $\beta > 0$, there exist γ and δ with $\delta < \beta$ and $\alpha = \gamma.\beta + \delta$.

(v) For every α, either there exists a cofinal map $f : \omega \to \alpha$ (that is, one such that $\alpha = \bigcup\{f(n)^+ \mid n \in \omega\}$), or there exists β such that $\alpha = \omega_1.\beta$.

Paper 1, Section II
16H Logic and Set Theory

State Zorn's lemma, and show how it may be deduced from the Axiom of Choice using the Bourbaki–Witt theorem (which should be clearly stated but not proved).

Show that, if a and b are distinct elements of a distributive lattice L, there is a lattice homomorphism $f : L \to \{0, 1\}$ with $f(a) \neq f(b)$. Indicate briefly how this result may be used to prove the completeness theorem for propositional logic.

Paper 1, Section II
16H Logic and Set Theory

Give the inductive and synthetic definitions of ordinal addition, and prove that they are equivalent.

Which of the following assertions about ordinals α, β and γ are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) $\alpha\beta = \beta\alpha$.

(ii) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.

(iii) If $\alpha \geqslant \omega^2$ then $\alpha + \omega^2 = \omega^2 + \alpha$.

(iv) If $\alpha \geqslant \omega_1$ then $\alpha\omega_1 = \omega_1\alpha$.

Paper 2, Section II
16H Logic and Set Theory

State and prove Zorn's Lemma. [You may assume Hartogs' Lemma.] Where in your argument have you made use of the Axiom of Choice?

Show that every real vector space has a basis.

Let V be a real vector space having a basis of cardinality $\aleph_1$. What is the cardinality of V? Justify your answer.

Paper 3, Section II
16H Logic and Set Theory

State and prove the Upward Löwenheim–Skolem Theorem.

[You may assume the Compactness Theorem, provided that you state it clearly.]

A total ordering $(X, <)$ is called *dense* if for any $x < y$ there exists z with $x < z < y$. Show that a dense total ordering (on more than one point) cannot be a well-ordering.

For each of the following theories, either give axioms, in the language of posets, for the theory or prove carefully that the theory is not axiomatisable in the language of posets.

(i) The theory of dense total orderings.

(ii) The theory of countable dense total orderings.

(iii) The theory of uncountable dense total orderings.

(iv) The theory of well-orderings.

Paper 4, Section II

16H Logic and Set Theory

Define the sets V_α for ordinals α. Show that each V_α is transitive. Show also that $V_\alpha \subseteq V_\beta$ whenever $\alpha \leqslant \beta$. Prove that every set x is a member of some V_α.

For which ordinals α does there exist a set x such that the power-set of x has rank α? [You may assume standard properties of rank.]

Paper 1, Section II
16G Logic and Set Theory

Show that $\aleph_\alpha{}^2 = \aleph_\alpha$ for all α.

An infinite cardinal m is called *regular* if it cannot be written as a sum of fewer than m cardinals each of which is smaller than m. Prove that $\aleph_0$ and $\aleph_1$ are regular.

Is $\aleph_2$ regular? Is $\aleph_\omega$ regular? Justify your answers.

Paper 2, Section II
16G Logic and Set Theory

Let α be a non-zero ordinal. Prove that there exists a greatest ordinal β such that $\omega^\beta \leqslant \alpha$. Explain why there exists an ordinal γ with $\omega^\beta + \gamma = \alpha$. Prove that γ is unique, and that $\gamma < \alpha$.

A non-zero ordinal α is called *decomposable* if it can be written as the sum of two smaller non-zero ordinals. Deduce that if α is not a power of ω then α is decomposable.

Conversely, prove that if α is a power of ω then α is not decomposable.

[*Hint: consider the cases $\alpha = \omega^\beta$ (β a successor) and $\alpha = \omega^\beta$ (β a limit) separately.*]

Paper 3, Section II
16G Logic and Set Theory

Define the sets V_α, $\alpha \in ON$. What is meant by the *rank* of a set?

Explain briefly why, for every α, there exists a set of rank α.

Let x be a transitive set of rank α. Show that x has an element of rank β for every $\beta < \alpha$.

For which α does there exist a finite set of rank α? For which α does there exist a finite transitive set of rank α? Justify your answers.

[Standard properties of rank may be assumed.]

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Paper 4, Section II

16G Logic and Set Theory

State and prove the Completeness Theorem for Propositional Logic.

[You do **not** need to give definitions of the various terms involved. You may assume that the set of primitive propositions is countable. You may also assume the Deduction Theorem.]

Explain briefly how your proof should be modified if the set of primitive propositions is allowed to be uncountable.

Paper 1, Section II
16G Logic and Set Theory
Prove that if $G : \text{On} \times V \to V$ is a definable function, then there is a definable function $F : \text{On} \to V$ satisfying

$$F(\alpha) = G(\alpha, \{F(\beta) : \beta < \alpha\}).$$

Define the notion of an initial ordinal, and explain its significance for cardinal arithmetic. State Hartogs' lemma. Using the recursion theorem define, giving justification, a function $\omega : \text{On} \to \text{On}$ which enumerates the infinite initial ordinals.

Is there an ordinal α with $\alpha = \omega_\alpha$? Justify your answer.

Paper 2, Section II
16G Logic and Set Theory
(i) Give an axiom system and rules of inference for the classical propositional calculus, and explain the notion of *syntactic entailment*. What does it mean to say that a set of propositions is consistent? Let P be a set of primitive propositions and let Φ be a maximal consistent set of propositional formulae in the language based on P. Show that there is a valuation $v : P \to \{T, F\}$ with respect to which all members of Φ are true.
[You should state clearly but need not prove those properties of syntactic entailment which you use.]

(ii) Exhibit a theory T which axiomatizes the collection of groups all of whose non-unit elements have infinite order. Is this theory finitely axiomatizable? Is the theory of groups all of whose elements are of finite order axiomatizable? Justify your answers.

Paper 3, Section II
16G Logic and Set Theory
Let $x \subseteq \alpha$ be a subset of a (von Neumann) ordinal α taken with the induced ordering. Using the recursion theorem or otherwise show that x is order isomorphic to a unique ordinal $\mu(x)$. Suppose that $x \subseteq y \subseteq \alpha$. Show that $\mu(x) \leqslant \mu(y) \leqslant \alpha$.

Suppose that $x_0 \subseteq x_1 \subseteq x_2 \subseteq \cdots$ is an increasing sequence of subsets of α, with x_i an initial segment of x_j whenever $i < j$. Show that $\mu(\bigcup_n x_n) = \bigcup_n \mu(x_n)$. Does this result still hold if the condition on initial segments is omitted? Justify your answer.

Suppose that $x_0 \supseteq x_1 \supseteq x_2 \supseteq \cdots$ is a decreasing sequence of subsets of α. Why is the sequence $\mu(x_n)$ eventually constant? Is it the case that $\mu(\bigcap_n x_n) = \bigcap_n \mu(x_n)$? Justify your answer.

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Paper 4, Section II

16G Logic and Set Theory

What is a *transitive class*? What is the significance of this notion for models of set theory?

Prove that for any set x there is a least transitive set $\mathrm{TC}(x)$, the transitive closure of x, with $x \subseteq \mathrm{TC}(x)$. Show that for any set x, one has $\mathrm{TC}(x) = x \cup \mathrm{TC}(\bigcup x)$, and deduce that $\mathrm{TC}(\{x\}) = \{x\} \cup \mathrm{TC}(x)$.

A set x is hereditarily countable when every member of $\mathrm{TC}(\{x\})$ is countable. Let $(\mathrm{HC}, \in)$ be the collection of hereditarily countable sets with the usual membership relation. Is HC transitive? Show that $(\mathrm{HC}, \in)$ satisfies the axiom of unions. Show that $(\mathrm{HC}, \in)$ satisfies the axiom of separation. What other axioms of ZF set theory are satisfied in $(\mathrm{HC}, \in)$?

1/II/16G Logic and Set Theory

What is a well-ordered set? Show that given any two well-ordered sets there is a unique order isomorphism between one and an initial segment of the other.

Show that for any ordinal α and for any non-zero ordinal β there are unique ordinals γ and δ with $\alpha = \beta.\gamma + \delta$ and $\delta < \beta$.

Show that a non-zero ordinal λ is a limit ordinal if and only if $\lambda = \omega.\gamma$ for some non-zero ordinal γ.

[You may assume standard properties of ordinal addition, multiplication and subtraction.]

2/II/16G Logic and Set Theory

(i) State the Completeness Theorem and the Compactness Theorem for the predicate calculus.

(ii) Show that if a theory has arbitrarily large finite models then it has an infinite model. Deduce that there is no first order theory whose models are just the finite fields of characteristic 2. Show that the theory of infinite fields of characteristic 2 does not have a finite axiomatisation.

(iii) Let $\mathcal{T}$ be the collection of closed terms in some first order language $\mathcal{L}$. Suppose that $\exists x.\phi(x)$ is a provable sentence of $\mathcal{L}$ with ϕ quantifier-free. Show that the set of sentences $\{\neg\phi(t) : t \in \mathcal{T}\}$ is inconsistent.

[*Hint: consider the minimal substructure of a model.*]

Deduce that there are $t_1, t_2, \ldots, t_n$ in $\mathcal{T}$ such that $\phi(t_1) \vee \phi(t_2) \vee \cdots \vee \phi(t_n)$ is provable.

3/II/16G Logic and Set Theory

What is a transitive set? Show that if x is transitive then so are the union $\bigcup x$ and the power set Px of x. If $\bigcup x$ is transitive, is x transitive? If Px is transitive, is x transitive? Justify your answers.

What is the transitive closure of a set? Show that any set x has a transitive closure $TC(x)$.

Suppose that x has rank α. What is the rank of Px? What is the rank of $TC(x)$?

[You may use standard properties of rank.]

4/II/16G   **Logic and Set Theory**

Prove Hartog's Lemma that for any set x there is an ordinal α which cannot be mapped injectively into x.

Now assume the Axiom of Choice. Prove Zorn's Lemma and the Well-ordering Principle.

[If you appeal to a fixed point theorem then you should state it clearly.]

1/II/16G **Set Theory and Logic**

By a *directed set* in a poset $(P, \leqslant)$, we mean a nonempty subset D such that any pair $\{x, y\}$ of elements of D has an upper bound in D. We say $(P, \leqslant)$ is *directed-complete* if each directed subset $D \subseteq P$ has a least upper bound in P. Show that a poset is complete if and only if it is directed-complete and has joins for all its finite subsets. Show also that, for any two sets A and B, the set $[A \rightharpoonup B]$ of partial functions from A to B, ordered by extension, is directed-complete.

Let $(P, \leqslant)$ be a directed-complete poset, and $f : P \to P$ an order-preserving map which is *inflationary*, i.e. satisfies $x \leqslant f(x)$ for all $x \in P$. We define a subset $C \subseteq P$ to be *closed* if it satisfies $(x \in C) \to (f(x) \in C)$, and is also closed under joins of directed sets (i.e., $D \subseteq C$ and D directed imply $\bigvee D \in C$). We write $x \ll y$ to mean that every closed set containing x also contains y. Show that $\ll$ is a partial order on P, and that $x \ll y$ implies $x \leqslant y$. Now consider the set H of all functions $h : P \to P$ which are order-preserving and satisfy $x \ll h(x)$ for all x. Show that H is closed under composition of functions, and deduce that, for each $x \in P$, the set $H_x = \{h(x) \mid h \in H\}$ is directed. Defining $h_0(x) = \bigvee H_x$ for each x, show that the function h_0 belongs to H, and deduce that $h_0(x)$ is the least fixed point of f lying above x, for each $x \in P$.

2/II/16G **Set Theory and Logic**

Explain carefully what is meant by a *deduction* in the propositional calculus. State the completeness theorem for the propositional calculus, and deduce the compactness theorem.

Let P, Q, R be three pairwise-disjoint sets of primitive propositions, and suppose given compound propositions $s \in \mathcal{L}(P \cup Q)$ and $t \in \mathcal{L}(Q \cup R)$ such that $(s \vdash t)$ holds. Let U denote the set

$$\{u \in \mathcal{L}(Q) \mid (s \vdash u)\} \ .$$

If $v : Q \to 2$ is any valuation making all the propositions in U true, show that the set

$$\{s\} \cup \{q \mid q \in Q, v(q) = 1\} \cup \{\neg q \mid q \in Q, v(q) = 0\}$$

is consistent. Deduce that $U \cup \{\neg t\}$ is inconsistent, and hence show that there exists $u \in \mathcal{L}(Q)$ such that $(s \vdash u)$ and $(u \vdash t)$ both hold.

3/II/16G   **Set Theory and Logic**

Write down the recursive definitions of ordinal addition, multiplication and exponentiation. Prove carefully that $\omega^\alpha \geqslant \alpha$ for all α, and hence show that for each non-zero ordinal α there exists a unique $\alpha_0 \leqslant \alpha$ such that

$$\omega^{\alpha_0} \leqslant \alpha < \omega^{\alpha_0+1} \ .$$

Deduce that any non-zero ordinal α has a unique representation of the form

$$\omega^{\alpha_0} \cdot a_0 + \omega^{\alpha_1} \cdot \alpha_1 + \cdots + \omega^{\alpha_n} \cdot a_n$$

where $\alpha \geqslant \alpha_0 > \alpha_1 > \cdots > \alpha_n$ and $a_0, a_1, \ldots, a_n$ are non-zero natural numbers.

Two ordinals β, γ are said to be *commensurable* if we have neither $\beta + \gamma = \gamma$ nor $\gamma + \beta = \beta$. Show that β and γ are commensurable if and only if there exists α such that both β and γ lie in the set
$$\{\delta \mid \omega^\alpha \leqslant \delta < \omega^{\alpha+1}\} \ .$$

4/II/16G   **Set Theory and Logic**

Explain what is meant by a *well-founded* binary relation on a set.

Given a set a, we say that a mapping $f: a \to \mathcal{P}a$ is *recursive* if, given any set b equipped with a mapping $g: \mathcal{P}b \to b$, there exists a unique $h: a \to b$ such that $h = g \circ h_* \circ f$, where $h_*: \mathcal{P}a \to \mathcal{P}b$ denotes the mapping $a' \mapsto \{h(x) \mid x \in a'\}$. Show that f is recursive if and only if the relation $\{\langle x, y \rangle \mid x \in f(y)\}$ is well-founded.

[*If you need to use any form of the recursion theorem, you should prove it.*]

1/II/16H   **Logic and Set Theory**

Explain what it means for a poset to be *chain-complete*. State Zorn's Lemma, and use it to prove that, for any two elements a and b of a distributive lattice L with $b \not\leqslant a$, there exists a lattice homomorphism $f \colon L \to \{0,1\}$ with $f(a) = 0$ and $f(b) = 1$. Explain briefly how this result implies the completeness theorem for propositional logic.

2/II/16H   **Logic and Set Theory**

Which of the following statements are true, and which false? Justify your answers.

(a) For any ordinals α and β with $\beta \neq 0$, there exist ordinals γ and δ with $\delta < \beta$ such that $\alpha = \beta.\gamma + \delta$.

(b) For any ordinals α and β with $\beta \neq 0$, there exist ordinals γ and δ with $\delta < \beta$ such that $\alpha = \gamma.\beta + \delta$.

(c) $\alpha.(\beta + \gamma) = \alpha.\beta + \alpha.\gamma$ for all α, β, γ.

(d) $(\alpha + \beta).\gamma = \alpha.\gamma + \beta.\gamma$ for all α, β, γ.

(e) Any ordinal of the form $\omega.\alpha$ is a limit ordinal.

(f) Any limit ordinal is of the form $\omega.\alpha$.

3/II/16H   **Logic and Set Theory**

Explain what is meant by a *structure* for a first-order signature Σ, and describe how first-order formulae over Σ are interpreted in a given structure. Show that if B is a substructure of A, and ϕ is a quantifier-free formula (with n free variables), then

$$\llbracket \phi \rrbracket_B = \llbracket \phi \rrbracket_A \cap B^n.$$

A first-order theory is said to be *inductive* if its axioms all have the form

$$(\forall x_1, \ldots, x_n)(\exists y_1, \ldots, y_m)\phi$$

where ϕ is quantifier-free (and either of the strings $x_1, \ldots, x_n$ or $y_1, \ldots, y_m$ may be empty). If T is an inductive theory, and A is a structure for the appropriate signature, show that the poset of those substructures of A which are T-models is chain-complete.

Which of the following can be expressed as inductive theories over the signature with one binary predicate symbol $\leqslant$? Justify your answers.

(a) The theory of totally ordered sets without greatest or least elements.

(b) The theory of totally ordered sets with greatest and least elements.

Part II   2006

4/II/16H   **Logic and Set Theory**

Explain carefully what is meant by a *well-founded* relation on a set. State the recursion theorem, and use it to prove that a binary relation r on a set a is well-founded if and only if there exists a function f from a to some ordinal α such that $(x, y) \in r$ implies $f(x) < f(y)$.

Deduce, using the axiom of choice, that any well-founded relation on a set may be extended to a well-ordering.

1/II/16F   Logic and Set Theory

State and prove Zorn's Lemma. [*You may assume Hartogs' Lemma.*] Where in your argument have you made use of the Axiom of Choice?

Show that $\mathbb{R}$, considered as a rational vector space, has a basis.

Prove that $\mathbb{R}$ and $\mathbb{R}^2$ are isomorphic as rational vector spaces.

2/II/16F   Logic and Set Theory

Give the inductive and the synthetic definitions of ordinal addition, and prove that they are equivalent. Give an example to show that ordinal addition is not commutative.

Which of the following assertions about ordinals α, β and γ are always true, and which can be false? Give proofs or counterexamples as appropriate.

(i) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.

(ii) If α and β are limit ordinals then $\alpha + \beta = \beta + \alpha$.

(iii) If $\alpha + \beta = \omega_1$ then $\alpha = 0$ or $\alpha = \omega_1$.

(iv) If $\alpha + \beta = \omega_1$ then $\beta = 0$ or $\beta = \omega_1$.

3/II/16F   Logic and Set Theory

State the Axiom of Foundation and the Principle of $\in$-Induction, and show that they are equivalent (in the presence of the other axioms of ZF). [*You may assume the existence of transitive closures.*]

Explain briefly how the Principle of $\in$-Induction implies that every set is a member of some V_α.

For each natural number n, find the cardinality of V_n. For which ordinals α is the cardinality of V_α equal to that of the reals?

4/II/16F   Logic and Set Theory

State and prove the Completeness Theorem for Propositional Logic. [*You do* **not** *need to give definitions of the various terms involved. You may assume that the set of primitive propositions is countable. You may also assume the Deduction Theorem, provided that you state it clearly.*]

Where in your argument have you used the third axiom, namely $(\neg\neg p) \Rightarrow p$?

State the Compactness Theorem, and deduce it from the Completeness Theorem.