## Part II

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### Integrable Systems

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Suppose $\psi^s : (x, u) \mapsto (\tilde{x}, \tilde{u})$ is a smooth one-parameter group of transformations acting on $\mathbb{R}^2$, with infinitesimal generator

$$V = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u},$$

(a) Define the $n^{th}$ prolongation $\text{Pr}^{(n)} V$ of $V$, and show that

$$\text{Pr}^{(n)} V = V + \sum_{i=1}^{n} \eta_i \frac{\partial}{\partial u^{(i)}},$$

where you should give an explicit formula to determine the $\eta_i$ recursively in terms of $\xi$ and $\eta$.

(b) Find the $n^{th}$ prolongation of each of the following generators:

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = x \frac{\partial}{\partial x}, \quad V_3 = x^2 \frac{\partial}{\partial x}.$$  

(c) Given a smooth, real-valued, function $u = u(x)$, the Schwarzian derivative is defined by

$$S = S[u] := \frac{u_{xx} u_{xxx} - \frac{3}{2} u_x^2}{u_x^2}.$$  

Show that,

$$\text{Pr}^{(3)} V_i (S) = c_i S,$$

for $i = 1, 2, 3$ where $c_i$ are real functions which you should determine. What can you deduce about the symmetries of the equations:

(i) $S[u] = 0$,

(ii) $S[u] = 1$,

(iii) $S[u] = \frac{1}{x^2}$?
Suppose \( p = p(x) \) is a smooth, real-valued, function of \( x \in \mathbb{R} \) which satisfies \( p(x) > 0 \) for all \( x \) and \( p(x) \to 1 \), \( p_x(x), p_{xx}(x) \to 0 \) as \( |x| \to \infty \). Consider the Sturm-Liouville operator:

\[
L\psi := -\frac{d}{dx} \left( p^2 \frac{d\psi}{dx} \right),
\]

which acts on smooth, complex-valued, functions \( \psi = \psi(x) \). You may assume that for any \( k > 0 \) there exists a unique function \( \varphi_k(x) \) which satisfies:

\[
L\varphi_k = k^2 \varphi_k,
\]

and has the asymptotic behaviour:

\[
\varphi_k(x) \sim \begin{cases} 
e^{-ikx} & \text{as } x \to -\infty, \\ a(k)e^{-ikx} + b(k)e^{ikx} & \text{as } x \to +\infty. \end{cases}
\]

(a) By analogy with the standard Schrödinger scattering problem, define the reflection and transmission coefficients: \( R(k), T(k) \). Show that \( |R(k)|^2 + |T(k)|^2 = 1 \).

[\text{Hint: You may wish to consider } W(x) = p(x)^2 [\psi_1(x)\psi'_2(x) - \psi_2(x)\psi'_1(x)] \text{ for suitable functions } \psi_1 \text{ and } \psi_2.]

(b) Show that, if \( \kappa > 0 \), there exists no non-trivial normalizable solution \( \psi \) to the equation

\[
L\psi = -\kappa^2 \psi.
\]

Assume now that \( p = p(x, t) \), such that \( p(x, t) > 0 \) and \( p(x, t) \to 1 \), \( p_x(x, t), p_{xx}(x, t) \to 0 \) as \( |x| \to \infty \). You are given that the operator \( A \) defined by:

\[
A\psi := -4p^3 \frac{d^3\psi}{dx^3} - 18p^2 p_x \frac{d^2\psi}{dx^2} - (12pp_x^2 + 6p^2 p_{xx}) \frac{d\psi}{dx},
\]

satisfies:

\[
(LA - AL)\psi = -\frac{d}{dx} \left( 2p^4 p_{xxx} \frac{d\psi}{dx} \right).
\]

(c) Show that \( L, A \) form a Lax pair if the Harry Dym equation,

\[
p_t = p^3 p_{xxx}
\]

is satisfied. [You may assume \( L = L^\dagger \), \( A = -A^\dagger \).]

(d) Assuming that \( p \) solves the Harry Dym equation, find how the transmission and reflection amplitudes evolve as functions of \( t \).
Let $M = \mathbb{R}^{2n} = \{(q, p) | q, p \in \mathbb{R}^n\}$ be equipped with its standard Poisson bracket.

(a) Given a Hamiltonian function $H = H(q, p)$, write down Hamilton’s equations for $(M, H)$. Define a first integral of the system and state what it means that the system is integrable.

(b) Show that if $n = 1$ then every Hamiltonian system is integrable whenever

$$\left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right) \neq 0.$$

Let $\tilde{M} = \mathbb{R}^{2m} = \{ (\tilde{q}, \tilde{p}) | \tilde{q}, \tilde{p} \in \mathbb{R}^m \}$ be another phase space, equipped with its standard Poisson bracket. Suppose that $\tilde{H} = \tilde{H}(\tilde{q}, \tilde{p})$ is a Hamiltonian function for $\tilde{M}$. Define $Q = (q_1, \ldots, q_n, \tilde{q}_1, \ldots, \tilde{q}_m)$, $P = (p_1, \ldots, p_n, \tilde{p}_1, \ldots, \tilde{p}_m)$ and let the combined phase space $\mathcal{M} = \mathbb{R}^{2(n+m)} = \{(Q, P)\}$ be equipped with the standard Poisson bracket.

(c) Show that if $(M, H)$ and $(\tilde{M}, \tilde{H})$ are both integrable, then so is $(\mathcal{M}, \mathcal{H})$, where the combined Hamiltonian is given by:

$$\mathcal{H}(Q, P) = H(q, p) + \tilde{H}(\tilde{q}, \tilde{p}).$$

(d) Consider the $n$–dimensional simple harmonic oscillator with phase space $M$ and Hamiltonian $H$ given by:

$$H = \frac{1}{2}p_1^2 + \ldots + \frac{1}{2}p_n^2 + \frac{1}{2}\omega_1^2 q_1^2 + \ldots + \frac{1}{2}\omega_n^2 q_n^2,$$

where $\omega_i > 0$. Using the results above, or otherwise, show that $(M, H)$ is integrable for $(q, p) \neq 0$.

(e) Is it true that every bounded orbit of an integrable system is necessarily periodic? You should justify your answer.
Paper 1, Section II

32A Integrable Systems

Let $M = \mathbb{R}^{2n} = \{(q, p) | q, p \in \mathbb{R}^n\}$ be equipped with the standard symplectic form so that the Poisson bracket is given by:

$$\{f, g\} = \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j},$$

for $f, g$ real-valued functions on $M$. Let $H = H(q, p)$ be a Hamiltonian function.

(a) Write down Hamilton’s equations for $(M, H)$, define a first integral of the system and state what it means that the system is integrable.

(b) State the Arnol’d–Liouville theorem.

(c) Define complex coordinates $z_j$ by $z_j = q_j + ip_j$, and show that if $f, g$ are real-valued functions on $M$ then:

$$\{f, g\} = -2i \frac{\partial f}{\partial z_j} \frac{\partial g}{\partial \overline{z}_j} + 2i \frac{\partial g}{\partial z_j} \frac{\partial f}{\partial \overline{z}_j}.$$

(d) For an $n \times n$ anti-Hermitian matrix $A$ with components $A_{jk}$, let $I_A := \frac{1}{2i} \sum_{j,k} A_{jk} z_k$. Show that:

$$\{I_A, I_B\} = -I_{[A, B]},$$

where $[A, B] = AB - BA$ is the usual matrix commutator.

(e) Consider the Hamiltonian:

$$H = \frac{1}{2} \overline{z}_j z_j.$$

Show that $(M, H)$ is integrable and describe the invariant tori.

[In this question $j, k = 1, \ldots, n$, and the summation convention is understood for these indices.]
(a) Let $L, A$ be two families of linear operators, depending on a parameter $t$, which act on a Hilbert space $H$ with inner product $(\cdot, \cdot)$. Suppose further that for each $t$, $L$ is self-adjoint and that $A$ is anti-self-adjoint. State Lax’s equation for the pair $L, A$, and show that if it holds then the eigenvalues of $L$ are independent of $t$.

(b) For $\psi, \phi : \mathbb{R} \to \mathbb{C}$, define the inner product:

$$(\psi, \phi) := \int_{-\infty}^{\infty} \overline{\psi(x)} \phi(x) dx.$$

Let $L, A$ be the operators:

$L\psi := i \frac{d^3 \psi}{dx^3} - i \left( q \frac{d \psi}{dx} + \frac{d}{dx}(q \psi) \right) + p\psi,$

$A\psi := 3i \frac{d^2 \psi}{dx^2} - 4iq\psi,$

where $p = p(x, t), q = q(x, t)$ are smooth, real-valued functions. You may assume that the normalised eigenfunctions of $L$ are smooth functions of $x, t$, which decay rapidly as $|x| \to \infty$ for all $t$.

(i) Show that if $\psi, \phi$ are smooth and rapidly decaying towards infinity then:

$$(L\psi, \phi) = (\psi, L\phi), \quad (A\psi, \phi) = -(\psi, A\phi).$$

Deduce that the eigenvalues of $L$ are real.

(ii) Show that if Lax’s equation holds for $L, A$, then $q$ must satisfy the Boussinesq equation:

$$q_{tt} = aq_{xxxx} + b(q^2)_{xx},$$

where $a, b$ are constants whose values you should determine. [You may assume without proof that the identity:

$$LA\psi = AL\psi - 3i \left( p_x \frac{d \psi}{dx} + \frac{d}{dx}(p_x \psi) \right) + [q_{xxx} - 4(q^2)_x] \psi,$$

holds for smooth, rapidly decaying $\psi$.]

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**Part II, 2018 List of Questions**
Paper 3, Section II
33A Integrable Systems

Suppose $\psi^s : (x, u) \mapsto (\tilde{x}, \tilde{u})$ is a smooth one-parameter group of transformations acting on $\mathbb{R}^2$.

(a) Define the generator of the transformation,

$$ V = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u}, $$

where you should specify $\xi$ and $\eta$ in terms of $\psi^s$.

(b) Define the $n$th prolongation of $V$, $\text{Pr}^{(n)} V$ and explicitly compute $\text{Pr}^{(1)} V$ in terms of $\xi, \eta$.

Recall that if $\psi^s$ is a Lie point symmetry of the ordinary differential equation:

$$ \Delta \left( x, u, \frac{du}{dx}, \ldots, \frac{d^n u}{dx^n} \right) = 0, $$

then it follows that $\text{Pr}^{(n)} V [\Delta] = 0$ whenever $\Delta = 0$.

(c) Consider the ordinary differential equation:

$$ \frac{du}{dx} = F(x, u), $$

for $F$ a smooth function. Show that if $V$ generates a Lie point symmetry of this equation, then:

$$ 0 = \eta_x + (\eta_u - \xi_x - F\xi_u) F - \xi F_x - \eta F_u. $$

(d) Find all the Lie point symmetries of the equation:

$$ \frac{du}{dx} = xG \left( \frac{u}{x^2} \right), $$

where $G$ is an arbitrary smooth function.
Define a Lie point symmetry of the first order ordinary differential equation $\Delta [t, x, \dot{x}] = 0$. Describe such a Lie point symmetry in terms of the vector field that generates it.

Consider the $2n$-dimensional Hamiltonian system $(M, H)$ governed by the differential equation

$$\frac{dx}{dt} = J \frac{\partial H}{\partial x}. \quad (\star)$$

Define the Poisson bracket $\{\cdot, \cdot\}$. For smooth functions $f, g : M \to \mathbb{R}$ show that the associated Hamiltonian vector fields $V_f, V_g$ satisfy

$$[V_f, V_g] = -V_{\{f, g\}}.$$

If $F : M \to \mathbb{R}$ is a first integral of $(M, H)$, show that the Hamiltonian vector field $V_F$ generates a Lie point symmetry of $(\star)$. Prove the converse is also true if $(\star)$ has a fixed point, i.e. a solution of the form $x(t) = x_0$. 
Let $U$ and $V$ be non-singular $N \times N$ matrices depending on $(x, t, \lambda)$ which are periodic in $x$ with period $2\pi$. Consider the associated linear problem

$$
\Psi_x = U\Psi, \quad \Psi_t = V\Psi,
$$

for the vector $\Psi = \Psi(x, t; \lambda)$. On the assumption that these equations are compatible, derive the zero curvature equation for $(U, V)$.

Let $W = W(x, t, \lambda)$ denote the $N \times N$ matrix satisfying

$$
W_x = UW, \quad W(0, t, \lambda) = I_N,
$$

where $I_N$ is the $N \times N$ identity matrix. You should assume $W$ is unique. By considering $(W_t - VW)_x$, show that the matrix $w(t, \lambda) = W(2\pi, t, \lambda)$ satisfies the Lax equation

$$
w_t = [v, w], \quad v(t, \lambda) \equiv V(2\pi, t, \lambda).
$$

Deduce that $\{\text{tr}(w^k)\}_{k \geq 1}$ are first integrals.

By considering the matrices

$$
\frac{1}{2\lambda} \begin{bmatrix} \cos u & -i \sin u \\ i \sin u & -\cos u \end{bmatrix}, \quad \frac{i}{2} \begin{bmatrix} 2\lambda & u_x \\ u_x & -2\lambda \end{bmatrix},
$$

show that the periodic Sine-Gordon equation $u_{xt} = \sin u$ has infinitely many first integrals. [You need not prove anything about independence.]
Let $u = u(x, t)$ be a smooth solution to the KdV equation
\[ u_t + u_{xxx} - 6uu_x = 0 \]
which decays rapidly as $|x| \to \infty$ and let $L = -\partial_x^2 + u$ be the associated Schrödinger operator. You may assume $L$ and $A = 4\partial_x^3 - 3(u\partial_x + \partial_x u)$ constitute a Lax pair for KdV.

Consider a solution to $L\varphi = k^2 \varphi$ which has the asymptotic form
\[
\varphi(x, k, t) = \begin{cases} e^{-ikx}, & \text{as } x \to -\infty, \\ a(k, t)e^{-ikx} + b(k, t)e^{ikx}, & \text{as } x \to +\infty. \end{cases}
\]
Find evolution equations for $a$ and $b$. Deduce that $a(k, t)$ is $t$-independent.

By writing $\varphi$ in the form
\[
\varphi(x, k, t) = \exp \left[ -ikx + \int_{-\infty}^{x} S(y, k, t) \, dy \right], \quad S(x, k, t) = \sum_{n=1}^{\infty} \frac{S_n(x, t)}{(2k)^n},
\]
show that
\[
a(k, t) = \exp \left[ \int_{-\infty}^{\infty} S(x, k, t) \, dx \right].
\]
Deduce that $\{ \int_{-\infty}^{\infty} S_n(x, t) \, dx \}_{n=1}^{\infty}$ are first integrals of KdV.

By writing a differential equation for $S = X + iY$ (with $X, Y$ real), show that these first integrals are trivial when $n$ is even.
What is meant by an auto-Bäcklund transformation?

The sine-Gordon equation in light-cone coordinates is

\[ \frac{\partial^2 \varphi}{\partial \xi \partial \tau} = \sin \varphi, \]  

(1)

where \( \xi = \frac{1}{2}(x - t) \), \( \tau = \frac{1}{2}(x + t) \) and \( \varphi \) is to be understood modulo \( 2\pi \). Show that the pair of equations

\[ \partial_\xi (\varphi_1 - \varphi_0) = 2\epsilon \sin \left( \frac{\varphi_1 + \varphi_0}{2} \right), \quad \partial_\tau (\varphi_1 + \varphi_0) = \frac{2}{\epsilon} \sin \left( \frac{\varphi_1 - \varphi_0}{2} \right) \]  

(2)

constitute an auto-Bäcklund transformation for (1).

By noting that \( \varphi = 0 \) is a solution to (1), use the transformation (2) to derive the soliton (or ‘kink’) solution to the sine-Gordon equation. Show that this solution can be expressed as

\[ \varphi(x, t) = 4 \arctan \left[ \exp \left( \pm \frac{x - ct}{\sqrt{1 - c^2}} + x_0 \right) \right], \]

for appropriate constants \( c \) and \( x_0 \).

[Hint: You may use the fact that \( \int \cosec x \, dx = \log \tanh(x/2) + \text{const.} \)]

The following function is a solution to the sine-Gordon equation:

\[ \varphi(x, t) = 4 \arctan \left[ \frac{\sinh(x/\sqrt{1 - c^2})}{\cosh(ct/\sqrt{1 - c^2})} \right] \quad (c > 0). \]

Verify that this represents two solitons travelling towards each other at the same speed by considering \( x \pm ct = \text{constant} \) and taking an appropriate limit.
What does it mean for an evolution equation $u_t = K(x, u, u_x, \ldots)$ to be in Hamiltonian form? Define the associated Poisson bracket.

An evolution equation $u_t = K(x, u, u_x, \ldots)$ is said to be bi-Hamiltonian if it can be written in Hamiltonian form in two distinct ways, i.e.

$$K = \mathcal{J} \delta H_0 = \mathcal{E} \delta H_1$$

for Hamiltonian operators $\mathcal{J}, \mathcal{E}$ and functionals $H_0, H_1$. By considering the sequence $\{H_m\}_{m \geq 0}$ defined by the recurrence relation

$$\mathcal{E} \delta H_{m+1} = \mathcal{J} \delta H_m,$$

(*

show that bi-Hamiltonian systems possess infinitely many first integrals in involution. [You may assume that (*) can always be solved for $H_{m+1}$, given $H_m$.]

The Harry Dym equation for the function $u = u(x, t)$ is

$$u_t = \frac{\partial^3}{\partial x^3} \left( u^{-1/2} \right).$$

This equation can be written in Hamiltonian form $u_t = \mathcal{E} \delta H_1$ with

$$\mathcal{E} = 2u \frac{\partial}{\partial x} + u_x, \quad H_1[u] = \frac{1}{8} \int u^{-5/2} u_x^2 \, dx.$$  

Show that the Harry Dym equation possesses infinitely many first integrals in involution. [You need not verify the Jacobi identity if your argument involves a Hamiltonian operator.]
31D Integrable Systems

What does it mean for \( g^\varepsilon : (x, u) \mapsto (\tilde{x}, \tilde{u}) \) to describe a 1-parameter group of transformations? Explain how to compute the vector field

\[
V = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u}
\]

that generates such a 1-parameter group of transformations.

Suppose now \( u = u(x) \). Define the \( n \)th prolongation, \( \text{pr}^{(n)}g^\varepsilon \), of \( g^\varepsilon \) and the vector field which generates it. If \( V \) is defined by (\#) show that

\[
\text{pr}^{(n)}V = V + \sum_{k=1}^{n} \eta_k \frac{\partial}{\partial u^{(k)}},
\]

where \( u^{(k)} = \frac{d^k u}{dx^k} \) and \( \eta_k \) are functions to be determined.

The curvature of the curve \( u = u(x) \) in the \((x, u)\)-plane is given by

\[
\kappa = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}.
\]

Rotations in the \((x, u)\)-plane are generated by the vector field

\[
W = x \frac{\partial}{\partial u} - u \frac{\partial}{\partial x}.
\]

Show that the curvature \( \kappa \) at a point along a plane curve is invariant under such rotations. Find two further transformations that leave \( \kappa \) invariant.
Let $u_t = K(x, u, u_x, \ldots)$ be an evolution equation for the function $u = u(x, t)$. Assume $u$ and all its derivatives decay rapidly as $|x| \to \infty$. What does it mean to say that the evolution equation for $u$ can be written in Hamiltonian form?

The modified KdV (mKdV) equation for $u$ is

$$u_t + u_{xxx} - 6u^2u_x = 0.$$ 

Show that small amplitude solutions to this equation are dispersive.

Demonstrate that the mKdV equation can be written in Hamiltonian form and define the associated Poisson bracket $\{ , \}$ on the space of functionals of $u$. Verify that the Poisson bracket is linear in each argument and anti-symmetric.

Show that a functional $I = I[u]$ is a first integral of the mKdV equation if and only if $\{I, H\} = 0$, where $H = H[u]$ is the Hamiltonian.

Show that if $u$ satisfies the mKdV equation then

$$\frac{\partial}{\partial t} (u^2) + \frac{\partial}{\partial x} (2uu_{xx} - u_x^2 - 3u^4) = 0.$$ 

Using this equation, show that the functional

$$I[u] = \int u^2 \, dx$$

Poisson-commutes with the Hamiltonian.
Paper 2, Section II
29D Integrable Systems

(a) Explain how a vector field

\[ V = \xi(x, u) \frac{\partial}{\partial x} + \eta(x, u) \frac{\partial}{\partial u} \]

generates a 1-parameter group of transformations \( g^\epsilon : (x, u) \mapsto (\tilde{x}, \tilde{u}) \) in terms of the solution to an appropriate differential equation. [You may assume the solution to the relevant equation exists and is unique.]

(b) Suppose now that \( u = u(x) \). Define what is meant by a Lie-point symmetry of the ordinary differential equation

\[ \Delta[x, u, u^{(1)}, \ldots, u^{(n)}] = 0, \quad \text{where} \quad u^{(k)} \equiv \frac{d^k u}{dx^k}, \quad k = 1, \ldots, n. \]

(c) Prove that every homogeneous, linear ordinary differential equation for \( u = u(x) \) admits a Lie-point symmetry generated by the vector field

\[ V = u \frac{\partial}{\partial u}. \]

By introducing new coordinates

\[ s = s(x, u), \quad t = t(x, u) \]

which satisfy \( V(s) = 1 \) and \( V(t) = 0 \), show that every differential equation of the form

\[ \frac{d^2 u}{dx^2} + p(x) \frac{du}{dx} + q(x) u = 0 \]

can be reduced to a first-order differential equation for an appropriate function.
Let $L = L(t)$ and $A = A(t)$ be real $N \times N$ matrices, with $L$ symmetric and $A$ antisymmetric. Suppose that

$$\frac{dL}{dt} = LA - AL.$$ 

Show that all eigenvalues of the matrix $L(t)$ are $t$-independent. Deduce that the coefficients of the polynomial

$$P(x) = \det(xI - L(t))$$

are first integrals of the system.

What does it mean for a $2n$-dimensional Hamiltonian system to be integrable?

Consider the Toda system with coordinates $(q_1, q_2, q_3)$ obeying

$$\frac{d^2q_i}{dt^2} = e^{q_{i-1} - q_i} - e^{q_i - q_{i+1}}, \quad i = 1, 2, 3$$

where here and throughout the subscripts are to be determined modulo 3 so that $q_4 \equiv q_1$ and $q_0 \equiv q_3$. Show that

$$H(q_i, p_i) = \frac{1}{2} \sum_{i=1}^{3} p_i^2 + \sum_{i=1}^{3} e^{q_i - q_{i+1}}$$

is a Hamiltonian for the Toda system.

Set $a_i = \frac{1}{2} \exp\left(\frac{a_i - q_{i+1}}{2}\right)$ and $b_i = -\frac{1}{2} p_i$. Show that

$$\frac{da_i}{dt} = (b_{i+1} - b_i) a_i, \quad \frac{db_i}{dt} = 2 \left( a_i^2 - a_{i-1}^2 \right), \quad i = 1, 2, 3.$$ 

Is this coordinate transformation canonical?

By considering the matrices

$$L = \begin{pmatrix} b_1 & a_1 & a_3 \\ a_1 & b_2 & a_2 \\ a_3 & a_2 & b_3 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & -a_1 & a_3 \\ a_1 & 0 & -a_2 \\ -a_3 & a_2 & 0 \end{pmatrix},$$

or otherwise, compute three independent first integrals of the Toda system. [Proof of independence is not required.]
What does it mean to say that a finite-dimensional Hamiltonian system is integrable?

State the Arnold–Liouville theorem.

A six-dimensional dynamical system with coordinates \((x_1, x_2, x_3, y_1, y_2, y_3)\) is governed by the differential equations

\[
\frac{dx_i}{dt} = -\frac{1}{2\pi} \sum_{j \neq i} \frac{\Gamma_j (y_i - y_j)}{(x_i - x_j)^2 + (y_i - y_j)^2}, \quad \frac{dy_i}{dt} = \frac{1}{2\pi} \sum_{j \neq i} \frac{\Gamma_j (x_i - x_j)}{(x_i - x_j)^2 + (y_i - y_j)^2}
\]

for \(i = 1, 2, 3\), where \(\{\Gamma_i\}_{i=1}^3\) are positive constants. Show that these equations can be written in the form

\[
\Gamma_i \frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \quad \Gamma_i \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \quad i = 1, 2, 3
\]

for an appropriate function \(F\). By introducing the coordinates \(q = (x_1, x_2, x_3)\), \(p = (\Gamma_1 y_1, \Gamma_2 y_2, \Gamma_3 y_3)\), show that the system can be written in Hamiltonian form

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}
\]

for some Hamiltonian \(H = H(q, p)\) which you should determine.

Show that the three functions

\[
A = \sum_{i=1}^3 \Gamma_i x_i, \quad B = \sum_{i=1}^3 \Gamma_i y_i, \quad C = \sum_{i=1}^3 \Gamma_i (x_i^2 + y_i^2)
\]

are first integrals of the Hamiltonian system.

Making use of the fundamental Poisson brackets \(\{q_i, q_j\} = \{p_i, p_j\} = 0\) and \(\{q_i, p_j\} = \delta_{ij}\), show that

\[
\{A, C\} = 2B, \quad \{B, C\} = -2A.
\]

Hence show that the Hamiltonian system is integrable.
Let \( u = u(x) \) be a smooth function that decays rapidly as \( |x| \to \infty \) and let \( L = -\partial_x^2 + u(x) \) denote the associated Schrödinger operator. Explain very briefly each of the terms appearing in the scattering data

\[
S = \left\{ \{\chi_n, c_n\}_{n=1}^N, R(k) \right\},
\]
associated with the operator \( L \). What does it mean to say \( u(x) \) is reflectionless?

Given \( S \), define the function

\[
F(x) = \sum_{n=1}^N c_n^2 e^{-\chi_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} R(k) \, dk.
\]

If \( K = K(x, y) \) is the unique solution to the GLM equation

\[
K(x, y) + F(x + y) + \int_x^\infty K(x, z) F(z + y) \, dz = 0,
\]
what is the relationship between \( u(x) \) and \( K(x, x) \)?

Now suppose that \( u = u(x, t) \) is time dependent and that it solves the KdV equation

\[
u_t + u_{xxx} - 6uu_x = 0.\]
Show that \( L = -\partial_x^2 + u(x, t) \) obeys the Lax equation

\[
L_t = [L, A], \quad \text{where } A = 4\partial_x^3 - 3(u\partial_x + \partial_x u).
\]
Show that the discrete eigenvalues of \( L \) are time independent.

In what follows you may assume the time-dependent scattering data take the form

\[
S(t) = \left\{ \{\chi_n, c_n e^{4\chi_n^3 t}\}_{n=1}^N, R(k, 0) e^{8ik^3 t} \right\}.
\]
Show that if \( u(x, 0) \) is reflectionless, then the solution to the KdV equation takes the form

\[
u(x, t) = -2 \frac{\partial^2}{\partial x^2} \log |\det A(x, t)|,
\]
where \( A \) is an \( N \times N \) matrix which you should determine.

Assume further that \( R(k, 0) = k^2 f(k) \), where \( f \) is smooth and decays rapidly at infinity. Show that, for any fixed \( x \),

\[
\int_{-\infty}^{\infty} e^{ikx} R(k, 0) e^{8ik^3 t} \, dk = O(t^{-1}) \quad \text{as } t \to \infty.
\]
Comment briefly on the significance of this result.

[You may assume \( \frac{1}{\det A} \frac{d}{dx} (\det A) = \text{tr} \left( A^{-1} \frac{dA}{dx} \right) \) for a non-singular matrix \( A(x) \).]
Consider the coordinate transformation
\[ g^\epsilon : (x, u) \mapsto (\bar{x}, \bar{u}) = (x \cos \epsilon - u \sin \epsilon, x \sin \epsilon + u \cos \epsilon). \]

Show that \( g^\epsilon \) defines a one-parameter group of transformations. Define what is meant by the generator \( V \) of a one-parameter group of transformations and compute it for the above case.

Now suppose \( u = u(x) \). Explain what is meant by the first prolongation \( \text{pr}^{(1)} g^\epsilon \) of \( g^\epsilon \). Compute \( \text{pr}^{(1)} g^\epsilon \) in this case and deduce that
\[ \text{pr}^{(1)} V = V + (1 + u_x^2) \frac{\partial}{\partial u_x}. \] (*)

Similarly find \( \text{pr}^{(2)} V \).

Define what is meant by a Lie point symmetry of the first-order differential equation \( \Delta[x, u, u_x] = 0 \). Describe this condition in terms of the vector field that generates the Lie point symmetry. Consider the case
\[ \Delta[x, u, u_x] \equiv u_x - \frac{u + xf(x^2 + u^2)}{x - uf(x^2 + u^2)}, \]
where \( f \) is an arbitrary smooth function of one variable. Using (*), show that \( g^\epsilon \) generates a Lie point symmetry of the corresponding differential equation.
Let $U = U(x, y)$ and $V = V(x, y)$ be two $n \times n$ complex-valued matrix functions, smoothly differentiable in their variables. We wish to explore the solution of the overdetermined linear system

$$\frac{\partial \mathbf{v}}{\partial y} = U(x, y)\mathbf{v}, \quad \frac{\partial \mathbf{v}}{\partial x} = V(x, y)\mathbf{v},$$

for some twice smoothly differentiable vector function $\mathbf{v}(x, y)$.

Prove that, if the overdetermined system holds, then the functions $U$ and $V$ obey the zero curvature representation

$$\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} + UV - VU = 0.$$

Let $u = u(x, y)$ and

$$U = \begin{bmatrix} i\lambda & i\bar{u} \\ iu & -i\lambda \end{bmatrix}, \quad V = \begin{bmatrix} 2i\lambda^2 - |u|^2 & 2i\lambda \bar{u} + \bar{u}_y \\ 2i\lambda u - u_y & -2i\lambda^2 + i|u|^2 \end{bmatrix},$$

where subscripts denote derivatives, $\bar{u}$ is the complex conjugate of $u$ and $\lambda$ is a constant. Find the compatibility condition on the function $u$ so that $U$ and $V$ obey the zero curvature representation.

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Consider the Hamiltonian system

$$p' = -\frac{\partial H}{\partial q}, \quad q' = \frac{\partial H}{\partial p},$$

where $H = H(p, q)$.

When is the transformation $P = P(p, q), Q = Q(p, q)$ canonical?

Prove that, if the transformation is canonical, then the equations in the new variables $(P, Q)$ are also Hamiltonian, with the same Hamiltonian function $H$.

Let $P = C^{-1}p + Bq, Q = Cq$, where $C$ is a symmetric nonsingular matrix. Determine necessary and sufficient conditions on $C$ for the transformation to be canonical.
Quoting carefully all necessary results, use the theory of inverse scattering to derive the 1-soliton solution of the KdV equation

\[ u_t = 6uu_x - u_{xxx}. \]
Consider a one-parameter group of transformations acting on $\mathbb{R}^4$

$$(x, y, t, u) \rightarrow (\exp (\epsilon \alpha)x, \exp (\epsilon \beta)y, \exp (\epsilon \gamma)t, \exp (\epsilon \delta)u),$$

where $\epsilon$ is a group parameter and $(\alpha, \beta, \gamma, \delta)$ are constants.

(a) Find a vector field $W$ which generates this group.

(b) Find two independent Lie point symmetries $S_1$ and $S_2$ of the PDE

$$(u_t - uu_x)_x = u_{yy}, \quad u = u(x, y, t),$$

which are of the form (1).

(c) Find three functionally-independent invariants of $S_1$, and do the same for $S_2$. Find a non-constant function $G = G(x, y, t, u)$ which is invariant under both $S_1$ and $S_2$.

(d) Explain why all the solutions of (2) that are invariant under a two-parameter group of transformations generated by vector fields

$$W = u \frac{\partial}{\partial u} + x \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial y}, \quad V = \frac{\partial}{\partial y},$$

are of the form $u = x F(t)$, where $F$ is a function of one variable. Find an ODE for $F$ characterising these group-invariant solutions.
Consider the KdV equation for the function $u(x, t)$

$$u_t = 6uu_x - u_{xxx}. \quad (1)$$

(a) Write equation (1) in the Hamiltonian form

$$u_t = \frac{\partial}{\partial x} \frac{\delta H[u]}{\delta u},$$

where the functional $H[u]$ should be given. Use equation (1), together with the boundary conditions $u \to 0$ and $u_x \to 0$ as $|x| \to \infty$, to show that $\int_{\mathbb{R}} u^2 dx$ is independent of $t$.

(b) Use the Gelfand–Levitan–Marchenko equation

$$K(x, y) + F(x + y) + \int_{\mathbb{R}} \infty K(x, z) F(z + y) dz = 0 \quad (2)$$

to find the one soliton solution of the KdV equation, i.e.

$$u(x, t) = -\frac{4\beta \chi \exp(-2\chi x)}{\left[1 + \frac{\beta}{\chi} \exp(-2\chi x)\right]^2}.$$  

[Hint. Consider $F(x) = \beta \exp(-\chi x)$, with $\beta = \beta_0 \exp(8\chi^3 t)$, where $\beta_0, \chi$ are constants, and $t$ should be regarded as a parameter in equation (2). You may use any facts about the Inverse Scattering Transform without proof.]

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Consider an integrable system with six-dimensional phase space, and assume that $\nabla \wedge p = 0$ on any Liouville tori $p_i = p_i(q_j, e_j)$, where $\nabla = (\partial/\partial q_1, \partial/\partial q_2, \partial/\partial q_3)$.

(a) Define the action variables and use Stokes’ theorem to show that the actions are independent of the choice of the cycles.

(b) Define the generating function, and show that the angle coordinates are periodic with period $2\pi$. 

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**Paper 1, Section II**

**32D Integrable Systems**

State the Arnold–Liouville theorem.

Consider an integrable system with six-dimensional phase space, and assume that $\nabla \wedge p = 0$ on any Liouville tori $p_i = p_i(q_j, e_j)$, where $\nabla = (\partial/\partial q_1, \partial/\partial q_2, \partial/\partial q_3)$.

(a) Define the action variables and use Stokes’ theorem to show that the actions are independent of the choice of the cycles.

(b) Define the generating function, and show that the angle coordinates are periodic with period $2\pi$. 

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**Part II, 2012 List of Questions**
Paper 1, Section II

32A Integrable Systems

Define a finite-dimensional integrable system and state the Arnold–Liouville theorem.

Consider a four-dimensional phase space with coordinates \((q_1, q_2, p_1, p_2)\), where \(q_2 > 0\) and \(q_1\) is periodic with period \(2\pi\). Let the Hamiltonian be

\[
H = \frac{(p_1)^2}{2(q_2)^2} + \frac{(p_2)^2}{2} - \frac{k}{q_2}, \quad \text{where } k > 0.
\]

Show that the corresponding Hamilton equations form an integrable system.

Determine the sign of the constant \(E\) so that the motion is periodic on the surface \(H = E\). Demonstrate that in this case, the action variables are given by

\[
I_1 = p_1, \quad I_2 = \gamma \int_\alpha^\beta \frac{\sqrt{(q_2 - \alpha)(\beta - q_2)}}{q_2} \, dq_2,
\]

where \(\alpha, \beta, \gamma\) are positive constants which you should determine.
Consider the Poisson structure

\[ \{ F, G \} = \int_{\mathbb{R}} \frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)} \, dx , \tag{1} \]

where \( F, G \) are polynomial functionals of \( u, u_x, u_{xx}, \ldots \). Assume that \( u, u_x, u_{xx}, \ldots \) tend to zero as \( |x| \to \infty \).

(i) Show that \( \{ F, G \} = -\{ G, F \} \).

(ii) Write down Hamilton’s equations for \( u = u(x, t) \) corresponding to the following Hamiltonians:

\[ H_0[u] = \int_{\mathbb{R}} \frac{1}{2} u^2 \, dx , \quad H[u] = \int_{\mathbb{R}} \left( \frac{1}{2} u_x^2 + u^3 + uu_x \right) \, dx . \]

(iii) Calculate the Poisson bracket \( \{ H_0, H \} \), and hence or otherwise deduce that the following overdetermined system of partial differential equations for \( u = u(x, t_0, t) \) is compatible:

\[ u_{t_0} = u_x , \tag{2} \]
\[ u_t = 6uu_x - u_{xxx} . \tag{3} \]

[You may assume that the Jacobi identity holds for (1).]

(iv) Find a symmetry of (3) generated by \( X = \partial/\partial u + \alpha t \partial/\partial x \) for some constant \( \alpha \in \mathbb{R} \) which should be determined. Construct a vector field \( Y \) corresponding to the one-parameter group

\[ x \to \beta x , \quad t \to \gamma t , \quad u \to \delta u , \]

where \( (\beta, \gamma, \delta) \) should be determined from the symmetry requirement. Find the Lie algebra generated by the vector fields \( (X, Y) \).
Let $U(\rho, \tau, \lambda)$ and $V(\rho, \tau, \lambda)$ be matrix-valued functions. Consider the following system of overdetermined linear partial differential equations:

$$\frac{\partial}{\partial \rho} \psi = U \psi, \quad \frac{\partial}{\partial \tau} \psi = V \psi,$$

where $\psi$ is a column vector whose components depend on $(\rho, \tau, \lambda)$. Using the consistency condition of this system, derive the associated zero curvature representation (ZCR)

$$\frac{\partial}{\partial \tau} U - \frac{\partial}{\partial \rho} V + [U, V] = 0, \quad (\ast)$$

where $[\cdot, \cdot]$ denotes the usual matrix commutator.

(i) Let

$$U = \frac{i}{2} \begin{pmatrix} 2\lambda & \partial_{\rho} \phi \\ \partial_{\rho} \phi & -2\lambda \end{pmatrix}, \quad V = \frac{1}{4i\lambda} \begin{pmatrix} \cos \phi & -i \sin \phi \\ i \sin \phi & -\cos \phi \end{pmatrix}.$$

Find a partial differential equation for $\phi = \phi(\rho, \tau)$ which is equivalent to the ZCR $(\ast)$.

(ii) Assuming that $U$ and $V$ in $(\ast)$ do not depend on $t := \rho - \tau$, show that the trace of $(U - V)^p$ does not depend on $x := \rho + \tau$, where $p$ is any positive integer. Use this fact to construct a first integral of the ordinary differential equation

$$\phi'' = \sin \phi, \quad \text{where} \quad \phi = \phi(x).$$
Define a Poisson structure on an open set \( U \subset \mathbb{R}^n \) in terms of an anti-symmetric matrix \( \omega_{ab} : U \rightarrow \mathbb{R} \), where \( a, b = 1, \cdots, n \). By considering the Poisson brackets of the coordinate functions \( x^a \) show that

\[
\sum_{d=1}^{n} \left( \omega_{dc} \frac{\partial \omega_{ab}}{\partial x^d} + \omega_{db} \frac{\partial \omega_{ca}}{\partial x^d} + \omega_{da} \frac{\partial \omega_{bc}}{\partial x^d} \right) = 0.
\]

Now set \( n = 3 \) and consider \( \omega_{ab} = \sum_{c=1}^{3} \varepsilon_{abc} x^c \), where \( \varepsilon_{abc} \) is the totally antisymmetric symbol on \( \mathbb{R}^3 \) with \( \varepsilon_{123} = 1 \). Find a non-constant function \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) such that

\[
\{ f, x^a \} = 0, \quad a = 1, 2, 3.
\]

Consider the Hamiltonian

\[
H(x^1, x^2, x^3) = \frac{1}{2} \sum_{a,b=1}^{3} M_{ab} x^a x^b,
\]

where \( M_{ab} \) is a constant symmetric matrix and show that the Hamilton equations of motion with \( \omega_{ab} = \sum_{c=1}^{3} \varepsilon_{abc} x^c \) are of the form

\[
\dot{x}^a = \sum_{b,c=1}^{3} Q_{abc} x^b x^c,
\]

where the constants \( Q_{abc} \) should be determined in terms of \( M_{ab} \).
Consider the Gelfand–Levitan–Marchenko (GLM) integral equation
\[ K(x, y) + F(x + y) + \int_x^\infty K(x, z)F(z + y) \, dz = 0, \]
with \( F(x) = \sum_1^N \beta_n e^{-c_n x} \), where \( c_1, \ldots, c_N \) are positive constants and \( \beta_1, \ldots, \beta_N \) are constants. Consider separable solutions of the form
\[ K(x, y) = \sum_{n=1}^N K_n(x) e^{-c_n y}, \]
and reduce the GLM equation to a linear system
\[ \sum_{m=1}^N A_{nm}(x)K_m(x) = B_n(x), \]
where the matrix \( A_{nm}(x) \) and the vector \( B_n(x) \) should be determined.

How is \( K \) related to solutions of the KdV equation?

Set \( N = 1, c_1 = c, \beta_1 = \beta \exp(8c^3t) \) where \( c, \beta \) are constants. Show that the corresponding one–soliton solution of the KdV equation is given by
\[ u(x, t) = -\frac{4\beta_1 c e^{-2cx}}{(1 + (\beta_1/2c)e^{-2cx})^2}. \]

[You may use any facts about the Inverse Scattering Transform without proof.]
Consider a vector field
\[ V = \alpha x \frac{\partial}{\partial x} + \beta t \frac{\partial}{\partial t} + \gamma v \frac{\partial}{\partial v}, \]
on \( \mathbb{R}^3 \), where \( \alpha, \beta \) and \( \gamma \) are constants. Find the one-parameter group of transformations generated by this vector field.

Find the values of the constants \((\alpha, \beta, \gamma)\) such that \( V \) generates a Lie point symmetry of the modified KdV equation (mKdV)
\[ v_t - 6v^2v_x + v_{xxx} = 0, \quad \text{where} \quad v = v(x, t). \]

Show that the function \( u = u(x, t) \) given by \( u = v^2 + v_x \) satisfies the KdV equation and find a Lie point symmetry of KdV corresponding to the Lie point symmetry of mKdV which you have determined from \( V \).
Let $H$ be a smooth function on a $2n$–dimensional phase space with local coordinates $(p_j, q_j)$. Write down the Hamilton equations with the Hamiltonian given by $H$ and state the Arnold–Liouville theorem.

By establishing the existence of sufficiently many first integrals demonstrate that the system of $n$ coupled harmonic oscillators with the Hamiltonian

$$H = \frac{1}{2} \sum_{k=1}^{n} (p_k^2 + \omega_k^2 q_k^2),$$

where $\omega_1, \ldots, \omega_n$ are constants, is completely integrable. Find the action variables for this system.

Let $L = -\partial_x^2 + u(x, t)$ be a Schrödinger operator and let $A$ be another differential operator which does not contain derivatives with respect to $t$ and such that

$$L_t = [L, A].$$

Show that the eigenvalues of $L$ are independent of $t$, and deduce that if $f$ is an eigenfunction of $L$ then so is $f_t + Af$. [You may assume that $L$ is self–adjoint.]

Let $f$ be an eigenfunction of $L$ corresponding to an eigenvalue $\lambda$ which is non-degenerate. Show that there exists a function $\hat{f} = \hat{f}(x, t, \lambda)$ such that

$$L\hat{f} = \lambda\hat{f}, \quad \hat{f}_t + A\hat{f} = 0. \quad (*)$$

Assume

$$A = \partial_x^3 + a_1 \partial_x + a_0,$$

where $a_k = a_k(x, t)$, $k = 0, 1$ are functions. Show that the system $(*)$ is equivalent to a pair of first order matrix PDEs

$$\partial_x F = UF, \quad \partial_t F = VF,$$

where $F = (\hat{f}, \partial_x \hat{f})^T$ and $U, V$ are $2 \times 2$ matrices which should be determined.
Consider the partial differential equation
\[
\frac{\partial u}{\partial t} = u^n \frac{\partial u}{\partial x} + \frac{\partial^{2k+1} u}{\partial x^{2k+1}},
\]
where \( u = u(x, t) \) and \( k, n \) are non-negative integers.

(i) Find a Lie point symmetry of (*) of the form
\[
(x, t, u) \rightarrow (\alpha x, \beta t, \gamma u),
\]
where \( (\alpha, \beta, \gamma) \) are non-zero constants, and find a vector field generating this symmetry. Find two more vector fields generating Lie point symmetries of (*) which are not of the form (**) and verify that the three vector fields you have found form a Lie algebra.

(ii) Put (*) in a Hamiltonian form.
1/II/31C Integrable Systems

Define an integrable system in the context of Hamiltonian mechanics with a finite number of degrees of freedom and state the Arnold–Liouville theorem.

Consider a six-dimensional phase space with its canonical coordinates \((p_j, q_j)\), \(j = 1, 2, 3\), and the Hamiltonian

\[
\frac{1}{2} \sum_{j=1}^{3} p_j^2 + F(r),
\]

where \(r = \sqrt{q_1^2 + q_2^2 + q_3^2}\) and where \(F\) is an arbitrary function. Show that both \(M_1 = q_2 p_3 - q_3 p_2\) and \(M_2 = q_3 p_1 - q_1 p_3\) are first integrals.

State the Jacobi identity and deduce that the Poisson bracket

\[
M_3 = \{M_1, M_2\}
\]

is also a first integral. Construct a suitable expression out of \(M_1, M_2, M_3\) to demonstrate that the system admits three first integrals in involution and thus satisfies the hypothesis of the Arnold–Liouville theorem.

2/II/31C Integrable Systems

Describe the inverse scattering transform for the KdV equation, paying particular attention to the Lax representation and the evolution of the scattering data.

[Hint: you may find it helpful to consider the operator

\[
A = 4 \frac{d^3}{dx^3} - 3 \left( u \frac{d}{dx} + \frac{d}{dx} u \right).
\]
3/II/31C  **Integrable Systems**

Let $U(\lambda)$ and $V(\lambda)$ be matrix-valued functions of $(x,y)$ depending on the auxiliary parameter $\lambda$. Consider a system of linear PDEs

$$\frac{\partial}{\partial x} \Phi = U(\lambda)\Phi, \hspace{1cm} \frac{\partial}{\partial y} \Phi = V(\lambda)\Phi$$

(1)

where $\Phi$ is a column vector whose components depend on $(x,y,\lambda)$. Derive the zero curvature representation as the compatibility conditions for this system.

Assume that

$$U(\lambda) = \begin{pmatrix} u_x & 0 & \lambda \\ 1 & -u_x & 0 \\ 0 & 1 & 0 \end{pmatrix}, \hspace{1cm} V(\lambda) = \begin{pmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^u \\ \lambda^{-1} e^u & 0 & 0 \end{pmatrix}$$

and show that (1) is compatible if the function $u = u(x,y)$ satisfies the PDE

$$\frac{\partial^2 u}{\partial x \partial y} = F(u)$$

(2)

for some $F(u)$ which should be determined.

Show that the transformation

$$(x,y) \rightarrow (cx, c^{-1}y), \hspace{1cm} c \in \mathbb{R} \setminus \{0\}$$

forms a symmetry group of the PDE (2) and find the vector field generating this group.

Find the ODE characterising the group-invariant solutions of (2).
1/II/31E Integrable Systems

(i) Using the Cole–Hopf transformation
\[ u = -\frac{2\nu}{\phi} \frac{\partial \phi}{\partial x}, \]
map the Burgers equation
\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} \]
to the heat equation
\[ \frac{\partial \phi}{\partial t} = \nu \frac{\partial^2 \phi}{\partial x^2}. \]

(ii) Given that the solution of the heat equation on the infinite line \( \mathbb{R} \) with initial condition \( \phi(x,0) = \Phi(x) \) is given by
\[ \phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \Phi(\xi) e^{-\frac{(x-\xi)^2}{4\nu t}} d\xi, \]
show that the solution of the analogous problem for the Burgers equation with initial condition \( u(x,0) = U(x) \) is given by
\[ u = \frac{\int_{-\infty}^{\infty} \frac{x-\xi}{t} e^{-\frac{1}{2\nu} G(x,\xi,t)} d\xi}{\int_{-\infty}^{\infty} e^{-\frac{1}{2\nu} G(x,\xi,t)} d\xi}, \]
where the function \( G \) is to be determined in terms of \( U \).

(iii) Determine the ODE characterising the scaling reduction of the spherical modified Korteweg–de Vries equation
\[ \frac{\partial u}{\partial t} + 6u^2 \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} + \frac{u}{t} = 0. \]

2/II/31E Integrable Systems

Solve the following linear singular equation
\[ (t + t^{-1}) \phi(t) + \frac{(t - t^{-1})}{\pi i} \oint_C \frac{\phi(\tau)}{\tau - t} d\tau - \frac{(t + t^{-1})}{2\pi i} \oint_C (\tau + 2\tau^{-1}) \phi(\tau) d\tau = 2t^{-1}, \]
where \( C \) denotes the unit circle, \( t \in C \) and \( \oint_C \) denotes the principal value integral.
3/II/31E  Integrable Systems

Find a Lax pair formulation for the linearised NLS equation

\[ iq_t + q_{xx} = 0. \]

Use this Lax pair formulation to show that the initial value problem on the infinite line of the linearised NLS equation is associated with the following Riemann–Hilbert problem

\[
M^+(x,t,k) = M^-(x,t,k) \begin{pmatrix} 1 & e^{ikx-ik^2t}q_0(k) \\ 0 & 1 \end{pmatrix}, \quad k \in \mathbb{R},
\]

\[
M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + O\left(\frac{1}{k}\right), \quad k \to \infty.
\]

By deforming the above problem obtain the Riemann–Hilbert problem and hence the linear integral equation associated with the following system of nonlinear evolution PDEs

\[
iq_t + q_{xx} - 2\vartheta q^2 = 0,
\]

\[-i\vartheta_t + \vartheta_{xx} - 2\vartheta^2 q = 0.\]
1/II/31E  Integrable Systems

(a) Let $q(x,t)$ satisfy the heat equation

$$\frac{\partial q}{\partial t} = \frac{\partial^2 q}{\partial x^2}.$$  

Find the function $X$, which depends linearly on $\frac{\partial q}{\partial x}$, $q$, $k$, such that the heat equation can be written in the form

$$\frac{\partial}{\partial t} \left( e^{-ikx+k^2t} q \right) + \frac{\partial}{\partial x} \left( e^{-ikx+k^2t} X \right) = 0, \quad k \in \mathbb{C}.$$  

Use this equation to construct a Lax pair for the heat equation.

(b) Use the above result, as well as the Cole–Hopf transformation, to construct a Lax pair for the Burgers equation

$$\frac{\partial Q}{\partial t} - 2Q \frac{\partial Q}{\partial x} = \frac{\partial^2 Q}{\partial x^2}.$$  

(c) Find the second-order ordinary differential equation satisfied by the similarity solution of the so-called cylindrical KdV equation:

$$\frac{\partial q}{\partial t} + \frac{\partial^3 q}{\partial x^3} + q \frac{\partial q}{\partial x} + \frac{q^3}{3t} = 0, \quad t \neq 0.$$  

2/II/31E  Integrable Systems

Let $\phi(t)$ satisfy the singular integral equation

$$(t^4 + t^3 - t^2) \frac{\phi(t)}{2} + \frac{(t^4 - t^3 - t^2)}{2\pi i} \oint_C \frac{\phi(\tau)}{\tau - t} d\tau = (A - 1)t^3 + t^2,$$  

where $C$ denotes the circle of radius 2 centred on the origin, $\oint$ denotes the principal value integral and $A$ is a constant. Derive the associated Riemann–Hilbert problem, and compute the canonical solution of the corresponding homogeneous problem.

Find the value of $A$ such that $\phi(t)$ exists, and compute the unique solution $\phi(t)$ if $A$ takes this value.
The solution of the initial value problem of the KdV equation is given by
\[ q(x, t) = -2i \lim_{k \to \infty} k \frac{\partial N(x, t, k)}{\partial x}, \]
where the scalar function \( N(x, t, k) \) can be obtained by solving the following Riemann–Hilbert problem:
\[ \frac{M(x, t, k)}{a(k)} = N(x, t, -k) + \frac{b(k)}{a(k)} \exp(2ikx + 8k^3t) N(x, t, k), \quad k \in \mathbb{R}, \]
\[ M, N, \text{and} a \text{ are the boundary values of functions of } k \text{ that are analytic for } \Im k > 0 \text{ and tend to unity as } k \to \infty. \] The functions \( a(k) \) and \( b(k) \) can be determined from the initial condition \( q(x, 0) \).

Assume that \( M \) can be written in the form
\[ \frac{M}{a} = M(x, t, k) + \frac{c \exp(-2px + 8p^3t)}{k - ip} N(x, t, ip), \quad \Im k \geq 0, \]
where \( M \) as a function of \( k \) is analytic for \( \Im k > 0 \) and tends to unity as \( k \to \infty \); \( c \) and \( p \) are constants and \( p > 0 \).

(a) By solving the above Riemann–Hilbert problem find a linear equation relating \( N(x, t, k) \) and \( N(x, t, ip) \).

(b) By solving this equation explicitly in the case that \( b = 0 \) and letting \( c = 2ipe^{-2x_0} \), compute the one-soliton solution.

(c) Assume that \( q(x, 0) \) is such that \( a(k) \) has a simple zero at \( k = ip \). Discuss the dominant form of the solution as \( t \to \infty \) and \( x/t = O(1) \).
1/II/31D  Integrable Systems

Let \( \phi(t) \) satisfy the linear singular integral equation

\[
(t^2 + t - 1)\phi(t) - t^2 - t - 1 = \frac{1}{\pi i} \int_L \phi(\tau) \frac{d\tau}{\tau - t} - \frac{1}{\pi i} \int_L \left( \tau + \frac{1}{\tau} \right) \phi(\tau) d\tau = t - 1, \quad t \in L,
\]

where \( \oint \) denotes the principal value integral and \( L \) denotes a counterclockwise smooth closed contour, enclosing the origin but not the points \( \pm 1 \).

(a) Formulate the associated Riemann–Hilbert problem.

(b) For this Riemann–Hilbert problem, find the index, the homogeneous canonical solution and the solvability condition.

(c) Find \( \phi(t) \).

2/II/31C  Integrable Systems

Suppose \( q(x, t) \) satisfies the mKdV equation

\[
q_t + q_{xxx} + 6q^2 q_x = 0,
\]

where \( q_t = \partial q / \partial t \) etc.

(a) Find the 1-soliton solution.

[You may use, without proof, the indefinite integral \( \int \frac{dx}{x\sqrt{1 - x^2}} = \arcsch x \).]

(b) Express the self-similar solution of the mKdV equation in terms of a solution, denoted by \( v(z) \), of the Painlevé II equation.

(c) Using the Ansatz

\[
\frac{dv}{dz} + iv^2 - \frac{i}{6} = 0,
\]

find a particular solution of the mKdV equation in terms of a solution of the Airy equation

\[
\frac{d^2\Psi}{dz^2} + \frac{z}{6} \Psi = 0.
\]
3/II/31A  Integrable Systems

Let $Q(x,t)$ be an off-diagonal $2 \times 2$ matrix. The matrix NLS equation

$$iQ_t - Q_{xx} \sigma_3 + 2Q^3 \sigma_3 = 0, \quad \sigma_3 = \text{diag}(1,-1),$$

admits the Lax pair

$$\begin{align*}
\mu_x + ik[\sigma_3, \mu] &= Q\mu, \\
\mu_t + 2ik^2[\sigma_3, \mu] &= (2kQ - iQ^2 \sigma_3 - iQ_x \sigma_3)\mu,
\end{align*}$$

where $k \in \mathbb{C}$, $\mu(x,t,k)$ is a $2 \times 2$ matrix and $[\sigma_3, \mu]$ denotes the matrix commutator.

Let $S(k)$ be a $2 \times 2$ matrix-valued function decaying as $|k| \to \infty$. Let $\mu(x,t,k)$ satisfy the $2 \times 2$ matrix Riemann–Hilbert problem

$$\begin{align*}
\mu^+(x,t,k) &= \mu^-(x,t,k)e^{-i(kx+2k^2t)\sigma_3}S(k)e^{i(kx+2k^2t)\sigma_3}, \quad k \in \mathbb{R}, \\
\mu &= \text{diag}(1,1) + O\left(\frac{1}{k}\right), \quad k \to \infty.
\end{align*}$$

(a) Find expressions for $Q(x,t), A(x,t)$ and $B(x,t)$, in terms of the coefficients in the large $k$ expansion of $\mu$, so that $\mu$ solves

$$\begin{align*}
\mu_x + ik[\sigma_3, \mu] - Q\mu &= 0, \\
\mu_t + 2ik^2[\sigma_3, \mu] - (kA + B)\mu &= 0.
\end{align*}$$

(b) Use the result of (a) to establish that

$$A = 2Q, \quad B = -i(Q^2 + Q_x)\sigma_3.$$

(c) Show that the above results provide a linearization of the matrix NLS equation. What is the disadvantage of this approach in comparison with the inverse scattering method?