Part IB

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Metric and Topological Spaces

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Paper 3, Section I
3E Metric and Topological Spaces

What does it mean to say that a topological space is connected? If $X$ is a topological space and $x \in X$, show that there is a connected subspace $K_x$ of $X$ so that if $S$ is any other connected subspace containing $x$ then $S \subseteq K_x$.

Show that the sets $K_x$ partition $X$.

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Paper 2, Section I
4E Metric and Topological Spaces

What does it mean to say that $d$ is a metric on a set $X$? What does it mean to say that a subset of $X$ is open with respect to the metric $d$? Show that the collection of subsets of $X$ that are open with respect to $d$ satisfies the axioms of a topology.

For $X = C[0, 1]$, the set of continuous functions $f : [0, 1] \to \mathbb{R}$, show that the metrics

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| \, dx$$
$$d_2(f, g) = \left[ \int_0^1 |f(x) - g(x)|^2 \, dx \right]^{1/2}$$

give different topologies.

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Paper 1, Section II
12E Metric and Topological Spaces

What does it mean to say that a topological space is compact? Prove directly from the definition that $[0, 1]$ is compact. Hence show that the unit circle $S^1 \subset \mathbb{R}^2$ is compact, proving any results that you use. [You may use without proof the continuity of standard functions.]

The set $\mathbb{R}^2$ has a topology $\mathcal{T}$ for which the closed sets are the empty set and the finite unions of vector subspaces. Let $X$ denote the set $\mathbb{R}^2 \setminus \{0\}$ with the subspace topology induced by $\mathcal{T}$. By considering the subspace topology on $S^1 \subset \mathbb{R}^2$, or otherwise, show that $X$ is compact.
Paper 4, Section II
13E Metric and Topological Spaces

Let \( X = \{2, 3, 4, 5, 6, 7, 8, \ldots\} \) and for each \( n \in X \) let

\[
U_n = \{d \in X \mid d \text{ divides } n\}.
\]

Prove that the set of unions of the sets \( U_n \) forms a topology on \( X \).

Prove or disprove each of the following:

(i) \( X \) is Hausdorff;

(ii) \( X \) is compact.

If \( Y \) and \( Z \) are topological spaces, \( Y \) is the union of closed subspaces \( A \) and \( B \), and \( f : Y \to Z \) is a function such that both \( f|_A : A \to Z \) and \( f|_B : B \to Z \) are continuous, show that \( f \) is continuous. Hence show that \( X \) is path-connected.
3E Metric and Topological Spaces

Let $X$ and $Y$ be topological spaces.

(a) Define what is meant by the product topology on $X \times Y$. Define the projection maps $p: X \times Y \to X$ and $q: X \times Y \to Y$ and show they are continuous.

(b) Consider $\Delta = \{ (x, x) \mid x \in X \}$ in $X \times X$. Show that $X$ is Hausdorff if and only if $\Delta$ is a closed subset of $X \times X$ in the product topology.

4E Metric and Topological Spaces

Let $f: (X, d) \to (Y, e)$ be a function between metric spaces.

(a) Give the $\varepsilon$-$\delta$ definition for $f$ to be continuous. Show that $f$ is continuous if and only if $f^{-1}(U)$ is an open subset of $X$ for each open subset $U$ of $Y$.

(b) Give an example of $f$ such that $f$ is not continuous but $f(V)$ is an open subset of $Y$ for every open subset $V$ of $X$.

12E Metric and Topological Spaces

Consider $\mathbb{R}$ and $\mathbb{R}^2$ with their usual Euclidean topologies.

(a) Show that a non-empty subset of $\mathbb{R}$ is connected if and only if it is an interval. Find a compact subset $K \subset \mathbb{R}$ for which $\mathbb{R} \setminus K$ has infinitely many connected components.

(b) Let $T$ be a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 \setminus T$ is path-connected. Deduce that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.
Let $f: X \to Y$ be a continuous map between topological spaces.

(a) Assume $X$ is compact and that $Z \subseteq X$ is a closed subset. Prove that $Z$ and $f(Z)$ are both compact.

(b) Suppose that

(i) $f^{-1}(\{y\})$ is compact for each $y \in Y$, and

(ii) if $A$ is any closed subset of $X$, then $f(A)$ is a closed subset of $Y$.

Show that if $K \subseteq Y$ is compact, then $f^{-1}(K)$ is compact.

[Hint: Given an open cover $f^{-1}(K) \subseteq \bigcup_{i \in I} U_i$, find a finite subcover, say $f^{-1}(\{y\}) \subseteq \bigcup_{i \in I_y} U_i$, for each $y \in K$; use closedness of $X \setminus \bigcup_{i \in I_y} U_i$ and property (ii) to produce an open cover of $K$.]
Let $X$ be a topological space and $A \subseteq X$ be a subset. A limit point of $A$ is a point $x \in X$ such that any open neighbourhood $U$ of $x$ intersects $A$. Show that $A$ is closed if and only if it contains all its limit points. Explain what is meant by the interior $\text{Int}(A)$ and the closure $\overline{A}$ of $A$. Show that if $A$ is connected, then $\overline{A}$ is connected.

Consider $\mathbb{R}$ and $\mathbb{Q}$ with their usual topologies.

(a) Show that compact subsets of a Hausdorff topological space are closed. Show that compact subsets of $\mathbb{R}$ are closed and bounded.

(b) Show that there exists a complete metric space $(X,d)$ admitting a surjective continuous map $f : X \to \mathbb{Q}$.

Let $p$ be a prime number. Define what is meant by the $p$-adic metric $d_p$ on $\mathbb{Q}$. Show that for $a, b, c \in \mathbb{Q}$ we have

$$d_p(a, b) \leq \max\{d_p(a, c), d_p(c, b)\}.$$  

Show that the sequence $(a_n)$, where $a_n = 1 + p + \cdots + p^{n-1}$, converges to some element in $\mathbb{Q}$.

For $a \in \mathbb{Q}$ define $|a|_p = d_p(a, 0)$. Show that if $a, b \in \mathbb{Q}$ and if $|a|_p \neq |b|_p$, then

$$|a + b|_p = \max\{|a|_p, |b|_p\}.$$  

Let $a \in \mathbb{Q}$ and let $B(a, \delta)$ be the open ball with centre $a$ and radius $\delta > 0$, with respect to the metric $d_p$. Show that $B(a, \delta)$ is a closed subset of $\mathbb{Q}$ with respect to the topology induced by $d_p$. 

Part IB, 2016 List of Questions
Paper 4, Section II
13E Metric and Topological Spaces

(a) Let $X$ be a topological space. Define what is meant by a quotient of $X$ and describe the quotient topology on the quotient space. Give an example in which $X$ is Hausdorff but the quotient space is not Hausdorff.

(b) Let $T^2$ be the 2-dimensional torus considered as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, and let $\pi : \mathbb{R}^2 \to T^2$ be the quotient map.

(i) Let $B(u, r)$ be the open ball in $\mathbb{R}^2$ with centre $u$ and radius $r < 1/2$. Show that $U = \pi(B(u, r))$ is an open subset of $T^2$ and show that $\pi^{-1}(U)$ has infinitely many connected components. Show each connected component is homeomorphic to $B(u, r)$.

(ii) Let $\alpha \in \mathbb{R}$ be an irrational number and let $L \subset \mathbb{R}^2$ be the line given by the equation $y = \alpha x$. Show that $\pi(L)$ is dense in $T^2$ but $\pi(L) \neq T^2$. 

Part IB, 2016 List of Questions
Paper 3, Section I
3E Metric and Topological Spaces
Define what it means for a topological space $X$ to be (i) connected (ii) path-connected.

Prove that any path-connected space $X$ is connected. [You may assume the interval $[0, 1]$ is connected.]

Give a counterexample (without justification) to the converse statement.

Paper 2, Section I
4E Metric and Topological Spaces
Let $X$ and $Y$ be topological spaces and $f : X \to Y$ a continuous map. Suppose $H$ is a subset of $X$ such that $f(H)$ is closed (where $\overline{H}$ denotes the closure of $H$). Prove that $f(\overline{H}) = \overline{f(H)}$.

Give an example where $f, X, Y$ and $H$ are as above but $f(\overline{H})$ is not closed.

Paper 1, Section II
12E Metric and Topological Spaces
Give the definition of a metric on a set $X$ and explain how this defines a topology on $X$.

Suppose $(X, d)$ is a metric space and $U$ is an open set in $X$. Let $x, y \in X$ and $\epsilon > 0$ such that the open ball $B_{\epsilon}(y) \subseteq U$ and $x \in B_{\epsilon/2}(y)$. Prove that $y \in B_{\epsilon/2}(x) \subseteq U$.

Explain what it means (i) for a set $S$ to be dense in $X$, (ii) to say $B$ is a base for a topology $\mathcal{T}$.

Prove that any metric space which contains a countable dense set has a countable basis.
13E Metric and Topological Spaces

Explain what it means for a metric space \((M, d)\) to be (i) compact, (ii) sequentially compact. Prove that a compact metric space is sequentially compact, stating clearly any results that you use.

Let \((M, d)\) be a compact metric space and suppose \(f: M \rightarrow M\) satisfies \(d(f(x), f(y)) = d(x, y)\) for all \(x, y \in M\). Prove that \(f\) is surjective, stating clearly any results that you use. [Hint: Consider the sequence \((f^n(x))\) for \(x \in M\).]

Give an example to show that the result does not hold if \(M\) is not compact.
Paper 3, Section I

3E Metric and Topological Spaces
Suppose \((X,d)\) is a metric space. Do the following necessarily define a metric on \(X\)? Give proofs or counterexamples.

(i) \(d_1(x,y) = kd(x,y)\) for some constant \(k > 0\), for all \(x, y \in X\).

(ii) \(d_2(x,y) = \min\{1, d(x,y)\}\) for all \(x, y \in X\).

(iii) \(d_3(x,y) = (d(x,y))^2\) for all \(x, y \in X\).

Paper 2, Section I

4E Metric and Topological Spaces
Let \(X\) and \(Y\) be topological spaces. What does it mean to say that a function \(f : X \to Y\) is continuous?

Are the following statements true or false? Give proofs or counterexamples.

(i) Every continuous function \(f : X \to Y\) is an open map, i.e. if \(U\) is open in \(X\) then \(f(U)\) is open in \(Y\).

(ii) If \(f : X \to Y\) is continuous and bijective then \(f\) is a homeomorphism.

(iii) If \(f : X \to Y\) is continuous, open and bijective then \(f\) is a homeomorphism.

Paper 1, Section II

12E Metric and Topological Spaces
Define what it means for a topological space to be compact. Define what it means for a topological space to be Hausdorff.

Prove that a compact subspace of a Hausdorff space is closed. Hence prove that if \(C_1\) and \(C_2\) are compact subspaces of a Hausdorff space \(X\) then \(C_1 \cap C_2\) is compact.

A subset \(U\) of \(\mathbb{R}\) is open in the cocountable topology if \(U\) is empty or its complement in \(\mathbb{R}\) is countable. Is \(\mathbb{R}\) Hausdorff in the cocountable topology? Which subsets of \(\mathbb{R}\) are compact in the cocountable topology?
Explain what it means for a metric space to be complete.

Let $X$ be a metric space. We say the subsets $A_i$ of $X$, with $i \in \mathbb{N}$, form a descending sequence in $X$ if $A_1 \supset A_2 \supset A_3 \supset \cdots$.

Prove that the metric space $X$ is complete if and only if any descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of $X$, such that the diameters of the subsets $A_i$ converge to zero, has an intersection $\bigcap_{i=1}^{\infty} A_i$ that is non-empty.

[Recall that the diameter $\text{diam}(S)$ of a set $S$ is the supremum of the set $\{d(x, y) : x, y \in S\}$.

Give examples of

(i) a metric space $X$, and a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of $X$, with $\text{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

(ii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty subsets in $\mathbb{R}$ with $\text{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

(iii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed sets in $\mathbb{R}$ with $\bigcap_{i=1}^{\infty} A_i = \emptyset$. 

Part IB, 2014 List of Questions
Paper 3, Section I
3G  Metric and Topological Spaces

Let $X$ be a metric space with the metric $d : X \times X \to \mathbb{R}$.

(i) Show that if $X$ is compact as a topological space, then $X$ is complete.

(ii) Show that the completeness of $X$ is not a topological property, i.e. give an example of two metrics $d, d'$ on a set $X$, such that the associated topologies are the same, but $(X, d)$ is complete and $(X, d')$ is not.

Paper 2, Section I
4G  Metric and Topological Spaces

Let $X$ be a topological space. Prove or disprove the following statements.

(i) If $X$ is discrete, then $X$ is compact if and only if it is a finite set.

(ii) If $Y$ is a subspace of $X$ and $X, Y$ are both compact, then $Y$ is closed in $X$.

Paper 1, Section II
12G  Metric and Topological Spaces

Consider the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, a subset of $\mathbb{R}^3$, as a subspace of $\mathbb{R}^3$ with the Euclidean metric.

(i) Show that $S^2$ is compact and Hausdorff as a topological space.

(ii) Let $X = S^2/\sim$ be the quotient set with respect to the equivalence relation identifying the antipodes, i.e.

$$(x, y, z) \sim (x', y', z') \iff (x', y', z') = (x, y, z) \text{ or } (x, y, z) \text{ or } (-x, -y, -z).$$

Show that $X$ is compact and Hausdorff with respect to the quotient topology.
Let $X$ be a topological space. A connected component of $X$ means an equivalence class with respect to the equivalence relation on $X$ defined as:

$$x \sim y \iff x, y \text{ belong to some connected subspace of } X.$$ 

(i) Show that every connected component is a connected and closed subset of $X$.

(ii) If $X, Y$ are topological spaces and $X \times Y$ is the product space, show that every connected component of $X \times Y$ is a direct product of connected components of $X$ and $Y$. 

Part IB, 2013 List of Questions
Paper 3, Section I

3F Metric and Topological Spaces

Define the notion of a connected component of a space $X$.

If $A_\alpha \subset X$ are connected subsets of $X$ such that $\bigcap_\alpha A_\alpha \neq \emptyset$, show that $\bigcup_\alpha A_\alpha$ is connected.

Prove that any point $x \in X$ is contained in a unique connected component.

Let $X \subset \mathbb{R}$ consist of the points $0, 1, \frac{1}{2}, 1, \frac{1}{3}, \ldots, 1, \ldots$. What are the connected components of $X$?

Paper 2, Section I

4F Metric and Topological Spaces

For each case below, determine whether the given metrics $d_1$ and $d_2$ induce the same topology on $X$. Justify your answers.

(i) $X = \mathbb{R}^2$, $d_1((x_1, y_1), (x_2, y_2)) = \sup\{|x_1 - x_2|, |y_1 - y_2|\}$

$$d_2((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$ 

(ii) $X = C[0,1]$, $d_1(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$

$$d_2(f, g) = \int_0^1 |f(t) - g(t)| \, dt.$$ 

Paper 1, Section II

12F Metric and Topological Spaces

A topological space $X$ is said to be normal if each point of $X$ is a closed subset of $X$ and for each pair of closed sets $C_1, C_2 \subset X$ with $C_1 \cap C_2 = \emptyset$ there are open sets $U_1, U_2 \subset X$ so that $C_i \subset U_i$ and $U_1 \cap U_2 = \emptyset$. In this case we say that the $U_i$ separate the $C_i$.

Show that a compact Hausdorff space is normal. [Hint: first consider the case where $C_2$ is a point.]

For $C \subset X$ we define an equivalence relation $\sim_C$ on $X$ by $x \sim_C y$ for all $x, y \in C$, $x \sim_C x$ for $x \notin C$. If $C, C_1$ and $C_2$ are pairwise disjoint closed subsets of a normal space $X$, show that $C_1$ and $C_2$ may be separated by open subsets $U_1$ and $U_2$ such that $U_i \cap C = \emptyset$. Deduce that the quotient space $X/\sim_C$ is also normal.
Paper 4, Section II

13F Metric and Topological Spaces

Suppose $A_1$ and $A_2$ are topological spaces. Define the product topology on $A_1 \times A_2$. Let $\pi_i : A_1 \times A_2 \to A_i$ be the projection. Show that a map $F : X \to A_1 \times A_2$ is continuous if and only if $\pi_1 \circ F$ and $\pi_2 \circ F$ are continuous.

Prove that if $A_1$ and $A_2$ are connected, then $A_1 \times A_2$ is connected.

Let $X$ be the topological space whose underlying set is $\mathbb{R}$, and whose open sets are of the form $(a, \infty)$ for $a \in \mathbb{R}$, along with the empty set and the whole space. Describe the open sets in $X \times X$. Are the maps $f, g : X \times X \to X$ defined by $f(x, y) = x + y$ and $g(x, y) = xy$ continuous? Justify your answers.
Paper 2, Section I
4G Metric and Topological Spaces
(i) Let $t > 0$. For $x = (x, y), x' = (x', y') \in \mathbb{R}^2$, let
\[ d(x, x') = |x' - x| + t|y' - y|, \]
\[ \delta(x, x') = \sqrt{(x' - x)^2 + (y' - y)^2}. \]
($\delta$ is the usual Euclidean metric on $\mathbb{R}^2$.) Show that $d$ is a metric on $\mathbb{R}^2$ and that the two metrics $d, \delta$ give rise to the same topology on $\mathbb{R}^2$.

(ii) Give an example of a topology on $\mathbb{R}^2$, different from the one in (i), whose induced topology (subspace topology) on the $x$-axis is the usual topology (the one defined by the metric $d(x, x') = |x' - x|$). Justify your answer.

Paper 3, Section I
3G Metric and Topological Spaces
Let $X, Y$ be topological spaces, and suppose $Y$ is Hausdorff.

(i) Let $f, g : X \to Y$ be two continuous maps. Show that the set
\[ E(f, g) := \{ x \in X \mid f(x) = g(x) \} \subset X \]
is a closed subset of $X$.

(ii) Let $W$ be a dense subset of $X$. Show that a continuous map $f : X \to Y$ is determined by its restriction $f|_W$ to $W$.

Paper 1, Section II
12G Metric and Topological Spaces
Let $X$ be a metric space with the distance function $d : X \times X \to \mathbb{R}$. For a subset $Y$ of $X$, its diameter is defined as $\delta(Y) := \sup\{d(y, y') \mid y, y' \in Y\}$.

Show that, if $X$ is compact and $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open covering of $X$, then there exists an $\epsilon > 0$ such that every subset $Y \subset X$ with $\delta(Y) < \epsilon$ is contained in some $U_\lambda$. 
Let $X, Y$ be topological spaces and $X \times Y$ their product set. Let $p_Y : X \times Y \to Y$ be the projection map.

(i) Define the product topology on $X \times Y$. Prove that if a subset $Z \subset X \times Y$ is open then $p_Y(Z)$ is open in $Y$.

(ii) Give an example of $X, Y$ and a closed set $Z \subset X \times Y$ such that $p_Y(Z)$ is not closed.

(iii) When $X$ is compact, show that if a subset $Z \subset X \times Y$ is closed then $p_Y(Z)$ is closed.
Paper 2, Section I
4H Metric and Topological Spaces

On the set $\mathbb{Q}$ of rational numbers, the 3-adic metric $d_3$ is defined as follows: for $x, y \in \mathbb{Q}$, define $d_3(x, x) = 0$ and $d_3(x, y) = 3^{-n}$, where $n$ is the integer satisfying $x - y = 3^n u$ where $u$ is a rational number whose denominator and numerator are both prime to 3.

(1) Show that this is indeed a metric on $\mathbb{Q}$.

(2) Show that in $(\mathbb{Q}, d_3)$, we have $3^n \to 0$ as $n \to \infty$ while $3^{-n} \not\to 0$ as $n \to \infty$.

Let $d$ be the usual metric $d(x, y) = |x - y|$ on $\mathbb{Q}$. Show that neither the identity map $(\mathbb{Q}, d) \to (\mathbb{Q}, d_3)$ nor its inverse is continuous.

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Paper 3, Section I
3H Metric and Topological Spaces

Let $X$ be a topological space and $Y$ be a set. Let $p : X \to Y$ be a surjection. The quotient topology on $Y$ is defined as follows: a subset $V \subset Y$ is open if and only if $p^{-1}(V)$ is open in $X$.

(1) Show that this does indeed define a topology on $Y$, and show that $p$ is continuous when we endow $Y$ with this topology.

(2) Let $Z$ be another topological space and $f : Y \to Z$ be a map. Show that $f$ is continuous if and only if $f \circ p : X \to Z$ is continuous.

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Paper 1, Section II
12H Metric and Topological Spaces

Let $f : X \to Y$ and $g : Y \to X$ be continuous maps of topological spaces with $f \circ g = \text{id}_Y$.

(1) Suppose that (i) $Y$ is path-connected, and (ii) for every $y \in Y$, its inverse image $f^{-1}(y)$ is path-connected. Prove that $X$ is path-connected.

(2) Prove the same statement when “path-connected” is everywhere replaced by “connected”.

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(1) Prove that if \( X \) is a compact topological space, then a closed subset \( Y \) of \( X \) endowed with the subspace topology is compact.

(2) Consider the following equivalence relation on \( \mathbb{R}^2 \):

\[
(x_1, y_1) \sim (x_2, y_2) \iff (x_1 - x_2, y_1 - y_2) \in \mathbb{Z}^2.
\]

Let \( X = \mathbb{R}^2 / \sim \) be the quotient space endowed with the quotient topology, and let \( p : \mathbb{R}^2 \to X \) be the canonical surjection mapping each element to its equivalence class. Let \( Z = \{(x, y) \in \mathbb{R}^2 \mid y = \sqrt{2}x\} \).

(i) Show that \( X \) is compact.

(ii) Assuming that \( p(Z) \) is dense in \( X \), show that \( p|_Z : Z \to p(Z) \) is bijective but not homeomorphic.
Paper 2, Section I

4F Metric and Topological Spaces

Explain what is meant by a Hausdorff (topological) space, and prove that every compact subset of a Hausdorff space is closed.

Let $X$ be an uncountable set, and consider the topology $T$ on $X$ defined by

$$U \in T \iff \text{either } U = \emptyset \text{ or } X \setminus U \text{ is countable}.$$ 


Paper 3, Section I

4F Metric and Topological Spaces

Are the following statements true or false? Give brief justifications for your answers.

(i) If $X$ is a connected open subset of $\mathbb{R}^n$ for some $n$, then $X$ is path-connected.

(ii) A cartesian product of two connected spaces is connected.

(iii) If $X$ is a Hausdorff space and the only connected subsets of $X$ are singletons $\{x\}$, then $X$ is discrete.

Paper 1, Section II

12F Metric and Topological Spaces

Given a function $f : X \to Y$ between metric spaces, we write $\Gamma_f$ for the subset $\{(x, f(x)) \mid x \in X\}$ of $X \times Y$.

(i) If $f$ is continuous, show that $\Gamma_f$ is closed in $X \times Y$.

(ii) If $Y$ is compact and $\Gamma_f$ is closed in $X \times Y$, show that $f$ is continuous.

(iii) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\Gamma_f$ is closed but $f$ is not continuous.
14F Metric and Topological Spaces

A nonempty subset $A$ of a topological space $X$ is called irreducible if, whenever we have open sets $U$ and $V$ such that $U \cap A$ and $V \cap A$ are nonempty, then we also have $U \cap V \cap A \neq \emptyset$. Show that the closure of an irreducible set is irreducible, and deduce that the closure of any singleton set $\{x\}$ is irreducible.

$X$ is said to be a sober topological space if, for any irreducible closed $A \subseteq X$, there is a unique $x \in X$ such that $A = \{x\}$. Show that any Hausdorff space is sober, but that an infinite set with the cofinite topology is not sober.

Given an arbitrary topological space $(X, \mathcal{T})$, let $\hat{X}$ denote the set of all irreducible closed subsets of $X$, and for each $U \in \mathcal{T}$ let

$$\hat{U} = \{ A \in \hat{X} \mid U \cap A \neq \emptyset \}.$$  

Show that the sets $\{ \hat{U} \mid U \in \mathcal{T} \}$ form a topology $\hat{\mathcal{T}}$ on $\hat{X}$, and that the mapping $U \mapsto \hat{U}$ is a bijection from $\mathcal{T}$ to $\hat{\mathcal{T}}$. Deduce that $(\hat{X}, \hat{\mathcal{T}})$ is sober. 

[Hint: consider the complement of an irreducible closed subset of $\hat{X}$.]
1/II/12F  Metric and Topological Spaces

Write down the definition of a topology on a set $X$.

For each of the following families $T$ of subsets of $Z$, determine whether $T$ is a topology on $Z$. In the cases where the answer is ‘yes’, determine also whether $(Z, T)$ is a Hausdorff space and whether it is compact.

(a) $T = \{U \subseteq Z : \text{either } U \text{ is finite or } 0 \in U\}$.
(b) $T = \{U \subseteq Z : \text{either } Z \setminus U \text{ is finite or } 0 \notin U\}$.
(c) $T = \{U \subseteq Z : \text{there exists } k > 0 \text{ such that, for all } n, n \in U \iff n + k \in U\}$.
(d) $T = \{U \subseteq Z : \text{for all } n \in U, \text{there exists } k > 0 \text{ such that } \{n + km : m \in Z\} \subseteq U\}$.

2/I/4F  Metric and Topological Spaces

Stating carefully any results on compactness which you use, show that if $X$ is a compact space, $Y$ is a Hausdorff space and $f : X \to Y$ is bijective and continuous, then $f$ is a homeomorphism.

Hence or otherwise show that the unit circle $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is homeomorphic to the quotient space $[0, 1]/\sim$, where $\sim$ is the equivalence relation defined by $x \sim y \iff \text{either } x = y \text{ or } \{x, y\} = \{0, 1\}$.

3/I/4F  Metric and Topological Spaces

Explain what it means for a topological space to be connected.

Are the following subspaces of the unit square $[0, 1] \times [0, 1]$ connected? Justify your answers.

(a) $\{(x, y) : x \neq 0, y \neq 0, \text{ and } x/y \in \mathbb{Q}\}$.
(b) $\{(x, y) : (x = 0) \text{ or } (x \neq 0 \text{ and } y \in \mathbb{Q})\}$.
4/II/14F  Metric and Topological Spaces

Explain what is meant by a base for a topology. Illustrate your definition by
describing bases for the topology induced by a metric on a set, and for the product
topology on the cartesian product of two topological spaces.

A topological space \((X, T)\) is said to be separable if there is a countable subset
\(C \subseteq X\) which is dense, i.e. such that \(C \cap U \neq \emptyset\) for every nonempty \(U \in T\). Show that a
product of two separable spaces is separable. Show also that a metric space is separable
if and only if its topology has a countable base, and deduce that every subspace of a
separable metric space is separable.

Now let \(X = \mathbb{R}\) with the topology \(T\) having as a base the set of all half-open
intervals
\([a, b) = \{x \in \mathbb{R} : a \leq x < b\}\)
with \(a < b\). Show that \(X\) is separable, but that the subspace \(Y = \{(x, -x) : x \in \mathbb{R}\}\) of
\(X \times X\) is not separable.

\[\text{You may assume standard results on countability.}\]
1/II/12A  Metric and Topological Spaces

Let $X$ and $Y$ be topological spaces. Define the product topology on $X \times Y$ and show that if $X$ and $Y$ are Hausdorff then so is $X \times Y$.

Show that the following statements are equivalent.

(i) $X$ is a Hausdorff space.

(ii) The diagonal $\Delta = \{(x, x) : x \in X\}$ is a closed subset of $X \times X$, in the product topology.

(iii) For any topological space $Y$ and any continuous maps $f, g : Y \to X$, the set $\{y \in Y : f(y) = g(y)\}$ is closed in $Y$.

2/I/4A  Metric and Topological Spaces

Are the following statements true or false? Give a proof or a counterexample as appropriate.

(i) If $f : X \to Y$ is a continuous map of topological spaces and $S \subseteq X$ is compact then $f(S)$ is compact.

(ii) If $f : X \to Y$ is a continuous map of topological spaces and $K \subseteq Y$ is compact then $f^{-1}(K) = \{x \in X : f(x) \in K\}$ is compact.

(iii) If a metric space $M$ is complete and a metric space $T$ is homeomorphic to $M$ then $T$ is complete.

3/I/4A  Metric and Topological Spaces

(a) Let $X$ be a connected topological space such that each point $x$ of $X$ has a neighbourhood homeomorphic to $\mathbb{R}^n$. Prove that $X$ is path-connected.

(b) Let $\tau$ denote the topology on $\mathbb{N} = \{1, 2, \ldots\}$, such that the open sets are $\mathbb{N}$, the empty set, and all the sets $\{1, 2, \ldots, n\}$, for $n \in \mathbb{N}$. Prove that any continuous map from the topological space $(\mathbb{N}, \tau)$ to the Euclidean $\mathbb{R}$ is constant.
Metric and Topological Spaces

(a) For a subset $A$ of a topological space $X$, define the closure $\text{cl}(A)$ of $A$. Let $f : X \rightarrow Y$ be a map to a topological space $Y$. Prove that $f$ is continuous if and only if $f(\text{cl}(A)) \subseteq \text{cl}(f(A))$, for each $A \subseteq X$.

(b) Let $M$ be a metric space. A subset $S$ of $M$ is called dense in $M$ if the closure of $S$ is equal to $M$.

Prove that if a metric space $M$ is compact then it has a countable subset which is dense in $M$. 
1/II/12F Metric and Topological Spaces

(i) Define the product topology on $X \times Y$ for topological spaces $X$ and $Y$, proving that your definition does define a topology.

(ii) Let $X$ be the logarithmic spiral defined in polar coordinates by $r = e^\theta$, where $-\infty < \theta < \infty$. Show that $X$ (with the subspace topology from $\mathbb{R}^2$) is homeomorphic to the real line.

2/I/4F Metric and Topological Spaces

Which of the following subspaces of Euclidean space are connected? Justify your answers.

(i) $\{ (x, y, z) \in \mathbb{R}^3 : z^2 - x^2 - y^2 = 1 \}$;
(ii) $\{ (x, y) \in \mathbb{R}^2 : x^2 = y^2 \}$;
(iii) $\{ (x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2 \}$.

3/I/4F Metric and Topological Spaces

Which of the following are topological spaces? Justify your answers.

(i) The set $X = \mathbb{Z}$ of the integers, with a subset $A$ of $X$ called “open” when $A$ is either finite or the whole set $X$;
(ii) The set $X = \mathbb{Z}$ of the integers, with a subset $A$ of $X$ called “open” when, for each element $x \in A$ and every even integer $n$, $x + n$ is also in $A$.

4/II/14F Metric and Topological Spaces

(a) Show that every compact subset of a Hausdorff topological space is closed.

(b) Let $X$ be a compact metric space. For $F$ a closed subset of $X$ and $p$ any point of $X$, show that there is a point $q$ in $F$ with

$$d(p, q) = \inf_{q' \in F} d(p, q').$$

Suppose that for every $x$ and $y$ in $X$ there is a point $m$ in $X$ with $d(x, m) = (1/2)d(x, y)$ and $d(y, m) = (1/2)d(x, y)$. Show that $X$ is connected.
1/II/12A  Metric and Topological Spaces

Suppose that \((X, d_X)\) and \((Y, d_Y)\) are metric spaces. Show that the definition

\[ d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2) \]

defines a metric on the product \(X \times Y\), under which the projection map \(\pi : X \times Y \to Y\) is continuous.

If \((X, d_X)\) is compact, show that every sequence in \(X\) has a subsequence converging to a point of \(X\). Deduce that the projection map \(\pi\) then has the property that, for any closed subset \(F \subset X \times Y\), the image \(\pi(F)\) is closed in \(Y\). Give an example to show that this fails if \((X, d_X)\) is not assumed compact.

2/I/4A  Metric and Topological Spaces

Let \(X\) be a topological space. Suppose that \(U_1, U_2, \ldots\) are connected subsets of \(X\) with \(U_j \cap U_1\) non-empty for all \(j > 0\). Prove that

\[ W = \bigcup_{j>0} U_j \]

is connected. If each \(U_j\) is path-connected, prove that \(W\) is path-connected.

3/I/4A  Metric and Topological Spaces

Show that a topology \(\tau_1\) is determined on the real line \(\mathbb{R}\) by specifying that a non-empty subset is open if and only if it is a union of half-open intervals \(\{a \leq x < b\}\), where \(a < b\) are real numbers. Determine whether \((\mathbb{R}, \tau_1)\) is Hausdorff.

Let \(\tau_2\) denote the cofinite topology on \(\mathbb{R}\) (that is, a non-empty subset is open if and only if its complement is finite). Prove that the identity map induces a continuous map \((\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)\).
Let \((M, d)\) be a metric space, and \(F\) a non-empty closed subset of \(M\). For \(x \in M\), set
\[
d(x, F) = \inf_{z \in F} d(x, z).
\]
Prove that \(d(x, F)\) is a continuous function of \(x\), and that it is strictly positive for \(x \notin F\).

A topological space is called \textit{normal} if for any pair of disjoint closed subsets \(F_1, F_2\), there exist disjoint open subsets \(U_1 \supset F_1, U_2 \supset F_2\). By considering the function
\[
d(x, F_1) - d(x, F_2),
\]
or otherwise, deduce that any metric space is normal.

Suppose now that \(X\) is a normal topological space, and that \(F_1, F_2\) are disjoint closed subsets in \(X\). Prove that there exist open subsets \(W_1 \supset F_1, W_2 \supset F_2\), whose \textit{closures} are disjoint. In the case when \(X = \mathbb{R}^2\) with the standard metric topology, \(F_1 = \{(x, -1/x) : x < 0\}\) and \(F_2 = \{(x, 1/x) : x > 0\}\), find explicit open subsets \(W_1, W_2\) with the above property.