Part IB

Variational Principles

Year

2023
2022
2021
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2010

Paper 1, Section I
4C Variational Principles

Briefly explain how to use a Lagrange multiplier to find the extrema of a function $f(\mathbf{x})$ subject to a constraint $g(\mathbf{x}) = 0$.

Find the maximum volume of a cuboid of side lengths $x \geqslant 0$, $y \geqslant 0$, and $z \geqslant 0$ whose space diagonal has length L.

Paper 3, Section I
4C Variational Principles

Consider a function $f\colon \mathbb{R}^n \to \mathbb{R}$, not necessarily differentiable. What does it mean for f to be *convex* in a domain D?

If f is once differentiable, state an equivalent condition involving ∇f at two points $\mathbf{x}$ and $\mathbf{y}$ in D.

If f is twice differentiable, state an equivalent condition involving the Hessian $\mathbf{H}$.

Compute the largest domain on which the function $f(x, y) = x^3 + y^3 + Axy$ is convex in $\mathbb{R}^2$ (A is a constant) and sketch it.

Paper 2, Section II
13C Variational Principles

(a) For a functional of the form

$$\mathcal{L}[y] = \int_a^b F(x, y, y', y'')\,\mathrm{d}x,$$

derive the Euler–Lagrange equation satisfied by the solution $y(x)$ leading to a stationary value of $\mathcal{L}$. Show that all boundary terms cancel if the solution is assumed to have fixed values for y and y' at the end points.

(b) A diving board of length L at a swimming pool takes the shape $y(x)$ that minimises the energy

$$\mathcal{E} = \int_0^L \left[\frac{1}{2}A(y'')^2 + \rho g y\right]\,\mathrm{d}x,$$

where $A > 0$ is the bending rigidity, $\rho > 0$ the mass density and $g > 0$ the acceleration due to gravity (A, ρ, g are constants).

(i) Derive the ODE satisfied by $y(x)$.

(ii) The board is clamped at the origin (i.e. $y(0) = 0$, $y'(0) = 0$) while at $x = L$, it is torque free (i.e. $y''(L) = 0$) and a vertical force of magnitude F is applied to it (i.e. $-Ay'''(L) = F$). Solve for $y(x)$ and show that it may be written as $y(x) = y_0 + y_F$, where y_0 is the solution when $F = 0$ and y_F is proportional to F.

(iii) Compute the vertical displacement at the end of the board, $\Delta = y(L)$, and show that it can be written as $\Delta = h_0 + h$, where h_0 is the displacement when $F = 0$ and h is proportional to F.

(iv) For the solution in part (ii) compute the corresponding value of the energy $\mathcal{E}$ and show that it can be written as $\mathcal{E} = E_0 + E$, with E_0 independent of F and E quadratic in F.

(v) Relate $\frac{\mathrm{d}E}{\mathrm{d}F}$ to h and interpret your result.

Paper 4, Section II
13C Variational Principles

(a) Consider a functional of the form

$$\mathcal{L}[u,v] = \iint_\Omega f(x,y,u,v,u_x,u_y,v_x,v_y)\,\mathrm{d}x\mathrm{d}y,$$

where u and v are functions of x and y [we use the notation a_b to denote the partial derivative $\partial a/\partial b$]. Assuming small variations $u \to u + \delta u$ and $v \to v + \delta v$ and using integration by parts, derive the two Euler–Lagrange equations satisfied by u and v in Ω associated with an extremum of $\mathcal{L}$ (you may ignore all contributions from boundary terms).

(b) An elastic material deforms in two dimensions with a displacement field $\mathbf{u}(\mathbf{x}) = [u(x,y), v(x,y)]$, that minimises the total elastic energy

$$\mathcal{J} = \iint_\Omega \left[\frac{1}{2}\mu(\nabla\mathbf{u} : \nabla\mathbf{u}^T) + \frac{1}{2}(\lambda+\mu)(\nabla\cdot\mathbf{u})^2\right]\mathrm{d}x\mathrm{d}y$$

where $\nabla\mathbf{u}$ is the displacement gradient tensor, defined as

$$\nabla\mathbf{u} = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix},$$

where μ and λ are two material constants and where we use the notation $\mathbf{A} : \mathbf{B}$ to refer to the trace of the matrix product $\mathbf{AB}$.

(i) Show that

$$\mathcal{J} = \iint_\Omega \left[\left(\frac{\lambda}{2}+\mu\right)(u_x^2+v_y^2) + \frac{\mu}{2}(u_y^2+v_x^2) + (\lambda+\mu)u_x v_y\right]\mathrm{d}x\mathrm{d}y.$$

(ii) Derive the two Euler–Lagrange equations satisfied by u and v and show that they can be combined into a single equation for $\mathbf{u}$.

(iii) In the one-dimensional limit where $v = 0$, $\partial u/\partial y = 0$ with boundary conditions $u(0) = 0$, $u(L) = \Delta$, show that the solution to the equation obtained in (ii) is linear in x.

END OF PAPER

Paper 1, Section I
4D Variational Principles

Write down the Euler-Lagrange equation for the functional

$$I[y] = \int_0^{\pi/2} \left[y'^2 - y^2 - 2y\sin(x) \right] \mathrm{d}x \ .$$

Solve it subject to the boundary conditions $y'(0) = y'(\pi/2) = 0$.

Paper 3, Section I
4D Variational Principles

Explain the method of Lagrange multipliers for finding the stationary values of a function $F(x, y, z)$ subject to the constraint $G(x, y, z) = 0$.

Use the method of Lagrange multipliers to find the minimum of $x^2 + y^2 + z^2$ subject to the constraint $z - xy = 1$.

Find the maximum of $z - xy$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Paper 2, Section II
13D Variational Principles

(a) A functional $I[z]$ of $z(x)$ is given by

$$I[z] = \int_a^b f(z, z'; x)\, dx$$

where $z' = dz/dx$. State the Euler-Lagrange equation that governs the extrema of I.

If f does not depend explicitly on x, construct a non-constant quantity that, when evaluated on the extrema of I, does not depend on x.

Explain how to determine the extrema of I subject to the further functional constraint that $J[z]$ is constant.

(b) A heavy, uniform rope of fixed length L is suspended between two points $(x_1, z_1) = (-a, 0)$ and $(x_2, z_2) = (+a, 0)$ with $L > 2a$. In a gravitational potential $\Phi(z)$, the potential energy is given by

$$V[z] = \rho \int_{-a}^{a} \Phi(z) \sqrt{1 + z'^2}\, dx \ .$$

where ρ is the mass per unit length.

(i) Show that, in a gravitational potential $\Phi(z) = gz$, the shape adopted by the rope is

$$z - z_0 = -B \cosh\left(\frac{x}{B}\right)$$

where z_0 and B are two constants. Find implicit expressions for z_0 and B in terms of a and L.

(ii) What is the gravitational potential $\Phi(z)$ if, for $L = \pi a$, the rope hangs in a semi-circle?

Paper 4, Section II
13D Variational Principles

(a) Derive the Euler-Lagrange equation for the functional

$$\int_a^b f(y, y', y''; x)\, dx \ ,$$

where prime denotes differentiation with respect to x, and both y and y' are specified at $x = a, b$.

(b) If f does not depend explicitly on x show that, when evaluated on the extremum,

$$f - \left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\right] y' - \frac{\partial f}{\partial y''} y'' = \text{constant} \ .$$

(c) Find $y(x)$ that extremises the integral

$$\int_0^{\pi/2} \left(-\frac{1}{2} y''^2 + y'^2 - \frac{1}{2} y^2\right) dx$$

subject to $y(0) = y'(0) = 0$ and $y(\pi/2) = \pi/2$ and $y'(\pi/2) = 1$.

END OF PAPER

Paper 1, Section I
4D Variational Principles

Let D be a bounded region of $\mathbb{R}^2$, with boundary ∂D. Let $u(x,y)$ be a smooth function defined on D, subject to the boundary condition that $u = 0$ on ∂D and the normalization condition that

$$\int_D u^2 \, dx \, dy = 1 \, .$$

Let $I[u]$ be the functional

$$I[u] = \int_D |\boldsymbol{\nabla} u|^2 \, dx \, dy \, .$$

Show that $I[u]$ has a stationary value, subject to the stated boundary and normalization conditions, when u satisfies a partial differential equation of the form

$$\nabla^2 u + \lambda u = 0$$

in D, where λ is a constant.

Determine how λ is related to the stationary value of the functional $I[u]$. [*Hint: Consider* $\boldsymbol{\nabla} \cdot (u \boldsymbol{\nabla} u)$.]

Paper 3, Section I
4D Variational Principles

Find the function $y(x)$ that gives a stationary value of the functional

$$I[y] = \int_0^1 \left(y'^2 + y y' + y' + y^2 + y x^2 \right) dx \, ,$$

subject to the boundary conditions $y(0) = -1$ and $y(1) = e - e^{-1} - \frac{3}{2}$.

Paper 2, Section II

13D Variational Principles

A particle of unit mass moves in a smooth one-dimensional potential $V(x)$. Its path $x(t)$ is such that the action integral

$$S[x] = \int_a^b L(x, \dot{x})\, dt$$

has a stationary value, where a and $b > a$ are constants, a dot denotes differentiation with respect to time t,

$$L(x, \dot{x}) = \frac{1}{2}\dot{x}^2 - V(x)$$

is the Lagrangian function and the initial and final positions $x(a)$ and $x(b)$ are fixed.

By considering $S[x + \epsilon\xi]$ for suitably restricted functions $\xi(t)$, derive the differential equation governing the motion of the particle and obtain an integral expression for the second variation $\delta^2 S$.

If $x(t)$ is a solution of the equation of motion and $x(t) + \epsilon u(t) + O(\epsilon^2)$ is also a solution of the equation of motion in the limit $\epsilon \to 0$, show that $u(t)$ satisfies the equation

$$\ddot{u} + V''(x)\, u = 0\,.$$

If $u(t)$ satisfies this equation and is non-vanishing for $a \leqslant t \leqslant b$, show that

$$\delta^2 S = \frac{1}{2}\int_a^b \left(\dot{\xi} - \frac{\dot{u}\xi}{u}\right)^2 dt\,.$$

Consider the simple harmonic oscillator, for which

$$V(x) = \frac{1}{2}\omega^2 x^2\,,$$

where $2\pi/\omega$ is the oscillation period. Show that the solution of the equation of motion is a local minimum of the action integral, provided that the time difference $b - a$ is less than half an oscillation period.

Paper 4, Section II

13D Variational Principles

(a) Consider the functional

$$I[y] = \int_a^b L(y, y'; x)\, dx\,,$$

where $0 < a < b$, and $y(x)$ is subject to the requirement that $y(a)$ and $y(b)$ are some fixed constants. Derive the equation satisfied by $y(x)$ when $\delta I = 0$ for all variations δy that respect the boundary conditions.

(b) Consider the function

$$L(y, y'; x) = \frac{\sqrt{1 + y'^2}}{x}\,.$$

Verify that, if $y(x)$ describes an arc of a circle, with centre on the y-axis, then $\delta I = 0$.

(c) Consider the function

$$L(y, y'; x) = \frac{\sqrt{1 + y'^2}}{y}\,.$$

Find $y(x)$ such that $\delta I = 0$ subject to the requirement that $y(a) = a$ and $y(b) = \sqrt{2ab - b^2}$, with $b < 2a$. Sketch the curve $y(x)$.

Paper 2, Section I

3D Variational Principles

Find the stationary points of the function $\phi = xyz$ subject to the constraint $x + a^2y^2 + z^2 = b^2$, with $a, b > 0$. What are the maximum and minimum values attained by ϕ, subject to this constraint, if we further restrict to $x \geqslant 0$?

Paper 1, Section II

13D Variational Principles

A motion sensor sits at the origin, in the middle of a field. The probability that you are detected as you sneak from one point to another along a path $\mathbf{x}(t) : 0 \leqslant t \leqslant T$ is

$$P[\mathbf{x}(t)] = \lambda \int_0^T \frac{v(t)}{r(t)}\, dt\,,$$

where λ is a positive constant, $r(t)$ is your distance to the sensor, and $v(t)$ is your speed. (If $P[\mathbf{x}(t)] \geqslant 1$ for some path then you are detected with certainty.)

You start at point $(x, y) = (A, 0)$, where $A > 0$. Your mission is to reach the point $(x, y) = (B\cos\alpha, B\sin\alpha)$, where $B > 0$. What path should you take to minimise the chance of detection? Should you tiptoe or should you run?

A new and improved sensor detects you with probability

$$\tilde{P}[\mathbf{x}(t)] = \lambda \int_0^T \frac{v(t)^2}{r(t)}\, dt\,.$$

Show that the optimal path now satisfies the equation

$$\left(\frac{dr}{dt}\right)^2 = Er - h^2$$

for some constants E and h that you should identify.

Paper 1, Section I

4A Variational Principles

A function $\phi = xy - yz$ is defined on the surface $x^2 + 2y^2 + z^2 = 1$. Find the location (x, y, z) of every stationary point of this function.

Paper 3, Section I

6A Variational Principles

The function f with domain $x > 0$ is defined by $f(x) = \frac{1}{a}x^a$, where $a > 1$. Verify that f is convex, using an appropriate sufficient condition.

Determine the Legendre transform f^* of f, specifying clearly its domain of definition, and find $(f^*)^*$.

Show that

$$\frac{x^r}{r} + \frac{y^s}{s} \geqslant xy$$

where x, $y > 0$ and r and s are positive real numbers such that $\frac{1}{r} + \frac{1}{s} = 1$.

Paper 2, Section II

15A Variational Principles

Write down the Euler–Lagrange (EL) equations for a functional

$$\int_a^b f(\,u,\,w,\,u',\,w',\,x\,)\,dx\,,$$

where $u(x)$ and $w(x)$ each take specified values at $x = a$ and $x = b$. Show that the EL equations imply that

$$\kappa \;=\; f \;-\; u'\frac{\partial f}{\partial u'} \;-\; w'\frac{\partial f}{\partial w'}$$

is independent of x provided f satisfies a certain condition, to be specified. State conditions under which there exist additional first integrals of the EL equations.

Consider

$$f \;=\; \left(1-\frac{m}{u}\right)w'^{\,2} \;-\; \left(1-\frac{m}{u}\right)^{-1}u'^{\,2}$$

where m is a positive constant. Show that solutions of the EL equations satisfy

$$u'^{\,2} \;=\; \lambda^2 \;+\; \kappa\left(1-\frac{m}{u}\right),$$

for some constant λ. Assuming that $\kappa = -\lambda^2$, find dw/du and hence determine the most general solution for w as a function of u subject to the conditions $u > m$ and $w \to -\infty$ as $u \to \infty$. Show that, for any such solution, $w \to \infty$ as $u \to m$.

[*Hint:*

$$\frac{d}{dz}\left\{\,\log\left(\frac{z^{1/2}-1}{z^{1/2}+1}\right)\right\} \;=\; \frac{1}{z^{1/2}\,(z-1)}\,. \quad]$$

Paper 4, Section II

16A Variational Principles

Consider the functional

$$I[y] = \int_{-\infty}^{\infty} \left(\tfrac{1}{2}\, y'^{\,2} + \tfrac{1}{2}\, U(y)^2 \right) dx\,,$$

where $y(x)$ is subject to boundary conditions $y(x) \to a_\pm$ as $x \to \pm\infty$ with $U(a_\pm) = 0$. [You may assume the integral converges.]

(a) Find expressions for the first-order and second-order variations δI and $\delta^2 I$ resulting from a variation δy that respects the boundary conditions.

(b) If $a_\pm = a$, show that $I[y] = 0$ if and only if $y(x) = a$ for all x. Explain briefly how this is consistent with your results for δI and $\delta^2 I$ in part (a).

(c) Now suppose that $U(y) = c^2 - y^2$ with $a_\pm = \pm c$ $(c > 0)$. By considering an integral of $U(y)y'$, show that

$$I[y] \geqslant \frac{4c^3}{3}\,,$$

with equality if and only if y satisfies a first-order differential equation. Deduce that global minima of $I[y]$ with the specified boundary conditions occur precisely for

$$y(x) = c \tanh\{\, c(x - x_0)\,\}\,,$$

where x_0 is a constant. How is the first-order differential equation that appears in this case related to your general result for δI in part (a)?

Paper 1, Section I

4B Variational Principles

Find, using a Lagrange multiplier, the four stationary points in $\mathbb{R}^3$ of the function $x^2 + y^2 + z^2$ subject to the constraint $x^2 + 2y^2 - z^2 = 1$. By sketching sections of the constraint surface in each of the coordinate planes, or otherwise, identify the nature of the constrained stationary points.

How would the location of the stationary points differ if, instead, the function $x^2 + 2y^2 - z^2$ were subject to the constraint $x^2 + y^2 + z^2 = 1$?

Paper 3, Section I

6B Variational Principles

For a particle of unit mass moving freely on a unit sphere, the Lagrangian in polar coordinates is

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\sin^2\theta\,\dot{\phi}^2\,.$$

Determine the equations of motion. Show that $l = \sin^2\theta\,\dot{\phi}$ is a conserved quantity, and use this result to simplify the equation of motion for θ. Deduce that

$$h = \dot{\theta}^2 + \frac{l^2}{\sin^2\theta}$$

is a conserved quantity. What is the interpretation of h?

Paper 2, Section II

15B Variational Principles

Derive the Euler-Lagrange equation for the integral

$$I[y] = \int_{x_0}^{x_1} f(y, y', y'', x)\,dx,$$

when $y(x)$ and $y'(x)$ take given values at the fixed endpoints.

Show that the only function $y(x)$ with $y(0) = 1$, $y'(0) = 2$ and $y(x) \to 0$ as $x \to \infty$ for which the integral

$$I[y] = \int_0^\infty \left(y^2 + (y')^2 + (y' + y'')^2\right) dx$$

is stationary is $(3x + 1)e^{-x}$.

Paper 4, Section II

16B Variational Principles

(a) A two-dimensional oscillator has action

$$S = \int_{t_0}^{t_1} \left\{ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\omega^2 y^2 \right\} dt\,.$$

Find the equations of motion as the Euler-Lagrange equations associated with S, and use them to show that

$$J = \dot{x}y - \dot{y}x$$

is conserved. Write down the general solution of the equations of motion in terms of $\sin \omega t$ and $\cos \omega t$, and evaluate J in terms of the coefficients that arise in the general solution.

(b) Another kind of oscillator has action

$$\widetilde{S} = \int_{t_0}^{t_1} \left\{ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{4}\alpha x^4 - \frac{1}{2}\beta x^2 y^2 - \frac{1}{4}\alpha y^4 \right\} dt\,,$$

where α and β are real constants. Find the equations of motion and use these to show that in general $J = \dot{x}y - \dot{y}x$ is not conserved. Find the special value of the ratio β/α for which J is conserved. Explain what is special about the action $\widetilde{S}$ in this case, and state the interpretation of J.

Paper 1, Section I

4D Variational Principles

Derive the Euler-Lagrange equation for the function $u(x, y)$ that gives a stationary value of

$$I[u] = \int_{\mathcal{D}} L\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) dx\, dy,$$

where $\mathcal{D}$ is a bounded domain in the (x, y)-plane and u is fixed on the boundary $\partial\mathcal{D}$.

Find the equation satisfied by the function u that gives a stationary value of

$$I = \int_{\mathcal{D}} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + k^2 u^2\right] dx\, dy,$$

where k is a constant and u is prescribed on $\partial\mathcal{D}$.

Paper 3, Section I

6D Variational Principles

(a) A *Pringle crisp* can be defined as the surface

$$z = xy \quad \text{with} \quad x^2 + y^2 \leqslant 1.$$

Use the method of Lagrange multipliers to find the minimum and maximum values of z on the boundary of the Pringle crisp and the (x, y) positions where these occur.

(b) A farmer wishes to construct a grain silo in the form of a hollow vertical cylinder of radius r and height h with a hollow hemispherical cap of radius r on top of the cylinder. The walls of the cylinder cost £x per unit area to construct and the surface of the cap costs £$2x$ per unit area to construct. Given that a total volume V is desired for the silo, what values of r and h should be chosen to minimise the cost?

Paper 2, Section II

15D Variational Principles

A proto-planet of mass m in a uniform galactic dust cloud has kinetic and potential energies

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2, \qquad V = kmr^2$$

where k is constant. State Hamilton's principle and use it to determine the equations of motion for the proto-planet.

Write down two conserved quantities of the motion and state why their existence illustrates Noether's theorem.

Determine the Hamiltonian $H(\mathbf{p}, \mathbf{x})$ of this system, where $\mathbf{p} = (p_r, p_\phi)$, $\mathbf{x} = (r, \phi)$ and (p_r, p_ϕ) are the conjugate momenta corresponding to (r, ϕ).

Write down Hamilton's equations for this system and use them to show that

$$m\ddot{r} = -V'_{\text{eff}}(r), \quad \text{where} \quad V_{\text{eff}}(r) = m\left(\frac{h^2}{2m^2r^2} + kr^2\right)$$

and h is a constant. With the aid of a diagram, explain why there is a stable circular orbit.

Paper 4, Section II

16D Variational Principles

Consider the functional

$$F[y] = \int_\alpha^\beta f(y, y', x)\, dx$$

of a function $y(x)$ defined for $x \in [\alpha, \beta]$, with y having fixed values at $x = \alpha$ and $x = \beta$.

By considering $F[y + \epsilon\xi]$, where $\xi(x)$ is an arbitrary function with $\xi(\alpha) = \xi(\beta) = 0$ and $\epsilon \ll 1$, determine that the second variation of F is

$$\delta^2 F[y, \xi] = \int_\alpha^\beta \left\{ \xi^2 \left[\frac{\partial^2 f}{\partial y^2} - \frac{d}{dx}\left(\frac{\partial^2 f}{\partial y \partial y'} \right) \right] + \xi'^2 \frac{\partial^2 f}{\partial y'^2} \right\} dx.$$

The surface area of an axisymmetric soap film joining two parallel, co-axial, circular rings of radius a distance $2L$ apart can be expressed by the functional

$$F[y] = \int_{-L}^{L} 2\pi y \sqrt{1 + y'^2}\, dx,$$

where x is distance in the axial direction and y is radial distance from the axis. Show that the surface area is stationary when

$$y = E \cosh \frac{x}{E},$$

where E is a constant that need not be determined, and that the stationary area is a local minimum if

$$\int_{-L/E}^{L/E} \left(\xi'^2 - \xi^2 \right) \operatorname{sech}^2 z\, dz > 0$$

for all functions $\xi(z)$ that vanish at $z = \pm L/E$, where $z = x/E$.

Paper 1, Section I

4C Variational Principles

(a) Consider the function $f(x_1, x_2) = 2x_1^2 + x_2^2 + \alpha x_1 x_2$, where α is a real constant. For what values of α is the function f convex?

(b) In the case $\alpha = -3$, calculate the extremum of x_1^2 on the set of points where $f(x_1, x_2) + 1 = 0$.

Paper 3, Section I

6C Variational Principles

Two points A and B are located on the curved surface of the circular cylinder of radius R with axis along the $z-$axis. We denote their locations by (R, ϕ_A, z_A) and (R, ϕ_B, z_B) using cylindrical polar coordinates and assume $\phi_A \neq \phi_B$, $z_A \neq z_B$. A path $\phi(z)$ is drawn on the cylinder to join A and B. Show that the path of minimum distance between the points A and B is a helix, and determine its pitch. [For a helix with axis parallel to the z axis, the pitch is the change in z after one complete helical turn.]

Paper 2, Section II

15C Variational Principles

A flexible wire filament is described by the curve $(x, y(x), z(x))$ in cartesian coordinates for $0 \leqslant x \leqslant L$. The filament is assumed to be almost straight and thus we assume $|y'| \ll 1$ and $|z'| \ll 1$ everywhere.

(a) Show that the total length of the filament is approximately $L + \Delta$ where

$$\Delta = \frac{1}{2} \int_0^L \left[(y')^2 + (z')^2 \right] dx.$$

(b) Under a uniform external axial force, $F > 0$, the filament adopts the shape which minimises the total energy, $\mathcal{E} = E_B - F\Delta$, where E_B is the bending energy given by

$$E_B[y, z] = \frac{1}{2} \int_0^L \left[A(x)(y'')^2 + B(x)(z'')^2 \right] dx,$$

and where $A(x)$ and $B(x)$ are x-dependent bending rigidities (both known and strictly positive). The filament satisfies the boundary conditions

$$y(0) = y'(0) = z(0) = z'(0) = 0, \quad y(L) = y'(L) = z(L) = z'(L) = 0.$$

Derive the Euler-Lagrange equations for $y(x)$ and $z(x)$.

(c) In the case where $A = B = 1$ and $L = 1$, show that below a critical force, F_c, which should be determined, the only energy-minimising solution for the filament is straight ($y = z = 0$), but that a new nonzero solution is admissible at $F = F_c$.

Paper 4, Section II

16C Variational Principles

A fish swims in the ocean along a straight line with speed $V(t)$. The fish starts its journey from rest (zero velocity at $t = 0$) and, during a given time T, swims subject to the constraint that the total distance travelled is L. The energy cost for swimming is $aV^2 + b\dot{V}^2$ per unit time, where $a, b \geqslant 0$ are known and $a^2 + b^2 \neq 0$.

(a) Derive the Euler-Lagrange condition on $V(t)$ for the journey to have minimum energetic cost.

(b) In the case $a \neq 0, b \neq 0$ solve for $V(t)$ assuming that the fish starts at $t = 0$ with zero acceleration (in addition to zero velocity).

(c) In the case $a = 0$, the fish can decide between three different boundary conditions for its journey. In addition to starting with zero velocity, it can:

(1) start at $t = 0$ with zero acceleration;

(2) end at $t = T$ with zero velocity; or

(3) end at $t = T$ with zero acceleration.

Which of (1), (2) or (3) is the best minimal-energy cost strategy?

Paper 1, Section I

4A Variational Principles

Consider a frictionless bead on a stationary wire. The bead moves under the action of gravity acting in the negative y-direction and the wire traces out a path $y(x)$, connecting points $(x, y) = (0, 0)$ and $(1, 0)$. Using a first integral of the Euler-Lagrange equations, find the choice of $y(x)$ which gives the shortest travel time, starting from rest. You may give your solution for y and x separately, in parametric form.

Paper 3, Section I

6A Variational Principles

(a) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be *convex*.

(b) Define the *Legendre transform* $f^*(p)$ of a convex function $f(x)$, where $x \in \mathbb{R}$. Show that $f^*(p)$ is a convex function.

(c) Find the Legendre transform $f^*(p)$ of the function $f(x) = e^x$, and the domain of f^*.

Paper 2, Section II

15A Variational Principles

A right circular cylinder of radius a and length l has volume V and total surface area A. Use Lagrange multipliers to do the following:

(a) Show that, for a given total surface area, the maximum volume is

$$V = \frac{1}{3}\sqrt{\frac{A^3}{C\pi}},$$

determining the integer C in the process.

(b) For a cylinder inscribed in the unit sphere, show that the value of l/a which maximises the area of the cylinder is

$$D + \sqrt{E},$$

determining the integers D and E as you do so.

(c) Consider the rectangular parallelepiped of largest volume which fits inside a hemisphere of fixed radius. Find the ratio of the parallelepiped's volume to the volume of the hemisphere.

[You need *not* show that suitable extrema you find are actually maxima.]

Paper 4, Section II

16A Variational Principles

Derive the Euler–Lagrange equation for the integral

$$\int_{x_0}^{x_1} f(x, u, u')\ dx$$

where $u(x_0)$ is allowed to float, $\partial f/\partial u'|_{x_0} = 0$ and $u(x_1)$ takes a given value.

Given that $y(0)$ is finite, $y(1) = 1$ and $y'(1) = 1$, find the stationary value of

$$J = \int_0^1 \left(x^4 (y'')^2 + 4x^2 (y')^2 \right)\ dx.$$

Paper 1, Section I

4C Variational Principles

Define the Legendre transform $f^*(\mathbf{p})$ of a function $f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$.

Show that for $g(\mathbf{x}) = \lambda\, f(\mathbf{x} - \mathbf{x}_0) - \mu$,

$$g^*(\mathbf{p}) = \lambda f^*\left(\frac{\mathbf{p}}{\lambda}\right) + \mathbf{p}^{\mathbf{T}}\mathbf{x}_0 + \mu.$$

Show that for $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^{\mathbf{T}}\mathbf{A}\,\mathbf{x}$ where $\mathbf{A}$ is a real, symmetric, invertible matrix with positive eigenvalues,

$$f^*(\mathbf{p}) = \tfrac{1}{2}\mathbf{p}^{\mathbf{T}}\mathbf{A}^{-\mathbf{1}}\,\mathbf{p}.$$

Paper 3, Section I

6C Variational Principles

Let $f(x, y, z) = xz + yz$. Using Lagrange multipliers, find the location(s) and value of the maximum of f on the intersection of the unit sphere ($x^2 + y^2 + z^2 = 1$) and the ellipsoid given by $\frac{1}{4}x^2 + \frac{1}{4}y^2 + 4z^2 = 1$.

Paper 2, Section II

15C Variational Principles

Write down the Euler–Lagrange equation for the integral

$$\int f(y, y', x)dx.$$

An ant is walking on the surface of a sphere, which is parameterised by $\theta \in [0, \pi]$ (angle from top of sphere) and $\phi \in [0, 2\pi)$ (azimuthal angle). The sphere is sticky towards the top and the bottom and so the ant's speed is proportional to $\sin\theta$. Show that the ant's fastest route between two points will be of the form

$$\sinh(A\phi + B) = \cot\theta$$

for some constants A and B. [A, B need not be determined.]

Paper 4, Section II

16C Variational Principles

Consider the integral

$$I = \int f(y, y')dx.$$

Show that if f satisfies the Euler–Lagrange equation, then

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}.$$

An axisymmetric soap film $y(x)$ is formed between two circular wires at $x = \pm l$. The wires both have radius r. Show that the shape that minimises the surface area takes the form

$$y(x) = k\cosh\frac{x}{k}.$$

Show that there exist two possible k that satisfy the boundary conditions for r/l sufficiently large.

Show that for these solutions the second variation is given by

$$\delta^2 I = \pi\int_{-l}^{+l}\left(k\eta'^2 - \frac{1}{k}\eta^2\right)\operatorname{sech}^2\left(\frac{x}{k}\right)dx$$

where η is an axisymmetric perturbation with $\eta(\pm l) = 0$.

Paper 1, Section I

4A Variational Principles

(a) Define what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be convex. Assuming g'' exists, state an equivalent condition. Let $f(x) = x \log x$, defined on $x > 0$. Show that $f(x)$ is convex.

(b) Find the Legendre transform $f^*(p)$ of $f(x) = x \log x$. State the domain of $f^*(p)$. Without further calculation, explain why $(f^*)^* = f$ in this case.

Paper 3, Section I

6A Variational Principles

A cylindrical drinking cup has thin curved sides with density ρ per unit area, and a disk-shaped base with density $k\rho$ per unit area. The cup has capacity to hold a fixed volume V of liquid. Use the method of Lagrange multipliers to find the minimum mass of the cup.

Paper 2, Section II

15A Variational Principles

Starting from the Euler–Lagrange equation, show that a condition for

$$\int f(y, y')dx$$

to be stationary is

$$f - y' \frac{\partial f}{\partial y'} = \text{constant}.$$

In the half-plane $y > 0$, light has speed $c(y) = y + c_0$ where $c_0 > 0$. Find the equation for a light ray between $(-a, 0)$ and $(a, 0)$. Sketch the solution.

Paper 4, Section II

16A Variational Principles

Derive the Euler–Lagrange equation for the integral

$$\int_a^b f(x, y, y', y'')\, dx$$

where prime denotes differentiation with respect to x, and both y and y' are specified at $x = a, b$.

Find $y(x)$ that extremises the integral

$$\int_0^\pi \left(y + \frac{1}{2}y^2 - \frac{1}{2}y''^2 \right) dx$$

subject to $y(0) = -1$, $y'(0) = 0$, $y(\pi) = \cosh \pi$ and $y'(\pi) = \sinh \pi$.

Show that your solution is a global maximum. You may use the result that

$$\int_0^\pi \phi^2(x) dx \leqslant \int_0^\pi \phi'^2(x) dx$$

for any (suitably differentiable) function ϕ which satisfies $\phi(0) = 0$ and $\phi(\pi) = 0$.

Paper 1, Section I

4B Variational Principles

State how to find the stationary points of a C^1 function $f(x, y)$ restricted to the circle $x^2 + y^2 = 1$, using the method of Lagrange multipliers. Explain why, in general, the method of Lagrange multipliers works, in the case where there is just one constraint.

Find the stationary points of $x^4 + 2y^3$ restricted to the circle $x^2 + y^2 = 1$.

Paper 3, Section I

6B Variational Principles

For a particle of unit mass moving freely on a unit sphere, the Lagrangian in polar coordinates is

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\sin^2\theta\,\dot{\phi}^2\,.$$

Find the equations of motion. Show that $l = \sin^2\theta\,\dot{\phi}$ is a conserved quantity, and use this result to simplify the equation of motion for θ. Deduce that

$$h = \dot{\theta}^2 + \frac{l^2}{\sin^2\theta}$$

is a conserved quantity. What is the interpretation of h?

Paper 2, Section II

15B Variational Principles

(i) A two-dimensional oscillator has action

$$S = \int_{t_0}^{t_1} \left\{ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{2}\omega^2 x^2 - \frac{1}{2}\omega^2 y^2 \right\} dt\,.$$

Find the equations of motion as the Euler-Lagrange equations associated to S, and use them to show that

$$J = \dot{x}y - \dot{y}x$$

is conserved. Write down the general solution of the equations of motion in terms of $\sin\omega t$ and $\cos\omega t$, and evaluate J in terms of the coefficients which arise in the general solution.

(ii) Another kind of oscillator has action

$$\widetilde{S} = \int_{t_0}^{t_1} \left\{ \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 - \frac{1}{4}\alpha x^4 - \frac{1}{2}\beta x^2 y^2 - \frac{1}{4}\alpha y^4 \right\} dt\,,$$

where α and β are real constants. Find the equations of motion and use these to show that in general $J = \dot{x}y - \dot{y}x$ is not conserved. Find the special value of the ratio β/α for which J is conserved. Explain what is special about the action $\widetilde{S}$ in this case, and state the interpretation of J.

Paper 4, Section II

16B Variational Principles

Consider a functional

$$I = \int_a^b F(x, y, y')dx$$

where F is smooth in all its arguments, $y(x)$ is a C^1 function and $y' = \frac{dy}{dx}$. Consider the function $y(x) + h(x)$ where $h(x)$ is a small C^1 function which vanishes at a and b. Obtain formulae for the first and second variations of I about the function $y(x)$. Derive the Euler-Lagrange equation from the first variation, and state its variational interpretation.

Suppose now that

$$I = \int_0^1 (y'^2 - 1)^2 dx$$

where $y(0) = 0$ and $y(1) = \beta$. Find the Euler-Lagrange equation and the formula for the second variation of I. Show that the function $y(x) = \beta x$ makes I stationary, and that it is a (local) minimizer if $\beta > \frac{1}{\sqrt{3}}$.

Show that when $\beta = 0$, the function $y(x) = 0$ is not a minimizer of I.

Paper 1, Section I

4D Variational Principles

(i) Write down the Euler-Lagrange equations for the volume integral

$$\int_V \big(\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} u + 12u\big) dV \,,$$

where V is the unit ball $x^2+y^2+z^2 \leqslant 1$, and verify that the function $u(x,y,z) = x^2+y^2+z^2$ gives a stationary value of the integral subject to the condition $u = 1$ on the boundary.

(ii) Write down the Euler-Lagrange equations for the integral

$$\int_0^1 \big(\dot{x}^2 + \dot{y}^2 + 4x + 4y\big)\, dt \,,$$

where the dot denotes differentiation with respect to t, and verify that the functions $x(t) = t^2$, $y(t) = t^2$ give a stationary value of the integral subject to the boundary conditions $x(0) = y(0) = 0$ and $x(1) = y(1) = 1$.

Paper 3, Section I

6D Variational Principles

Find, using a Lagrange multiplier, the four stationary points in $\mathbb{R}^3$ of the function $x^2 + y^2 + z^2$ subject to the constraint $x^2 + 2y^2 - z^2 = 1$. By considering the situation geometrically, or otherwise, identify the nature of the constrained stationary points.

How would your answers differ if, instead, the stationary points of the function $x^2 + 2y^2 - z^2$ were calculated subject to the constraint $x^2 + y^2 + z^2 = 1$?

Paper 2, Section II

15D Variational Principles

(i) Let $I[y] = \int_0^1 \big((y')^2 - y^2\big)dx$, where y is twice differentiable and $y(0) = y(1) = 0$. Write down the associated Euler-Lagrange equation and show that the only solution is $y(x) = 0$.

(ii) Let $J[y] = \int_0^1 \big(y' + y\tan x\big)^2 dx$, where y is twice differentiable and $y(0) = y(1) = 0$. Show that $J[y] = 0$ only if $y(x) = 0$.

(iii) Show that $I[y] = J[y]$ and deduce that the extremal value of $I[y]$ is a global minimum.

(iv) Use the second variation of $I[y]$ to verify that the extremal value of $I[y]$ is a local minimum.

(v) How would your answers to part (i) differ in the case $I[y] = \int_0^{2\pi} \big((y')^2 - y^2\big)dx$, where $y(0) = y(2\pi) = 0$? Show that the solution $y(x) = 0$ is not a global minimizer in this case. (You may use without proof the result $I[x(2\pi - x)] = -\frac{8}{15}(2\pi^2 - 5)$.) Explain why the arguments of parts (iii) and (iv) cannot be used.

Paper 4, Section II

16D Variational Principles

Derive the Euler-Lagrange equation for the integral

$$\int_{x_0}^{x_1} f(y, y', y'', x)\, dx$$

where the endpoints are fixed, and $y(x)$ and $y'(x)$ take given values at the endpoints.

Show that the only function $y(x)$ with $y(0) = 1$, $y'(0) = 2$ and $y(x) \to 0$ as $x \to \infty$ for which the integral

$$\int_0^\infty \big(y^2 + (y')^2 + (y' + y'')^2\big)dx$$

is stationary is $(3x + 1)e^{-x}$.

Paper 1, Section I

4D Variational Principles

(a) Define what it means for a function $f : \mathbb{R}^n \to \mathbb{R}$ to be convex and strictly convex.

(b) State a necessary and sufficient first-order condition for strict convexity of $f \in C^1(\mathbb{R}^n)$, and give, with proof, an example of a function which is strictly convex but with second derivative which is not everywhere strictly positive.

Paper 3, Section I

6D Variational Principles

Derive the Euler-Lagrange equation for the function $u(x, y)$ which gives a stationary value of

$$I = \int_{\mathcal{D}} L\Big(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big)\, dxdy,$$

where $\mathcal{D}$ is a bounded domain in the (x, y) plane, with u fixed on the boundary $\partial\mathcal{D}$.

Find the equation satisfied by the function u which gives a stationary value of

$$I = \int_{\mathcal{D}} \Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2\Big]\, dxdy,$$

with u given on $\partial\mathcal{D}$.

Paper 2, Section II

15D Variational Principles

Describe briefly the method of Lagrange multipliers for finding the stationary points of a function $f(x, y)$ subject to a constraint $\phi(x, y) = 0$.

A tent manufacturer wants to maximize the volume of a new design of tent, subject only to a constant weight (which is directly proportional to the amount of fabric used). The models considered have either equilateral-triangular or semi-circular vertical cross–section, with vertical planar ends in both cases and with floors of the same fabric. Which shape maximizes the volume for a given area A of fabric?

[Hint: $(2\pi)^{-1/2}3^{-3/4}(2 + \pi) < 1$.]

Paper 4, Section II

16D Variational Principles

A function $\theta(\phi)$ with given values of $\theta(\phi_1)$ and $\theta(\phi_2)$ makes the integral

$$I = \int_{\phi_1}^{\phi_2} \mathcal{L}(\theta, \theta')\, d\phi$$

stationary with respect to small variations of θ which vanish at ϕ_1 and ϕ_2. Show that $\theta(\phi)$ satisfies the first integral of the Euler–Lagrange equation,

$$\mathcal{L}(\theta, \theta') - \theta'(\partial\mathcal{L}/\partial\theta') = C,$$

for some constant C. You may state the Euler–Lagrange equation without proof.

It is desired to tow an iceberg across open ocean from a point on the Antarctic coast (longitude ϕ_1) to a place in Australia (longitude ϕ_2), to provide fresh water for irrigation. The iceberg will melt at a rate proportional to the difference between its temperature (the constant T_0, measured in degrees Celsius and therefore negative) and the sea temperature $T(\theta) > T_0$, where θ is the colatitude (the usual spherical polar coordinate θ). Assume that the iceberg is towed at a constant speed along a path $\theta(\phi)$, where ϕ is the longitude. Given that the infinitesimal arc length on the unit sphere is $(d\theta^2 + \sin^2\theta\ d\phi^2)^{1/2}$, show that the total ice melted along the path from ϕ_1 to ϕ_2 is proportional to

$$I = \int_{\phi_1}^{\phi_2} (T(\theta) - T_0)(\theta'^2 + \sin^2\theta)^{1/2}\, d\phi.$$

Now suppose that in the relevant latitudes, the sea temperature may be approximated by $T(\theta) = T_0(1+3\tan\theta)$. (*Note that* $(1+3\tan\theta)$ *is negative in the relevant latitudes.*) Show that any smooth path $\theta(\phi)$ which minimizes the total ice melted must satisfy

$$\theta'^2 = \sin^2\theta\,(\tfrac{1}{4}k^2\tan^2\theta\sin^2\theta - 1),$$

and hence that

$$\sin^2\theta = \frac{2}{1 - (1+k^2)^{1/2}\sin 2(\phi - \phi_0)},$$

where k and ϕ_0 are constants.

[Hint:

$$\int \frac{dx}{x(\alpha^2x^4 + x^2 - 1)^{1/2}} = \frac{1}{2}\arcsin\left[\frac{x^2-2}{x^2(1+4\alpha^2)^{1/2}}\right].\ ]$$