Part IB

Linear Algebra

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Paper 4, Section I
1F Linear Algebra

What is an eigenvalue of a matrix $A$? What is the eigenspace corresponding to an eigenvalue $\lambda$ of $A$?

Consider the matrix

$$A = \begin{pmatrix} a & a & b & d \\ b & b & c & d \\ c & a & c & d \\ d & b & c & d \end{pmatrix}$$

for $(a, b, c, d) \in \mathbb{R}^4$ a non-zero vector. Show that $A$ has rank 1. Find the eigenvalues of $A$ and describe the corresponding eigenspaces. Is $A$ diagonalisable?

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Paper 2, Section I
1F Linear Algebra

If $U$ and $W$ are finite-dimensional subspaces of a vector space $V$, prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Let

$$U = \{x \in \mathbb{R}^4 | x_1 = 7x_3 + 8x_4, \ x_2 + 5x_3 + 6x_4 = 0\},$$

$$W = \{x \in \mathbb{R}^4 | x_1 + 2x_2 + 3x_3 = 0, \ x_4 = 0\}.$$

Show that $U + W$ is 3-dimensional and find a linear map $\ell : \mathbb{R}^4 \to \mathbb{R}$ such that

$$U + W = \{x \in \mathbb{R}^4 | \ell(x) = 0\}.$$

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Paper 1, Section I
1F Linear Algebra

Define a basis of a vector space $V$.

If $V$ has a finite basis $B$, show using only the definition that any other basis $B'$ has the same cardinality as $B$. 

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Part IB, 2019 List of Questions
What is the adjugate $\text{adj}(A)$ of an $n \times n$ matrix $A$? How is it related to $\det(A)$?

(a) Define matrices $B_0, B_1, \ldots, B_{n-1}$ by

$$\text{adj}(tI - A) = \sum_{i=0}^{n-1} B_i t^{n-1-i}$$

and scalars $c_0, c_1, \ldots, c_n$ by

$$\det(tI - A) = \sum_{j=0}^{n} c_j t^{n-j}.$$ 

Find a recursion for the matrices $B_i$ in terms of $A$ and the $c_j$’s.

(b) By considering the partial derivatives of the multivariable polynomial

$$p(t_1, t_2, \ldots, t_n) = \det\left(\begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix} - A\right),$$

show that

$$\frac{d}{dt}(\det(tI - A)) = \text{Tr}(\text{adj}(tI - A)).$$

(c) Hence show that the $c_j$’s may be expressed in terms of $\text{Tr}(A), \text{Tr}(A^2), \ldots, \text{Tr}(A^n)$.
Paper 4, Section II
10F Linear Algebra

If \( U \) is a finite-dimensional real vector space with inner product \( \langle \cdot, \cdot \rangle \), prove that the linear map \( \phi : U \to U^* \) given by \( \phi(u)(u') = \langle u, u' \rangle \) is an isomorphism. [You do not need to show that it is linear.]

If \( V \) and \( W \) are inner product spaces and \( \alpha : V \to W \) is a linear map, what is meant by the adjoint \( \alpha^* \) of \( \alpha \)? If \( \{e_1, e_2, \ldots, e_n\} \) is an orthonormal basis for \( V \), \( \{f_1, f_2, \ldots, f_m\} \) is an orthonormal basis for \( W \), and \( A \) is the matrix representing \( \alpha \) in these bases, derive a formula for the matrix representing \( \alpha^* \) in these bases.

Prove that \( \text{Im}(\alpha) = \text{Ker}(\alpha^*)^\perp \).

If \( w_0 \notin \text{Im}(\alpha) \) then the linear equation \( \alpha(v) = w_0 \) has no solution, but we may instead search for a \( v_0 \in V \) minimising \( \|\alpha(v) - w_0\|^2 \), known as a least-squares solution. Show that \( v_0 \) is such a least-squares solution if and only if it satisfies \( \alpha^*\alpha(v_0) = \alpha^*(w_0) \). Hence find a least-squares solution to the linear equation

\[
\begin{pmatrix}
1 & 0 \\
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
=
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}.
\]

Paper 3, Section II
10F Linear Algebra

If \( q \) is a quadratic form on a finite-dimensional real vector space \( V \), what is the associated symmetric bilinear form \( \varphi(\cdot, \cdot) \)? Prove that there is a basis for \( V \) with respect to which the matrix for \( \varphi \) is diagonal. What is the signature of \( q \)?

If \( R \subseteq V \) is a subspace such that \( \varphi(r, v) = 0 \) for all \( r \in R \) and all \( v \in V \), show that \( q'(v + R) = q(v) \) defines a quadratic form on the quotient vector space \( V/R \). Show that the signature of \( q' \) is the same as that of \( q \).

If \( e, f \in V \) are vectors such that \( \varphi(e, e) = 0 \) and \( \varphi(e, f) = 1 \), show that there is a direct sum decomposition \( V = \text{span}(e, f) \oplus U \) such that the signature of \( q|_U \) is the same as that of \( q \).
Let $A$ and $B$ be $n \times n$ matrices over $\mathbb{C}$.

(a) Assuming that $A$ is invertible, show that $AB$ and $BA$ have the same characteristic polynomial.

(b) By considering the matrices $A - sI$, show that $AB$ and $BA$ have the same characteristic polynomial even when $A$ is singular.

(c) Give an example to show that the minimal polynomials $m_{AB}(t)$ and $m_{BA}(t)$ of $AB$ and $BA$ may be different.

(d) Show that $m_{AB}(t)$ and $m_{BA}(t)$ differ at most by a factor of $t$. Stating carefully any results which you use, deduce that if $AB$ is diagonalisable then so is $(BA)^2$. 


Paper 1, Section I
1E Linear Algebra

State the Rank-Nullity Theorem.

If $\alpha : V \rightarrow W$ and $\beta : W \rightarrow X$ are linear maps and $W$ is finite dimensional, show that

$$\dim \text{Im}(\alpha) = \dim \text{Im}(\beta \alpha) + \dim (\text{Im}(\alpha) \cap \text{Ker}(\beta)).$$

If $\gamma : U \rightarrow V$ is another linear map, show that

$$\dim \text{Im}(\beta \alpha) + \dim \text{Im}(\alpha \gamma) \leq \dim \text{Im}(\alpha) + \dim \text{Im}(\beta \alpha \gamma).$$

Paper 2, Section I
1E Linear Algebra

Let $V$ be a real vector space. Define the dual vector space $V^*$ of $V$. If $U$ is a subspace of $V$, define the annihilator $U^0$ of $U$. If $x_1, x_2, \ldots, x_n$ is a basis for $V$, define its dual $x_1^*, x_2^*, \ldots, x_n^*$ and prove that it is a basis for $V^*$.

If $V$ has basis $x_1, x_2, x_3, x_4$ and $U$ is the subspace spanned by

$$x_1 + 2x_2 + 3x_3 + 4x_4 \quad \text{and} \quad 5x_1 + 6x_2 + 7x_3 + 8x_4,$$

give a basis for $U^0$ in terms of the dual basis $x_1^*, x_2^*, x_3^*, x_4^*$.

Paper 4, Section I
1E Linear Algebra

Define a quadratic form on a finite dimensional real vector space. What does it mean for a quadratic form to be positive definite?

Find a basis with respect to which the quadratic form

$$x^2 + 2xy + 2y^2 + 2yz + 3z^2$$

is diagonal. Is this quadratic form positive definite?
Define a Jordan block $J_m(\lambda)$. What does it mean for a complex $n \times n$ matrix to be in Jordan normal form?

If $A$ is a matrix in Jordan normal form for an endomorphism $\alpha : V \to V$, prove that

$$\dim \ker((\alpha - \lambda I)^r) - \dim \ker((\alpha - \lambda I)^{r-1})$$

is the number of Jordan blocks $J_m(\lambda)$ of $A$ with $m \geq r$.

Find a matrix in Jordan normal form for $J_m(\lambda)^2$. [Consider all possible values of $\lambda$.]

Find a matrix in Jordan normal form for the complex matrix

$$\begin{bmatrix}
0 & 0 & 0 & a_1 \\
0 & 0 & a_2 & 0 \\
0 & a_3 & 0 & 0 \\
a_4 & 0 & 0 & 0
\end{bmatrix}$$

assuming it is invertible.

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If $X$ is an $n \times m$ matrix over a field, show that there are invertible matrices $P$ and $Q$ such that

$$Q^{-1}XP = \begin{bmatrix} I_r & 0 \\
0 & 0 
\end{bmatrix}$$

for some $0 \leq r \leq \min(m, n)$, where $I_r$ is the identity matrix of dimension $r$.

For a square matrix of the form $A = \begin{bmatrix} B & D \\
0 & C \end{bmatrix}$ with $B$ and $C$ square matrices, prove that $\det(A) = \det(B) \det(C)$.

If $A \in M_{n \times n}(\mathbb{C})$ and $B \in M_{m \times m}(\mathbb{C})$ have no common eigenvalue, show that the linear map

$$L : M_{n \times m}(\mathbb{C}) \to M_{n \times m}(\mathbb{C})$$

$$X \mapsto AX - XB$$

is injective.
Paper 4, Section II
10E Linear Algebra

Let $V$ be a finite dimensional inner-product space over $\mathbb{C}$. What does it mean to say that an endomorphism of $V$ is self-adjoint? Prove that a self-adjoint endomorphism has real eigenvalues and may be diagonalised.

An endomorphism $\alpha : V \to V$ is called positive definite if it is self-adjoint and satisfies $\langle \alpha(x), x \rangle > 0$ for all non-zero $x \in V$; it is called negative definite if $-\alpha$ is positive definite. Characterise the property of being positive definite in terms of eigenvalues, and show that the sum of two positive definite endomorphisms is positive definite.

Show that a self-adjoint endomorphism $\alpha : V \to V$ has all eigenvalues in the interval $[a, b]$ if and only if $\alpha - \lambda I$ is positive definite for all $\lambda < a$ and negative definite for all $\lambda > b$.

Let $\alpha, \beta : V \to V$ be self-adjoint endomorphisms whose eigenvalues lie in the intervals $[a, b]$ and $[c, d]$ respectively. Show that all of the eigenvalues of $\alpha + \beta$ lie in the interval $[a + c, b + d]$.

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Paper 3, Section II
10E Linear Algebra

State and prove the Cayley–Hamilton Theorem.

Let $A$ be an $n \times n$ complex matrix. Using division of polynomials, show that if $p(x)$ is a polynomial then there is another polynomial $r(x)$ of degree at most $(n - 1)$ such that $p(\lambda) = r(\lambda)$ for each eigenvalue $\lambda$ of $A$ and such that $p(A) = r(A)$.

Hence compute the $(1, 1)$ entry of the matrix $A^{1000}$ when

$$
A = \begin{bmatrix}
2 & -1 & 0 \\
1 & -1 & 1 \\
-1 & -1 & 1
\end{bmatrix}.
$$
Paper 2, Section I
1F Linear Algebra
State and prove the Rank–Nullity theorem.

Let \( \alpha \) be a linear map from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) of rank 2. Give an example to show that \( \mathbb{R}^3 \) may be the direct sum of the kernel of \( \alpha \) and the image of \( \alpha \), and also an example where this is not the case.

Paper 1, Section I
1F Linear Algebra
State and prove the Steinitz Exchange Lemma.

Deduce that, for a subset \( S \) of \( \mathbb{R}^n \), any two of the following imply the third:

(i) \( S \) is linearly independent
(ii) \( S \) is spanning
(iii) \( S \) has exactly \( n \) elements

Let \( e_1, e_2 \) be a basis of \( \mathbb{R}^2 \). For which values of \( \lambda \) do \( \lambda e_1 + e_2, \ e_1 + \lambda e_2 \) form a basis of \( \mathbb{R}^2 \)?

Paper 4, Section I
1F Linear Algebra
Briefly explain the Gram–Schmidt orthogonalisation process in a real finite-dimensional inner product space \( V \).

For a subspace \( U \) of \( V \), define \( U^\perp \), and show that \( V = U \oplus U^\perp \).

For which positive integers \( n \) does

\[
(f, g) = f(1)g(1) + f(2)g(2) + f(3)g(3)
\]

define an inner product on the space of all real polynomials of degree at most \( n \)?
Paper 1, Section II
9F Linear Algebra

Let $U$ and $V$ be finite-dimensional real vector spaces, and let $\alpha : U \to V$ be a surjective linear map. Which of the following are always true and which can be false? Give proofs or counterexamples as appropriate.

(i) There is a linear map $\beta : V \to U$ such that $\beta \alpha$ is the identity map on $U$.

(ii) There is a linear map $\beta : V \to U$ such that $\alpha \beta$ is the identity map on $V$.

(iii) There is a subspace $W$ of $U$ such that the restriction of $\alpha$ to $W$ is an isomorphism from $W$ to $V$.

(iv) If $X$ and $Y$ are subspaces of $U$ with $U = X \oplus Y$ then $V = \alpha(X) \oplus \alpha(Y)$.

(v) If $X$ and $Y$ are subspaces of $U$ with $V = \alpha(X) \oplus \alpha(Y)$ then $U = X \oplus Y$.

Paper 2, Section II
10F Linear Algebra

Let $\alpha : U \to V$ and $\beta : V \to W$ be linear maps between finite-dimensional real vector spaces.

Show that the rank $r(\beta \alpha)$ satisfies $r(\beta \alpha) \leq \min(r(\beta), r(\alpha))$. Show also that $r(\beta \alpha) \geq r(\alpha) + r(\beta) - \dim V$. For each of these two inequalities, give examples to show that we may or may not have equality.

Now let $V$ have dimension $2n$ and let $\alpha : V \to V$ be a linear map of rank $2n - 2$ such that $\alpha^n = 0$. Find the rank of $\alpha^k$ for each $1 \leq k \leq n - 1$. 

Part IB, 2017 List of Questions
Paper 4, Section II
10F Linear Algebra

What is the dual $X^*$ of a finite-dimensional real vector space $X$? If $X$ has a basis $e_1, \ldots, e_n$, define the dual basis, and prove that it is indeed a basis of $X^*$.

[No results on the dimension of duals may be assumed without proof.]

Write down (without making a choice of basis) an isomorphism from $X$ to $X^{**}$. Prove that your map is indeed an isomorphism.

Does every basis of $X^*$ arise as the dual basis of some basis of $X$? Justify your answer.

A subspace $W$ of $X^*$ is called separating if for every non-zero $x \in X$ there is a $T \in W$ with $T(x) \neq 0$. Show that the only separating subspace of $X^*$ is $X^*$ itself.

Now let $X$ be the (infinite-dimensional) space of all real polynomials. Explain briefly how we may identify $X^*$ with the space of all real sequences. Give an example of a proper subspace of $X^*$ that is separating.

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Paper 3, Section II
10F Linear Algebra

Let $f$ be a quadratic form on a finite-dimensional real vector space $V$. Prove that there exists a diagonal basis for $f$, meaning a basis with respect to which the matrix of $f$ is diagonal.

Define the rank $r$ and signature $s$ of $f$ in terms of this matrix. Prove that $r$ and $s$ are independent of the choice of diagonal basis.

In terms of $r$, $s$, and the dimension $n$ of $V$, what is the greatest dimension of a subspace on which $f$ is zero?

Now let $f$ be the quadratic form on $\mathbb{R}^3$ given by $f(x, y, z) = x^2 - y^2$. For which points $v$ in $\mathbb{R}^3$ is it the case that there is some diagonal basis for $f$ containing $v$?
Paper 4, Section I
1F Linear Algebra

For which real numbers $x$ do the vectors 

$$(x, 1, 1, 1), \quad (1, x, 1, 1), \quad (1, 1, x, 1), \quad (1, 1, 1, x),$$

not form a basis of $\mathbb{R}^4$? For each such value of $x$, what is the dimension of the subspace of $\mathbb{R}^4$ that they span? For each such value of $x$, provide a basis for the spanned subspace, and extend this basis to a basis of $\mathbb{R}^4$.

Paper 2, Section I
1F Linear Algebra

Find a linear change of coordinates such that the quadratic form

$$2x^2 + 8xy - 6xz + y^2 - 4yz + 2z^2$$

takes the form

$$\alpha x^2 + \beta y^2 + \gamma z^2,$$

for real numbers $\alpha, \beta$ and $\gamma$.

Paper 1, Section I
1F Linear Algebra

(a) Consider the linear transformation $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$
\begin{pmatrix}
5 & -6 & -6 \\
-1 & 4 & 2 \\
3 & -6 & -4 \\
\end{pmatrix}.
$$

Find a basis of $\mathbb{R}^3$ in which $\alpha$ is represented by a diagonal matrix.

(b) Give a list of $6 \times 6$ matrices such that any linear transformation $\beta : \mathbb{R}^6 \to \mathbb{R}^6$ with characteristic polynomial

$$(x - 2)^4(x + 7)^2$$
and minimal polynomial

$$(x - 2)^2(x + 7)$$
is similar to one of the matrices on your list. No two distinct matrices on your list should be similar. [No proof is required.]
Paper 1, Section II
9F Linear Algebra
Let $M_{n,n}$ denote the vector space over $F = \mathbb{R}$ or $\mathbb{C}$ of $n \times n$ matrices with entries in $F$. Let $\text{Tr} : M_{n,n} \to F$ denote the trace functional, i.e., if $A = (a_{ij})_{1 \leq i,j \leq n} \in M_{n,n}$, then

$$\text{Tr}(A) = \sum_{i=1}^{n} a_{ii}.$$ 

(a) Show that $\text{Tr}$ is a linear functional.

(b) Show that $\text{Tr}(AB) = \text{Tr}(BA)$ for $A,B \in M_{n,n}$.

(c) Show that $\text{Tr}$ is unique in the following sense: If $f : M_{n,n} \to F$ is a linear functional such that $f(AB) = f(BA)$ for each $A,B \in M_{n,n}$, then $f$ is a scalar multiple of the trace functional. If, in addition, $f(I) = n$, then $f = \text{Tr}$.

(d) Let $W \subseteq M_{n,n}$ be the subspace spanned by matrices $C$ of the form $C = AB - BA$ for $A,B \in M_{n,n}$. Show that $W$ is the kernel of $\text{Tr}$.

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Paper 4, Section II
10F Linear Algebra
(a) Let $\alpha : V \to W$ be a linear transformation between finite dimensional vector spaces over a field $F = \mathbb{R}$ or $\mathbb{C}$.

Define the dual map of $\alpha$. Let $\delta$ be the dual map of $\alpha$. Given a subspace $U \subseteq V$, define the annihilator $U^\circ$ of $U$. Show that $(\ker \alpha)^\circ$ and the image of $\delta$ coincide. Conclude that the dimension of the image of $\alpha$ is equal to the dimension of the image of $\delta$. Show that $\dim \ker(\alpha) - \dim \ker(\delta) = \dim V - \dim W$.

(b) Now suppose in addition that $V,W$ are inner product spaces. Define the adjoint $\alpha^*$ of $\alpha$. Let $\beta : U \to V, \gamma : V \to W$ be linear transformations between finite dimensional inner product spaces. Suppose that the image of $\beta$ is equal to the kernel of $\gamma$. Then show that $\beta \beta^* + \gamma^* \gamma$ is an isomorphism.
Paper 3, Section II
10F Linear Algebra

Let \( \alpha : V \to V \) be a linear transformation defined on a finite dimensional inner product space \( V \) over \( \mathbb{C} \). Recall that \( \alpha \) is normal if \( \alpha \) and its adjoint \( \alpha^* \) commute. Show that \( \alpha \) being normal is equivalent to each of the following statements:

(i) \( \alpha = \alpha_1 + i \alpha_2 \) where \( \alpha_1, \alpha_2 \) are self-adjoint operators and \( \alpha_1 \alpha_2 = \alpha_2 \alpha_1 \);

(ii) there is an orthonormal basis for \( V \) consisting of eigenvectors of \( \alpha \);

(iii) there is a polynomial \( g \) with complex coefficients such that \( \alpha^* = g(\alpha) \).

Paper 2, Section II
10F Linear Algebra

Let \( M_{n,n} \) denote the vector space over a field \( F = \mathbb{R} \) or \( \mathbb{C} \) of \( n \times n \) matrices with entries in \( F \). Given \( B \in M_{n,n} \), consider the two linear transformations \( R_B, L_B : M_{n,n} \to M_{n,n} \) defined by

\[
L_B(A) = BA, \quad R_B(A) = AB.
\]

(a) Show that \( \det L_B = (\det B)^n \).

[For parts (b) and (c), you may assume the analogous result \( \det R_B = (\det B)^n \) without proof.]

(b) Now let \( F = \mathbb{C} \). For \( B \in M_{n,n} \), write \( B^* \) for the conjugate transpose of \( B \), i.e., \( B^* := \overline{B}^T \). For \( B \in M_{n,n} \), define the linear transformation \( M_B : M_{n,n} \to M_{n,n} \) by

\[
M_B(A) = BAB^*.
\]

Show that \( \det M_B = |\det B|^{2n} \).

(c) Again let \( F = \mathbb{C} \). Let \( W \subseteq M_{n,n} \) be the set of Hermitian matrices. [Note that \( W \) is not a vector space over \( \mathbb{C} \) but only over \( \mathbb{R} \).] For \( B \in M_{n,n} \) and \( A \in W \), define \( T_B(A) = BAB^* \). Show that \( T_B \) is an \( \mathbb{R} \)-linear operator on \( W \), and show that as such,

\[
\det T_B = |\det B|^{2n}.
\]
Paper 4, Section I
1E Linear Algebra

Define the dual space $V^*$ of a vector space $V$. Given a basis $\{x_1, \ldots, x_n\}$ of $V$ define its dual and show it is a basis of $V^*$.

Let $V$ be a 3-dimensional vector space over $\mathbb{R}$ and let $\{\zeta_1, \zeta_2, \zeta_3\}$ be the basis of $V^*$ dual to the basis $\{x_1, x_2, x_3\}$ for $V$. Determine, in terms of the $\zeta_i$, the bases dual to each of the following:

(a) $\{x_1 + x_2, x_2 + x_3, x_3\}$,
(b) $\{x_1 + x_2, x_2 + x_3, x_3 + x_1\}$.

Paper 2, Section I
1E Linear Algebra

Let $q$ denote a quadratic form on a real vector space $V$. Define the rank and signature of $q$.

Find the rank and signature of the following quadratic forms.

(a) $q(x, y, z) = x^2 + y^2 + z^2 - 2xz - 2yz$.
(b) $q(x, y, z) = xy - xz$.
(c) $q(x, y, z) = xy - 2z^2$.

Paper 1, Section I
1E Linear Algebra

Let $U$ and $V$ be finite dimensional vector spaces and $\alpha : U \to V$ a linear map. Suppose $W$ is a subspace of $U$. Prove that

$$r(\alpha) \geq r(\alpha|_W) \geq r(\alpha) - \dim(U) + \dim(W)$$

where $r(\alpha)$ denotes the rank of $\alpha$ and $\alpha|_W$ denotes the restriction of $\alpha$ to $W$. Give examples showing that each inequality can be both a strict inequality and an equality.
Paper 1, Section II
9E Linear Algebra

Determine the characteristic polynomial of the matrix
\[
M = \begin{pmatrix}
    x & 1 & 1 & 0 \\
    1 - x & 0 & -1 & 0 \\
    2 & 2x & 1 & 0 \\
    0 & 0 & 0 & 1
\end{pmatrix}.
\]

For which values of \( x \in \mathbb{C} \) is \( M \) invertible? When \( M \) is not invertible determine (i) the Jordan normal form \( J \) of \( M \), (ii) the minimal polynomial of \( M \).

Find a basis of \( \mathbb{C}^4 \) such that \( J \) is the matrix representing the endomorphism \( M : \mathbb{C}^4 \to \mathbb{C}^4 \) in this basis. Give a change of basis matrix \( P \) such that \( P^{-1}MP = J \).

Paper 4, Section II
10E Linear Algebra

Suppose \( U \) and \( W \) are subspaces of a vector space \( V \). Explain what is meant by \( U \cap W \) and \( U + W \) and show that both of these are subspaces of \( V \).

Show that if \( U \) and \( W \) are subspaces of a finite dimensional space \( V \) then
\[
\dim U + \dim W = \dim(U \cap W) + \dim(U + W).
\]

Determine the dimension of the subspace \( W \) of \( \mathbb{R}^5 \) spanned by the vectors
\[
\begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \\ -1 \\ 1 \end{pmatrix}.
\]

Write down a \( 5 \times 5 \) matrix which defines a linear map \( \mathbb{R}^5 \to \mathbb{R}^5 \) with \( (1, 1, 1, 1, 1)^T \) in the kernel and with image \( W \).

What is the dimension of the space spanned by all linear maps \( \mathbb{R}^5 \to \mathbb{R}^5 \)
(i) with \( (1, 1, 1, 1, 1)^T \) in the kernel and with image contained in \( W \),
(ii) with \( (1, 1, 1, 1, 1)^T \) in the kernel or with image contained in \( W \)?
Let $A_1, A_2, \ldots, A_k$ be $n \times n$ matrices over a field $F$. We say $A_1, A_2, \ldots, A_k$ are simultaneously diagonalisable if there exists an invertible matrix $P$ such that $P^{-1}A_iP$ is diagonal for all $1 \leq i \leq k$. We say the matrices are commuting if $A_iA_j = A_jA_i$ for all $i, j$.

(i) Suppose $A_1, A_2, \ldots, A_k$ are simultaneously diagonalisable. Prove that they are commuting.

(ii) Define an eigenspace of a matrix. Suppose $B_1, B_2, \ldots, B_k$ are commuting $n \times n$ matrices over a field $F$. Let $E$ denote an eigenspace of $B_1$. Prove that $B_i(E) \subseteq E$ for all $i$.

(iii) Suppose $B_1, B_2, \ldots, B_k$ are commuting diagonalisable matrices. Prove that they are simultaneously diagonalisable.

(iv) Are the $2 \times 2$ diagonalisable matrices over $\mathbb{C}$ simultaneously diagonalisable? Explain your answer.

\begin{itemize}
  \item (i) Suppose $A$ is a matrix that does not have $-1$ as an eigenvalue. Show that $A + I$ is non-singular. Further, show that $A$ commutes with $(A + I)^{-1}$.
  \item (ii) A matrix $A$ is called skew-symmetric if $A^T = -A$. Show that a real skew-symmetric matrix does not have $-1$ as an eigenvalue.
  \item (iii) Suppose $A$ is a real skew-symmetric matrix. Show that $U = (I - A)(I + A)^{-1}$ is orthogonal with determinant 1.
  \item (iv) Verify that every orthogonal matrix $U$ with determinant 1 which does not have $-1$ as an eigenvalue can be expressed as $(I - A)(I + A)^{-1}$ where $A$ is a real skew-symmetric matrix.
\end{itemize}
Paper 4, Section I
1G Linear Algebra
Let $V$ denote the vector space of all real polynomials of degree at most 2. Show that

$$(f, g) = \int_{-1}^{1} f(x)g(x) \, dx$$

defines an inner product on $V$.

Find an orthonormal basis for $V$.

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Paper 2, Section I
1G Linear Algebra
State and prove the Rank–Nullity Theorem.

Let $\alpha$ be a linear map from $\mathbb{R}^5$ to $\mathbb{R}^3$. What are the possible dimensions of the kernel of $\alpha$? Justify your answer.

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Paper 1, Section I
1G Linear Algebra
State and prove the Steinitz Exchange Lemma. Use it to prove that, in a finite-dimensional vector space: any two bases have the same size, and every linearly independent set extends to a basis.

Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. Is $e_1 + e_2, e_2 + e_3, e_3 + e_1$ a basis for $\mathbb{R}^3$? Is $e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4 + e_1$ a basis for $\mathbb{R}^4$? Justify your answers.
Paper 1, Section II

9G Linear Algebra

Let \( V \) be an \( n \)-dimensional real vector space, and let \( T \) be an endomorphism of \( V \). We say that \( T \) acts on a subspace \( W \) if \( T(W) \subseteq W \).

(i) For any \( x \in V \), show that \( T \) acts on the linear span of \( \{ x, T(x), T^2(x), \ldots, T^{n-1}(x) \} \).

(ii) If \( \{ x, T(x), T^2(x), \ldots, T^{n-1}(x) \} \) spans \( V \), show directly (i.e. without using the Cayley–Hamilton Theorem) that \( T \) satisfies its own characteristic equation.

(iii) Suppose that \( T \) acts on a subspace \( W \) with \( W \neq \{ 0 \} \) and \( W \neq V \). Let \( e_1, \ldots, e_k \) be a basis for \( W \), and extend to a basis \( e_1, \ldots, e_n \) for \( V \). Describe the matrix of \( T \) with respect to this basis.

(iv) Using (i), (ii) and (iii) and induction, give a proof of the Cayley–Hamilton Theorem.

[Simple properties of determinants may be assumed without proof.]

Paper 4, Section II

10G Linear Algebra

Let \( V \) be a real vector space. What is the dual \( V^* \) of \( V \)? If \( e_1, \ldots, e_n \) is a basis for \( V \), define the dual basis \( e_1^*, \ldots, e_n^* \) for \( V^* \), and show that it is indeed a basis for \( V^* \).

[No result about dimensions of dual spaces may be assumed.]

For a subspace \( U \) of \( V \), what is the annihilator of \( U \)? If \( V \) is \( n \)-dimensional, how does the dimension of the annihilator of \( U \) relate to the dimension of \( U \)?

Let \( \alpha : V \to W \) be a linear map between finite-dimensional real vector spaces. What is the dual map \( \alpha^* \)? Explain why the rank of \( \alpha^* \) is equal to the rank of \( \alpha \). Prove that the kernel of \( \alpha^* \) is the annihilator of the image of \( \alpha \), and also that the image of \( \alpha^* \) is the annihilator of the kernel of \( \alpha \).

[Results about the matrices representing a map and its dual may be used without proof, provided they are stated clearly.]

Now let \( V \) be the vector space of all real polynomials, and define elements \( L_0, L_1, \ldots \) of \( V^* \) by setting \( L_i(p) \) to be the coefficient of \( X^i \) in \( p \) (for each \( p \in V \)). Do the \( L_i \) form a basis for \( V^* \)?
Let \( q \) be a nonsingular quadratic form on a finite-dimensional real vector space \( V \). Prove that we may write \( V = P \oplus N \), where the restriction of \( q \) to \( P \) is positive definite, the restriction of \( q \) to \( N \) is negative definite, and \( q(x + y) = q(x) + q(y) \) for all \( x \in P \) and \( y \in N \). [No result on diagonalisability may be assumed.]

Show that the dimensions of \( P \) and \( N \) are independent of the choice of \( P \) and \( N \). Give an example to show that \( P \) and \( N \) are not themselves uniquely defined.

Find such a decomposition \( V = P \oplus N \) when \( V = \mathbb{R}^3 \) and \( q \) is the quadratic form \( q((x, y, z)) = x^2 + 2y^2 - 2xy - 2xz \).

Define the determinant of an \( n \times n \) complex matrix \( A \). Explain, with justification, how the determinant of \( A \) changes when we perform row and column operations on \( A \).

Let \( A, B, C \) be complex \( n \times n \) matrices. Prove the following statements.

(i) \[ \det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det A \, \det B . \]

(ii) \[ \det \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \det(A + iB) \, \det(A - iB) . \]
Paper 4, Section I

1E  Linear Algebra

What is a quadratic form on a finite dimensional real vector space \( V \)? What does it mean for two quadratic forms to be isomorphic (\( i.e. \) congruent)? State Sylvester’s law of inertia and explain the definition of the quantities which appear in it. Find the signature of the quadratic form on \( \mathbb{R}^3 \) given by \( q(v) = v^T Av \), where

\[
A = \begin{pmatrix}
-2 & 1 & 6 \\
1 & -1 & -3 \\
6 & -3 & 1
\end{pmatrix}.
\]

Paper 2, Section I

1E  Linear Algebra

If \( A \) is an \( n \times n \) invertible Hermitian matrix, let

\[
U_A = \{ U \in M_{n \times n}(\mathbb{C}) \mid U^T AU = A \}.
\]

Show that \( U_A \) with the operation of matrix multiplication is a group, and that \( \text{det} U \) has norm 1 for any \( U \in U_A \). What is the relation between \( U_A \) and the complex Hermitian form defined by \( A \)?

If \( A = I_n \) is the \( n \times n \) identity matrix, show that any element of \( U_A \) is diagonalizable.

Paper 1, Section I

1E  Linear Algebra

What is the adjugate of an \( n \times n \) matrix \( A \)? How is it related to \( A^{-1} \)? Suppose all the entries of \( A \) are integers. Show that all the entries of \( A^{-1} \) are integers if and only if \( \text{det} A = \pm 1 \).
Paper 1, Section II

9E Linear Algebra

If \( V_1 \) and \( V_2 \) are vector spaces, what is meant by \( V_1 \oplus V_2 \)? If \( V_1 \) and \( V_2 \) are subspaces of a vector space \( V \), what is meant by \( V_1 + V_2 \)?

Stating clearly any theorems you use, show that if \( V_1 \) and \( V_2 \) are subspaces of a finite dimensional vector space \( V \), then

\[
\dim V_1 + \dim V_2 = \dim V_1 \cap V_2 + \dim (V_1 + V_2).
\]

Let \( V_1, V_2 \subset \mathbb{R}^4 \) be subspaces with bases

\[
V_1 = \langle (3, 2, 4, -1), (1, 2, 1, -2), (-2, 3, 3, 2) \rangle
\]

\[
V_2 = \langle (1, 4, 2, 4), (-1, 1, -1, -1), (3, 1, 2, 0) \rangle.
\]

Find a basis \( \langle v_1, v_2 \rangle \) for \( V_1 \cap V_2 \) such that the first component of \( v_1 \) and the second component of \( v_2 \) are both 0.

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Paper 4, Section II

10E Linear Algebra

What does it mean for an \( n \times n \) matrix to be in Jordan form? Show that if \( A \in M_{n \times n}(\mathbb{C}) \) is in Jordan form, there is a sequence \( (A_m) \) of diagonalizable \( n \times n \) matrices which converges to \( A \), in the sense that the \((ij)\)th component of \( A_m \) converges to the \((ij)\)th component of \( A \) for all \( i \) and \( j \). [Hint: A matrix with distinct eigenvalues is diagonalizable.]

Deduce that the same statement holds for all \( A \in M_{n \times n}(\mathbb{C}) \).

Let \( V = M_{2 \times 2}(\mathbb{C}) \). Given \( A \in V \), define a linear map \( T_A : V \to V \) by \( T_A(B) = AB + BA \). Express the characteristic polynomial of \( T_A \) in terms of the trace and determinant of \( A \). [Hint: First consider the case where \( A \) is diagonalizable.]

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Paper 3, Section II

10E Linear Algebra

Let \( V \) and \( W \) be finite dimensional real vector spaces and let \( T : V \to W \) be a linear map. Define the dual space \( V^* \) and the dual map \( T^* \). Show that there is an isomorphism \( \iota : V \to (V^*)^* \) which is canonical, in the sense that \( \iota \circ S = (S^*)^* \circ \iota \) for any automorphism \( S \) of \( V \).

Now let \( W \) be an inner product space. Use the inner product to show that there is an injective map from \( \mathrm{im} \ T \) to \( \mathrm{im} \ T^* \). Deduce that the row rank of a matrix is equal to its column rank.
Paper 2, Section II

10E Linear Algebra

Define what it means for a set of vectors in a vector space \( V \) to be linearly dependent. Prove from the definition that any set of \( n + 1 \) vectors in \( \mathbb{R}^n \) is linearly dependent.

Using this or otherwise, prove that if \( V \) has a finite basis consisting of \( n \) elements, then any basis of \( V \) has exactly \( n \) elements.

Let \( V \) be the vector space of bounded continuous functions on \( \mathbb{R} \). Show that \( V \) is infinite dimensional.
Paper 4, Section I

1F Linear Algebra

Let $V$ be a complex vector space with basis $\{e_1, \ldots, e_n\}$. Define $T : V \rightarrow V$ by $T(e_i) = e_i - e_{i+1}$ for $i < n$ and $T(e_n) = e_n - e_1$. Show that $T$ is diagonalizable and find its eigenvalues. [You may use any theorems you wish, as long as you state them clearly.]

Paper 2, Section I

1F Linear Algebra

Define the determinant $\det A$ of an $n \times n$ real matrix $A$. Suppose that $X$ is a matrix with block form

$$X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where $A$, $B$ and $C$ are matrices of dimensions $n \times n$, $n \times m$ and $m \times m$ respectively. Show that $\det X = (\det A)(\det C)$.

Paper 1, Section I

1F Linear Algebra

Define the notions of basis and dimension of a vector space. Prove that two finite-dimensional real vector spaces with the same dimension are isomorphic.

In each case below, determine whether the set $S$ is a basis of the real vector space $V$:

(i) $V = \mathbb{C}$ is the complex numbers; $S = \{1, i\}$.

(ii) $V = \mathbb{R}[x]$ is the vector space of all polynomials in $x$ with real coefficients; $S = \{1, (x - 1), (x - 1)(x - 2), (x - 1)(x - 2)(x - 3), \ldots\}$.

(iii) $V = \{f : [0,1] \rightarrow \mathbb{R}\}; S = \{\chi_p | p \in [0,1]\}$, where

$$\chi_p(x) = \begin{cases} 1 & x = p \\ 0 & x \neq p. \end{cases}$$
Define what it means for two $n \times n$ matrices to be similar to each other. Show that if two $n \times n$ matrices are similar, then the linear transformations they define have isomorphic kernels and images.

If $A$ and $B$ are $n \times n$ real matrices, we define $[A, B] = AB - BA$. Let

\[
K_A = \{ X \in M_{n \times n}(\mathbb{R}) \mid [A, X] = 0 \}
\]
\[
L_A = \{ [A, X] \mid X \in M_{n \times n}(\mathbb{R}) \}.
\]

Show that $K_A$ and $L_A$ are linear subspaces of $M_{n \times n}(\mathbb{R})$. If $A$ and $B$ are similar, show that $K_A \cong K_B$ and $L_A \cong L_B$.

Suppose that $A$ is diagonalizable and has characteristic polynomial

\[
(x - \lambda_1)^{m_1}(x - \lambda_2)^{m_2},
\]
where $\lambda_1 \neq \lambda_2$. What are $\dim K_A$ and $\dim L_A$?

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Let $V$ be a finite-dimensional real vector space of dimension $n$. A bilinear form $B : V \times V \to \mathbb{R}$ is nondegenerate if for all $v \neq 0$ in $V$, there is some $w \in V$ with $B(v, w) \neq 0$. For $v \in V$, define $\langle v \rangle^\perp = \{ w \in V \mid B(v, w) = 0 \}$. Assuming $B$ is nondegenerate, show that $V = \langle v \rangle \oplus \langle v \rangle^\perp$ whenever $B(v, v) \neq 0$.

Suppose that $B$ is a nondegenerate, symmetric bilinear form on $V$. Prove that there is a basis $\{ v_1, \ldots, v_n \}$ of $V$ with $B(v_i, v_j) = 0$ for $i \neq j$. [If you use the fact that symmetric matrices are diagonalizable, you must prove it.]

Define the signature of a quadratic form. Explain how to determine the signature of the quadratic form associated to $B$ from the basis you constructed above.

A linear subspace $V' \subset V$ is said to be isotropic if $B(v, w) = 0$ for all $v, w \in V'$. Show that if $B$ is nondegenerate, the maximal dimension of an isotropic subspace of $V$ is $(n - |\sigma|)/2$, where $\sigma$ is the signature of the quadratic form associated to $B$. 

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Part IB, 2012 List of Questions
Paper 3, Section II

10F Linear Algebra

What is meant by the Jordan normal form of an \( n \times n \) complex matrix?

Find the Jordan normal forms of the following matrices:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
-1 & -1 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
3 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 \\
9 & 6 & 3 & 0 \\
15 & 12 & 9 & 3
\end{pmatrix}.
\]

Suppose \( A \) is an invertible \( n \times n \) complex matrix. Explain how to derive the characteristic and minimal polynomials of \( A^n \) from the characteristic and minimal polynomials of \( A \). Justify your answer. [Hint: write each polynomial as a product of linear factors.]

Paper 2, Section II

10F Linear Algebra

(i) Define the transpose of a matrix. If \( V \) and \( W \) are finite-dimensional real vector spaces, define the dual of a linear map \( T : V \rightarrow W \). How are these two notions related?

Now suppose \( V \) and \( W \) are finite-dimensional inner product spaces. Use the inner product on \( V \) to define a linear map \( V \rightarrow V^* \) and show that it is an isomorphism. Define the adjoint of a linear map \( T : V \rightarrow W \). How are the adjoint of \( T \) and its dual related? If \( A \) is a matrix representing \( T \), under what conditions is the adjoint of \( T \) represented by the transpose of \( A \)?

(ii) Let \( V = C[0, 1] \) be the vector space of continuous real-valued functions on \([0, 1]\), equipped with the inner product

\[
\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt.
\]

Let \( T : V \rightarrow V \) be the linear map

\[
T f(t) = \int_0^t f(s) \, ds.
\]

What is the adjoint of \( T \)?
Paper 1, Section I
1G Linear Algebra

(i) State the rank-nullity theorem for a linear map between finite-dimensional vector spaces.

(ii) Show that a linear transformation \( f : V \to V \) of a finite-dimensional vector space \( V \) is bijective if it is injective or surjective.

(iii) Let \( V \) be the \( \mathbb{R} \)-vector space \( \mathbb{R}[X] \) of all polynomials in \( X \) with coefficients in \( \mathbb{R} \). Give an example of a linear transformation \( f : V \to V \) which is surjective but not bijective.

Paper 2, Section I
1G Linear Algebra

Let \( V \) be an \( n \)-dimensional \( \mathbb{R} \)-vector space with an inner product. Let \( W \) be an \( m \)-dimensional subspace of \( V \) and \( W^\perp \) its orthogonal complement, so that every element \( v \in V \) can be uniquely written as \( v = w + w' \) for \( w \in W \) and \( w' \in W^\perp \).

The reflection map with respect to \( W \) is defined as the linear map

\[
f_W : V \ni w + w' \mapsto w - w' \in V.
\]

Show that \( f_W \) is an orthogonal transformation with respect to the inner product, and find its determinant.

Paper 4, Section I
1G Linear Algebra

(i) Let \( V \) be a vector space over a field \( F \), and \( W_1, W_2 \) subspaces of \( V \). Define the subset \( W_1 + W_2 \) of \( V \), and show that \( W_1 + W_2 \) and \( W_1 \cap W_2 \) are subspaces of \( V \).

(ii) When \( W_1, W_2 \) are finite-dimensional, state a formula for \( \dim(W_1 + W_2) \) in terms of \( \dim W_1 \), \( \dim W_2 \) and \( \dim(W_1 \cap W_2) \).

(iii) Let \( V \) be the \( \mathbb{R} \)-vector space of all \( n \times n \) matrices over \( \mathbb{R} \). Let \( S \) be the subspace of all symmetric matrices and \( T \) the subspace of all upper triangular matrices (the matrices \( (a_{ij}) \) such that \( a_{ij} = 0 \) whenever \( i > j \)). Find \( \dim S \), \( \dim T \), \( \dim(S \cap T) \) and \( \dim(S + T) \). Briefly justify your answer.
Paper 1, Section II
9G Linear Algebra

Let $V, W$ be finite-dimensional vector spaces over a field $F$ and $f : V \to W$ a linear map.

(i) Show that $f$ is injective if and only if the image of every linearly independent subset of $V$ is linearly independent in $W$.

(ii) Define the dual space $V^*$ of $V$ and the dual map $f^* : W^* \to V^*$.

(iii) Show that $f$ is surjective if and only if the image under $f^*$ of every linearly independent subset of $W^*$ is linearly independent in $V^*$.

Paper 2, Section II
10G Linear Algebra

Let $n$ be a positive integer, and let $V$ be a $\mathbb{C}$-vector space of complex-valued functions on $\mathbb{R}$, generated by the set $\{ \cos kx, \sin kx; k = 0, 1, \ldots, n - 1 \}$.

(i) Let $\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}\,dx$ for $f, g \in V$. Show that this is a positive definite Hermitian form on $V$.

(ii) Let $\Delta(f) = \frac{d^2}{dx^2}f(x)$. Show that $\Delta$ is a self-adjoint linear transformation of $V$ with respect to the form defined in (i).

(iii) Find an orthonormal basis of $V$ with respect to the form defined in (i), which consists of eigenvectors of $\Delta$.

Paper 3, Section II
10G Linear Algebra

(i) Let $A$ be an $n \times n$ complex matrix and $f(X)$ a polynomial with complex coefficients. By considering the Jordan normal form of $A$ or otherwise, show that if the eigenvalues of $A$ are $\lambda_1, \ldots, \lambda_n$ then the eigenvalues of $f(A)$ are $f(\lambda_1), \ldots, f(\lambda_n)$.

(ii) Let $B = \begin{pmatrix} a & d & c & b \\ b & a & d & c \\ c & b & a & d \\ d & c & b & a \end{pmatrix}$. Write $B$ as $B = f(A)$ for a polynomial $f$ with $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, and find the eigenvalues of $B$.

[Hint: compute the powers of $A$.]
Let $V$ be an $n$-dimensional $\mathbb{R}$-vector space and $f, g : V \to V$ linear transformations. Suppose $f$ is invertible and diagonalisable, and $f \circ g = t \cdot (g \circ f)$ for some real number $t > 1$.

(i) Show that $g$ is nilpotent, i.e. some positive power of $g$ is 0.

(ii) Suppose that there is a non-zero vector $v \in V$ with $f(v) = v$ and $g^{n-1}(v) \neq 0$. Determine the diagonal form of $f$. 

Part IB, 2011 List of Questions
Paper 1, Section I
1F Linear Algebra
Suppose that $V$ is the complex vector space of polynomials of degree at most $n - 1$ in the variable $z$. Find the Jordan normal form for each of the linear transformations $\frac{d}{dz}$ and $z \frac{d}{dz}$ acting on $V$.

Paper 2, Section I
1F Linear Algebra
Suppose that $\phi$ is an endomorphism of a finite-dimensional complex vector space.

(i) Show that if $\lambda$ is an eigenvalue of $\phi$, then $\lambda^2$ is an eigenvalue of $\phi^2$.

(ii) Show conversely that if $\mu$ is an eigenvalue of $\phi^2$, then there is an eigenvalue $\lambda$ of $\phi$ with $\lambda^2 = \mu$.

Paper 4, Section I
1F Linear Algebra
Define the notion of an inner product on a finite-dimensional real vector space $V$, and the notion of a self-adjoint linear map $\alpha : V \rightarrow V$.

Suppose that $V$ is the space of real polynomials of degree at most $n$ in a variable $t$. Show that

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t) \, dt$$

is an inner product on $V$, and that the map $\alpha : V \rightarrow V$:

$$\alpha(f)(t) = (1 - t^2)f''(t) - 2tf'(t)$$

is self-adjoint.
Paper 1, Section II

9F Linear Algebra

Let $V$ denote the vector space of $n \times n$ real matrices.

(1) Show that if $\psi(A, B) = \text{tr}(AB^T)$, then $\psi$ is a positive-definite symmetric bilinear form on $V$.

(2) Show that if $q(A) = \text{tr}(A^2)$, then $q$ is a quadratic form on $V$. Find its rank and signature.

[Hint: Consider symmetric and skew-symmetric matrices.]

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Paper 2, Section II

10F Linear Algebra

(i) Show that two $n \times n$ complex matrices $A, B$ are similar (i.e. there exists invertible $P$ with $A = P^{-1}BP$) if and only if they represent the same linear map $\mathbb{C}^n \to \mathbb{C}^n$ with respect to different bases.

(ii) Explain the notion of Jordan normal form of a square complex matrix.

(iii) Show that any square complex matrix $A$ is similar to its transpose.

(iv) If $A$ is invertible, describe the Jordan normal form of $A^{-1}$ in terms of that of $A$. Justify your answers.

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Paper 3, Section II

10F Linear Algebra

Suppose that $V$ is a finite-dimensional vector space over $\mathbb{C}$, and that $\alpha : V \to V$ is a $\mathbb{C}$-linear map such that $\alpha^n = 1$ for some $n > 1$. Show that if $V_1$ is a subspace of $V$ such that $\alpha(V_1) \subset V_1$, then there is a subspace $V_2$ of $V$ such that $V = V_1 \oplus V_2$ and $\alpha(V_2) \subset V_2$.

[Hint: Show, for example by picking bases, that there is a linear map $\pi : V \to V_1$ with $\pi(x) = x$ for all $x \in V_1$. Then consider $\rho : V \to V_1$ with $\rho(y) = \frac{1}{n} \sum_{i=0}^{n-1} \alpha^i\alpha^{-i}(y)$.]
(i) Show that the group \( O_n(\mathbb{R}) \) of orthogonal \( n \times n \) real matrices has a normal subgroup
\[
SO_n(\mathbb{R}) = \{ A \in O_n(\mathbb{R}) \mid \det A = 1 \}.
\]

(ii) Show that \( O_n(\mathbb{R}) = SO_n(\mathbb{R}) \times \{ \pm I_n \} \) if and only if \( n \) is odd.

(iii) Show that if \( n \) is even, then \( O_n(\mathbb{R}) \) is not the direct product of \( SO_n(\mathbb{R}) \) with any normal subgroup.

[You may assume that the only elements of \( O_n(\mathbb{R}) \) that commute with all elements of \( O_n(\mathbb{R}) \) are \( \pm I_n \).]
(1) Let $V$ be a finite-dimensional vector space and let $T : V \to V$ be a non-zero endomorphism of $V$. If $\ker(T) = \im(T)$ show that the dimension of $V$ is an even integer. Find the minimal polynomial of $T$. [You may assume the rank-nullity theorem.]

(2) Let $A_i$, $1 \leq i \leq 3$, be non-zero subspaces of a vector space $V$ with the property that

$$V = A_1 \oplus A_2 = A_2 \oplus A_3 = A_1 \oplus A_3.$$ 

Show that there is a 2-dimensional subspace $W \subset V$ for which all the $W \cap A_i$ are one-dimensional.

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Paper 2, Section I

1G Linear Algebra

Let $V$ denote the vector space of polynomials $f(x, y)$ in two variables of total degree at most $n$. Find the dimension of $V$.

If $S : V \to V$ is defined by

$$(Sf)(x, y) = x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2},$$

find the kernel of $S$ and the image of $S$. Compute the trace of $S$ for each $n$ with $1 \leq n \leq 4$.

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Paper 4, Section I

1G Linear Algebra

Show that every endomorphism of a finite-dimensional vector space satisfies some polynomial, and define the minimal polynomial of such an endomorphism.

Give a linear transformation of an eight-dimensional complex vector space which has minimal polynomial $x^2(x-1)^3$. 

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Part IB, 2009 List of Questions [TURN OVER
Paper 1, Section II
9G Linear Algebra
Define the dual of a vector space V. State and prove a formula for its dimension.

Let V be the vector space of real polynomials of degree at most n. If \{a_0, \ldots, a_n\} are distinct real numbers, prove that there are unique real numbers \{\lambda_0, \ldots, \lambda_n\} with

$$\frac{dp}{dx}(0) = \sum_{j=0}^{n} \lambda_j p(a_j)$$

for every \(p(x) \in V\).

Paper 2, Section II
10G Linear Algebra
Let V be a finite-dimensional vector space and let \(T : V \to V\) be an endomorphism of V. Show that there is a positive integer l such that \(V = \ker(T^l) \oplus \im(T^l)\). Hence, or otherwise, show that if \(T\) has zero determinant there is some non-zero endomorphism \(S\) with \(TS = 0 = ST\).

Suppose \(T_1\) and \(T_2\) are endomorphisms of V for which \(T_1^2 = T_1, i = 1, 2\). Show that \(T_1\) is similar to \(T_2\) if and only if they have the same rank.

Paper 3, Section II
10G Linear Algebra
For each of the following, provide a proof or counterexample.

1. If \(A, B\) are complex \(n \times n\) matrices and \(AB = BA\), then \(A\) and \(B\) have a common eigenvector.

2. If \(A, B\) are complex \(n \times n\) matrices and \(AB = BA\), then \(A\) and \(B\) have a common eigenvalue.

3. If \(A, B\) are complex \(n \times n\) matrices and \((AB)^n = 0\) then \((BA)^n = 0\).

4. If \(T : V \to V\) is an endomorphism of a finite-dimensional vector space V and \(\lambda\) is an eigenvalue of \(T\), then the dimension of \(\{v \in V \mid (T - \lambda I)v = 0\}\) equals the multiplicity of \(\lambda\) as a root of the minimal polynomial of \(T\).

5. If \(T : V \to V\) is an endomorphism of a finite-dimensional complex vector space V, \(\lambda\) is an eigenvalue of \(T\), and \(W_i = \{v \in V \mid (T - \lambda I)^i(v) = 0\}\), then \(W_c = W_{c+1}\) where \(c\) is the multiplicity of \(\lambda\) as a root of the minimal polynomial of \(T\).
What does it mean to say two real symmetric bilinear forms $A$ and $B$ on a vector space $V$ are congruent?

State and prove Sylvester’s law of inertia, and deduce that the rank and signature determine the congruence class of a real symmetric bilinear form. [You may use without proof a result on diagonalisability of real symmetric matrices, provided it is clearly stated.]

How many congruence classes of symmetric bilinear forms on a real $n$-dimensional vector space are there? Such a form $\psi$ defines a family of subsets $\{x \in \mathbb{R}^n \mid \psi(x, x) = t\}$, for $t \in \mathbb{R}$. For how many of the congruence classes are these associated subsets all bounded subsets of $\mathbb{R}^n$? Is the quadric surface

$$\{3x^2 + 6y^2 + 5z^2 + 4xy + 2xz + 8yz = 1\}$$

a bounded or unbounded subset of $\mathbb{R}^3$? Justify your answers.
Linear Algebra

Let $A$ be an $n \times n$ matrix over $\mathbb{C}$. What does it mean to say that $\lambda$ is an eigenvalue of $A$? Show that $A$ has at least one eigenvalue. For each of the following statements, provide a proof or a counterexample as appropriate.

(i) If $A$ is Hermitian, all eigenvalues of $A$ are real.
(ii) If all eigenvalues of $A$ are real, $A$ is Hermitian.
(iii) If all entries of $A$ are real and positive, all eigenvalues of $A$ have positive real part.
(iv) If $A$ and $B$ have the same trace and determinant then they have the same eigenvalues.

Linear Algebra

Let $A$ be an $m \times n$ matrix of real numbers. Define the row rank and column rank of $A$ and show that they are equal.

Show that if a matrix $A'$ is obtained from $A$ by elementary row and column operations then $\text{rank}(A') = \text{rank}(A)$.

Let $P, Q$ and $R$ be $n \times n$ matrices. Show that the $2n \times 2n$ matrices

\[
\begin{pmatrix}
PQ & 0 \\
Q & QR
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & PQR \\
Q & 0
\end{pmatrix}
\]

have the same rank.

Hence, or otherwise, prove that

\[
\text{rank}(PQ) + \text{rank}(QR) \leq \text{rank}(Q) + \text{rank}(PQR).
\]

Linear Algebra

Suppose that $V$ and $W$ are finite-dimensional vector spaces over $\mathbb{R}$. What does it mean to say that $\psi : V \to W$ is a linear map? State the rank-nullity formula. Using it, or otherwise, prove that a linear map $\psi : V \to V$ is surjective if, and only if, it is injective.

Suppose that $\psi : V \to V$ is a linear map which has a right inverse, that is to say there is a linear map $\phi : V \to V$ such that $\psi \phi = \text{id}_V$, the identity map. Show that $\phi \psi = \text{id}_V$.

Suppose that $A$ and $B$ are two $n \times n$ matrices over $\mathbb{R}$ such that $AB = I$. Prove that $BA = I$. 
2/II/10E  Linear Algebra

Define the determinant $\det(A)$ of an $n \times n$ square matrix $A$ over the complex numbers. If $A$ and $B$ are two such matrices, show that $\det(AB) = \det(A)\det(B)$.

Write $p_M(\lambda) = \det(M - \lambda I)$ for the characteristic polynomial of a matrix $M$. Let $A, B, C$ be $n \times n$ matrices and suppose that $C$ is nonsingular. Show that $p_{BC} = p_{CB}$.

Show that if $p_A = p_B$ then $\text{tr}(A) = \text{tr}(B)$. Which of the following statements is true for all $n \times n$ matrices $A, B, C$? Justify your answers.

(i) $p_{ABC} = p_{ACB}$;
(ii) $p_{ABC} = p_{BCA}$.

3/II/10E  Linear Algebra

Let $k = \mathbb{R}$ or $\mathbb{C}$. What is meant by a quadratic form $q : k^n \to k$? Show that there is a basis $\{v_1, \ldots, v_n\}$ for $k^n$ such that, writing $x = x_1 v_1 + \ldots + x_n v_n$, we have $q(x) = a_1 x_1^2 + \ldots + a_n x_n^2$ for some scalars $a_1, \ldots, a_n \in \{-1, 0, 1\}$.

Suppose that $k = \mathbb{R}$. Define the rank and signature of $q$ and compute these quantities for the form $q : \mathbb{R}^3 \to \mathbb{R}$ given by $q(x) = -3x_1^2 + x_2^2 + 2x_1 x_2 - 2x_1 x_3 + 2x_2 x_3$.

Suppose now that $k = \mathbb{C}$ and that $q_1, \ldots, q_d : \mathbb{C}^n \to \mathbb{C}$ are quadratic forms. If $n \geq 2^d$, show that there is some nonzero $x \in \mathbb{C}^n$ such that $q_1(x) = \ldots = q_d(x) = 0$.

4/1/1E  Linear Algebra

Describe (without proof) what it means to put an $n \times n$ matrix of complex numbers into Jordan normal form. Explain (without proof) the sense in which the Jordan normal form is unique.

Put the following matrix in Jordan normal form:

$$
\begin{pmatrix}
-7 & 3 & -5 \\
7 & -1 & 5 \\
17 & -6 & 12
\end{pmatrix}.
$$
What is meant by a Hermitian matrix? Show that if $A$ is Hermitian then all its eigenvalues are real and that there is an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of $A$.

A Hermitian matrix is said to be positive definite if $\langle Ax, x \rangle > 0$ for all $x \neq 0$. We write $A > 0$ in this case. Show that $A$ is positive definite if, and only if, all of its eigenvalues are positive. Show that if $A > 0$ then $A$ has a unique positive definite square root $\sqrt{A}$.

Let $A, B$ be two positive definite Hermitian matrices with $A - B > 0$. Writing $C = \sqrt{A}$ and $X = \sqrt{A} - \sqrt{B}$, show that $CX + XC > 0$. By considering eigenvalues of $X$, or otherwise, show that $X > 0$. 

4/II/10E  Linear Algebra
1/I/1G  Linear Algebra
Suppose that \( \{e_1, \ldots, e_3\} \) is a basis of the complex vector space \( \mathbb{C}^3 \) and that \( A : \mathbb{C}^3 \to \mathbb{C}^3 \) is the linear operator defined by \( A(e_1) = e_2, A(e_2) = e_3, \) and \( A(e_3) = e_1. \)

By considering the action of \( A \) on column vectors of the form \( (1, \xi, \xi^2)^T \), where \( \xi^3 = 1 \), or otherwise, find the diagonalization of \( A \) and its characteristic polynomial.

1/II/9G  Linear Algebra
State and prove Sylvester’s law of inertia for a real quadratic form.
[You may assume that for each real symmetric matrix \( A \) there is an orthogonal matrix \( U \), such that \( U^{-1}AU \) is diagonal.]

Suppose that \( V \) is a real vector space of even dimension \( 2m \), that \( Q \) is a non-singular quadratic form on \( V \) and that \( U \) is an \( m \)-dimensional subspace of \( V \) on which \( Q \) vanishes. What is the signature of \( Q \)?

2/I/1G  Linear Algebra
Suppose that \( S,T \) are endomorphisms of the 3-dimensional complex vector space \( \mathbb{C}^3 \) and that the eigenvalues of each of them are 1, 2, 3. What are their characteristic and minimal polynomials? Are they conjugate?

2/II/10G  Linear Algebra
Suppose that \( P \) is the complex vector space of complex polynomials in one variable, \( z. \)

(i) Show that the form \( \langle , \rangle \) defined by

\[
\langle f, g \rangle = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) \cdot \overline{g(e^{i\theta})} \, d\theta
\]

is a positive definite Hermitian form on \( P. \)

(ii) Find an orthonormal basis of \( P \) for this form, in terms of the powers of \( z. \)

(iii) Generalize this construction to complex vector spaces of complex polynomials in any finite number of variables.
3/II/10G  Linear Algebra
(i) Define the terms row-rank, column-rank and rank of a matrix, and state a relation between them.
(ii) Fix positive integers $m$, $n$, $p$ with $m, n \geq p$. Suppose that $A$ is an $m \times p$ matrix and $B$ a $p \times n$ matrix. State and prove the best possible upper bound on the rank of the product $AB$.

4/I/1G  Linear Algebra
Suppose that $\alpha : V \to W$ is a linear map of finite-dimensional complex vector spaces. What is the dual map $\alpha^*$ of the dual vector spaces?
Suppose that we choose bases of $V, W$ and take the corresponding dual bases of the dual vector spaces. What is the relation between the matrices that represent $\alpha$ and $\alpha^*$ with respect to these bases? Justify your answer.

4/II/10G  Linear Algebra
(i) State and prove the Cayley–Hamilton theorem for square complex matrices.
(ii) A square matrix $A$ is of order $n$ for a strictly positive integer $n$ if $A^n = I$ and no smaller positive power of $A$ is equal to $I$.
Determine the order of a complex $2 \times 2$ matrix $A$ of trace zero and determinant 1.
1/I/1H Linear Algebra

Define what is meant by the minimal polynomial of a complex \( n \times n \) matrix, and show that it is unique. Deduce that the minimal polynomial of a real \( n \times n \) matrix has real coefficients.

For \( n > 2 \), find an \( n \times n \) matrix with minimal polynomial \((t - 1)^2(t + 1)\).

1/II/9H Linear Algebra

Let \( U, V \) be finite-dimensional vector spaces, and let \( \theta \) be a linear map of \( U \) into \( V \). Define the rank \( r(\theta) \) and the nullity \( n(\theta) \) of \( \theta \), and prove that

\[
r(\theta) + n(\theta) = \dim U.
\]

Now let \( \theta, \phi \) be endomorphisms of a vector space \( U \). Define the endomorphisms \( \theta + \phi \) and \( \theta\phi \), and prove that

\[
r(\theta + \phi) \leq r(\theta) + r(\phi)
\]

\[
n(\theta\phi) \leq n(\theta) + n(\phi).
\]

Prove that equality holds in both inequalities if and only if \( \theta + \phi \) is an isomorphism and \( \theta\phi \) is zero.

2/I/1E Linear Algebra

State Sylvester’s law of inertia.

Find the rank and signature of the quadratic form \( q \) on \( \mathbb{R}^n \) given by

\[
q(x_1, \ldots, x_n) = \left( \sum_{i=1}^{n} x_i \right)^2 - \sum_{i=1}^{n} x_i^2.
\]
2/II/10E  Linear Algebra

Suppose that $V$ is the set of complex polynomials of degree at most $n$ in the variable $x$. Find the dimension of $V$ as a complex vector space.

Define
\[ e_k : V \to \mathbb{C} \quad \text{by} \quad e_k(\phi) = \frac{d^k \phi}{dx^k}(0). \]

Find a subset of $\{e_k \mid k \in \mathbb{N}\}$ that is a basis of the dual vector space $V^*$. Find the corresponding dual basis of $V$.

Define
\[ D : V \to V \quad \text{by} \quad D(\phi) = \frac{d\phi}{dx}. \]

Write down the matrix of $D$ with respect to the basis of $V$ that you have just found, and the matrix of the map dual to $D$ with respect to the dual basis.

3/II/10H  Linear Algebra

(a) Define what is meant by the trace of a complex $n \times n$ matrix $A$. If $T$ denotes an $n \times n$ invertible matrix, show that $A$ and $TAT^{-1}$ have the same trace.

(b) If $\lambda_1, \ldots, \lambda_r$ are distinct non-zero complex numbers, show that the endomorphism of $\mathbb{C}^n$ defined by the matrix
\[ \Lambda = \begin{pmatrix} \lambda_1 & \ldots & \lambda_1^r \\ \vdots & \ddots & \vdots \\ \lambda_r & \ldots & \lambda_r^r \end{pmatrix} \]
has trivial kernel, and hence that the same is true for the transposed matrix $\Lambda^t$.

For arbitrary complex numbers $\lambda_1, \ldots, \lambda_n$, show that the vector $(1, \ldots, 1)^t$ is not in the kernel of the endomorphism of $\mathbb{C}^n$ defined by the matrix
\[ \begin{pmatrix} \lambda_1 & \ldots & \lambda_n \\ \vdots & \ddots & \vdots \\ \lambda_1^n & \ldots & \lambda_n^n \end{pmatrix}, \]
unless all the $\lambda_i$ are zero.

[Hint: reduce to the case when $\lambda_1, \ldots, \lambda_r$ are distinct non-zero complex numbers, with $r \leq n$, and each $\lambda_j$ for $j > r$ is either zero or equal to some $\lambda_i$ with $i \leq r$. If the kernel of the endomorphism contains $(1, \ldots, 1)^t$, show that it also contains a vector of the form $(m_1, \ldots, m_r, 0, \ldots, 0)^t$ with the $m_i$ strictly positive integers.]

(c) Assuming the fact that any complex $n \times n$ matrix is conjugate to an upper-triangular one, prove that if $A$ is an $n \times n$ matrix such that $A^k$ has zero trace for all $1 \leq k \leq n$, then $A^n = 0$. 

Part IB  2006
4/I/1H  Linear Algebra

Suppose $V$ is a vector space over a field $k$. A finite set of vectors is said to be a basis for $V$ if it is both linearly independent and spanning. Prove that any two finite bases for $V$ have the same number of elements.

4/II/10E  Linear Algebra

Suppose that $\alpha$ is an orthogonal endomorphism of the finite-dimensional real inner product space $V$. Suppose that $V$ is decomposed as a direct sum of mutually orthogonal $\alpha$-invariant subspaces. How small can these subspaces be made, and how does $\alpha$ act on them? Justify your answer.

Describe the possible matrices for $\alpha$ with respect to a suitably chosen orthonormal basis of $V$ when $\dim V = 3$. 
1/1/1C  Linear Algebra

Let $V$ be an $n$-dimensional vector space over $\mathbb{R}$, and let $\beta : V \to V$ be a linear map. Define the minimal polynomial of $\beta$. Prove that $\beta$ is invertible if and only if the constant term of the minimal polynomial of $\beta$ is non-zero.

1/II/9C  Linear Algebra

Let $V$ be a finite dimensional vector space over $\mathbb{R}$, and $V^*$ be the dual space of $V$. If $W$ is a subspace of $V$, we define the subspace $\alpha(W)$ of $V^*$ by

$$
\alpha(W) = \{ f \in V^* : f(w) = 0 \text{ for all } w \text{ in } W \}.
$$

Prove that $\dim(\alpha(W)) = \dim(V) - \dim(W)$. Deduce that, if $A = (a_{ij})$ is any real $m \times n$-matrix of rank $r$, the equations

$$
\sum_{j=1}^{n} a_{ij} x_j = 0 \quad (i = 1, \ldots, m)
$$

have $n - r$ linearly independent solutions in $\mathbb{R}^n$.

2/1/1C  Linear Algebra

Let $\Omega$ be the set of all $2 \times 2$ matrices of the form $\alpha = aI + bJ + cK + dL$, where $a, b, c, d$ are in $\mathbb{R}$, and

$$
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad (i^2 = -1).
$$

Prove that $\Omega$ is closed under multiplication and determine its dimension as a vector space over $\mathbb{R}$. Prove that

$$
(aI + bJ + cK + dL)(aI - bJ - cK - dL) = (a^2 + b^2 + c^2 + d^2)I,
$$

and deduce that each non-zero element of $\Omega$ is invertible.
2/II/10C  Linear Algebra

(i) Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in $\mathbb{C}$. Define the determinant of $A$, the cofactor of each $a_{ij}$, and the adjugate matrix $\text{adj}(A)$. Assuming the expansion of the determinant of a matrix in terms of its cofactors, prove that

$$\text{adj}(A) A = \det(A) I_n,$$

where $I_n$ is the $n \times n$ identity matrix.

(ii) Let

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Show the eigenvalues of $A$ are $\pm 1, \pm i$, where $i^2 = -1$, and determine the diagonal matrix to which $A$ is similar. For each eigenvalue, determine a non-zero eigenvector.

3/II/10B  Linear Algebra

Let $S$ be the vector space of functions $f : \mathbb{R} \to \mathbb{R}$ such that the $n$th derivative of $f$ is defined and continuous for every $n \geq 0$. Define linear maps $A, B : S \to S$ by $A(f) = df/dx$ and $B(f)(x) = xf(x)$. Show that

$$[A, B] = 1_S,$$

where in this question $[A, B]$ means $AB - BA$ and $1_S$ is the identity map on $S$.

Now let $V$ be any real vector space with linear maps $A, B : V \to V$ such that $[A, B] = 1_V$. Suppose that there is a nonzero element $y \in V$ with $Ay = 0$. Let $W$ be the subspace of $V$ spanned by $y$, $By$, $B^2y$, and so on. Show that $A(By)$ is in $W$ and give a formula for it. More generally, show that $A(B^iy)$ is in $W$ for each $i \geq 0$, and give a formula for it.

Show, using your formula or otherwise, that $\{y, By, B^2y, \ldots\}$ are linearly independent. (Or, equivalently: show that $y, By, B^2y, \ldots, B^n y$ are linearly independent for every $n \geq 0$.)

4/I/1B  Linear Algebra

Define what it means for an $n \times n$ complex matrix to be unitary or Hermitian. Show that every eigenvalue of a Hermitian matrix is real. Show that every eigenvalue of a unitary matrix has absolute value 1.

Show that two eigenvectors of a Hermitian matrix that correspond to different eigenvalues are orthogonal, using the standard inner product on $\mathbb{C}^n$. 

Part IB 2005
Linear Algebra

(i) Let $V$ be a finite-dimensional real vector space with an inner product. Let $e_1, \ldots, e_n$ be a basis for $V$. Prove by an explicit construction that there is an orthonormal basis $f_1, \ldots, f_n$ for $V$ such that the span of $e_1, \ldots, e_i$ is equal to the span of $f_1, \ldots, f_i$ for every $1 \leq i \leq n$.

(ii) For any real number $a$, consider the quadratic form

$$q_a(x, y, z) = xy + yz + zx + ax^2$$

on $\mathbb{R}^3$. For which values of $a$ is $q_a$ nondegenerate? When $q_a$ is nondegenerate, compute its signature in terms of $a$. 

Part IB 2005
1/I/1H Linear Algebra

Suppose that \( \{e_1, \ldots, e_{r+1}\} \) is a linearly independent set of distinct elements of a vector space \( V \) and \( \{e_1, \ldots, e_r, f_{r+1}, \ldots, f_m\} \) spans \( V \). Prove that \( f_{r+1}, \ldots, f_m \) may be reordered, as necessary, so that \( \{e_1, \ldots, e_r, f_{r+2}, \ldots, f_m\} \) spans \( V \).

Suppose that \( \{e_1, \ldots, e_n\} \) is a linearly independent set of distinct elements of \( V \) and that \( \{f_1, \ldots, f_m\} \) spans \( V \). Show that \( n \leq m \).

1/II/12H Linear Algebra

Let \( U \) and \( W \) be subspaces of the finite-dimensional vector space \( V \). Prove that both the sum \( U + W \) and the intersection \( U \cap W \) are subspaces of \( V \). Prove further that
\[
\dim U + \dim W = \dim (U + W) + \dim (U \cap W).
\]

Let \( U, W \) be the kernels of the maps \( A, B : \mathbb{R}^4 \to \mathbb{R}^2 \) given by the matrices \( A \) and \( B \) respectively, where
\[
A = \begin{pmatrix} 1 & 2 & -1 & -3 \\ -1 & 1 & 2 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & -4 \end{pmatrix}.
\]

Find a basis for the intersection \( U \cap W \), and extend this first to a basis of \( U \), and then to a basis of \( U + W \).

2/I/1E Linear Algebra

For each \( n \) let \( A_n \) be the \( n \times n \) matrix defined by
\[
(A_n)_{ij} = \begin{cases} i & i \leq j, \\ j & i > j. \end{cases}
\]

What is \( \det A_n \)? Justify your answer.

[It may be helpful to look at the cases \( n = 1, 2, 3 \) before tackling the general case.]
2/II/12E  Linear Algebra

Let $Q$ be a quadratic form on a real vector space $V$ of dimension $n$. Prove that there is a basis $e_1, \ldots, e_n$ with respect to which $Q$ is given by the formula

$$Q\left(\sum_{i=1}^{n} x_i e_i\right) = x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2.$$

Prove that the numbers $p$ and $q$ are uniquely determined by the form $Q$. By means of an example, show that the subspaces $\langle e_1, \ldots, e_p \rangle$ and $\langle e_{p+1}, \ldots, e_{p+q} \rangle$ need not be uniquely determined by $Q$.

3/I/1E  Linear Algebra

Let $V$ be a finite-dimensional vector space over $\mathbb{R}$. What is the dual space of $V$? Prove that the dimension of the dual space is the same as that of $V$.

3/II/13E  Linear Algebra

(i) Let $V$ be an $n$-dimensional vector space over $\mathbb{C}$ and let $\alpha : V \to V$ be an endomorphism. Suppose that the characteristic polynomial of $\alpha$ is $\Pi_{i=1}^{k} (x - \lambda_i)^{n_i}$, where the $\lambda_i$ are distinct and $n_i > 0$ for every $i$.

Describe all possibilities for the minimal polynomial and prove that there are no further ones.

(ii) Give an example of a matrix for which both the characteristic and the minimal polynomial are $(x - 1)^3(x - 3)$.

(iii) Give an example of two matrices $A, B$ with the same rank and the same minimal and characteristic polynomials such that there is no invertible matrix $P$ with $PAP^{-1} = B$.

4/I/1E  Linear Algebra

Let $V$ be a real $n$-dimensional inner-product space and let $W \subset V$ be a $k$-dimensional subspace. Let $e_1, \ldots, e_k$ be an orthonormal basis for $W$. In terms of this basis, give a formula for the orthogonal projection $\pi : V \to W$.

Let $v \in V$. Prove that $\pi v$ is the closest point in $W$ to $v$.

[You may assume that the sequence $e_1, \ldots, e_k$ can be extended to an orthonormal basis $e_1, \ldots, e_n$ of $V$.]
Linear Algebra

(i) Let $V$ be an $n$-dimensional inner-product space over $\mathbb{C}$ and let $\alpha : V \to V$ be a Hermitian linear map. Prove that $V$ has an orthonormal basis consisting of eigenvectors of $\alpha$.

(ii) Let $\beta : V \to V$ be another Hermitian map. Prove that $\alpha \beta$ is Hermitian if and only if $\alpha \beta = \beta \alpha$.

(iii) A Hermitian map $\alpha$ is positive-definite if $\langle \alpha v, v \rangle > 0$ for every non-zero vector $v$. If $\alpha$ is a positive-definite Hermitian map, prove that there is a unique positive-definite Hermitian map $\beta$ such that $\beta^2 = \alpha$. 
1/I/5E  Linear Mathematics

Let $V$ be the subset of $\mathbb{R}^5$ consisting of all quintuples $(a_1, a_2, a_3, a_4, a_5)$ such that

$$ a_1 + a_2 + a_3 + a_4 + a_5 = 0 $$

and

$$ a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0. $$

Prove that $V$ is a subspace of $\mathbb{R}^5$. Solve the above equations for $a_1$ and $a_2$ in terms of $a_3, a_4$ and $a_5$. Hence, exhibit a basis for $V$, explaining carefully why the vectors you give form a basis.

1/II/14E  Linear Mathematics

(a) Let $U, U'$ be subspaces of a finite-dimensional vector space $V$. Prove that $\dim(U + U') = \dim U + \dim U' - \dim(U \cap U')$.

(b) Let $V$ and $W$ be finite-dimensional vector spaces and let $\alpha$ and $\beta$ be linear maps from $V$ to $W$. Prove that

$$ \text{rank}(\alpha + \beta) \leq \text{rank} \alpha + \text{rank} \beta. $$

(c) Deduce from this result that

$$ \text{rank}(\alpha + \beta) \geq |\text{rank} \alpha - \text{rank} \beta|. $$

(d) Let $V = W = \mathbb{R}^n$ and suppose that $1 \leq r \leq s \leq n$. Exhibit linear maps $\alpha, \beta : V \to W$ such that $\text{rank} \alpha = r$, $\text{rank} \beta = s$ and $\text{rank}(\alpha + \beta) = s - r$. Suppose that $r + s \geq n$. Exhibit linear maps $\alpha, \beta : V \to W$ such that $\text{rank} \alpha = r$, $\text{rank} \beta = s$ and $\text{rank}(\alpha + \beta) = n$.

2/I/6E  Linear Mathematics

Let $a_1, a_2, \ldots, a_n$ be distinct real numbers. For each $i$ let $v_i$ be the vector $(1, a_i, a_i^2, \ldots, a_i^{n-1})$. Let $A$ be the $n \times n$ matrix with rows $v_1, v_2, \ldots, v_n$ and let $c$ be a column vector of size $n$. Prove that $Ac = 0$ if and only if $c = 0$. Deduce that the vectors $v_1, v_2, \ldots, v_n$ span $\mathbb{R}^n$.

[You may use general facts about matrices if you state them clearly.]
2/II/15 E  Linear Mathematics

(a) Let \( A = (a_{ij}) \) be an \( m \times n \) matrix and for each \( k \leq n \) let \( A_k \) be the \( m \times k \) matrix formed by the first \( k \) columns of \( A \). Suppose that \( n > m \). Explain why the nullity of \( A \) is non-zero. Prove that if \( k \) is minimal such that \( A_k \) has non-zero nullity, then the nullity of \( A_k \) is \( k \).

(b) Suppose that no column of \( A \) consists entirely of zeros. Deduce from (a) that there exist scalars \( b_1, \ldots, b_k \) (where \( k \) is defined as in (a)) such that \( \sum_{j=1}^{k} a_{ij} b_j = 0 \) for every \( i \leq m \), but whenever \( \lambda_1, \ldots, \lambda_k \) are distinct real numbers there is some \( i \leq m \) such that \( \sum_{j=1}^{k} a_{ij} \lambda_j b_j \neq 0 \).

(c) Now let \( v_1, v_2, \ldots, v_m \) and \( w_1, w_2, \ldots, w_m \) be bases for the same real \( m \)-dimensional vector space. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be distinct real numbers such that for every \( j \) the vectors \( v_1 + \lambda_j w_1, \ldots, v_m + \lambda_j w_m \) are linearly dependent. For each \( j \), let \( a_{1j}, \ldots, a_{mj} \) be scalars, not all zero, such that \( \sum_{i=1}^{m} a_{ij} (v_i + \lambda_j w_i) = 0 \). By applying the result of (b) to the matrix \((a_{ij})\), deduce that \( n \leq m \).

(d) It follows that the vectors \( v_1 + \lambda w_1, \ldots, v_m + \lambda w_m \) are linearly dependent for at most \( m \) values of \( \lambda \). Explain briefly how this result can also be proved using determinants.

3/I/7 G  Linear Mathematics

Let \( \alpha \) be an endomorphism of a finite-dimensional real vector space \( U \) and let \( \beta \) be another endomorphism of \( U \) that commutes with \( \alpha \). If \( \lambda \) is an eigenvalue of \( \alpha \), show that \( \beta \) maps the kernel of \( \alpha - \lambda I \) into itself, where \( I \) is the identity map. Suppose now that \( \alpha \) is diagonalizable with \( n \) distinct real eigenvalues where \( n = \dim U \). Prove that if there exists an endomorphism \( \beta \) of \( U \) such that \( \alpha = \beta^2 \), then \( \lambda \geq 0 \) for all eigenvalues \( \lambda \) of \( \alpha \).

3/II/17 G  Linear Mathematics

Define the \textit{determinant} \( \det(A) \) of an \( n \times n \) complex matrix \( A \). Let \( A_1, \ldots, A_n \) be the columns of \( A \), let \( \sigma \) be a permutation of \( \{1, \ldots, n\} \) and let \( A^\sigma \) be the matrix whose columns are \( A_{\sigma(1)}, \ldots, A_{\sigma(n)} \). Prove from your definition of determinant that \( \det(A^\sigma) = \epsilon(\sigma) \det(A) \), where \( \epsilon(\sigma) \) is the sign of the permutation \( \sigma \). Prove also that \( \det(A) = \det(A^t) \).

Define the \textit{adjugate} matrix \( \text{adj}(A) \) and prove from your definitions that \( A \text{adj}(A) = \text{adj}(A) A = \det(A) I \), where \( I \) is the identity matrix. Hence or otherwise, prove that if \( \det(A) \neq 0 \), then \( A \) is invertible.

Let \( C \) and \( D \) be real \( n \times n \) matrices such that the complex matrix \( C + iD \) is invertible. By considering \( \det(C + \lambda D) \) as a function of \( \lambda \) or otherwise, prove that there exists a real number \( \lambda \) such that \( C + \lambda D \) is invertible. [\textit{You may assume that if a matrix \( A \) is invertible, then \( \det(A) \neq 0 \).}]

Deduce that if two real matrices \( A \) and \( B \) are such that there exists an invertible complex matrix \( P \) with \( P^{-1} A P = B \), then there exists an invertible \textbf{real} matrix \( Q \) such that \( Q^{-1} A Q = B \).
4/1/6G  Linear Mathematics

Let \( \alpha \) be an endomorphism of a finite-dimensional real vector space \( U \) such that \( \alpha^2 = \alpha \). Show that \( U \) can be written as the direct sum of the kernel of \( \alpha \) and the image of \( \alpha \). Hence or otherwise, find the characteristic polynomial of \( \alpha \) in terms of the dimension of \( U \) and the rank of \( \alpha \). Is \( \alpha \) diagonalizable? Justify your answer.

4/II/15G  Linear Mathematics

Let \( \alpha \in L(U,V) \) be a linear map between finite-dimensional vector spaces. Let

\[
M^I(\alpha) = \{ \beta \in L(V,U) : \beta \alpha = 0 \}
\]
and

\[
M^R(\alpha) = \{ \beta \in L(V,U) : \alpha \beta = 0 \}.
\]

(a) Prove that \( M^I(\alpha) \) and \( M^R(\alpha) \) are subspaces of \( L(V,U) \) of dimensions

\[
\dim M^I(\alpha) = (\dim V - \text{rank } \alpha) \dim U \quad \text{and}
\]
\[
\dim M^R(\alpha) = \dim \ker(\alpha) \dim V.
\]

[You may use the result that there exist bases in \( U \) and \( V \) so that \( \alpha \) is represented by

\[
\begin{pmatrix}
I_r & 0 \\
0 & 0
\end{pmatrix},
\]

where \( I_r \) is the \( r \times r \) identity matrix and \( r \) is the rank of \( \alpha \).]

(b) Let \( \Phi : L(U,V) \to L(V^*,U^*) \) be given by \( \Phi(\alpha) = \alpha^* \), where \( \alpha^* \) is the dual map induced by \( \alpha \). Prove that \( \Phi \) is an isomorphism. [You may assume that \( \Phi \) is linear, and you may use the result that a finite-dimensional vector space and its dual have the same dimension.]

(c) Prove that

\[
\Phi(M^I(\alpha)) = M^*(\alpha^*) \quad \text{and} \quad \Phi(M^R(\alpha)) = M^I(\alpha^*).
\]

[You may use the results that \( (\beta \alpha)^* = \alpha^* \beta^* \) and that \( \beta^{**} \) can be identified with \( \beta \) under the canonical isomorphism between a vector space and its double dual.]

(d) Conclude that \( \text{rank}(\alpha) = \text{rank}(\alpha^*) \).
1/I/8G Quadratic Mathematics

Let $U$ and $V$ be finite-dimensional vector spaces. Suppose that $b$ and $c$ are bilinear forms on $U \times V$ and that $b$ is non-degenerate. Show that there exist linear endomorphisms $S$ of $U$ and $T$ of $V$ such that $c(x, y) = b(S(x), y) = b(x, T(y))$ for all $(x, y) \in U \times V$.

1/II/17G Quadratic Mathematics

(a) Suppose $p$ is an odd prime and $a$ an integer coprime to $p$. Define the Legendre symbol $\left( \frac{a}{p} \right)$ and state Euler’s criterion.

(b) Compute $\left( \frac{-1}{p} \right)$ and prove that

$$ \left( \frac{ab}{p} \right) = \left( \frac{a}{p} \right) \left( \frac{b}{p} \right) $$

whenever $a$ and $b$ are coprime to $p$.

(c) Let $n$ be any integer such that $1 \leq n \leq p - 2$. Let $m$ be the unique integer such that $1 \leq m \leq p - 2$ and $mn \equiv 1 \pmod{p}$. Prove that

$$ \left( \frac{n(n+1)}{p} \right) = \left( \frac{1 + m}{p} \right). $$

(d) Find

$$ \sum_{n=1}^{p-2} \left( \frac{n(n+1)}{p} \right). $$

2/I/8G Quadratic Mathematics

Let $U$ be a finite-dimensional real vector space and $b$ a positive definite symmetric bilinear form on $U \times U$. Let $\psi : U \to U$ be a linear map such that $b(\psi(x), y) + b(x, \psi(y)) = 0$ for all $x$ and $y$ in $U$. Prove that if $\psi$ is invertible, then the dimension of $U$ must be even. By considering the restriction of $\psi$ to its image or otherwise, prove that the rank of $\psi$ is always even.
2/II/17G  Quadratic Mathematics

Let $S$ be the set of all $2 \times 2$ complex matrices $A$ which are hermitian, that is, $A^* = A$, where $A^* = \bar{A}$.

(a) Show that $S$ is a real 4-dimensional vector space. Consider the real symmetric bilinear form $b$ on this space defined by

$$b(A, B) = \frac{1}{2} \left( \text{tr}(AB) - \text{tr}(A) \text{tr}(B) \right) .$$

Prove that $b(A, A) = -\det A$ and $b(A, I) = -\frac{1}{2} \text{tr}(A)$, where $I$ denotes the identity matrix.

(b) Consider the three matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} .$$

Prove that the basis $I, A_1, A_2, A_3$ of $S$ diagonalizes $b$. Hence or otherwise find the rank and signature of $b$.

(c) Let $Q$ be the set of all $2 \times 2$ complex matrices $C$ which satisfy $C + C^* = \text{tr}(C) I$. Show that $Q$ is a real 4-dimensional vector space. Given $C \in Q$, put

$$\Phi(C) = \frac{1-i}{2} \text{tr}(C) I + i C .$$

Show that $\Phi$ takes values in $S$ and is a linear isomorphism between $Q$ and $S$.

(d) Define a real symmetric bilinear form on $Q$ by setting $c(C, D) = -\frac{1}{2} \text{tr}(CD)$, $C, D \in Q$. Show that $b(\Phi(C), \Phi(D)) = c(C, D)$ for all $C, D \in Q$. Find the rank and signature of the symmetric bilinear form $c$ defined on $Q$.

3/I/9G  Quadratic Mathematics

Let $f(x, y) = ax^2 + bxy + cy^2$ be a binary quadratic form with integer coefficients. Explain what is meant by the discriminant $d$ of $f$. State a necessary and sufficient condition for some form of discriminant $d$ to represent an odd prime number $p$. Using this result or otherwise, find the primes $p$ which can be represented by the form $x^2 + 3y^2$.  

Part IB 2003
Let $U$ be a finite-dimensional real vector space endowed with a positive definite inner product. A linear map $\tau : U \rightarrow U$ is said to be an *orthogonal projection* if $\tau$ is self-adjoint and $\tau^2 = \tau$.

(a) Prove that for every orthogonal projection $\tau$ there is an orthogonal decomposition

$$U = \ker(\tau) \oplus \text{im}(\tau).$$

(b) Let $\phi : U \rightarrow U$ be a linear map. Show that if $\phi^2 = \phi$ and $\phi \phi^* = \phi^* \phi$, where $\phi^*$ is the adjoint of $\phi$, then $\phi$ is an orthogonal projection. [You may find it useful to prove first that if $\phi \phi^* = \phi^* \phi$, then $\phi$ and $\phi^*$ have the same kernel.]

(c) Show that given a subspace $W$ of $U$ there exists a unique orthogonal projection $\tau$ such that $\text{im}(\tau) = W$. If $W_1$ and $W_2$ are two subspaces with corresponding orthogonal projections $\tau_1$ and $\tau_2$, show that $\tau_2 \circ \tau_1 = 0$ if and only if $W_1$ is orthogonal to $W_2$.

(d) Let $\phi : U \rightarrow U$ be a linear map satisfying $\phi^2 = \phi$. Prove that one can define a positive definite inner product on $U$ such that $\phi$ becomes an orthogonal projection.
1/I/5G Linear Mathematics

Define \( f : \mathbb{C}^3 \to \mathbb{C}^3 \) by
\[
f(a, b, c) = (a + 3b - c, 2b + c, -4b - c).
\]

Find the characteristic polynomial and the minimal polynomial of \( f \). Is \( f \) diagonalisable? Are \( f \) and \( f^2 \) linearly independent endomorphisms of \( \mathbb{C}^3 \)? Justify your answers.

1/II/14G Linear Mathematics

Let \( \alpha \) be an endomorphism of a vector space \( V \) of finite dimension \( n \).

(a) What is the dimension of the vector space of linear endomorphisms of \( V \)? Show that there exists a non-trivial polynomial \( p(X) \) such that \( p(\alpha) = 0 \). Define what is meant by the minimal polynomial \( m_\alpha \) of \( \alpha \).

(b) Show that the eigenvalues of \( \alpha \) are precisely the roots of the minimal polynomial of \( \alpha \).

(c) Let \( W \) be a subspace of \( V \) such that \( \alpha(W) \subseteq W \) and let \( \beta \) be the restriction of \( \alpha \) to \( W \). Show that \( m_\beta \) divides \( m_\alpha \).

(d) Give an example of an endomorphism \( \alpha \) and a subspace \( W \) as in (c) not equal to \( V \) for which \( m_\alpha = m_\beta \), and \( \deg(m_\alpha) > 1 \).

2/I/6G Linear Mathematics

Let \( A \) be a complex \( 4 \times 4 \) matrix such that \( A^3 = A^2 \). What are the possible minimal polynomials of \( A \)? If \( A \) is not diagonalisable and \( A^2 \neq 0 \), list all possible Jordan normal forms of \( A \).

2/II/15G Linear Mathematics

(a) A complex \( n \times n \) matrix is said to be unipotent if \( U - I \) is nilpotent, where \( I \) is the identity matrix. Show that \( U \) is unipotent if and only if 1 is the only eigenvalue of \( U \).

(b) Let \( T \) be an invertible complex matrix. By considering the Jordan normal form of \( T \) show that there exists an invertible matrix \( P \) such that
\[
PTP^{-1} = D_0 + N,
\]
where \( D_0 \) is an invertible diagonal matrix, \( N \) is an upper triangular matrix with zeros in the diagonal and \( D_0N = ND_0 \).

(c) Set \( D = P^{-1}D_0P \) and show that \( U = D^{-1}T \) is unipotent.

(d) Conclude that any invertible matrix \( T \) can be written as \( T = DU \) where \( D \) is diagonalisable, \( U \) is unipotent and \( DU = UD \).

Part IB
3/I/7F  Linear Mathematics

Which of the following statements are true, and which false? Give brief justifications for your answers.

(a) If $U$ and $W$ are subspaces of a vector space $V$, then $U \cap W$ is always a subspace of $V$.

(b) If $U$ and $W$ are distinct subspaces of a vector space $V$, then $U \cup W$ is never a subspace of $V$.

(c) If $U$, $W$ and $X$ are subspaces of a vector space $V$, then $U \cap (W + X) = (U \cap W) + (U \cap X)$.

(d) If $U$ is a subspace of a finite-dimensional space $V$, then there exists a subspace $W$ such that $U \cap W = \{0\}$ and $U + W = V$.

3/II/17F  Linear Mathematics

Define the determinant of an $n \times n$ matrix $A$, and prove from your definition that if $A'$ is obtained from $A$ by an elementary row operation (i.e. by adding a scalar multiple of the $i$th row of $A$ to the $j$th row, for some $j \neq i$), then $\det A' = \det A$.

Prove also that if $X$ is a $2n \times 2n$ matrix of the form

$$
\begin{pmatrix}
A & B \\
O & C
\end{pmatrix},
$$

where $O$ denotes the $n \times n$ zero matrix, then $\det X = \det A \det C$. Explain briefly how the $2n \times 2n$ matrix

$$
\begin{pmatrix}
B & I \\
O & A
\end{pmatrix}
$$

can be transformed into the matrix

$$
\begin{pmatrix}
B & I \\
-AB & O
\end{pmatrix}
$$

by a sequence of elementary row operations. Hence or otherwise prove that $\det AB = \det A \det B$.

4/I/6F  Linear Mathematics

Define the rank and nullity of a linear map between finite-dimensional vector spaces. State the rank–nullity formula.

Let $\alpha : U \to V$ and $\beta : V \to W$ be linear maps. Prove that

$$
\operatorname{rank}(\alpha) + \operatorname{rank}(\beta) - \dim V \leq \operatorname{rank}(\beta \alpha) \leq \min\{\operatorname{rank}(\alpha), \operatorname{rank}(\beta)\}.
$$
Define the dual space $V^*$ of a finite-dimensional real vector space $V$, and explain what is meant by the basis of $V^*$ dual to a given basis of $V$. Explain also what is meant by the statement that the second dual $V^{**}$ is naturally isomorphic to $V$.

Let $V_n$ denote the space of real polynomials of degree at most $n$. Show that, for any real number $x$, the function $e_x$ mapping $p$ to $p(x)$ is an element of $V_n^*$. Show also that, if $x_1, x_2, \ldots, x_{n+1}$ are distinct real numbers, then \{${e_{x_1}, e_{x_2}, \ldots, e_{x_{n+1}}}$\} is a basis of $V_n^*$, and find the basis of $V_n$ dual to it.

Deduce that, for any $(n + 1)$ distinct points $x_1, \ldots, x_{n+1}$ of the interval $[-1, 1]$, there exist scalars $\lambda_1, \ldots, \lambda_{n+1}$ such that

$$\int_{-1}^{1} p(t) \, dt = \sum_{i=1}^{n+1} \lambda_i p(x_i)$$

for all $p \in V_n$. For $n = 4$ and $(x_1, x_2, x_3, x_4, x_5) = (-1, -\frac{1}{2}, 0, \frac{1}{2}, 1)$, find the corresponding scalars $\lambda_i$.
1/I/8F Quadratic Mathematics

Define the rank and signature of a symmetric bilinear form \( \phi \) on a finite-dimensional real vector space. (If your definitions involve a matrix representation of \( \phi \), you should explain why they are independent of the choice of representing matrix.)

Let \( V \) be the space of all \( n \times n \) real matrices (where \( n \geq 2 \)), and let \( \phi \) be the bilinear form on \( V \) defined by 
\[
\phi(A, B) = \text{tr} AB - \text{tr} A \text{tr} B.
\]
Find the rank and signature of \( \phi \).

[Hint: You may find it helpful to consider the subspace of symmetric matrices having trace zero, and a suitable complement for this subspace.]

1/II/17F Quadratic Mathematics

Let \( A \) and \( B \) be \( n \times n \) real symmetric matrices, such that the quadratic form \( x^T A x \) is positive definite. Show that it is possible to find an invertible matrix \( P \) such that \( P^T A P = I \) and \( P^T B P \) is diagonal. Show also that the diagonal entries of the matrix \( P^T B P \) may be calculated directly from \( A \) and \( B \), without finding the matrix \( P \). If
\[
A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
find the diagonal entries of \( P^T B P \).

2/I/8F Quadratic Mathematics

Explain what is meant by a sesquilinear form on a complex vector space \( V \). If \( \phi \) and \( \psi \) are two such forms, and \( \phi(v, v) = \psi(v, v) \) for all \( v \in V \), prove that \( \phi(v, w) = \psi(v, w) \) for all \( v, w \in V \). Deduce that if \( \alpha : V \to V \) is a linear map satisfying \( \phi(\alpha(v), \alpha(v)) = \phi(v, v) \) for all \( v \in V \), then \( \phi(\alpha(v), \alpha(w)) = \phi(v, w) \) for all \( v, w \in V \).
Define the adjoint $\alpha^*$ of an endomorphism $\alpha$ of a complex inner-product space $V$. Show that if $W$ is a subspace of $V$, then $\alpha(W) \subseteq W$ if and only if $\alpha^*(W^\perp) \subseteq W^\perp$.

An endomorphism of a complex inner-product space is said to be normal if it commutes with its adjoint. Prove the following facts about a normal endomorphism $\alpha$ of a finite-dimensional space $V$.

(i) $\alpha$ and $\alpha^*$ have the same kernel.

(ii) $\alpha$ and $\alpha^*$ have the same eigenvectors, with complex conjugate eigenvalues.

(iii) If $E_\lambda = \{x \in V : \alpha(x) = \lambda x\}$, then $\alpha(E_\lambda^\perp) \subseteq E_\lambda^\perp$.

(iv) There is an orthonormal basis of $V$ consisting of eigenvectors of $\alpha$.

Deduce that an endomorphism $\alpha$ is normal if and only if it can be written as a product $\beta\gamma$, where $\beta$ is Hermitian, $\gamma$ is unitary and $\beta$ and $\gamma$ commute with each other. [Hint: Given $\alpha$, define $\beta$ and $\gamma$ in terms of their effect on the basis constructed in (iv).]

Explain what is meant by a quadratic residue modulo an odd prime $p$, and show that $a$ is a quadratic residue modulo $p$ if and only if $a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$. Hence characterize the odd primes $p$ for which $-1$ is a quadratic residue.

State the law of quadratic reciprocity, and use it to determine whether $73$ is a quadratic residue (mod $127$).

Explain what is meant by saying that a positive definite integral quadratic form $f(x, y) = ax^2 + bxy + cy^2$ is reduced, and show that every positive definite form is equivalent to a reduced form.

State a criterion for a prime number $p$ to be representable by some form of discriminant $d$, and deduce that $p$ is representable by a form of discriminant $-32$ if and only if $p \equiv 1, 2$ or $3 \pmod{8}$. Find the reduced forms of discriminant $-32$, and hence or otherwise show that a prime $p$ is representable by the form $3x^2 + 2xy + 3y^2$ if and only if $p \equiv 3 \pmod{8}$.

[Standard results on when $-1$ and $2$ are squares (mod $p$) may be assumed.]

Part IB
1/I/5C Linear Mathematics

Determine for which values of $x \in \mathbb{C}$ the matrix

$$M = \begin{pmatrix}
  x & 1 & 1 \\
  1-x & 0 & -1 \\
  2 & 2x & 1
\end{pmatrix}$$

is invertible. Determine the rank of $M$ as a function of $x$. Find the adjugate and hence the inverse of $M$ for general $x$.

1/II/14C Linear Mathematics

(a) Find a matrix $M$ over $\mathbb{C}$ with both minimal polynomial and characteristic polynomial equal to $(x - 2)^3(x + 1)^2$. Furthermore find two matrices $M_1$ and $M_2$ over $\mathbb{C}$ which have the same characteristic polynomial, $(x - 3)^5(x - 1)^2$, and the same minimal polynomial, $(x - 3)^2(x - 1)^2$, but which are not conjugate to one another. Is it possible to find a third such matrix, $M_3$, neither conjugate to $M_1$ nor to $M_2$? Justify your answer.

(b) Suppose $A$ is an $n \times n$ matrix over $\mathbb{R}$ which has minimal polynomial of the form $(x - \lambda_1)(x - \lambda_2)$ for distinct roots $\lambda_1 \neq \lambda_2$ in $\mathbb{R}$. Show that the vector space $V = \mathbb{R}^n$ on which $A$ defines an endomorphism $\alpha : V \to V$ decomposes as a direct sum into $V = \ker(\alpha - \lambda_1\iota) \oplus \ker(\alpha - \lambda_2\iota)$, where $\iota$ is the identity.

[Hint: Express $v \in V$ in terms of $(\alpha - \lambda_1\iota)(v)$ and $(\alpha - \lambda_2\iota)(v).$]

Now suppose that $A$ has minimal polynomial $(x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_m)$ for distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. By induction or otherwise show that

$$V = \ker(\alpha - \lambda_1\iota) \oplus \ker(\alpha - \lambda_2\iota) \oplus \ldots \oplus \ker(\alpha - \lambda_m\iota).$$

Use this last statement to prove that an arbitrary matrix $A \in M_{n \times n}(\mathbb{R})$ is diagonalizable if and only if all roots of its minimal polynomial lie in $\mathbb{R}$ and have multiplicity 1.

2/I/6C Linear Mathematics

Show that right multiplication by $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2 \times 2}(\mathbb{C})$ defines a linear transformation $\rho_A : M_{2 \times 2}(\mathbb{C}) \to M_{2 \times 2}(\mathbb{C})$. Find the matrix representing $\rho_A$ with respect to the basis

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

of $M_{2 \times 2}(\mathbb{C})$. Prove that the characteristic polynomial of $\rho_A$ is equal to the square of the characteristic polynomial of $A$, and that $A$ and $\rho_A$ have the same minimal polynomial.

Part IB
2/II/15C  **Linear Mathematics**

Define the dual $V^*$ of a vector space $V$. Given a basis $\{v_1, \ldots, v_n\}$ of $V$ define its dual and show it is a basis of $V^*$. For a linear transformation $\alpha : V \to W$ define the dual $\alpha^* : W^* \to V^*$.

Explain (with proof) how the matrix representing $\alpha : V \to W$ with respect to given bases of $V$ and $W$ relates to the matrix representing $\alpha^* : W^* \to V^*$ with respect to the corresponding dual bases of $V^*$ and $W^*$.

Prove that $\alpha$ and $\alpha^*$ have the same rank.

Suppose that $\alpha$ is an invertible endomorphism. Prove that $(\alpha^*)^{-1} = (\alpha^{-1})^*$.

3/1/7C  **Linear Mathematics**

Determine the dimension of the subspace $W$ of $\mathbb{R}^5$ spanned by the vectors
\[
\begin{pmatrix} 1 \\ 2 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 5 \\ 3 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 4 \\ 0 \\ 5 \\ -1 \end{pmatrix}.
\]

Write down a $5 \times 5$ matrix $M$ which defines a linear map $\mathbb{R}^5 \to \mathbb{R}^5$ whose image is $W$ and which contains $(1,1,1,1,1)^T$ in its kernel. What is the dimension of the space of all linear maps $\mathbb{R}^5 \to \mathbb{R}^5$ with $(1,1,1,1,1)^T$ in the kernel, and image contained in $W$?

3/II/17C  **Linear Mathematics**

Let $V$ be a vector space over $\mathbb{R}$. Let $\alpha : V \to V$ be a nilpotent endomorphism of $V$, i.e. $\alpha^m = 0$ for some positive integer $m$. Prove that $\alpha$ can be represented by a strictly upper-triangular matrix (with zeros along the diagonal). \[You may wish to consider the subspaces ker(\alpha^j) for j = 1, \ldots, m.\]

Show that if $\alpha$ is nilpotent, then $\alpha^n = 0$ where $n$ is the dimension of $V$. Give an example of a $4 \times 4$ matrix $M$ such that $M^4 = 0$ but $M^3 \neq 0$.

Let $A$ be a nilpotent matrix and $I$ the identity matrix. Prove that $I + A$ has all eigenvalues equal to 1. Is the same true of $(I + A)(I + B)$ if $A$ and $B$ are nilpotent? Justify your answer.
4/I/6C  Linear Mathematics

Find the Jordan normal form $J$ of the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

and determine both the characteristic and the minimal polynomial of $M$.

Find a basis of $\mathbb{C}^4$ such that $J$ (the Jordan normal form of $M$) is the matrix representing the endomorphism $M : \mathbb{C}^4 \to \mathbb{C}^4$ in this basis. Give a change of basis matrix $P$ such that $P^{-1}MP = J$.

4/II/15C  Linear Mathematics

Let $A$ and $B$ be $n \times n$ matrices over $\mathbb{C}$. Show that $AB$ and $BA$ have the same characteristic polynomial. [Hint: Look at $\det(CBC - xC)$ for $C = A + yI$, where $x$ and $y$ are scalar variables.]

Show by example that $AB$ and $BA$ need not have the same minimal polynomial.

Suppose that $AB$ is diagonalizable, and let $p(x)$ be its minimal polynomial. Show that the minimal polynomial of $BA$ must divide $xp(x)$. Using this and the first part of the question prove that $(AB)^2$ and $(BA)^2$ are conjugate.
1/I/8B Quadratic Mathematics

Let \( q(x, y) = ax^2 + bxy + cy^2 \) be a binary quadratic form with integer coefficients. Define what is meant by the discriminant \( d \) of \( q \), and show that \( q \) is positive-definite if and only if \( a > 0 > d \). Define what it means for the form \( q \) to be reduced. For any integer \( d < 0 \), we define the class number \( h(d) \) to be the number of positive-definite reduced binary quadratic forms (with integer coefficients) with discriminant \( d \). Show that \( h(d) \) is always finite (for negative \( d \)). Find \( h(-39) \), and exhibit the corresponding reduced forms.

1/II/17B Quadratic Mathematics

Let \( \phi \) be a symmetric bilinear form on a finite dimensional vector space \( V \) over a field \( k \) of characteristic \( \neq 2 \). Prove that the form \( \phi \) may be diagonalized, and interpret the rank \( r \) of \( \phi \) in terms of the resulting diagonal form.

For \( \phi \) a symmetric bilinear form on a real vector space \( V \) of finite dimension \( n \), define the signature \( \sigma \) of \( \phi \), proving that it is well-defined. A subspace \( U \) of \( V \) is called null if \( \phi|_U \equiv 0 \); show that \( V \) has a null subspace of dimension \( n - \frac{1}{2}(r + |\sigma|) \), but no null subspace of higher dimension.

Consider now the quadratic form \( q \) on \( \mathbb{R}^5 \) given by

\[
2(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1).
\]

Write down the matrix \( A \) for the corresponding symmetric bilinear form, and calculate \( \det A \). Hence, or otherwise, find the rank and signature of \( q \).

2/1/B Quadratic Mathematics

Let \( V \) be a finite-dimensional vector space over a field \( k \). Describe a bijective correspondence between the set of bilinear forms on \( V \), and the set of linear maps of \( V \) to its dual space \( V^* \). If \( \phi_1, \phi_2 \) are non-degenerate bilinear forms on \( V \), prove that there exists an isomorphism \( \alpha : V \to V \) such that \( \phi_2(u, v) = \phi_1(u, \alpha v) \) for all \( u, v \in V \). If furthermore both \( \phi_1, \phi_2 \) are symmetric, show that \( \alpha \) is self-adjoint (i.e. equals its adjoint) with respect to \( \phi_1 \).
2/II/17B  Quadratic Mathematics

Suppose \( p \) is an odd prime and \( a \) an integer coprime to \( p \). Define the Legendre symbol \( \left( \frac{a}{p} \right) \), and state (without proof) Euler’s criterion for its calculation.

For \( j \) any positive integer, we denote by \( r_j \) the (unique) integer with \( |r_j| \leq (p-1)/2 \) and \( r_j \equiv aj \mod p \). Let \( l \) be the number of integers \( 1 \leq j \leq (p-1)/2 \) for which \( r_j \) is negative. Prove that

\[
\left( \frac{a}{p} \right) = (-1)^l.
\]

Hence determine the odd primes for which \( 2 \) is a quadratic residue.

Suppose that \( p_1, \ldots, p_m \) are primes congruent to 7 modulo 8, and let

\[
N = 8(p_1 \cdots p_m)^2 - 1.
\]

Show that \( 2 \) is a quadratic residue for any prime dividing \( N \). Prove that \( N \) is divisible by some prime \( p \equiv 7 \mod 8 \). Hence deduce that there are infinitely many primes congruent to 7 modulo 8.

3/I/9B  Quadratic Mathematics

Let \( A \) be the Hermitian matrix

\[
\begin{pmatrix}
1 & i & 2i \\
-i & 3 & -i \\
-2i & i & 5
\end{pmatrix}.
\]

Explaining carefully the method you use, find a diagonal matrix \( D \) with rational entries, and an invertible (complex) matrix \( T \) such that \( T^*DT = A \), where \( T^* \) here denotes the conjugated transpose of \( T \).

Explain briefly why we cannot find \( T, D \) as above with \( T \) unitary.

[You may assume that if a monic polynomial \( t^3 + a_2t^2 + a_1t + a_0 \) with integer coefficients has all its roots rational, then all its roots are in fact integers.]
Let $J_1$ denote the $2 \times 2$ matrix \[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]. Suppose that $T$ is a $2 \times 2$ upper-triangular real matrix with strictly positive diagonal entries and that $J_1^{-1}TJ_1T^{-1}$ is orthogonal. Verify that $J_1T = TJ_1$.

Prove that any real invertible matrix $A$ has a decomposition $A = BC$, where $B$ is an orthogonal matrix and $C$ is an upper-triangular matrix with strictly positive diagonal entries.

Let $A$ now denote a $2n \times 2n$ real matrix, and $A = BC$ be the decomposition of the previous paragraph. Let $K$ denote the $2n \times 2n$ matrix with $n$ copies of $J_1$ on the diagonal, and zeros elsewhere, and suppose that $KA = AK$. Prove that $K^{-1}CKC^{-1}$ is orthogonal. From this, deduce that the entries of $K^{-1}CKC^{-1}$ are zero, apart from $n$ orthogonal $2 \times 2$ blocks $E_1, \ldots, E_n$ along the diagonal. Show that each $E_i$ has the form $J_1^{-1}C_iJ_1C_i^{-1}$, for some $2 \times 2$ upper-triangular matrix $C_i$ with strictly positive diagonal entries. Deduce that $KC = CK$ and $KB = BK$.

[Hint: The invertible $2n \times 2n$ matrices $S$ with $2 \times 2$ blocks $S_1, \ldots, S_n$ along the diagonal, but with all other entries below the diagonal zero, form a group under matrix multiplication.]