Part IB

Metric and Topological Spaces

Year
2019
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Paper 3, Section I
3G Metric and Topological Spaces
Let $X$ be a metric space.

(a) What does it mean for $X$ to be compact? What does it mean for $X$ to be sequentially compact?

(b) Prove that if $X$ is compact then $X$ is sequentially compact.

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Paper 2, Section I
4G Metric and Topological Spaces
(a) Let $f : X \to Y$ be a continuous surjection of topological spaces. Prove that, if $X$ is connected, then $Y$ is also connected.

(b) Let $g : [0, 1] \to [0, 1]$ be a continuous map. Deduce from part (a) that, for every $y$ between $g(0)$ and $g(1)$, there is $x \in [0, 1]$ such that $g(x) = y$. [You may not assume the Intermediate Value Theorem, but you may use the fact that suprema exist in $\mathbb{R}$.]

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Paper 1, Section II
12G Metric and Topological Spaces
Consider the set of sequences of integers

$$X = \{(x_1, x_2, \ldots) \mid x_n \in \mathbb{Z} \text{ for all } n\}.$$  

Define

$$n_{\min}((x_n), (y_n)) = \begin{cases} \infty & x_n = y_n \text{ for all } n \\ \min\{n \mid x_n \neq y_n\} & \text{otherwise} \end{cases}$$

for two sequences $(x_n), (y_n) \in X$. Let

$$d((x_n), (y_n)) = 2^{-n_{\min}((x_n), (y_n))}$$

where, as usual, we adopt the convention that $2^{-\infty} = 0$.

(a) Prove that $d$ defines a metric on $X$.

(b) What does it mean for a metric space to be complete? Prove that $(X, d)$ is complete.

(c) Is $(X, d)$ path connected? Justify your answer.
Paper 4, Section II
13G Metric and Topological Spaces
(a) Define the subspace, quotient and product topologies.

(b) Let $X$ be a compact topological space and $Y$ a Hausdorff topological space. Prove that a continuous bijection $f : X \to Y$ is a homeomorphism.

(c) Let $S = [0, 1] \times [0, 1]$, equipped with the product topology. Let $\sim$ be the smallest equivalence relation on $S$ such that $(s, 0) \sim (s, 1)$ and $(0, t) \sim (1, t)$, for all $s, t \in [0, 1]$. Let

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1\}$$

equipped with the subspace topology from $\mathbb{R}^3$. Prove that $S/\sim$ and $T$ are homeomorphic.

[You may assume without proof that $S$ is compact.]
Paper 3, Section I
3E Metric and Topological Spaces

What does it mean to say that a topological space is connected? If $X$ is a topological space and $x \in X$, show that there is a connected subspace $K_x$ of $X$ so that if $S$ is any other connected subspace containing $x$ then $S \subseteq K_x$.

Show that the sets $K_x$ partition $X$.

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Paper 2, Section I
4E Metric and Topological Spaces

What does it mean to say that $d$ is a metric on a set $X$? What does it mean to say that a subset of $X$ is open with respect to the metric $d$? Show that the collection of subsets of $X$ that are open with respect to $d$ satisfies the axioms of a topology.

For $X = C[0, 1]$, the set of continuous functions $f : [0, 1] \to \mathbb{R}$, show that the metrics

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| \, dx$$
$$d_2(f, g) = \left[ \int_0^1 |f(x) - g(x)|^2 \, dx \right]^{1/2}$$

give different topologies.

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Paper 1, Section II
12E Metric and Topological Spaces

What does it mean to say that a topological space is compact? Prove directly from the definition that $[0, 1]$ is compact. Hence show that the unit circle $S^1 \subset \mathbb{R}^2$ is compact, proving any results that you use. [You may use without proof the continuity of standard functions.]

The set $\mathbb{R}^2$ has a topology $\mathcal{T}$ for which the closed sets are the empty set and the finite unions of vector subspaces. Let $X$ denote the set $\mathbb{R}^2 \setminus \{0\}$ with the subspace topology induced by $\mathcal{T}$. By considering the subspace topology on $S^1 \subset \mathbb{R}^2$, or otherwise, show that $X$ is compact.
Let $X = \{2, 3, 4, 5, 6, 7, 8, \ldots\}$ and for each $n \in X$ let

$$U_n = \{ d \in X \mid d \text{ divides } n \}.$$

Prove that the set of unions of the sets $U_n$ forms a topology on $X$.

Prove or disprove each of the following:

(i) $X$ is Hausdorff;

(ii) $X$ is compact.

If $Y$ and $Z$ are topological spaces, $Y$ is the union of closed subspaces $A$ and $B$, and $f : Y \to Z$ is a function such that both $f|_A : A \to Z$ and $f|_B : B \to Z$ are continuous, show that $f$ is continuous. Hence show that $X$ is path-connected.
Paper 3, Section I
3E Metric and Topological Spaces
Let X and Y be topological spaces.

(a) Define what is meant by the product topology on $X \times Y$. Define the projection maps $p: X \times Y \to X$ and $q: X \times Y \to Y$ and show they are continuous.

(b) Consider $\Delta = \{(x, x) \mid x \in X\}$ in $X \times X$. Show that X is Hausdorff if and only if $\Delta$ is a closed subset of $X \times X$ in the product topology.

Paper 2, Section I
4E Metric and Topological Spaces
Let $f: (X, d) \to (Y, e)$ be a function between metric spaces.

(a) Give the $\epsilon$-$\delta$ definition for $f$ to be continuous. Show that $f$ is continuous if and only if $f^{-1}(U)$ is an open subset of $X$ for each open subset $U$ of $Y$.

(b) Give an example of $f$ such that $f$ is not continuous but $f(V)$ is an open subset of $Y$ for every open subset $V$ of $X$.

Paper 1, Section II
12E Metric and Topological Spaces
Consider $\mathbb{R}$ and $\mathbb{R}^2$ with their usual Euclidean topologies.

(a) Show that a non-empty subset of $\mathbb{R}$ is connected if and only if it is an interval. Find a compact subset $K \subset \mathbb{R}$ for which $\mathbb{R} \setminus K$ has infinitely many connected components.

(b) Let $T$ be a countable subset of $\mathbb{R}^2$. Show that $\mathbb{R}^2 \setminus T$ is path-connected. Deduce that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.
Let $f : X \to Y$ be a continuous map between topological spaces.

(a) Assume $X$ is compact and that $Z \subseteq X$ is a closed subset. Prove that $Z$ and $f(Z)$ are both compact.

(b) Suppose that

(i) $f^{-1}(\{y\})$ is compact for each $y \in Y$, and

(ii) if $A$ is any closed subset of $X$, then $f(A)$ is a closed subset of $Y$.

Show that if $K \subseteq Y$ is compact, then $f^{-1}(K)$ is compact.

[Hint: Given an open cover $f^{-1}(K) \subseteq \bigcup_{i \in I} U_i$, find a finite subcover, say $f^{-1}(\{y\}) \subseteq \bigcup_{i \in I_y} U_i$, for each $y \in K$; use closedness of $X \setminus \bigcup_{i \in I_y} U_i$ and property (ii) to produce an open cover of $K$.]

\[ \text{Part IB, 2017 List of Questions} \]
Paper 3, Section I
3E Metric and Topological Spaces

Let $X$ be a topological space and $A \subseteq X$ be a subset. A limit point of $A$ is a point $x \in X$ such that any open neighbourhood $U$ of $x$ intersects $A$. Show that $A$ is closed if and only if it contains all its limit points. Explain what is meant by the interior $\text{Int}(A)$ and the closure $\overline{A}$ of $A$. Show that if $A$ is connected, then $\overline{A}$ is connected.

Paper 2, Section I
4E Metric and Topological Spaces

Consider $\mathbb{R}$ and $\mathbb{Q}$ with their usual topologies.

(a) Show that compact subsets of a Hausdorff topological space are closed. Show that compact subsets of $\mathbb{R}$ are closed and bounded.

(b) Show that there exists a complete metric space $(X,d)$ admitting a surjective continuous map $f : X \to \mathbb{Q}$.

Paper 1, Section II
12E Metric and Topological Spaces

Let $p$ be a prime number. Define what is meant by the $p$-adic metric $d_p$ on $\mathbb{Q}$. Show that for $a, b, c \in \mathbb{Q}$ we have

$$d_p(a,b) \leq \max\{d_p(a,c), d_p(c,b)\}.$$

Show that the sequence $(a_n)$, where $a_n = 1 + p + \cdots + p^{n-1}$, converges to some element in $\mathbb{Q}$.

For $a \in \mathbb{Q}$ define $|a|_p = d_p(a,0)$. Show that if $a, b \in \mathbb{Q}$ and if $|a|_p \neq |b|_p$, then

$$|a + b|_p = \max\{|a|_p, |b|_p\}.$$

Let $a \in \mathbb{Q}$ and let $B(a, \delta)$ be the open ball with centre $a$ and radius $\delta > 0$, with respect to the metric $d_p$. Show that $B(a, \delta)$ is a closed subset of $\mathbb{Q}$ with respect to the topology induced by $d_p$. 
Paper 4, Section II
13E Metric and Topological Spaces

(a) Let $X$ be a topological space. Define what is meant by a quotient of $X$ and describe the quotient topology on the quotient space. Give an example in which $X$ is Hausdorff but the quotient space is not Hausdorff.

(b) Let $T^2$ be the 2-dimensional torus considered as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, and let $\pi : \mathbb{R}^2 \to T^2$ be the quotient map.

(i) Let $B(u, r)$ be the open ball in $\mathbb{R}^2$ with centre $u$ and radius $r < 1/2$. Show that $U = \pi(B(u, r))$ is an open subset of $T^2$ and show that $\pi^{-1}(U)$ has infinitely many connected components. Show each connected component is homeomorphic to $B(u, r)$.

(ii) Let $\alpha \in \mathbb{R}$ be an irrational number and let $L \subset \mathbb{R}^2$ be the line given by the equation $y = \alpha x$. Show that $\pi(L)$ is dense in $T^2$ but $\pi(L) \neq T^2$. 

Part IB, 2016 List of Questions [TURN OVER

2016
Paper 3, Section I
3E Metric and Topological Spaces
Define what it means for a topological space $X$ to be (i) connected (ii) path-connected.

Prove that any path-connected space $X$ is connected. [You may assume the interval $[0, 1]$ is connected.]

Give a counterexample (without justification) to the converse statement.

Paper 2, Section I
4E Metric and Topological Spaces
Let $X$ and $Y$ be topological spaces and $f : X \rightarrow Y$ a continuous map. Suppose $H$ is a subset of $X$ such that $f(\overline{H})$ is closed (where $\overline{H}$ denotes the closure of $H$). Prove that $f(\overline{H}) = \overline{f(H)}$.

Give an example where $f, X, Y$, and $H$ are as above but $f(\overline{H})$ is not closed.

Paper 1, Section II
12E Metric and Topological Spaces
Give the definition of a metric on a set $X$ and explain how this defines a topology on $X$.

Suppose $(X, d)$ is a metric space and $U$ is an open set in $X$. Let $x, y \in X$ and $\epsilon > 0$ such that the open ball $B_\epsilon(y) \subseteq U$ and $x \in B_{\epsilon/2}(y)$. Prove that $y \in B_{\epsilon/2}(x) \subseteq U$.

Explain what it means (i) for a set $S$ to be dense in $X$, (ii) to say $B$ is a base for a topology $\mathcal{T}$.

Prove that any metric space which contains a countable dense set has a countable basis.
Explain what it means for a metric space \((M, d)\) to be (i) \textit{compact}, (ii) \textit{sequentially compact}. Prove that a compact metric space is sequentially compact, stating clearly any results that you use.

Let \((M, d)\) be a compact metric space and suppose \(f : M \to M\) satisfies \(d(f(x), f(y)) = d(x, y)\) for all \(x, y \in M\). Prove that \(f\) is surjective, stating clearly any results that you use. [\textit{Hint: Consider the sequence \(f^n(x)\) for } x \in M.]

Give an example to show that the result does not hold if \(M\) is not compact.
3E Metric and Topological Spaces

Suppose \((X, d)\) is a metric space. Do the following necessarily define a metric on \(X\)? Give proofs or counterexamples.

(i) \(d_1(x, y) = kd(x, y)\) for some constant \(k > 0\), for all \(x, y \in X\).

(ii) \(d_2(x, y) = \min\{1, d(x, y)\}\) for all \(x, y \in X\).

(iii) \(d_3(x, y) = (d(x, y))^2\) for all \(x, y \in X\).

4E Metric and Topological Spaces

Let \(X\) and \(Y\) be topological spaces. What does it mean to say that a function \(f : X \to Y\) is continuous? Are the following statements true or false? Give proofs or counterexamples.

(i) Every continuous function \(f : X \to Y\) is an open map, i.e. if \(U\) is open in \(X\) then \(f(U)\) is open in \(Y\).

(ii) If \(f : X \to Y\) is continuous and bijective then \(f\) is a homeomorphism.

(iii) If \(f : X \to Y\) is continuous, open and bijective then \(f\) is a homeomorphism.

12E Metric and Topological Spaces

Define what it means for a topological space to be compact. Define what it means for a topological space to be Hausdorff.

Prove that a compact subspace of a Hausdorff space is closed. Hence prove that if \(C_1\) and \(C_2\) are compact subspaces of a Hausdorff space \(X\) then \(C_1 \cap C_2\) is compact.

A subset \(U\) of \(\mathbb{R}\) is open in the cocountable topology if \(U\) is empty or its complement in \(\mathbb{R}\) is countable. Is \(\mathbb{R}\) Hausdorff in the cocountable topology? Which subsets of \(\mathbb{R}\) are compact in the cocountable topology?
Explain what it means for a metric space to be complete.

Let $X$ be a metric space. We say the subsets $A_i$ of $X$, with $i \in \mathbb{N}$, form a descending sequence in $X$ if $A_1 \supset A_2 \supset A_3 \supset \cdots$.

Prove that the metric space $X$ is complete if and only if any descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of $X$, such that the diameters of the subsets $A_i$ converge to zero, has an intersection $\bigcap_{i=1}^{\infty} A_i$ that is non-empty.

[Recall that the diameter $\text{diam}(S)$ of a set $S$ is the supremum of the set $\{d(x, y) : x, y \in S\}$.]

Give examples of
(i) a metric space $X$, and a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed subsets of $X$, with $\text{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.
(ii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty sets in $\mathbb{R}$ with $\text{diam}(A_i)$ converging to 0 but $\bigcap_{i=1}^{\infty} A_i = \emptyset$.
(iii) a descending sequence $A_1 \supset A_2 \supset \cdots$ of non-empty closed sets in $\mathbb{R}$ with $\bigcap_{i=1}^{\infty} A_i = \emptyset$. 
Paper 3, Section I
3G Metric and Topological Spaces
Let $X$ be a metric space with the metric $d : X \times X \to \mathbb{R}$.

(i) Show that if $X$ is compact as a topological space, then $X$ is complete.

(ii) Show that the completeness of $X$ is not a topological property, i.e. give an example of two metrics $d, d'$ on a set $X$, such that the associated topologies are the same, but $(X, d)$ is complete and $(X, d')$ is not.

Paper 2, Section I
4G Metric and Topological Spaces
Let $X$ be a topological space. Prove or disprove the following statements.

(i) If $X$ is discrete, then $X$ is compact if and only if it is a finite set.

(ii) If $Y$ is a subspace of $X$ and $X, Y$ are both compact, then $Y$ is closed in $X$.

Paper 1, Section II
12G Metric and Topological Spaces
Consider the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, a subset of $\mathbb{R}^3$, as a subspace of $\mathbb{R}^3$ with the Euclidean metric.

(i) Show that $S^2$ is compact and Hausdorff as a topological space.

(ii) Let $X = S^2/\sim$ be the quotient set with respect to the equivalence relation identifying the antipodes, i.e.

$$(x, y, z) \sim (x', y', z') \iff (x', y', z') = (x, y, z) \text{ or } (-x, -y, -z).$$

Show that $X$ is compact and Hausdorff with respect to the quotient topology.
Let $X$ be a topological space. A connected component of $X$ means an equivalence class with respect to the equivalence relation on $X$ defined as:

$$x \sim y \iff x, y \text{ belong to some connected subspace of } X.$$  

(i) Show that every connected component is a connected and closed subset of $X$.

(ii) If $X, Y$ are topological spaces and $X \times Y$ is the product space, show that every connected component of $X \times Y$ is a direct product of connected components of $X$ and $Y$. 
3F Metric and Topological Spaces

Define the notion of a connected component of a space $X$.

If $A_\alpha \subset X$ are connected subsets of $X$ such that $\bigcap_\alpha A_\alpha \neq \emptyset$, show that $\bigcup_\alpha A_\alpha$ is connected.

Prove that any point $x \in X$ is contained in a unique connected component.

Let $X \subset \mathbb{R}$ consist of the points $0, 1, 1\frac{1}{2}, 1\frac{1}{3}, \ldots, 1\frac{1}{n}, \ldots$. What are the connected components of $X$?

4F Metric and Topological Spaces

For each case below, determine whether the given metrics $d_1$ and $d_2$ induce the same topology on $X$. Justify your answers.

(i) $X = \mathbb{R}^2$, $d_1((x_1, y_1), (x_2, y_2)) = \sup\{|x_1 - x_2|, |y_1 - y_2|\}$

\[ d_2((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|. \]

(ii) $X = C[0, 1]$, $d_1(f, g) = \sup_{t \in [0, 1]} |f(t) - g(t)|$

\[ d_2(f, g) = \int_0^1 |f(t) - g(t)| \, dt. \]

12F Metric and Topological Spaces

A topological space $X$ is said to be normal if each point of $X$ is a closed subset of $X$ and for each pair of closed sets $C_1, C_2 \subset X$ with $C_1 \cap C_2 = \emptyset$ there are open sets $U_1, U_2 \subset X$ so that $C_1 \subset U_1$ and $U_1 \cap U_2 = \emptyset$. In this case we say that the $U_i$ separate the $C_i$.

Show that a compact Hausdorff space is normal. [Hint: first consider the case where $C_2$ is a point.]

For $C \subset X$ we define an equivalence relation $\sim_C$ on $X$ by $x \sim_C y$ for all $x, y \in C$, $x \sim_C x$ for $x \notin C$. If $C, C_1$ and $C_2$ are pairwise disjoint closed subsets of a normal space $X$, show that $C_1$ and $C_2$ may be separated by open subsets $U_1$ and $U_2$ such that $U_i \cap C = \emptyset$. Deduce that the quotient space $X/\sim_C$ is also normal.
Suppose $A_1$ and $A_2$ are topological spaces. Define the product topology on $A_1 \times A_2$. Let $\pi_i : A_1 \times A_2 \rightarrow A_i$ be the projection. Show that a map $F : X \rightarrow A_1 \times A_2$ is continuous if and only if $\pi_1 \circ F$ and $\pi_2 \circ F$ are continuous.

Prove that if $A_1$ and $A_2$ are connected, then $A_1 \times A_2$ is connected.

Let $X$ be the topological space whose underlying set is $\mathbb{R}$, and whose open sets are of the form $(a, \infty)$ for $a \in \mathbb{R}$, along with the empty set and the whole space. Describe the open sets in $X \times X$. Are the maps $f, g : X \times X \rightarrow X$ defined by $f(x, y) = x + y$ and $g(x, y) = xy$ continuous? Justify your answers.
Paper 2, Section I

4G Metric and Topological Spaces
(i) Let $t > 0$. For $x = (x, y), x' = (x', y') \in \mathbb{R}^2$, let
\[
d(x, x') = |x' - x| + t|y' - y|,
\]
\[
\delta(x, x') = \sqrt{(x' - x)^2 + (y' - y)^2}.
\]
($\delta$ is the usual Euclidean metric on $\mathbb{R}^2$.) Show that $d$ is a metric on $\mathbb{R}^2$ and that the two metrics $d, \delta$ give rise to the same topology on $\mathbb{R}^2$.

(ii) Give an example of a topology on $\mathbb{R}^2$, different from the one in (i), whose induced topology (subspace topology) on the $x$-axis is the usual topology (the one defined by the metric $d(x, x') = |x' - x|$). Justify your answer.

Paper 3, Section I

3G Metric and Topological Spaces
Let $X, Y$ be topological spaces, and suppose $Y$ is Hausdorff.

(i) Let $f, g : X \to Y$ be two continuous maps. Show that the set
\[
E(f, g) := \{x \in X \mid f(x) = g(x)\} \subset X
\]
is a closed subset of $X$.

(ii) Let $W$ be a dense subset of $X$. Show that a continuous map $f : X \to Y$ is determined by its restriction $f|_W$ to $W$.

Paper 1, Section II

12G Metric and Topological Spaces
Let $X$ be a metric space with the distance function $d : X \times X \to \mathbb{R}$. For a subset $Y$ of $X$, its diameter is defined as $\delta(Y) := \sup\{d(y, y') \mid y, y' \in Y\}$.

Show that, if $X$ is compact and $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open covering of $X$, then there exists an $\epsilon > 0$ such that every subset $Y \subset X$ with $\delta(Y) < \epsilon$ is contained in some $U_\lambda$. 
Let $X, Y$ be topological spaces and $X \times Y$ their product set. Let $p_Y : X \times Y \to Y$ be the projection map.

(i) Define the product topology on $X \times Y$. Prove that if a subset $Z \subset X \times Y$ is open then $p_Y(Z)$ is open in $Y$.

(ii) Give an example of $X, Y$ and a closed set $Z \subset X \times Y$ such that $p_Y(Z)$ is not closed.

(iii) When $X$ is compact, show that if a subset $Z \subset X \times Y$ is closed then $p_Y(Z)$ is closed.
On the set $\mathbb{Q}$ of rational numbers, the $3$-adic metric $d_3$ is defined as follows: for $x, y \in \mathbb{Q}$, define $d_3(x, x) = 0$ and $d_3(x, y) = 3^{-n}$, where $n$ is the integer satisfying $x - y = 3^n u$ where $u$ is a rational number whose denominator and numerator are both prime to $3$.

(1) Show that this is indeed a metric on $\mathbb{Q}$.

(2) Show that in $(\mathbb{Q}, d_3)$, we have $3^n \to 0$ as $n \to \infty$ while $3^{-n} \not\to 0$ as $n \to \infty$.

Let $d$ be the usual metric $d(x, y) = |x - y|$ on $\mathbb{Q}$. Show that neither the identity map $(\mathbb{Q}, d) \to (\mathbb{Q}, d_3)$ nor its inverse is continuous.

Let $X$ be a topological space and $Y$ be a set. Let $p : X \to Y$ be a surjection. The quotient topology on $Y$ is defined as follows: a subset $V \subset Y$ is open if and only if $p^{-1}(V)$ is open in $X$.

(1) Show that this does indeed define a topology on $Y$, and show that $p$ is continuous when we endow $Y$ with this topology.

(2) Let $Z$ be another topological space and $f : Y \to Z$ be a map. Show that $f$ is continuous if and only if $f \circ p : X \to Z$ is continuous.

Let $f : X \to Y$ and $g : Y \to X$ be continuous maps of topological spaces with $f \circ g = \text{id}_Y$.

(1) Suppose that (i) $Y$ is path-connected, and (ii) for every $y \in Y$, its inverse image $f^{-1}(y)$ is path-connected. Prove that $X$ is path-connected.

(2) Prove the same statement when “path-connected” is everywhere replaced by “connected”.
(1) Prove that if $X$ is a compact topological space, then a closed subset $Y$ of $X$ endowed with the subspace topology is compact.

(2) Consider the following equivalence relation on $\mathbb{R}^2$:

\[(x_1, y_1) \sim (x_2, y_2) \iff (x_1 - x_2, y_1 - y_2) \in \mathbb{Z}^2.\]

Let $X = \mathbb{R}^2/\sim$ be the quotient space endowed with the quotient topology, and let $p : \mathbb{R}^2 \to X$ be the canonical surjection mapping each element to its equivalence class. Let $Z = \{(x, y) \in \mathbb{R}^2 \mid y = \sqrt{2}x\}$.

(i) Show that $X$ is compact.

(ii) Assuming that $p(Z)$ is dense in $X$, show that $p|_Z : Z \to p(Z)$ is bijective but not homeomorphic.
Paper 2, Section I
4F Metric and Topological Spaces

Explain what is meant by a Hausdorff (topological) space, and prove that every compact subset of a Hausdorff space is closed.

Let $X$ be an uncountable set, and consider the topology $\mathcal{T}$ on $X$ defined by

$$U \in \mathcal{T} \iff \text{either } U = \emptyset \text{ or } X \setminus U \text{ is countable}.$$ 

Is $(X, \mathcal{T})$ Hausdorff? Is every compact subset of $X$ closed? Justify your answers.

Paper 3, Section I
4F Metric and Topological Spaces

Are the following statements true or false? Give brief justifications for your answers.

(i) If $X$ is a connected open subset of $\mathbb{R}^n$ for some $n$, then $X$ is path-connected.

(ii) A cartesian product of two connected spaces is connected.

(iii) If $X$ is a Hausdorff space and the only connected subsets of $X$ are singletons $\{x\}$, then $X$ is discrete.

Paper 1, Section II
12F Metric and Topological Spaces

Given a function $f : X \to Y$ between metric spaces, we write $\Gamma_f$ for the subset $\{(x, f(x)) \mid x \in X\}$ of $X \times Y$.

(i) If $f$ is continuous, show that $\Gamma_f$ is closed in $X \times Y$.

(ii) If $Y$ is compact and $\Gamma_f$ is closed in $X \times Y$, show that $f$ is continuous.

(iii) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\Gamma_f$ is closed but $f$ is not continuous.
A nonempty subset \( A \) of a topological space \( X \) is called \emph{irreducible} if, whenever we have open sets \( U \) and \( V \) such that \( U \cap A \) and \( V \cap A \) are nonempty, then we also have \( U \cap V \cap A \neq \emptyset \). Show that the closure of an irreducible set is irreducible, and deduce that the closure of any singleton set \( \{ x \} \) is irreducible.

\( X \) is said to be a \emph{sober} topological space if, for any irreducible closed \( A \subseteq X \), there is a unique \( x \in X \) such that \( A = \overline{\{ x \}} \). Show that any Hausdorff space is sober, but that an infinite set with the cofinite topology is not sober.

Given an arbitrary topological space \((X, \mathcal{T})\), let \( \hat{X} \) denote the set of all irreducible closed subsets of \( X \), and for each \( U \in \mathcal{T} \) let
\[
\hat{U} = \{ A \in \hat{X} \mid U \cap A \neq \emptyset \}.
\]
Show that the sets \( \{ \hat{U} \mid U \in \mathcal{T} \} \) form a topology \( \hat{\mathcal{T}} \) on \( \hat{X} \), and that the mapping \( U \mapsto \hat{U} \) is a bijection from \( \mathcal{T} \) to \( \hat{\mathcal{T}} \). Deduce that \((\hat{X}, \hat{\mathcal{T}})\) is sober. [\textit{Hint: consider the complement of an irreducible closed subset of} \( \hat{X} \).]
1/II/12F  Metric and Topological Spaces

Write down the definition of a topology on a set $X$.

For each of the following families $\mathcal{T}$ of subsets of $\mathbb{Z}$, determine whether $\mathcal{T}$ is a topology on $\mathbb{Z}$. In the cases where the answer is ‘yes’, determine also whether $(\mathbb{Z}, \mathcal{T})$ is a Hausdorff space and whether it is compact.

(a) $\mathcal{T} = \{U \subseteq \mathbb{Z} : \text{either } U \text{ is finite or } 0 \in U \}$.
(b) $\mathcal{T} = \{U \subseteq \mathbb{Z} : \text{either } \mathbb{Z} \setminus U \text{ is finite or } 0 \notin U \}$.
(c) $\mathcal{T} = \{U \subseteq \mathbb{Z} : \text{there exists } k > 0 \text{ such that, for all } n, n \in U \iff n + k \in U \}$.
(d) $\mathcal{T} = \{U \subseteq \mathbb{Z} : \text{for all } n \in U, \text{there exists } k > 0 \text{ such that } \{n + km : m \in \mathbb{Z}\} \subseteq U \}$.

2/I/4F  Metric and Topological Spaces

Stating carefully any results on compactness which you use, show that if $X$ is a compact space, $Y$ is a Hausdorff space and $f : X \to Y$ is bijective and continuous, then $f$ is a homeomorphism.

Hence or otherwise show that the unit circle $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is homeomorphic to the quotient space $[0, 1]/\sim$, where $\sim$ is the equivalence relation defined by

$$x \sim y \iff \text{either } x = y \text{ or } \{x, y\} = \{0, 1\}.$$ 

3/I/4F  Metric and Topological Spaces

Explain what it means for a topological space to be connected.

Are the following subspaces of the unit square $[0, 1] \times [0, 1]$ connected? Justify your answers.

(a) $\{(x, y) : x \neq 0, y \neq 0, \text{ and } x/y \in \mathbb{Q}\}$.
(b) $\{(x, y) : (x = 0) \text{ or } (x \neq 0 \text{ and } y \in \mathbb{Q})\}$.
4/II/14F Metric and Topological Spaces

Explain what is meant by a base for a topology. Illustrate your definition by describing bases for the topology induced by a metric on a set, and for the product topology on the cartesian product of two topological spaces.

A topological space \((X, T)\) is said to be separable if there is a countable subset \(C \subseteq X\) which is dense, i.e. such that \(C \cap U \neq \emptyset\) for every nonempty \(U \in T\). Show that a product of two separable spaces is separable. Show also that a metric space is separable if and only if its topology has a countable base, and deduce that every subspace of a separable metric space is separable.

Now let \(X = \mathbb{R}\) with the topology \(T\) having as a base the set of all half-open intervals

\[
[a, b) = \{x \in \mathbb{R} : a \leq x < b\}
\]

with \(a < b\). Show that \(X\) is separable, but that the subspace \(Y = \{(x, -x) : x \in \mathbb{R}\}\) of \(X \times X\) is not separable.

[You may assume standard results on countability.]
1/II/12A  Metric and Topological Spaces

Let \(X\) and \(Y\) be topological spaces. Define the product topology on \(X \times Y\) and show that if \(X\) and \(Y\) are Hausdorff then so is \(X \times Y\).

Show that the following statements are equivalent.

(i) \(X\) is a Hausdorff space.

(ii) The diagonal \(\Delta = \{(x, x) : x \in X\}\) is a closed subset of \(X \times X\), in the product topology.

(iii) For any topological space \(Y\) and any continuous maps \(f, g : Y \to X\), the set \(\{y \in Y : f(y) = g(y)\}\) is closed in \(Y\).

2/I/4A  Metric and Topological Spaces

Are the following statements true or false? Give a proof or a counterexample as appropriate.

(i) If \(f : X \to Y\) is a continuous map of topological spaces and \(S \subseteq X\) is compact then \(f(S)\) is compact.

(ii) If \(f : X \to Y\) is a continuous map of topological spaces and \(K \subseteq Y\) is compact then \(f^{-1}(K) = \{x \in X : f(x) \in K\}\) is compact.

(iii) If a metric space \(M\) is complete and a metric space \(T\) is homeomorphic to \(M\) then \(T\) is complete.

3/I/4A  Metric and Topological Spaces

(a) Let \(X\) be a connected topological space such that each point \(x\) of \(X\) has a neighbourhood homeomorphic to \(\mathbb{R}^n\). Prove that \(X\) is path-connected.

(b) Let \(\tau\) denote the topology on \(\mathbb{N} = \{1, 2, \ldots\}\), such that the open sets are \(\mathbb{N}\), the empty set, and all the sets \(\{1, 2, \ldots, n\}\), for \(n \in \mathbb{N}\). Prove that any continuous map from the topological space \((\mathbb{N}, \tau)\) to the Euclidean \(\mathbb{R}\) is constant.
Metric and Topological Spaces

(a) For a subset $A$ of a topological space $X$, define the closure $\text{cl}(A)$ of $A$. Let $f : X \to Y$ be a map to a topological space $Y$. Prove that $f$ is continuous if and only if $f(\text{cl}(A)) \subseteq \text{cl}(f(A))$, for each $A \subseteq X$.

(b) Let $M$ be a metric space. A subset $S$ of $M$ is called dense in $M$ if the closure of $S$ is equal to $M$.

Prove that if a metric space $M$ is compact then it has a countable subset which is dense in $M$. 
1/II/12F Metric and Topological Spaces

(i) Define the product topology on $X \times Y$ for topological spaces $X$ and $Y$, proving that your definition does define a topology.

(ii) Let $X$ be the logarithmic spiral defined in polar coordinates by $r = e^\theta$, where $-\infty < \theta < \infty$. Show that $X$ (with the subspace topology from $\mathbb{R}^2$) is homeomorphic to the real line.

2/I/4F Metric and Topological Spaces

Which of the following subspaces of Euclidean space are connected? Justify your answers.

(i) $\{(x, y, z) \in \mathbb{R}^3 : z^2 - x^2 - y^2 = 1\}$;

(ii) $\{(x, y) \in \mathbb{R}^2 : x^2 = y^2\}$;

(iii) $\{(x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2\}$.

3/I/4F Metric and Topological Spaces

Which of the following are topological spaces? Justify your answers.

(i) The set $X = \mathbb{Z}$ of the integers, with a subset $A$ of $X$ called “open” when $A$ is either finite or the whole set $X$;

(ii) The set $X = \mathbb{Z}$ of the integers, with a subset $A$ of $X$ called “open” when, for each element $x \in A$ and every even integer $n$, $x + n$ is also in $A$.

4/II/14F Metric and Topological Spaces

(a) Show that every compact subset of a Hausdorff topological space is closed.

(b) Let $X$ be a compact metric space. For $F$ a closed subset of $X$ and $p$ any point of $X$, show that there is a point $q$ in $F$ with

$$d(p, q) = \inf_{q' \in F} d(p, q').$$

Suppose that for every $x$ and $y$ in $X$ there is a point $m$ in $X$ with $d(x, m) = (1/2)d(x, y)$ and $d(y, m) = (1/2)d(x, y)$. Show that $X$ is connected.
1/II/12A  Metric and Topological Spaces

Suppose that \((X, d_X)\) and \((Y, d_Y)\) are metric spaces. Show that the definition
\[
d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2)
\]
defines a metric on the product \(X \times Y\), under which the projection map \(\pi : X \times Y \to Y\) is continuous.

If \((X, d_X)\) is compact, show that every sequence in \(X\) has a subsequence converging to a point of \(X\). Deduce that the projection map \(\pi\) then has the property that, for any closed subset \(F \subset X \times Y\), the image \(\pi(F)\) is closed in \(Y\). Give an example to show that this fails if \((X, d_X)\) is not assumed compact.

2/I/4A  Metric and Topological Spaces

Let \(X\) be a topological space. Suppose that \(U_1, U_2, \ldots\) are connected subsets of \(X\) with \(U_j \cap U_1\) non-empty for all \(j > 0\). Prove that
\[
W = \bigcup_{j>0} U_j
\]
is connected. If each \(U_j\) is path-connected, prove that \(W\) is path-connected.

3/I/4A  Metric and Topological Spaces

Show that a topology \(\tau_1\) is determined on the real line \(\mathbb{R}\) by specifying that a non-empty subset is open if and only if it is a union of half-open intervals \(\{a \leq x < b\}\), where \(a < b\) are real numbers. Determine whether \((\mathbb{R}, \tau_1)\) is Hausdorff.

Let \(\tau_2\) denote the cofinite topology on \(\mathbb{R}\) (that is, a non-empty subset is open if and only if its complement is finite). Prove that the identity map induces a continuous map \((\mathbb{R}, \tau_1) \to (\mathbb{R}, \tau_2)\).
Let \((M, d)\) be a metric space, and \(F\) a non-empty closed subset of \(M\). For \(x \in M\),
set
\[
d(x, F) = \inf_{z \in F} d(x, z).
\]
Prove that \(d(x, F)\) is a continuous function of \(x\), and that it is strictly positive for \(x \notin F\).

A topological space is called normal if for any pair of disjoint closed subsets \(F_1, F_2\),
there exist disjoint open subsets \(U_1 \supset F_1, U_2 \supset F_2\). By considering the function
\[
d(x, F_1) - d(x, F_2),
\]
or otherwise, deduce that any metric space is normal.

Suppose now that \(X\) is a normal topological space, and that \(F_1, F_2\) are disjoint
closed subsets in \(X\). Prove that there exist open subsets \(W_1 \supset F_1, W_2 \supset F_2\), whose
\(\text{closures}\) are disjoint. In the case when \(X = \mathbb{R}^2\) with the standard metric topology,
\(F_1 = \{(x, -1/x) : x < 0\}\) and \(F_2 = \{(x, 1/x) : x > 0\}\), find explicit open subsets \(W_1, W_2\)
with the above property.