Part IB

Topological Spaces

Year
2026

Paper 1, Section I

2F Topological Spaces

Let $(X, \mathcal{T})$ be a topological space. What is a *neighbourhood* of a point $x \in X$? What does it mean for a sequence in X to *converge* to some $x \in X$?

Recall that $(X, \mathcal{T})$ is metrisable if $\mathcal{T}$ is induced by a metric d on X. Given metrics d and d' on X, show that the following are equivalent:

(1) d and d' induce the same topology on X;

(2) the convergent sequences with respect to d and d' are the same, i.e. for every sequence (x_n) in X, $\exists x \in X$ such that $x_n \to x$ with respect to d $\iff$ $\exists y \in X$ such that $x_n \to y$ with respect to d'.

It follows that the convergent sequences uniquely determine the topology of a metrisable space. Does the same hold for a general topological space? Briefly justify your answer.

Paper 2, Section I

11F Topological Spaces

(a) What does it mean for a topological space to be *connected*?

Consider $\mathbb{R}$ with the usual Euclidean topology. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bijection with the property that, for every connected $X \subset \mathbb{R}$, $f(X)$ is connected. Show that f must be monotone. [You may assume the following fact: $g\colon \mathbb{R} \to \mathbb{R}$ is monotone $\iff$ for every triple (a,b,c) with $a<b<c$, $g(b)$ lies between $g(a)$ and $g(c)$.]

Show that f must be continuous.

(b) What does it mean for a topological space to be *path-connected*? Show that a path-connected space is connected.

Consider the unit square $Q = [0,1] \times [0,1]$ with the usual Euclidean topology. The boundary of Q is $\partial Q = ([0,1]\times[0,1]) \setminus ((0,1)\times(0,1))$. Let γ and δ be two continuous paths on ∂Q starting at $(0,0)$ and ending at $(1,1)$ that only intersect at these two points.

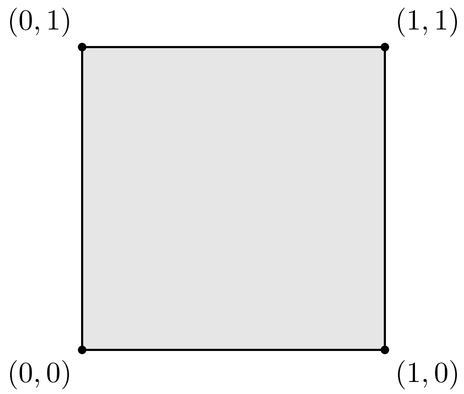

Figure: the unit square Q with its four corners.

(i) Give an explicit example of one possible such pair (γ, δ).

(ii) Let $\alpha, \beta\colon [0,1] \to Q$ be continuous paths on Q satisfying

$$\alpha(0) = (0,0), \quad \alpha(1) = (1,1),$$
$$\beta(0) = (0,1), \quad \beta(1) = (1,0).$$

Show that the images of α and β must intersect in Q. [You may assume the following fact: there is no continuous $F : Q \to S^1$ such that the images of $F \circ \gamma$ and of $F \circ \delta$ are opposite hemispheres of the unit circle S^1.]

(iii) Suppose that $A \subset Q$ is path-connected, contains $(0,0)$ and $(1,1)$ and does not contain the other two corners of Q. Deduce that $(0,1)$ and $(1,0)$ must lie in different path components of $Q \setminus A$.

[*You may assume results from the lectures about intervals in $\mathbb{R}$ if stated clearly.*]

Paper 3, Section II

2F Topological Spaces

Let $(X, \mathcal{T})$ be a topological space, and $A \subset X$. What is the *subspace topology* on A? If $Y = X/{\sim}$, for $\sim$ an equivalence relation on X, what is the *quotient topology* on Y? If $(Y, \mathcal{T}')$ is another topological space, what is a *homeomorphism* between X and Y?

Consider $\mathbb{R}^2$ with the Euclidean topology, and let $S = [-1,1]^2 \subset \mathbb{R}^2$. Let $\sim$ be the equivalence relation on S generated by $(x,y) \sim (-x,y)$ for all $(x,y) \in S$, and let $Y = S/{\sim}$. Determine the quotient map $q\colon S \to Y$.

By first showing that Y is homeomorphic to $S^+ = [0,1] \times [-1,1]$, or otherwise, show that Y is homeomorphic to S. [You may assume results from lectures if carefully stated.]

Paper 3, Section II

12F Topological Spaces

What is a *base* for a topology on a topological space?

Let X be a topological space whose topology is generated by a metric ρ. Fix $k \in \mathbb{N}$, $k \geqslant 2$ and let $Y = \{1, \ldots, k\}$. Consider the product space X^k, whose elements are of the form $(f(1), \ldots, f(k))$ for some $f\colon Y \to X$. Determine the product topology on X^k.

Explain briefly why

$$d(f,g) = \sum_{n=1}^{k} \frac{1}{2^n} \min\{1, \rho(f(n), g(n))\},$$

where $f, g\colon Y \to X$, defines a metric on the space of functions from Y to X. Show that the product topology on X^k is the same as that induced by d. What are the convergent sequences with respect to this topology?

Let $X^{\mathbb{N}} = X \times X \times \ldots$ with a topology that has as base the family of finite intersections of sets of the form

$$\{\pi_n^{-1}(U_n)\colon n \in \mathbb{N} \text{ and } U_n \subset X \text{ open in } X\},$$

where π_n is the projection map from $X^{\mathbb{N}}$ to the nth factor. Is this topology induced by a metric? Justify your answer. [*Hint: It may help to rewrite the base for the topology in terms of functions from* $\mathbb{N}$ *to* X *as in the previous part.*]

Paper 4, Section II

11F Topological Spaces

For $c = (c_1, c_2) \in \mathbb{R}^2$ and $r > 0$, let $S_r(c) = (c_1 - \frac{r}{2}, c_1 + \frac{r}{2}) \times (c_2 - \frac{r}{2}, c_2 + \frac{r}{2})$ be the open square of centre c and side length r. Consider the space

$$P = S_2((0,0)) \setminus \left(\overline{S_{\frac{1}{4}}((-\tfrac{1}{2},0))} \cup \overline{S_{\frac{1}{4}}((\tfrac{1}{2},0))} \right) \subset \mathbb{R}^2$$

endowed with the subspace topology.

(i) Define the *interior* and *closure* of a subset of a topological space. Identify the topological boundary of P inside $\mathbb{R}^2$, which is defined as $\partial P = \overline{P} \setminus \mathring{P}$.

(ii) Construct a polyhedron homeomorphic to $\overline{P}$. How many vertices, edges and triangles does it have?

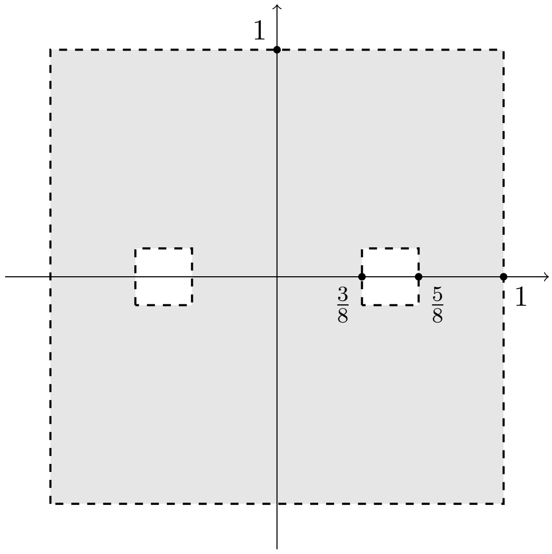

Figure: the set P.

Define $\sim$ to be the smallest equivalence relation on $(0 \times \overline{P}) \sqcup (1 \times \overline{P})$ such that

$$(0, x) \sim (1, x) \quad \text{for all} \quad x \in \partial P.$$

Consider the space $D = ((0 \times \overline{P}) \sqcup (1 \times \overline{P}))/\sim$ endowed with the quotient topology.

(iii) Show carefully that D is compact and connected. [You may assume results from lectures if clearly stated.]

(iv) What does it mean for a topological space to be *locally Euclidean*? What is a *topological manifold*? Explain briefly why D is locally Euclidean. [You do not need to construct explicit homeomorphisms.]

(v) Define the *Euler characteristic* of a topological surface. What is the image of the triangulation you found in part (ii) under the quotient map? Is it a triangulation of D? Justify your answer.

(vi) State the classification theorem for topological surfaces. Assume that D is homeomorphic to a connected sum of k tori. Write down, without proof, the value of k.