Part IB

—

Fluid Dynamics

—

Paper 2, Section I
5C   Fluid Dynamics

A three-dimensional flow has a velocity field $\mathbf{u}(\mathbf{x}) = \boldsymbol{\Gamma}\cdot\mathbf{x}+\mathbf{U}_0$, where $\boldsymbol{\Gamma}$ is a constant second-rank tensor and $\mathbf{U}_0$ is a constant vector, with components

$$\boldsymbol{\Gamma} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

(a) What are the conditions on the components of $\boldsymbol{\Gamma}$ and $\mathbf{U}_0$ for the flow to be:

  (i) incompressible?

  (ii) irrotational?

(b) In the case where

$$\boldsymbol{\Gamma} = \begin{pmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{U}_0 = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}, \quad (\alpha \neq 0),$$

compute the streamline passing through the point $(1,0,0)$.

Paper 3, Section I
7C   Fluid Dynamics

A two-dimensional cylinder of radius a is stationary in a uniform flow of velocity $U\mathbf{e}_x$. The flow is assumed to be steady, inviscid, two-dimensional and irrotational. There is no circulation around the cylinder.

Using a velocity potential, solve for the flow $\mathbf{u}(r,\theta)$ around the cylinder. Use Bernoulli's equation to compute the pressure on its surface as a function of the polar angle θ.

Paper 1, Section II

16C Fluid Dynamics

An incompressible viscous fluid of constant uniform viscosity μ and density ρ undergoes unidirectional flow of the form $\mathbf{u} = u(y,t)\mathbf{e}_x$ in two dimensions. Gravity is negligible.

(a) Use a small control fluid volume of size $\delta x \times \delta y$,

   (i) to show that this flow satisfies mass conservation;

   (ii) to derive the momentum conservation equation satisfied by $u(y,t)$ and the pressure $p(x)$.

(b) The flow is steady, is subject to a uniform pressure gradient $G = \mathrm{d}p/\mathrm{d}x$ and occurs between two rigid surfaces at $y = 0$ and $y = h$. The surface at $y = 0$ is stationary while the surface at $y = h$ translates with velocity $U\mathbf{e}_x$, where U is a constant parameter.

   (i) Solve for the flow $u(y)$ in terms of G and U.

   (ii) Compute the value G_0 of the applied pressure gradient G for which the shear stress at $y = 0$ is zero.

   (iii) For $G = G_0$, deduce the volume flux in the x direction.

   (iv) For $G = G_0$, use $u(y)$ to compute the shear stress exerted by the flow on the top plate. Show that it can also be obtained by using a force balance on a small control fluid volume of size $\delta x \times h$.

Paper 3, Section II
16C Fluid Dynamics

(a) Starting from the Euler equation for an inviscid fluid with no body force, derive the unsteady Bernoulli equation relating the pressure and the velocity potential in a time-dependent irrotational, incompressible flow.

(b) A liquid occupies the two-dimensional annular region $a(t) < r < b(t)$ between a gas bubble occupying $0 \leqslant r < a(t)$ and an infinite gas in $r > b(t)$. The flow is incompressible, irrotational and radially symmetric.

  (i) If the radius of the gas bubble is prescribed (i.e. the function $a(t)$ is known), solve for the potential flow in the liquid. Deduce the time-variation of $b(t)$ and interpret your result physically.

  (ii) The pressure in the gas in $r > b$ is a constant p_∞. Compute the time-varying pressure $p(r,t)$ in the liquid at $r = a(t)$.

  (iii) Assuming small perturbations for the bubble radius $a(t) = a_0[1 + \epsilon(t)]$ with $|\epsilon| \ll 1$, deduce the linearised variation of the radius $b(t)$. Find the linearised variation of the pressure $p(a,t)$.

  (iv) The pressure $p_0(t)$ in the bubble is uniform in space and satisfies $p_0 V = const$, where $V(t)$ is the volume of the bubble. Deduce the relationship between ϵ and $p(a,t) - p_\infty$.

  (v) Show that the bubble undergoes oscillations and compute its frequency ω.

Paper 4, Section II

16C   Fluid Dynamics

(a) A body of fluid has a free surface given by $z = \eta(x, y, t)$ in Cartesian coordinates and the fluid velocity is denoted by $\mathbf{u} = (u, v, w)$. Applying the kinematic boundary condition at the free surface, derive the relationship between the value of w at the free surface and $\mathrm{D}\eta/\mathrm{D}t$.

(b) An inviscid fluid is confined in a box with sides at $x = 0, L$ and $y = 0, L$. The fluid is semi-infinite in the $-z$ direction and is bounded above by a free surface at $z = \eta(x, y, t)$. The fluid is forced to oscillate by applying a prescribed variation in the air pressure just above the free surface,

$$p(x, y, t) = p_0 \cos\left(\pi x/L\right) \cos\left(2\pi y/L\right) \cos(\omega t),$$

with ω a prescribed constant frequency.

(i) Assuming irrotational flow and small-amplitude motion of the interface, state the equation satisfied by the velocity potential ϕ in the fluid and state all the boundary conditions.

(ii) Show that a separable solution for ϕ of the form

$$\phi = Z(z) \cos\left(\pi x/L\right) \cos\left(2\pi y/L\right) F(t)$$

is consistent with the dynamic boundary condition and that it satisfies the boundary conditions at $x = 0, L$ and $y = 0, L$.

(iii) Solve for the function $Z(z)$.

(iv) Using the kinematic boundary condition, show that the shape of the interface is of the form

$$\eta(x, y, t) = \cos\left(\pi x/L\right) \cos\left(2\pi y/L\right) H(t),$$

and derive the relationship between $H(t)$ and $F(t)$.

(v) Use the dynamic boundary condition to solve for $H(t)$ and $F(t)$.

(vi) Deduce that the amplitudes H and F do not remain bounded for a specific value of the frequency ω which you should determine, and briefly interpret this phenomenon physically.

Paper 2, Section I
5C Fluid Dynamics

An unsteady fluid flow has velocity field given in cartesian coordinates (x, y, z) by $\mathbf{u} = (2t, xt, 0)$, where $t > 0$ denotes time. Dye is continuously released into the fluid from the origin.

(a) Determine if this fluid flow is incompressible.

(b) Find the distance from the origin at time t of the dye particle that was released at time s, where $s < t$.

(c) Determine the equation of the curve formed by the dye streak in the (x, y)-plane.

Paper 3, Section I
7C Fluid Dynamics

A two-dimensional flow has velocity given by

$$\mathbf{u}(\mathbf{x}) = 2\frac{\mathbf{x}(\mathbf{d} \cdot \mathbf{x})}{r^4} - \frac{\mathbf{d}}{r^2}$$

as a function of the position vector $\mathbf{x}$, with $r = |\mathbf{x}|$, where $\mathbf{d}$ is a fixed vector.

(a) Show that this flow is incompressible for $r \neq 0$.

(b) Compute the stream function ψ for this flow in polar coordinates (r, θ) with $\theta = 0$ aligned with the vector $\mathbf{d}$.

[*Hint: in polar coordinates*

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta}$$

for a vector $\mathbf{F} = (F_r, F_\theta)$.]

Paper 1, Section II
16C Fluid Dynamics

Consider a steady viscous flow (with viscosity μ) of constant density ρ through a long pipe of circular cross-section with radius R. The flow is driven by a constant pressure gradient $\partial p/\partial z$ along the pipe (z is the coordinate along the pipe).

The Navier-Stokes equation describing this flow is

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] = -\nabla p + \mu\nabla^2\mathbf{u}.$$

(a) Using cylindrical coordinates (r,θ,z) aligned with the pipe, determine the velocity $\mathbf{u} = (0,0,w(r))$ of the flow.

[*Hint: in cylindrical coordinates*

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.]$$

(b) The viscous stress exerted on the flow by the pipe boundaries is equal to

$$\mu\left(\frac{\partial w}{\partial r}\right)\bigg|_{r=R}.$$

Demonstrate the overall force balance for the (cylindrical) volume of the fluid enclosed within the section of the pipe $z_0 \leqslant z \leqslant z_0 + L$.

(c) Compute the mass flux through the pipe.

Paper 3, Section II
16C Fluid Dynamics

Consider an axisymmetric, two-dimensional, incompressible flow $\mathbf{u}(r) = (u_r, u_\theta)$ in polar coordinates (r, θ).

(a) Determine the behaviour of u_r if it is finite everywhere in space.

(b) Representing $u_\theta = \Omega(r)r$, express the vorticity of the flow $\boldsymbol{\omega}$ in terms of Ω.

(c) Starting from the Navier-Stokes equation

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = -\nabla p + \mu \nabla^2 \mathbf{u}$$

derive the vorticity evolution equation

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}$$

for a general incompressible flow with kinematic viscosity $\nu = \mu/\rho$.

(d) Deduce the form of the evolution equation for the scalar vorticity $\omega = |\boldsymbol{\omega}|$ for the axisymmetric two-dimensional flow of part (a).

(e) Show that the equation derived in part (d) adopts a self-similar form $\omega(r, t) = \omega(\xi)$, where $\xi = r/\sqrt{\nu t}$ is the similarity variable.

[*You may use the fact that, in polar coordinates,*

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

and

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta} \right] \mathbf{e}_z$$

for a vector $\mathbf{F} = (F_r, F_\theta)$, *where* $\mathbf{e}_z$ *is a unit vector normal to the flow plane.*]

Paper 4, Section II
16C Fluid Dynamics

A fluid of density ρ_1 occupies the region $z > 0$ and a second fluid of density ρ_2 occupies the region $z < 0$. The system is perturbed so that the subsequent motion is irrotational and the interface is at $z = \zeta(x, t)$. State the equations and nonlinear boundary conditions that are satisfied by the corresponding velocity potentials ϕ_1 and ϕ_2 and pressures p_1 and p_2.

Obtain a set of linearised equations and boundary conditions when the perturbations are small and proportional to $e^{i(kx-\omega t)}$. Hence derive the dispersion relation

$$\omega^2 = gk\, F\left(\frac{\rho_1}{\rho_2} \right),$$

where g is the gravitational acceleration and F is a function to be determined.

Paper 2, Section I

5A   Fluid Dynamics

Consider an axisymmetric container, initially filled with water to a depth h_I. A small circular hole of radius r_0 is opened in the base of the container at $z = 0$.

(a) Determine how the radius r of the container should vary with $z < h_I$ so that the depth of the water will decrease at a constant rate.

(b) For such a container, determine how the cross-sectional area A of the free surface should decrease with time.

[*You may assume that the flow rate through the opening is sufficiently small that Bernoulli's theorem for steady flows can be applied.*]

Paper 3, Section I

7A   Fluid Dynamics

A two-dimensional flow $\mathbf{u} = (u, v)$ has a velocity field given by

$$u = \frac{x^2 - y^2}{(x^2 + y^2)^2} \quad \text{and} \quad v = \frac{2xy}{(x^2 + y^2)^2}.$$

(a) Show explicitly that this flow is incompressible and irrotational away from the origin.

(b) Find the stream function for this flow.

(c) Find the velocity potential for this flow.

Paper 1, Section II

16A Fluid Dynamics

A two-dimensional flow is given by a velocity potential

$$\phi(x,y,t) = \epsilon y \sin(x - t),$$

where ϵ is a constant.

(a) Find the corresponding velocity field $\mathbf{u}(x,y,t)$. Determine $\boldsymbol{\nabla}\cdot\mathbf{u}$.

(b) The time-average $\langle\psi\rangle(x,y)$ of a quantity $\psi(x,y,t)$ is defined as

$$\langle\psi\rangle(x,y) = \frac{1}{2\pi}\int_0^{2\pi}\psi(x,y,t)dt.$$

Show that the time-average of this velocity field is zero everywhere. Write down an expression for the acceleration of fluid particles, and find the time-average of this expression at a fixed point (x,y).

(c) Now assume that $|\epsilon| \ll 1$. The material particle at $(0,0)$ at $t=0$ is marked with dye. Write down equations for its subsequent motion. Verify that its position (x,y) for $t>0$ is given (correct to terms of order ϵ^2) by

$$\begin{aligned}
x &= \epsilon^2\left(\frac{1}{4}\sin 2t + \frac{t}{2} - \sin t\right), \\
y &= \epsilon(\cos t - 1).
\end{aligned}$$

Deduce the time-average velocity of the dyed particle correct to this order.

Paper 3, Section II
16A Fluid Dynamics

A two-dimensional layer of viscous fluid lies between two rigid boundaries at $y = \pm L_0$. The boundary at $y = L_0$ oscillates in its own plane with velocity $(U_0 \cos \omega t, 0)$, while the boundary at $y = -L_0$ oscillates in its own plane with velocity $(-U_0 \cos \omega t, 0)$. Assume that there is no pressure gradient and that the fluid flows parallel to the boundary with velocity $(u(y,t), 0)$, where $u(y,t)$ can be written as $u(y,t) = \text{Re}[U_0 f(y) \exp(i\omega t)]$.

(a) By exploiting the symmetry of the system or otherwise, show that

$$f(y) \;=\; \frac{\sinh[(1+i)\Delta \hat{y}]}{\sinh[(1+i)\Delta]}, \quad \text{where} \;\; \hat{y} = \frac{y}{L_0} \;\; \text{and} \;\; \Delta = \left(\frac{\omega L_0^2}{2\nu} \right)^{1/2}.$$

(b) Hence or otherwise, show that

$$\frac{u(y,t)}{U_0} \;=\; \frac{\cos \omega t \, [\cosh \Delta_+ \cos \Delta_- - \cosh \Delta_- \cos \Delta_+]}{(\cosh 2\Delta - \cos 2\Delta)}$$
$$+ \frac{\sin \omega t \, [\sinh \Delta_+ \sin \Delta_- - \sinh \Delta_- \sin \Delta_+]}{(\cosh 2\Delta - \cos 2\Delta)},$$

where $\Delta_\pm = \Delta(1 \pm \hat{y})$.

(c) Show that, for $\Delta \ll 1$,

$$u(y,t) \simeq \frac{U_0 y}{L_0} \cos \omega t,$$

and briefly interpret this result physically.

Paper 4, Section II

16A Fluid Dynamics

Consider the spherically symmetric motion induced by the collapse of a spherical cavity of radius $a(t)$, centred on the origin. For $r < a$, there is a vacuum, while for $r > a$, there is an inviscid incompressible fluid with constant density ρ. At time $t = 0$, $a = a_0$, and the fluid is at rest and at constant pressure p_0.

(a) Consider the radial volume transport in the fluid $Q(R, t)$, defined as

$$Q(R, t) = \int_{r=R} u \, dS,$$

where u is the radial velocity, and dS is an infinitesimal element of the surface of a sphere of radius $R \geqslant a$. Use the incompressibility condition to establish that Q is a function of time alone.

(b) Using the expression for pressure in potential flow or otherwise, establish that

$$\frac{1}{4\pi a}\frac{dQ}{dt} - \frac{(\dot{a})^2}{2} = -\frac{p_0}{\rho},$$

where $\dot{a}(t)$ is the radial velocity of the cavity boundary.

(c) By expressing $Q(t)$ in terms of a and $\dot{a}$, show that

$$\dot{a} = -\sqrt{\frac{2p_0}{3\rho}\left(\frac{a_0^3}{a^3} - 1\right)}.$$

[*Hint: You may find it useful to assume $\dot{a}(t)$ is an explicit function of a from the outset.*]

(d) Hence write down an integral expression for the implosion time τ, i.e. the time for the radius of the cavity $a \to 0$. [Do not attempt to evaluate the integral.]

Paper 2, Section I

6C Fluid Dynamics

Incompressible fluid of constant viscosity μ is confined to the region $0 < y < h$ between two parallel rigid plates. Consider two parallel viscous flows: flow A is driven by the motion of one plate in the x-direction with the other plate at rest; flow B is driven by a constant pressure gradient in the x-direction with both plates at rest. The velocity mid-way between the plates is the same for both flows.

The viscous friction in these flows is known to produce heat locally at a rate

$$Q = \mu \left(\frac{\partial u}{\partial y} \right)^2$$

per unit volume, where u is the x-component of the velocity. Determine the ratio of the total rate of heat production in flow A to that in flow B.

Paper 1, Section II

17C Fluid Dynamics

Steady two-dimensional potential flow of an incompressible fluid is confined to the wedge $0 < \theta < \alpha$, where (r, θ) are polar coordinates centred on the vertex of the wedge and $0 < \alpha < \pi$.

(a) Show that a velocity potential ϕ of the form

$$\phi(r, \theta) = A r^\gamma \cos(\lambda \theta),$$

where A, γ and λ are positive constants, satisfies the condition of incompressible flow, provided that γ and λ satisfy a certain relation to be determined.

Assuming that u_θ, the θ-component of velocity, does not change sign within the wedge, determine the values of γ and λ by using the boundary conditions.

(b) Calculate the shape of the streamlines of this flow, labelling them by the distance $r_{\min}$ of closest approach to the vertex. Sketch the streamlines.

(c) Show that the speed $|\mathbf{u}|$ and pressure p are independent of θ. Assuming that at some radius $r = r_0$ the speed and pressure are u_0 and p_0, respectively, find the pressure difference in the flow between the vertex of the wedge and r_0.

[*Hint: In polar coordinates* (r, θ),

$$\boldsymbol{\nabla} f = \left(\frac{\partial f}{\partial r} , \; \frac{1}{r} \frac{\partial f}{\partial \theta} \right) \quad and \quad \boldsymbol{\nabla} \cdot \mathbf{F} = \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta}$$

for a scalar f and a vector $\mathbf{F} = (F_r, F_\theta)$.]

Paper 2, Section II

16C Fluid Dynamics

A vertical cylindrical container of radius R is partly filled with fluid of constant density to depth h. The free surface is perturbed so that the fluid occupies the region

$$0 < r < R, \quad -h < z < \zeta(r, \theta, t),$$

where (r, θ, z) are cylindrical coordinates and ζ is the perturbed height of the free surface. For small perturbations, a linearised description of surface waves in the cylinder yields the following system of equations for ζ and the velocity potential ϕ:

$$\nabla^2 \phi = 0, \qquad 0 < r < R, \quad -h < z < 0, \tag{1}$$

$$\frac{\partial \phi}{\partial t} + g\zeta = 0 \quad \text{on} \quad z = 0, \tag{2}$$

$$\frac{\partial \zeta}{\partial t} - \frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = 0, \tag{3}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h, \tag{4}$$

$$\frac{\partial \phi}{\partial r} = 0 \quad \text{on} \quad r = R. \tag{5}$$

(a) Describe briefly the physical meaning of each equation.

(b) Consider axisymmetric normal modes of the form

$$\phi = \mathrm{Re}\left(\hat{\phi}(r, z) e^{-i\sigma t} \right), \qquad \zeta = \mathrm{Re}\left(\hat{\zeta}(r) e^{-i\sigma t} \right).$$

Show that the system of equations (1)–(5) admits a solution for $\hat{\phi}$ of the form

$$\hat{\phi}(r, z) = A J_0(k_n r) Z(z),$$

where A is an arbitrary amplitude, $J_0(x)$ satisfies the equation

$$\frac{d^2 J_0}{dx^2} + \frac{1}{x}\frac{dJ_0}{dx} + J_0 = 0,$$

the wavenumber k_n, $n = 1, 2, \ldots$ is such that $x_n = k_n R$ is one of the zeros of the function dJ_0/dx, and the function $Z(z)$ should be determined explicitly.

(c) Show that the frequency σ_n of the n-th mode is given by

$$\sigma_n^2 = \frac{g}{h}\Psi(k_n h),$$

where the function $\Psi(x)$ is to be determined.

[*Hint: In cylindrical coordinates* (r, θ, z),

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.]$$

Paper 1, Section I

5C Fluid Dynamics

A viscous fluid flows steadily down a plane that is inclined at an angle α to the horizontal. The fluid layer is of uniform thickness and has a free upper surface. Determine the velocity profile in the direction perpendicular to the plane and also the volume flux (per unit width), in terms of the gravitational acceleration g, the angle α, the kinematic viscosity ν and the thickness h of the fluid layer.

Show that the volume flux is reduced if the free upper surface is replaced by a stationary plane boundary, and give a physical explanation for this.

Paper 2, Section I

7C Fluid Dynamics

Consider the steady flow

$$u_x = \sin x \cos y\,, \qquad u_y = -\cos x \sin y\,, \qquad u_z = 0\,,$$

where (x, y, z) are Cartesian coordinates. Show that $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and determine the streamfunction. Calculate the vorticity and verify that the vorticity equation is satisfied in the absence of viscosity. Sketch the streamlines in the region $0 < x < 2\pi$, $0 < y < 2\pi$.

Paper 1, Section II

17C Fluid Dynamics

Explain why the irrotational flow of an incompressible fluid can be expressed in terms of a velocity potential ϕ that satisfies Laplace's equation.

The axis of a stationary cylinder of radius a coincides with the z-axis of a Cartesian coordinate system (x, y, z) with unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. A fluid of density ρ flows steadily past the cylinder such that the velocity field $\mathbf{u}$ is independent of z and has no component in the z-direction. The flow is irrotational but there is a constant non-zero circulation

$$\oint \mathbf{u} \cdot d\mathbf{r} = \kappa$$

around every closed curve that encloses the cylinder once in a positive sense. Far from the cylinder, the velocity field tends towards the uniform flow $\mathbf{u} = U \, \mathbf{e}_x$, where U is a constant.

State the boundary conditions on the velocity potential, in terms of polar coordinates (r, θ) in the (x, y)-plane. Explain why the velocity potential is not required to be a single-valued function of position. Hence obtain the appropriate solution $\phi(r, \theta)$, in terms of a, U and κ.

Neglecting gravity, show that the net force on the cylinder, per unit length in the z-direction, is

$$-\rho \kappa U \, \mathbf{e}_y \, .$$

Determine the number and location of stagnation points in the flow as a function of the dimensionless parameter

$$\lambda = \frac{\kappa}{4\pi U a} \, .$$

Paper 4, Section II

18C Fluid Dynamics

The linear shallow-water equations governing the motion of a fluid layer in the neighbourhood of a point on the Earth's surface in the northern hemisphere are

$$
\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x},
$$
$$
\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y},
$$
$$
\frac{\partial \eta}{\partial t} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),
$$

where $u(x,y,t)$ and $v(x,y,t)$ are the horizontal velocity components and $\eta(x,y,t)$ is the perturbation of the height of the free surface.

(a) Explain the meaning of the three positive constants f, g and h appearing in the equations above and outline the assumptions made in deriving these equations.

(b) Show that ζ, the z-component of vorticity, satisfies

$$
\frac{\partial \zeta}{\partial t} = -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right),
$$

and deduce that the potential vorticity

$$
q = \zeta - \frac{f}{h}\eta
$$

satisfies

$$
\frac{\partial q}{\partial t} = 0.
$$

(c) Consider a steady geostrophic flow that is uniform in the latitudinal (y) direction. Show that

$$
\frac{d^2\eta}{dx^2} - \frac{f^2}{gh}\eta = \frac{f}{g}q.
$$

Given that the potential vorticity has the piecewise constant profile

$$
q = \begin{cases} q_1, & x < 0, \\ q_2, & x > 0, \end{cases}
$$

where q_1 and q_2 are constants, and that $v \to 0$ as $x \to \pm\infty$, solve for $\eta(x)$ and $v(x)$ in terms of the Rossby radius $R = \sqrt{gh}/f$. Sketch the functions $\eta(x)$ and $v(x)$ in the case $q_1 > q_2$.

Paper 3, Section II

18C Fluid Dynamics

A cubic box of side $2h$, enclosing the region $0 < x < 2h$, $0 < y < 2h$, $-h < z < h$, contains equal volumes of two incompressible fluids that remain distinct. The system is initially at rest, with the fluid of density ρ_1 occupying the region $0 < z < h$ and the fluid of density ρ_2 occupying the region $-h < z < 0$, and with gravity $(0, 0, -g)$. The interface between the fluids is then slightly perturbed. Derive the linearized equations and boundary conditions governing small disturbances to the initial state.

In the case $\rho_2 > \rho_1$, show that the angular frequencies ω of the normal modes are given by

$$\omega^2 = \left(\frac{\rho_2 - \rho_1}{\rho_1 + \rho_2} \right) gk \tanh(kh)$$

and express the allowable values of the wavenumber k in terms of h. Identify the lowest-frequency non-trivial mode(s). Comment on the limit $\rho_1 \ll \rho_2$. What physical behaviour is expected in the case $\rho_1 > \rho_2$?

UNIVERSITY OF
CAMBRIDGE

14

Paper 1, Section I
5D   Fluid Dynamics
Show that the flow with velocity potential

$$\phi = \frac{q}{2\pi} \ln r$$

in two-dimensional, plane-polar coordinates (r, θ) is incompressible in $r > 0$. Determine the flux of fluid across a closed contour C that encloses the origin. What does this flow represent?

Show that the flow with velocity potential

$$\phi = \frac{q}{4\pi} \ln \left(x^2 + (y - a)^2 \right) + \frac{q}{4\pi} \ln \left(x^2 + (y + a)^2 \right)$$

has no normal flow across the line $y = 0$. What fluid flow does this represent in the unbounded plane? What flow does it represent for fluid occupying the domain $y > 0$?

Paper 2, Section I
7D   Fluid Dynamics
The Euler equations for steady fluid flow $\mathbf{u}$ in a rapidly rotating system can be written

$$\rho \mathbf{f} \times \mathbf{u} = -\nabla p + \rho \mathbf{g},$$

where ρ is the density of the fluid, p is its pressure, $\mathbf{g}$ is the acceleration due to gravity and $\mathbf{f} = (0, 0, f)$ is the constant Coriolis parameter in a Cartesian frame of reference (x, y, z), with z pointing vertically upwards.

Fluid occupies a layer of slowly-varying height $h(x, y)$. Given that the pressure $p = p_0$ is constant at $z = h$ and that the flow is approximately horizontal with components $\mathbf{u} = (u, v, 0)$, show that the contours of h are streamlines of the horizontal flow. What is the leading-order horizontal volume flux of fluid between two locations at which $h = h_0$ and $h = h_0 + \Delta h$, where $\Delta h \ll h_0$?

Identify the dimensions of all the quantities involved in your expression for the volume flux and show that your expression is dimensionally consistent.

Paper 1, Section II
17D Fluid Dynamics

A layer of fluid of dynamic viscosity μ, density ρ and uniform thickness h flows down a rigid vertical plane. The adjacent air has uniform pressure p_0 and exerts a tangential stress on the fluid that is proportional to the surface velocity and opposes the flow, with constant of proportionality k. The acceleration due to gravity is g.

(a) Draw a diagram of this situation, including indications of the applied stresses and body forces, a suitable coordinate system and a representation of the expected velocity profile.

(b) Write down the equations and boundary conditions governing the flow, with a brief description of each, paying careful attention to signs. Solve these equations to determine the pressure and velocity fields in terms of the parameters given above.

(c) Show that the surface velocity of the fluid layer is $\dfrac{\rho g h^2}{2\mu}\left(1+\dfrac{kh}{\mu}\right)^{-1}$.

(d) Determine the volume flux per unit width of the plane for general values of k and its limiting values when $k \to 0$ and $k \to \infty$.

Paper 4, Section II
18D Fluid Dynamics

A deep layer of inviscid fluid is initially confined to the region $0 < x < a$, $0 < y < a$, $z < 0$ in Cartesian coordinates, with z directed vertically upwards. An irrotational disturbance is caused to the fluid so that its upper surface takes position $z = \eta(x, y, t)$. Determine the linear normal modes of the system and the dispersion relation between the frequencies of the normal modes and their wavenumbers.

If the interface is initially displaced to position $z = \epsilon \cos \dfrac{3\pi x}{a} \cos \dfrac{4\pi y}{a}$ and released from rest, where ϵ is a small constant, determine its position for subsequent times. How far below the surface will the velocity have decayed to $1/e$ times its surface value?

Paper 3, Section II

18D Fluid Dynamics

A soap bubble of radius $a(t)$ is attached to the end of a long, narrow straw of internal radius ϵ and length L, the other end of which is open to the atmosphere. The pressure difference between the inside and outside of the bubble is $2\gamma/a$, where γ is the surface tension of the soap bubble. At time $t = 0$, $a = a_0$ and the air in the straw is at rest. Assume that the flow of air through the straw is irrotational and consider the pressure drop along the straw to show that subsequently

$$a^3 \ddot{a} + 2a^2 \dot{a}^2 = -\frac{\gamma \epsilon^2}{2\rho L},$$

where ρ is the density of air.

By multiplying the equation by $2a\dot{a}$ and integrating, or otherwise, determine an implicit equation for $a(t)$ and show that the bubble disappears in a time

$$t = \frac{\pi}{2} \frac{a_0^2}{\epsilon} \left(\frac{\rho L}{2\gamma} \right)^{1/2}.$$

[*Hint: The substitution $a = a_0 \sin\theta$ can be used.*]

Paper 1, Section I

5D   Fluid Dynamics

For each of the flows

(i) $\mathbf{u} = (2xy, x^2 + y^2)$

(ii) $\mathbf{u} = (-2y, -2x)$

determine whether or not the flow is incompressible and/or irrotational. Find the associated velocity potential and/or stream function when appropriate. For either **one** of the flows, sketch the streamlines of the flow, indicating the direction of the flow.

Paper 2, Section I

7D   Fluid Dynamics

From Euler's equations describing steady inviscid fluid flow under the action of a conservative force, derive Bernoulli's equation for the pressure along a streamline of the flow, defining all variables that you introduce.

Water fills an inverted, open, circular cone (radius increasing upwards) of half angle $\pi/4$ to a height h_0 above its apex. At time $t = 0$, the tip of the cone is removed to leave a small hole of radius $\epsilon \ll h_0$. Assuming that the flow is approximately steady while the depth of water $h(t)$ is much larger than ϵ, show that the time taken for the water to drain is approximately

$$\left(\frac{2}{25} \frac{h_0^5}{\epsilon^4 g} \right)^{1/2} .$$

Paper 1, Section II

17D Fluid Dynamics

A layer of thickness h of fluid of density ρ and dynamic viscosity μ flows steadily down and parallel to a rigid plane inclined at angle α to the horizontal. Wind blows over the surface of the fluid and exerts a stress S on the surface of the fluid in the upslope direction.

(a) Draw a diagram of this situation, including indications of the applied stresses and body forces, a suitable coordinate system and a representation of the expected velocity profile.

(b) Write down the equations and boundary conditions governing the flow, with a brief description of each, paying careful attention to signs. Solve these equations to determine the pressure and velocity fields.

(c) Determine the volume flux and show that there is no net flux if

$$S = \frac{2}{3}\rho g h \sin \alpha.$$

Draw a sketch of the corresponding velocity profile.

(d) Determine the value of S for which the shear stress on the rigid plane is zero and draw a sketch of the corresponding velocity profile.

Paper 4, Section II
18D Fluid Dynamics

The linearised equations governing the horizontal components of flow $\mathbf{u}(x, y, t)$ in a rapidly rotating shallow layer of depth $h = h_0 + \eta(x, y, t)$, where $\eta \ll h_0$, are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{f} \times \mathbf{u} = -g\nabla\eta,$$

$$\frac{\partial \eta}{\partial t} + h_0 \nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{f} = f\mathbf{e}_z$ is the constant Coriolis parameter, and $\mathbf{e}_z$ is the unit vector in the vertical direction.

Use these equations, either in vector form or using Cartesian components, to show that the potential vorticity

$$\mathbf{Q} = \boldsymbol{\zeta} - \frac{\eta}{h_0}\mathbf{f}$$

is independent of time, where $\boldsymbol{\zeta} = \nabla \times \mathbf{u}$ is the relative vorticity.

Derive the equation

$$\frac{\partial^2 \eta}{\partial t^2} - gh_0\nabla^2\eta + f^2\eta = -h_0\mathbf{f} \cdot \mathbf{Q}.$$

In the case that $\mathbf{Q} \equiv 0$, determine and sketch the dispersion relation $\omega(k)$ for plane waves with $\eta = Ae^{i(kx+\omega t)}$, where A is constant. Discuss the nature of the waves qualitatively: do long waves propagate faster or slower than short waves; how does the phase speed depend on wavelength; does rotation have more effect on long waves or short waves; how dispersive are the waves?

Paper 3, Section II
18D Fluid Dynamics

Use Euler's equations to derive the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla\mathbf{u},$$

where $\mathbf{u}$ is the fluid velocity and $\boldsymbol{\omega}$ is the vorticity.

Consider axisymmetric, incompressible, inviscid flow between two rigid plates at $z = h(t)$ and $z = -h(t)$ in cylindrical polar coordinates (r, θ, z), where t is time. Using mass conservation, or otherwise, find the complete flow field whose radial component is independent of z.

Now suppose that the flow has angular velocity $\boldsymbol{\Omega} = \Omega(t)\mathbf{e}_z$ and that $\Omega = \Omega_0$ when $h = h_0$. Use the vorticity equation to determine the angular velocity for subsequent times as a function of h. What physical principle does your result illustrate?

Paper 1, Section I

5C   Fluid Dynamics

Consider the flow field in cartesian coordinates (x, y, z) given by

$$\mathbf{u} = \left(-\frac{Ay}{x^2 + y^2}, \frac{Ax}{x^2 + y^2}, U(z) \right),$$

where A is a constant. Let $\mathcal{D}$ denote the whole of $\mathbb{R}^3$ excluding the z axis.

(a) Determine the conditions on A and $U(z)$ for the flow to be both incompressible and irrotational in $\mathcal{D}$.

(b) Calculate the circulation along any closed curve enclosing the z axis.

Paper 2, Section I

7C   Fluid Dynamics

A steady, two-dimensional unidirectional flow of a fluid with dynamic viscosity μ is set up between two plates at $y = 0$ and $y = h$. The plate at $y = 0$ is stationary while the plate at $y = h$ moves with constant speed $U\mathbf{e}_x$. The fluid is also subject to a constant pressure gradient $-G\mathbf{e}_x$. You may assume that the fluid velocity $\mathbf{u}$ has the form $\mathbf{u} = u(y)\mathbf{e}_x$.

(a) State the equation satisfied by $u(y)$ and its boundary conditions.

(b) Calculate $u(y)$.

(c) Show that the value of U may be chosen to lead to zero viscous shear stress acting on the bottom plate and calculate the resulting flow rate.

Paper 1, Section II

17C Fluid Dynamics

(a) For a velocity field $\mathbf{u}$, show that $\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{u} = \boldsymbol{\nabla}\left(\frac{1}{2}\mathbf{u}^2\right) - \mathbf{u} \times \boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the flow vorticity.

(b) For a scalar field $H(\mathbf{r})$, show that if $\mathbf{u} \cdot \boldsymbol{\nabla}H = 0$, then H is constant along the flow streamlines.

(c) State the Euler equations satisfied by an inviscid fluid of constant density subject to conservative body forces.

(i) If the flow is irrotational, show that an exact first integral of the Euler equations may be obtained.

(ii) If the flow is not irrotational, show that although an exact first integral of the Euler equations may not be obtained, a similar quantity is constant along the flow streamlines provided the flow is steady.

(iii) If the flow is now in a frame rotating with steady angular velocity $\Omega\mathbf{e}_z$, establish that a similar quantity is constant along the flow streamlines with an extra term due to the centrifugal force when the flow is steady.

Paper 4, Section II

18C Fluid Dynamics

(a) Show that for an incompressible fluid, $\boldsymbol{\nabla} \times \boldsymbol{\omega} = -\nabla^2\mathbf{u}$, where $\boldsymbol{\omega}$ is the flow vorticity.

(b) State the equation of motion for an inviscid flow of constant density in a rotating frame subject to gravity. Show that, on Earth, the local vertical component of the centrifugal force is small compared to gravity. Present a scaling argument to justify the linearisation of the Euler equations for sufficiently large rotation rates, and hence deduce the linearised version of the Euler equations in a rapidly rotating frame.

(c) Denoting the rotation rate of the frame as $\boldsymbol{\Omega} = \Omega\mathbf{e}_z$, show that the linearised Euler equations may be manipulated to obtain an equation for the velocity field $\mathbf{u}$ in the form

$$\frac{\partial^2 \nabla^2\mathbf{u}}{\partial t^2} + 4\Omega^2\frac{\partial^2\mathbf{u}}{\partial z^2} = \mathbf{0}.$$

(d) Assume that there exist solutions of the form $\mathbf{u} = \mathbf{U}_0 \exp\left[i(\mathbf{k} \cdot \mathbf{x} - \omega t)\right]$. Show that $\omega = \pm 2\Omega\cos\theta$ where the angle θ is to be determined.

Paper 3, Section II

18C Fluid Dynamics

A layer of thickness h_1 of a fluid of density ρ_1 is located above a layer of thickness h_2 of a fluid of density $\rho_2 > \rho_1$. The two-fluid system is bounded by two impenetrable surfaces at $y = h_1$ and $y = -h_2$ and is assumed to be two-dimensional (i.e. independent of z). The fluid is subsequently perturbed, and the interface between the two fluids is denoted $y = \eta(x, t)$.

(a) Assuming irrotational motion in each fluid, state the equations and boundary conditions satisfied by the flow potentials, φ_1 and φ_2.

(b) The interface is perturbed by small-amplitude waves of the form $\eta = \eta_0 e^{i(kx - \omega t)}$, with $\eta_0 k \ll 1$. State the equations and boundary conditions satisfied by the linearised system.

(c) Calculate the dispersion relation of the waves relating the frequency ω to the wavenumber k.

UNIVERSITY OF
CAMBRIDGE

Paper 1, Section I
5B   Fluid Dynamics
       Consider a spherical bubble of radius a in an inviscid fluid in the absence of gravity. The flow at infinity is at rest and the bubble undergoes translation with velocity $\mathbf{U} = U(t)\hat{\mathbf{x}}$. We assume that the flow is irrotational and derives from a potential given in spherical coordinates by

$$\phi(r,\theta) = U(t)\frac{a^3}{2r^2}\cos\theta,$$

where θ is measured with respect to $\hat{\mathbf{x}}$. Compute the force, $\mathbf{F}$, acting on the bubble. Show that the formula for $\mathbf{F}$ can be interpreted as the acceleration force of a fraction $\alpha < 1$ of the fluid displaced by the bubble, and determine the value of α.

Paper 2, Section I
7B   Fluid Dynamics
       Consider the two-dimensional velocity field $\mathbf{u} = (u, v)$ with

$$u(x,y) = x^2 - y^2, \quad v(x,y) = -2xy.$$

   (i) Show that the flow is incompressible and irrotational.

   (ii) Derive the velocity potential, ϕ, and the streamfunction, ψ.

   (iii) Plot all streamlines passing through the origin.

   (iv) Show that the complex function $w = \phi + i\psi$ (where $i^2 = -1$) can be written solely as a function of the complex coordinate $z = x + iy$ and determine that function.

Paper 1, Section II
17B Fluid Dynamics

A fluid layer of depth h_1 and dynamic viscosity μ_1 is located underneath a fluid layer of depth h_2 and dynamic viscosity μ_2. The total fluid system of depth $h = h_1 + h_2$ is positioned between a stationary rigid plate at $y = 0$ and a rigid plate at $y = h$ moving with speed $\mathbf{U} = U\hat{\mathbf{x}}$, where U is constant. Ignore the effects of gravity.

(i) Using dimensional analysis only, and the fact that the stress should be linear in U, derive the expected form of the shear stress acted by the fluid on the plate at $y = 0$ as a function of U, h_1, h_2, μ_1 and μ_2.

(ii) Solve for the unidirectional velocity profile between the two plates. State clearly all boundary conditions you are using to solve this problem.

(iii) Compute the exact value of the shear stress acted by the fluid on the plate at $y = 0$. Compare with the results in (i).

(iv) What is the condition on the viscosity of the bottom layer, μ_1, for the stress in (iii) to be *smaller* than it would be if the fluid had constant viscosity μ_2 in both layers?

(v) Show that the stress acting on the plate at $y = h$ is equal and opposite to the stress on the plate at $y = 0$ and justify this result physically.

Paper 4, Section II
18B Fluid Dynamics

Consider a steady inviscid, incompressible fluid of constant density ρ in the absence of external body forces. A cylindrical jet of area A and speed U impinges fully on a stationary sphere of radius R with $A < \pi R^2$. The flow is assumed to remain axisymmetric and be deflected into a conical sheet of vertex angle $\alpha > 0$.

(i) Show that the speed of the fluid in the conical sheet is constant.

(ii) Use conservation of mass to show that the width $d(r)$ of the fluid sheet at a distance $r \gg R$ from point of impact is given by

$$d(r) = \frac{A}{2\pi r \sin \alpha}.$$

(iii) Use Euler's equation to derive the momentum integral equation

$$\iint_S (pn_i + \rho n_j u_j u_i)\, dS = 0,$$

for a closed surface S with normal $\mathbf{n}$ where u_m is the mth component of the velocity field in cartesian coordinates and p is the pressure.

(iv) Use the result from (iii) to calculate the net force on the sphere.

Paper 3, Section II

18B Fluid Dynamics

A source of sound induces a travelling wave of pressure above the free surface of a fluid located in the $z < 0$ domain as

$$p = p_{atm} + p_0 \cos(kx - \omega t),$$

with $p_0 \ll p_{atm}$. Here k and ω are fixed real numbers. We assume that the flow induced in the fluid is irrotational.

(i) State the linearized equation of motion for the fluid and the free surface, $z = h(x,t)$, with all boundary conditions.

(ii) Solve for the velocity potential, $\phi(x,z,t)$, and the height of the free surface, $h(x,t)$. Verify that your solutions are dimensionally correct.

(iii) Interpret physically the behaviour of the solution when $\omega^2 = gk$.

UNIVERSITY OF
CAMBRIDGE

11

Paper 1, Section I
5B   Fluid Dynamics
 Constant density viscous fluid with dynamic viscosity μ flows in a two-dimensional horizontal channel of depth h. There is a constant pressure gradient $G > 0$ in the horizontal x−direction. The upper horizontal boundary at $y = h$ is driven at constant horizontal speed $U > 0$, with the lower boundary being held at rest. Show that the steady fluid velocity u in the x−direction is

$$u = \frac{-G}{2\mu}y(h - y) + \frac{Uy}{h}.$$

Show that it is possible to have $du/dy < 0$ at some point in the flow for sufficiently large pressure gradient. Derive a relationship between G and U so that there is no net volume flux along the channel. For the flow with no net volume flux, sketch the velocity profile.

Paper 2, Section I
7B   Fluid Dynamics
 Consider the steady two-dimensional fluid velocity field

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \epsilon & -\gamma \\ \gamma & -\epsilon \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\epsilon \geqslant 0$ and $\gamma \geqslant 0$. Show that the fluid is incompressible. The streamfunction ψ is defined by $\mathbf{u} = \mathbf{\nabla} \times \mathbf{\Psi}$, where $\mathbf{\Psi} = (0, 0, \psi)$. Show that ψ is given by

$$\psi = \epsilon xy - \frac{\gamma}{2}(x^2 + y^2).$$

Hence show that the streamlines are defined by

$$(\epsilon - \gamma)(x + y)^2 - (\epsilon + \gamma)(x - y)^2 = C,$$

for C a constant. For each of the three cases below, sketch the streamlines and briefly describe the flow.

(i) $\epsilon = 1$, $\gamma = 0$,

(ii) $\epsilon = 0$, $\gamma = 1$,

(iii) $\epsilon = 1$, $\gamma = 1$.

Paper 1, Section II

17B   Fluid Dynamics

Consider the purely two-dimensional steady flow of an inviscid incompressible constant density fluid in the absence of body forces. For velocity $\mathbf{u}$, the vorticity is $\boldsymbol{\nabla} \times \mathbf{u} = \boldsymbol{\omega} = (0, 0, \omega)$. Show that

$$\mathbf{u} \times \boldsymbol{\omega} = \boldsymbol{\nabla} \left[\frac{p}{\rho} + \frac{1}{2} |\mathbf{u}|^2 \right],$$

where p is the pressure and ρ is the fluid density. Hence show that, if ω is a constant in both space and time,

$$\frac{1}{2} |\mathbf{u}|^2 + \omega \, \psi + \frac{p}{\rho} = C,$$

where C is a constant and ψ is the streamfunction. Here, ψ is defined by $\mathbf{u} = \boldsymbol{\nabla} \times \boldsymbol{\Psi}$, where $\boldsymbol{\Psi} = (0, 0, \psi)$.

Fluid in the annular region $a < r < 2a$ has constant (in both space and time) vorticity ω. The streamlines are concentric circles, with the fluid speed zero on $r = 2a$ and $V > 0$ on $r = a$. Calculate the velocity field, and hence show that

$$\omega = \frac{-2V}{3a}.$$

Deduce that the pressure difference between the outer and inner edges of the annular region is

$$\Delta p = \left(\frac{15 - 16 \ln 2}{18} \right) \rho V^2.$$

[*Hint: Note that in cylindrical polar coordinates (r, ϕ, z), the curl of a vector field*

$\mathbf{A}(r, \phi) = [a(r, \phi), b(r, \phi), c(r, \phi)]$ *is*

$$\boldsymbol{\nabla} \times \mathbf{A} = \left[\frac{1}{r} \frac{\partial c}{\partial \phi}, -\frac{\partial c}{\partial r}, \frac{1}{r} \left(\frac{\partial (rb)}{\partial r} - \frac{\partial a}{\partial \phi} \right) \right]. \quad]$$

Paper 4, Section II

18B Fluid Dynamics

Consider a layer of fluid of constant density ρ and equilibrium depth h_0 in a rotating frame of reference, rotating at constant angular velocity Ω about the vertical z-axis. The equations of motion are

$$
\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x}, \\
\frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y}, \\
0 &= -\frac{\partial p}{\partial z} - \rho g,
\end{aligned}
$$

where p is the fluid pressure, u and v are the fluid velocities in the x-direction and y-direction respectively, $f = 2\Omega$, and g is the constant acceleration due to gravity. You may also assume that the horizontal extent of the layer is sufficiently large so that the layer may be considered to be shallow, such that vertical velocities may be neglected.

By considering mass conservation, show that the depth $h(x, y, t)$ of the layer satisfies

$$
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0.
$$

Now assume that $h = h_0 + \eta(x, y, t)$, where $|\eta| \ll h_0$. Show that the (linearised) potential vorticity $\mathbf{Q} = Q\hat{\mathbf{z}}$, defined by

$$
Q = \zeta - \eta\frac{f}{h_0}, \text{ where } \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
$$

and $\hat{\mathbf{z}}$ is the unit vector in the vertical z-direction, is a constant in time, i.e. $Q = Q_0(x, y)$.

When $Q_0 = 0$ everywhere, establish that the surface perturbation η satisfies

$$
\frac{\partial^2 \eta}{\partial t^2} - gh_0\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) + f^2\eta = 0,
$$

and show that this equation has wave-like solutions $\eta = \eta_0\cos[k(x - ct)]$ when c and k are related through a dispersion relation to be determined. Show that, to leading order, the trajectories of fluid particles for these waves are ellipses. Assuming that $\eta_0 > 0$, $k > 0$, $c > 0$ and $f > 0$, sketch the fluid velocity when $k(x - ct) = n\pi/2$ for $n = 0, 1, 2, 3$.

Paper 3, Section II

18B Fluid Dynamics

A bubble of gas occupies the spherical region $r \leqslant R(t)$, and an incompressible irrotational liquid of constant density ρ occupies the outer region $r \geqslant R$, such that as $r \to \infty$ the liquid is at rest with constant pressure p_∞. Briefly explain why it is appropriate to use a velocity potential $\phi(r, t)$ to describe the liquid velocity $\mathbf{u}$.

By applying continuity of velocity across the gas-liquid interface, show that the liquid pressure (for $r \geqslant R$) satisfies

$$\frac{p}{\rho} + \frac{1}{2}\left(\frac{R^2 \dot{R}}{r^2}\right)^2 - \frac{1}{r}\frac{d}{dt}\left(R^2 \dot{R}\right) = \frac{p_\infty}{\rho}, \quad \text{where} \quad \dot{R} = \frac{dR}{dt}.$$

Show that the excess pressure $p_s - p_\infty$ at the bubble surface $r = R$ is

$$p_s - p_\infty = \frac{\rho}{2}\left(3\dot{R}^2 + 2R\ddot{R}\right), \quad \text{where} \quad \ddot{R} = \frac{d^2 R}{dt^2},$$

and hence that

$$p_s - p_\infty = \frac{\rho}{2R^2}\frac{d}{dR}\left(R^3 \dot{R}^2\right).$$

The pressure $p_g(t)$ inside the gas bubble satisfies the equation of state

$$p_g V^{4/3} = C,$$

where C is a constant, and $V(t)$ is the bubble volume. At time $t = 0$ the bubble is at rest with radius $R = a$. If the bubble then expands and comes to rest at $R = 2a$, determine the required gas pressure p_0 at $t = 0$ in terms of p_∞.

[You may assume that there is contact between liquid and gas for all time, that all motion is spherically symmetric about the origin $r = 0$, and that there is no body force. You may also assume Bernoulli's integral of the equation of motion to determine the liquid pressure p:

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 = A(t),$$

where $\phi(r, t)$ is the velocity potential.]

Paper 1, Section I
5A   Fluid Dynamics

A two-dimensional flow is given by

$$\mathbf{u} = (x, -y + t).$$

Show that the flow is both irrotational and incompressible. Find a stream function $\psi(x, y)$ such that $\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$. Sketch the streamlines at $t = 0$.

Find the pathline of a fluid particle that passes through (x_0, y_0) at $t = 0$ in the form $y = f(x, x_0, y_0)$ and sketch the pathline for $x_0 = 1$, $y_0 = 1$.

Paper 2, Section I
7A   Fluid Dynamics

An incompressible, inviscid fluid occupies the region beneath the free surface $y = \eta(x, t)$ and moves with a velocity field determined by the velocity potential $\phi(x, y, t)$. Gravity acts in the $-y$ direction. You may assume Bernoulli's integral of the equation of motion:

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 + gy = F(t).$$

Give the kinematic and dynamic boundary conditions that must be satisfied by ϕ on $y = \eta(x, t)$.

In the absence of waves, the fluid has constant uniform velocity U in the x direction. Derive the linearised form of the boundary conditions for small amplitude waves.

Assume that the free surface and velocity potential are of the form:

$$\eta = ae^{i(kx-\omega t)}$$
$$\phi = Ux + ibe^{ky}e^{i(kx-\omega t)}$$

(where implicitly the real parts are taken). Show that

$$(\omega - kU)^2 = gk.$$

Paper 1, Section II

17A Fluid Dynamics

    Starting from the Euler momentum equation, derive the form of Bernoulli's equation appropriate for an unsteady irrotational motion of an inviscid incompressible fluid.

    Water of density ρ is driven through a horizontal tube of length L and internal radius a from a water-filled balloon attached to one end of the tube. Assume that the pressure exerted by the balloon is proportional to its current volume (in excess of atmospheric pressure). Also assume that water exits the tube at atmospheric pressure, and that gravity may be neglected. Show that the time for the balloon to empty does not depend on its initial volume. Find the maximum speed of water exiting the pipe.

Paper 4, Section II

18A Fluid Dynamics

    The axisymmetric, irrotational flow generated by a solid sphere of radius a translating at velocity U in an inviscid, incompressible fluid is represented by a velocity potential $\phi(r,\theta)$. Assume the fluid is at rest far away from the sphere. Explain briefly why $\nabla^2\phi = 0$.

    By trying a solution of the form $\phi(r,\theta) = f(r)\,g(\theta)$, show that

$$\phi = -\frac{Ua^3\cos\theta}{2r^2}$$

and write down the fluid velocity.

    Show that the total kinetic energy of the fluid is $kMU^2/4$ where M is the mass of the sphere and k is the ratio of the density of the fluid to the density of the sphere.

    A heavy sphere (i.e. $k < 1$) is released from rest in an inviscid fluid. Determine its speed after it has fallen a distance h in terms of M, k, g and h.

    Note, in spherical polars:

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial r}\mathbf{e_r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)\,.$$

UNIVERSITY OF
CAMBRIDGE

16

Paper 3, Section II

18A   Fluid Dynamics

A layer of incompressible fluid of density ρ and viscosity μ flows steadily down a plane inclined at an angle θ to the horizontal. The layer is of uniform thickness h measured perpendicular to the plane and the viscosity of the overlying air can be neglected. Using coordinates x parallel to the plane (in steepest downwards direction) and y normal to the plane, write down the equations of motion and the boundary conditions on the plane and on the free top surface. Determine the pressure and velocity fields and show that the volume flux down the plane is

$$\frac{\rho g h^3 \sin\theta}{3\mu}.$$

Consider now the case where a second layer of fluid, of uniform thickness αh, viscosity $\beta\mu$ and density ρ, flows steadily on top of the first layer. Explain why one of the appropriate boundary conditions between the two fluids is

$$\mu\frac{\partial}{\partial y}u(h_-) = \beta\mu\frac{\partial}{\partial y}u(h_+)\,,$$

where u is the component of velocity in the x direction and h_- and h_+ refer to just below and just above the boundary respectively. Determine the velocity field in each layer.

Paper 1, Section I
5A   Fluid Dynamics
   Viscous fluid, with viscosity μ and density ρ flows along a straight circular pipe of radius R. The average velocity of the flow is U. Define a Reynolds number for the flow.

   The flow is driven by a constant pressure gradient $-G > 0$ along the pipe and the velocity is parallel to the axis of the pipe with magnitude $u(r)$ that satisfies

$$\frac{\mu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = -G,$$

where r is the radial distance from the axis.

   State the boundary conditions on u and find the velocity as a function of r assuming that it is finite on the axis $r = 0$. Hence, show that the shear stress τ at the pipe wall is independent of the viscosity. Why is this the case?

Paper 2, Section I
7A   Fluid Dynamics
   Starting from Euler's equation for the motion of an inviscid fluid, derive the vorticity equation in the form

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u}.$$

Deduce that an initially irrotational flow remains irrotational.

   Consider a plane flow that at time $t = 0$ is described by the streamfunction

$$\psi = x^2 + y^2.$$

Calculate the vorticity everywhere at times $t > 0$.

Paper 1, Section II

17A Fluid Dynamics

Consider inviscid, incompressible fluid flow confined to the (x, y) plane. The fluid has density ρ, and gravity can be neglected. Using the conservation of volume flux, determine the velocity potential $\phi(r)$ of a point source of strength m, in terms of the distance r from the source.

Two point sources each of strength m are located at $\boldsymbol{x}_+ = (0, a)$ and $\boldsymbol{x}_- = (0, -a)$. Find the velocity potential of the flow.

Show that the flow in the region $y \geqslant 0$ is equivalent to the flow due to a source at $\boldsymbol{x}_+$ and a fixed boundary at $y = 0$.

Find the pressure on the boundary $y = 0$ and hence determine the force on the boundary.

[*Hint: you may find the substitution $x = a \tan \theta$ useful for the calculation of the pressure.*]

Paper 4, Section II

18A Fluid Dynamics

The equations governing the flow of a shallow layer of inviscid liquid of uniform depth H rotating with angular velocity $\frac{1}{2}f$ about the vertical z-axis are

$$
\begin{aligned}
\frac{\partial u}{\partial t} - fv &= -g\frac{\partial \eta}{\partial x}\,, \\
\frac{\partial v}{\partial t} + fu &= -g\frac{\partial \eta}{\partial y}\,, \\
\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0\,,
\end{aligned}
$$

where u, v are the x- and y-components of velocity, respectively, and η is the elevation of the free surface. Show that these equations imply that

$$
\frac{\partial q}{\partial t} = 0\,, \quad \text{where} \quad q = \omega - \frac{f\eta}{H} \quad \text{and} \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,.
$$

Consider an initial state where there is flow in the y-direction given by

$$
u = \eta = 0\,, \quad -\infty < x < \infty
$$

$$
v = \begin{cases} \frac{g}{2f}e^{2x}, & x < 0 \\ -\frac{g}{2f}e^{-2x}, & x > 0\,. \end{cases}
$$

Find the initial potential vorticity.

Show that when this initial state adjusts, there is a final steady state independent of y in which η satisfies

$$
\frac{\partial^2 \eta}{\partial x^2} - \frac{\eta}{a^2} = \begin{cases} e^{2x}, & x < 0 \\ e^{-2x}, & x > 0\,, \end{cases}
$$

where $a^2 = gH/f^2$.

In the case $a = 1$, find the final free surface elevation that is finite at large $|x|$ and which is continuous and has continuous slope at $x = 0$, and show that it is negative for all x.

Paper 3, Section II

18A Fluid Dynamics

A rigid circular cylinder of radius a executes small amplitude oscillations with velocity $U(t)$ in a direction perpendicular to its axis, while immersed in an inviscid fluid of density ρ contained within a larger concentric fixed cylinder of radius b. Gravity is negligible. Neglecting terms quadratic in the amplitude, determine the boundary condition on the velocity on the inner cylinder, and calculate the velocity potential of the induced flow.

With the same approximations show that the difference in pressures on the surfaces of the two cylinders has magnitude

$$\rho \frac{\mathrm{d}U}{\mathrm{d}t} \frac{a(b-a)}{b+a} \cos\theta,$$

where θ is the azimuthal angle measured from the direction of U.

Paper 1, Section I

5B   Fluid Dynamics

Inviscid fluid is contained in a square vessel with sides of length πL lying between $x = 0, \pi L$, $y = 0, \pi L$. The base of the container is at $z = -H$ where $H \gg L$ and the horizontal surface is at $z = 0$ when the fluid is at rest. The variation of pressure of the air above the fluid may be neglected.

Small amplitude surface waves are excited in the vessel.

(i) Now let $H \to \infty$. Explain why on dimensional grounds the frequencies ω of such waves are of the form

$$\omega = \left(\frac{\gamma g}{L}\right)^{\frac{1}{2}}$$

for some positive dimensionless constants γ, where g is the gravitational acceleration.

It is given that the velocity potential ϕ is of the form

$$\phi(x, y, z) \approx C \cos(mx/L) \cos(ny/L) e^{\gamma z/L}$$

where m and n are integers and C is a constant.

(ii) Why do cosines, rather than sines, appear in this expression?

(iii) Give an expression for γ in terms of m and n.

(iv) Give all possible values that γ^2 can take between 1 and 10 inclusive. How many different solutions for ϕ correspond to each of these values of γ^2?

Paper 2, Section I

7D   Fluid Dynamics

A body of volume V lies totally submerged in a motionless fluid of uniform density ρ. Show that the force $\mathbf{F}$ on the body is given by

$$\mathbf{F} = -\int_S (p - p_0) \, \mathbf{n} \, dS$$

where p is the pressure in the fluid and p_0 is atmospheric pressure. You may use without proof the generalised divergence theorem in the form

$$\int_S \phi \, \mathbf{n} \, dS = \int_V \boldsymbol{\nabla} \phi \, dV.$$

Deduce that

$$\mathbf{F} = \rho g V \hat{\mathbf{z}},$$

where $\hat{\mathbf{z}}$ is the vertically upward unit vector. Interpret this result.

Paper 1, Section II
17B Fluid Dynamics

A spherical bubble in an incompressible fluid of density ρ has radius $a(t)$. Write down an expression for the velocity field at a radius $R \geqslant a$.

The pressure far from the bubble is p_∞. What is the pressure at radius R?

Find conditions on a and its time derivatives that ensure that the maximum pressure in the fluid is reached at a radius $R_{\max}$ where $a < R_{\max} < \infty$. Give an expression for this maximum pressure when the conditions hold.

Give the most general form of $a(t)$ that ensures that the pressure at $R = a(t)$ is p_∞ for all time.

Paper 3, Section II
18D Fluid Dynamics

Water of constant density ρ flows steadily through a long cylindrical tube, the wall of which is elastic. The exterior radius of the tube at a distance z along the tube, $r(z)$, is determined by the pressure in the tube, $p(z)$, according to

$$r(z) = r_0 + b(p(z) - p_0),$$

where r_0 and p_0 are the radius and pressure far upstream ($z \to -\infty$), and b is a positive constant.

The interior radius of the tube is $r(z) - h(z)$, where $h(z)$, the thickness of the wall, is a given slowly varying function of z which is zero at both ends of the pipe. The velocity of the water in the pipe is $u(z)$ and the water enters the pipe at velocity V.

Show that $u(z)$ satisfies

$$H = 1 - v^{-\frac{1}{2}} + \tfrac{1}{4}k(1 - v^2),$$

where $H = \dfrac{h}{r_0}$, $v = \dfrac{u}{V}$ and $k = \dfrac{2b\rho V^2}{r_0}$. Sketch the graph of H against v.

Let H_m be the maximum value of H in the tube. Show that the flow is only possible if H_m does not exceed a certain critical value H_c. Find H_c in terms of k.

Show that, under conditions to be determined (which include a condition on the value of k), the water can leave the pipe with speed less than V.

Paper 4, Section II

18D Fluid Dynamics

Show that an irrotational incompressible flow can be determined from a velocity potential ϕ that satisfies $\nabla^2\phi = 0$.

Given that the general solution of $\nabla^2\phi = 0$ in plane polar coordinates is

$$\phi = \sum_{n=-\infty}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)r^n + c\log r + b\theta \,,$$

obtain the corresponding fluid velocity.

A two-dimensional irrotational incompressible fluid flows past the circular disc with boundary $r = a$. For large r, the flow is uniform and parallel to the x-axis ($x = r\cos\theta$). Write down the boundary conditions for large r and on $r = a$, and hence derive the velocity potential in the form

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{\kappa\theta}{2\pi}\,,$$

where κ is the circulation.

Show that the acceleration of the fluid at $r = a$ and $\theta = 0$ is

$$\frac{\kappa}{2\pi a^2}\left(-\frac{\kappa}{2\pi a}\mathbf{e}_r - 2U\mathbf{e}_\theta\right).$$

Paper 1, Section I
5B   Fluid Dynamics

A planar solenoidal velocity field has the velocity potential

$$\phi(x, y, t) = xe^{-t} + ye^{t}.$$

Find and sketch (i) the streamlines at $t = 0$; (ii) the pathline that passes through the origin at $t = 0$; (iii) the locus at $t = 0$ of points that pass through the origin at earlier times (streakline).

Paper 2, Section I
7B   Fluid Dynamics

Write down an expression for the velocity field of a line vortex of strength κ.

Consider N identical line vortices of strength κ arranged at equal intervals round a circle of radius a. Show that the vortices all move around the circle at constant angular velocity $(N - 1)\kappa/(4\pi a^2)$.

Paper 1, Section II
17B   Fluid Dynamics

Starting with the Euler equations for an inviscid incompressible fluid, derive Bernoulli's theorem for unsteady irrotational flow.

Inviscid fluid of density ρ is contained within a U-shaped tube with the arms vertical, of height h and with the same (unit) cross-section. The ends of the tube are closed. In the equilibrium state the pressures in the two arms are p_1 and p_2 and the heights of the fluid columns are ℓ_1, ℓ_2.

The fluid in arm 1 is displaced upwards by a distance ξ (and in the other arm downward by the same amount). In the subsequent evolution the pressure above each column may be taken as inversely proportional to the length of tube above the fluid surface. Using Bernoulli's theorem, show that $\xi(t)$ obeys the equation

$$\rho(\ell_1 + \ell_2)\ddot{\xi} + \frac{p_1\xi}{h - \ell_1 - \xi} + \frac{p_2\xi}{h - \ell_2 + \xi} + 2\rho g\xi = 0.$$

Now consider the special case $\ell_1 = \ell_2 = \ell_0$, $p_1 = p_2 = p_0$. Construct a first integral of this equation and hence give an expression for the total kinetic energy $\rho\ell_0\dot{\xi}^2$ of the flow in terms of ξ and the maximum displacement ξ_{max}.

Paper 3, Section II

18B Fluid Dynamics

Write down the exact kinematic and dynamic boundary conditions that apply at the free surface $z = \eta(x,t)$ of a fluid layer in the presence of gravity in the z-direction. Show how these may be approximated for small disturbances of a hydrostatic state about $z = 0$. (The flow of the fluid is in the (x,z)-plane and may be taken to be irrotational, and the pressure at the free surface may be assumed to be constant.)

Fluid of density ρ fills the region $0 > z > -h$. At $z = -h$ the z-component of the velocity is $\epsilon \mathrm{Re}(e^{i\omega t} \cos kx)$, where $|\epsilon| \ll 1$. Find the resulting disturbance of the free surface, assuming this to be small. Explain physically why your answer has a singularity for a particular value of ω^2.

Paper 4, Section II

18B Fluid Dynamics

Write down the velocity potential for a line source flow of strength m located at $(r,\theta) = (d,0)$ in polar coordinates (r,θ) and derive the velocity components u_r, u_θ.

A two-dimensional flow field consists of such a source in the presence of a circular cylinder of radius a $(a < d)$ centred at the origin. Show that the flow field outside the cylinder is the sum of the original source flow, together with that due to a source of the same strength at $(a^2/d, 0)$ and another at the origin, of a strength to be determined.

Use Bernoulli's law to find the pressure distribution on the surface of the cylinder, and show that the total force exerted on it is in the x-direction and of magnitude

$$\frac{m^2 \rho}{2\pi^2} \int_0^{2\pi} \frac{a d^2 \sin^2\theta \, \cos\theta}{(a^2 + d^2 - 2 \, a d \, \cos\theta)^2} \ d\theta \,,$$

where ρ is the density of the fluid. Without evaluating the integral, show that it is positive. Comment on the fact that the force on the cylinder is therefore *towards* the source.

Paper 1, Section I
5D   Fluid Dynamics

A steady velocity field $\mathbf{u} = (u_r, u_\theta, u_z)$ is given in cylindrical polar coordinates (r, θ, z) by

$$u_r = -\alpha r, \quad u_\theta = \frac{\gamma}{r}(1 - e^{-\beta r^2}), \quad u_z = 2\alpha z \,,$$

where α, β, γ are positive constants.

Show that this represents a possible flow of an incompressible fluid, and find the vorticity $\boldsymbol{\omega}$.

Show further that

$$\operatorname{curl}(\mathbf{u} \wedge \boldsymbol{\omega}) = -\nu \nabla^2 \boldsymbol{\omega}$$

for a constant ν which should be calculated.

[*The divergence and curl operators in cylindrical polars are given by*

$$
\begin{aligned}
\operatorname{div} \mathbf{u} &= \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \\
\operatorname{curl} \mathbf{u} &= \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right) \\
and, when \quad \boldsymbol{\omega} &= [0, 0, \omega(r, z)], \\
\nabla^2 \boldsymbol{\omega} &= \left[0, 0, \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r} \right) + \frac{\partial^2 \omega}{\partial z^2} \right]. \quad]
\end{aligned}
$$

Paper 2, Section I
8D   Fluid Dynamics

A fireman's hose full of water has cross-sectional area A_0, apart from a smooth contraction to the outlet nozzle which has cross-sectional area $A_1 < A_0$. The volume flow rate of water through the hose is Q.

Use Bernoulli's equation to calculate the pressure in the main part of the tube (relative to atmospheric pressure). Then use the integral momentum equation in the direction of the flow to show that the force F that the fireman has to exert on the nozzle to keep it still is given by

$$F = \frac{\rho Q^2}{2 A_0} \left(\frac{A_0}{A_1} - 1 \right)^2 ,$$

where ρ is the density of water.

Paper 1, Section II
17D Fluid Dynamics

A canal has uniform width and a bottom that is horizontal apart from a localised slowly-varying hump of height $D(x)$ whose maximum value is D_{max}. Far upstream the water has depth h_1 and velocity u_1. Show that the depth $h(x)$ of the water satisfies the following equation:

$$\frac{D(x)}{h_1} = 1 - \frac{h}{h_1} - \frac{F}{2} \left(\frac{h_1^2}{h^2} - 1 \right) ,$$

where $F = u_1^2 / g h_1$.

Describe qualitatively how $h(x)$ varies as the flow passes over the hump in the three cases

$$
\begin{array}{lll}
\text{(i)} & F < 1 & \text{and } D_{max} < D^* \\
\text{(ii)} & F > 1 & \text{and } D_{max} < D^* \\
\text{(iii)} & D_{max} & = D^*,
\end{array}
$$

where $\quad D^* = h_1 \left(1 - \frac{3}{2} F^{1/3} + \frac{1}{2} F \right).$

Calculate the water depth far downstream in case (iii) when $F < 1$.

Paper 3, Section II
18D Fluid Dynamics

Starting from Euler's equations for an inviscid incompressible fluid of density ρ with no body force, undergoing irrotational motion, show that the pressure p is given by

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\nabla \phi\right)^2 = F(t),$$

for some function $F(t)$, where ϕ is the velocity potential.

The fluid occupies an infinite domain and contains a spherical gas bubble of radius $R(t)$ in which the pressure is p_g. The pressure in the fluid at infinity is p_∞.

Show that

$$\ddot{R}\,R + \frac{3}{2}\,\dot{R}^2 \;=\; \frac{p_g - p_\infty}{\rho}\,.$$

The bubble contains a fixed mass M of gas in which

$$p_g = C\left(M/R^3\right)^2$$

for a constant C. At time $t = 0, R = R_0, \dot{R} = 0$ and $p_g = p_\infty/2$. Show that

$$\dot{R}^2 R^3 = \frac{p_\infty}{\rho}\left[R_0^3 - \frac{R_0^6}{3R^3} - \frac{2}{3}\,R^3\right],$$

and deduce that the bubble radius oscillates between R_0 and $R_0/2^{1/3}$.

Paper 4, Section II

18D Fluid Dynamics

An inviscid incompressible fluid occupies a rectangular tank with vertical sides at $x = 0, a$ and $y = 0, b$ and a horizontal bottom at $z = -h$. The undisturbed free surface is at $z = 0$.

(i) Write down the equations and boundary conditions governing small amplitude free oscillations of the fluid, neglecting surface tension, and show that the frequencies ω of such oscillations are given by

$$\frac{\omega^2}{g} = k \tanh kh, \text{ where } k^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \tag{1}$$

for non-negative integers m, n, which cannot both be zero.

(ii) The free surface is now acted on by a small external pressure

$$p = \epsilon \rho g h \, \sin \Omega t \cos \frac{m\pi x}{a} \, \cos \frac{n\pi y}{b},$$

where $\epsilon \ll 1$. Calculate the corresponding oscillation of the free surface when Ω is not equal to the quantity ω given by (1).

Why does your solution break down as $\Omega \to \omega$?

1/I/5B   **Fluid Dynamics**

Verify that the two-dimensional flow given in Cartesian coordinates by

$$\mathbf{u} = (e^y \sinh x, -e^y \cosh x)$$

satisfies $\nabla \cdot \mathbf{u} = 0$. Find the stream function $\psi(x, y)$. Sketch the streamlines.

1/II/17B   **Fluid Dynamics**

Two incompressible fluids flow in infinite horizontal streams, the plane of contact being $z = 0$, with z positive upwards. The flow is given by

$$\mathbf{U}(\mathbf{r}) = \begin{cases} U_2\hat{\mathbf{e}}_x, & z > 0; \\ U_1\hat{\mathbf{e}}_x, & z < 0, \end{cases}$$

where $\hat{\mathbf{e}}_x$ is the unit vector in the positive x direction. The upper fluid has density ρ_2 and pressure $p_0 - g\rho_2 z$, the lower has density ρ_1 and pressure $p_0 - g\rho_1 z$, where p_0 is a constant and g is the acceleration due to gravity.

(i) Consider a perturbation to the flat surface $z = 0$ of the form

$$z \equiv \zeta(x, y, t) = \zeta_0 e^{i(kx + \ell y) + st} \ .$$

State the kinematic boundary conditions on the velocity potentials ϕ_i that hold on the interface in the two domains, and show by linearising in ζ that they reduce to

$$\frac{\partial \phi_i}{\partial z} = \frac{\partial \zeta}{\partial t} + U_i \frac{\partial \zeta}{\partial x} \quad (z = 0, \ i = 1, 2) \,.$$

(ii) State the dynamic boundary condition on the perturbed interface, and show by linearising in ζ that it reduces to

$$\rho_1 \left(U_1 \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_1}{\partial t} + g\zeta \right) = \rho_2 \left(U_2 \frac{\partial \phi_2}{\partial x} + \frac{\partial \phi_2}{\partial t} + g\zeta \right) \quad (z = 0) \,.$$

(iii) Use the velocity potentials

$$\phi_1 = U_1 x + A_1 e^{qz} e^{i(kx + \ell y) + st} \ , \quad \phi_2 = U_2 x + A_2 e^{-qz} e^{i(kx + \ell y) + st},$$

where $q = \sqrt{k^2 + \ell^2}$, and the conditions in (i) and (ii) to perform a stability analysis. Show that the relation between s, k and ℓ is

$$s = -ik \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[\frac{k^2 \rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} - \frac{qg(\rho_1 - \rho_2)}{\rho_1 + \rho_2} \right]^{1/2} \,.$$

Find the criterion for instability.

UNIVERSITY OF
CAMBRIDGE   32

2/I/8B   Fluid Dynamics

(i) Show that for a two-dimensional incompressible flow $(u(x,y), v(x,y), 0)$, the vorticity is given by $\boldsymbol{\omega} \equiv \omega_z \hat{\mathbf{e}}_z = (0, 0, -\nabla^2 \psi)$ where ψ is the stream function.

(ii) Express the z-component of the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$$

in terms of the stream function ψ.

3/II/18B   Fluid Dynamics

An ideal liquid contained within a closed circular cylinder of radius a rotates about the axis of the cylinder (assume this axis to be in the vertical z-direction).

(i) Prove that the equation of continuity and the boundary conditions are satisfied by the velocity $\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}$, where $\boldsymbol{\Omega} = \Omega \hat{\mathbf{e}}_z$ is the angular velocity, with $\hat{\mathbf{e}}_z$ the unit vector in the z-direction, which depends only on time, and $\mathbf{r}$ is the position vector measured from a point on the axis of rotation.

(ii) Calculate the angular momentum $\mathbf{M} = \rho \int (\mathbf{r} \times \mathbf{v})dV$ per unit length of the cylinder.

(iii) Suppose the the liquid starts from rest and flows under the action of an external force per unit mass $\mathbf{f} = (\alpha x + \beta y, \gamma x + \delta y, 0)$. By taking the curl of the Euler equation, prove that

$$\frac{d\Omega}{dt} = \frac{1}{2}(\gamma - \beta)\ .$$

(iv) Find the pressure.

4/II/18B   Fluid Dynamics

(i) Starting from Euler's equation for an incompressible fluid show that for potential flow with $\mathbf{u} = \boldsymbol{\nabla}\phi$,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}u^2 + \chi = f(t)\,,$$

where $u = |\mathbf{u}|$, $\chi = p/\rho + V$, the body force per unit mass is $-\boldsymbol{\nabla}V$ and $f(t)$ is an arbitrary function of time.

(ii) Hence show that, for the steady flow of a liquid of density ρ through a pipe of varying cross-section that is subject to a pressure difference $\Delta p = p_1 - p_2$ between its two ends, the mass flow through the pipe per unit time is given by

$$m \equiv \frac{dM}{dt} = S_1 S_2 \sqrt{\frac{2\rho\Delta p}{S_1^2 - S_2^2}}\,,$$

where S_1 and S_2 are the cross-sectional areas of the two ends.

1/I/5D   **Fluid Dynamics**

A steady two-dimensional velocity field is given by

$$\mathbf{u}(x, y) = (\alpha x - \beta y, \, \beta x - \alpha y), \quad \alpha > 0, \quad \beta > 0.$$

(i) Calculate the vorticity of the flow.

(ii) Verify that $\mathbf{u}$ is a possible flow field for an incompressible fluid, and calculate the stream function.

(iii) Show that the streamlines are bounded if and only if $\alpha < \beta$.

(iv) What are the streamlines in the case $\alpha = \beta$?

1/II/17D   **Fluid Dynamics**

Write down the Euler equation for the steady motion of an inviscid, incompressible fluid in a constant gravitational field. From this equation, derive (a) Bernoulli's equation and (b) the integral form of the momentum equation for a fixed control volume V with surface S.

(i) A circular jet of water is projected vertically upwards with speed U_0 from a nozzle of cross-sectional area A_0 at height $z = 0$. Calculate how the speed U and cross-sectional area A of the jet vary with z, for $z \ll U_0^2/2g$.

(ii) A circular jet of speed U and cross-sectional area A impinges axisymmetrically on the vertex of a cone of semi-angle α, spreading out to form an almost parallel-sided sheet on the surface. Choose a suitable control volume and, neglecting gravity, show that the force exerted by the jet on the cone is

$$\rho A U^2 (1 - \cos \alpha).$$

(iii) A cone of mass M is supported, axisymmetrically and vertex down, by the jet of part (i), with its vertex at height $z = h$, where $h \ll U_0^2/2g$. Assuming that the result of part (ii) still holds, show that h is given by

$$\rho A_0 U_0^2 \left(1 - \frac{2gh}{U_0^2} \right)^{\frac{1}{2}} (1 - \cos \alpha) = Mg.$$

2/I/8D   **Fluid Dynamics**

An incompressible, inviscid fluid occupies the region beneath the free surface $y = \eta(x,t)$ and moves with a velocity field given by the velocity potential $\phi(x,y,t)$; gravity acts in the $-y$ direction. Derive the kinematic and dynamic boundary conditions that must be satisfied by ϕ on $y = \eta(x,t)$.

[*You may assume Bernoulli's integral of the equation of motion:*

$$\frac{p}{\rho} + \frac{\partial \phi}{\partial t} + \frac{1}{2}\,|\nabla\phi|^2 + gy \;=\; F(t)\,.]$$

In the absence of waves, the fluid has uniform velocity U in the x direction. Derive the linearised form of the above boundary conditions for small amplitude waves, and verify that they and Laplace's equation are satisfied by the velocity potential

$$\phi \;=\; Ux + Re\{be^{ky}e^{i(kx-\omega t)}\}\,,$$

where $|kb| \ll U$, with a corresponding expression for η, as long as

$$(\omega - kU)^2 \;=\; gk\,.$$

What are the propagation speeds of waves with a given wave-number k?

3/II/18D   Fluid Dynamics

Given that the circulation round every closed material curve in an inviscid, incompressible fluid remains constant in time, show that the velocity field of such a fluid started from rest can be written as the gradient of a potential, ϕ, that satisfies Laplace's equation.

A rigid sphere of radius a moves in a straight line at speed U in a fluid that is at rest at infinity. Using axisymmetric spherical polar coordinates (r, θ), with $\theta = 0$ in the direction of motion, write down the boundary conditions on ϕ and, by looking for a solution of the form $\phi = f(r) \cos \theta$, show that the velocity potential is given by

$$\phi = \frac{-Ua^3 \cos \theta}{2r^2} \, .$$

Calculate the kinetic energy of the fluid.

A rigid sphere of radius a and uniform density ρ_b is submerged in an infinite fluid of density ρ, under the action of gravity. Show that, when the sphere is released from rest, its initial upwards acceleration is

$$\frac{2(\rho - \rho_b)g}{\rho + 2\rho_b} \, .$$

[*Laplace's equation for an axisymmetric scalar field in spherical polars is:*

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) = 0 \, .]$$

4/II/18D   Fluid Dynamics

Starting from Euler's equation for an inviscid, incompressible fluid in the absence of body forces,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho} \nabla p \, ,$$

derive the equation for the vorticity $\boldsymbol{\omega} = \nabla_\wedge \mathbf{u}$.

[*You may assume that* $\nabla_\wedge(\mathbf{a}_\wedge \mathbf{b}) = \mathbf{a}\nabla.\mathbf{b} - \mathbf{b}\nabla.\mathbf{a} + (\mathbf{b}.\nabla)\mathbf{a} - (\mathbf{a}.\nabla)\mathbf{b} \,.]$

Show that, in a two-dimensional flow, vortex lines keep their strength and move with the fluid.

Show that a two-dimensional flow driven by a line vortex of circulation Γ at distance b from a rigid plane wall is the same as if the wall were replaced by another vortex of circulation $-\Gamma$ at the image point, distance b from the wall on the other side. Deduce that the first vortex will move at speed $\Gamma/4\pi b$ parallel to the wall.

A line vortex of circulation Γ moves in a quarter-plane, bounded by rigid plane walls at $x = 0$, $y > 0$ and $y = 0$, $x > 0$. Show that the vortex follows a trajectory whose equation in plane polar coordinates is $r \sin 2\theta = $ constant.

1/I/5A **Fluid Dynamics**

Use the Euler equation for the motion of an inviscid fluid to derive the vorticity equation in the form

$$D\boldsymbol{\omega}/Dt = \boldsymbol{\omega} \cdot \nabla \mathbf{u}.$$

Give a physical interpretation of the terms in this equation and deduce that irrotational flows remain irrotational.

In a plane flow the vorticity at time $t = 0$ has the uniform value $\boldsymbol{\omega}_0 \neq \mathbf{0}$. Find the vorticity everywhere at times $t > 0$.

1/II/17A **Fluid Dynamics**

A point source of fluid of strength m is located at $\mathbf{x}_s = (0, 0, a)$ in inviscid fluid of density ρ. Gravity is negligible. The fluid is confined to the region $z \geqslant 0$ by the fixed boundary $z = 0$. Write down the equation and boundary conditions satisfied by the velocity potential ϕ. Find ϕ.

[*Hint: consider the flow generated in unbounded fluid by the source m together with an 'image source' of equal strength at $\bar{\mathbf{x}}_s = (0, 0, -a)$.*]

Use Bernoulli's theorem, which may be stated without proof, to find the fluid pressure everywhere on $z = 0$. Deduce the magnitude of the hydrodynamic force on the boundary $z = 0$. Determine whether the boundary is attracted toward the source or repelled from it.

2/I/8A **Fluid Dynamics**

Explain what is meant by a *material time derivative*, D/Dt. Show that if the material velocity is $\mathbf{u}(\mathbf{x}, t)$ then

$$D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla.$$

When glass is processed in its liquid state, its temperature, $\theta(\mathbf{x}, t)$, satisfies the equation

$$D\theta/Dt = -\theta.$$

The glass flows in a two-dimensional channel $-1 < y < 1$, $x > 0$ with steady velocity $\mathbf{u} = (1 - y^2, 0)$. At $x = 0$ the glass temperature is maintained at the constant value θ_0. Find the steady temperature distribution throughout the channel.

UNIVERSITY OF
CAMBRIDGE

3/II/18A   **Fluid Dynamics**

State and prove Bernoulli's theorem for a time-dependent irrotational flow of an inviscid fluid.

A large vessel is part-filled with inviscid liquid of density ρ. The pressure in the air above the liquid is maintained at the constant value $P + p_a$, where p_a is atmospheric pressure and $P > 0$. Liquid can flow out of the vessel along a cylindrical tube of length L. The radius a of the tube is much smaller than both L and the linear dimensions of the vessel. Initially the tube is sealed and is full of liquid. At time $t = 0$ the tube is opened and the liquid starts to flow. Assuming that the tube remains full of liquid, that the pressure at the open end of the tube is atmospheric and that P is so large that gravity is negligible, determine the flux of liquid along the tube at time t.

4/II/18A   **Fluid Dynamics**

A rectangular tank has a horizontal base and vertical sides. Viewed from above, the cross-section of the tank is a square of side a. At rest, the depth of water in the tank is h. Suppose that the free-surface is disturbed in such a way that the flow in the water is irrotational. Take the pressure at the free surface as atmospheric. Starting from the appropriate non-linear expressions, obtain free-surface boundary conditions for the velocity potential appropriate for small-amplitude disturbances of the surface.

Show that the governing equations and boundary conditions admit small-amplitude normal mode solutions for which the free-surface elevation above its equilibrium level is everywhere proportional to $e^{i\omega t}$, and find the frequencies, ω, of such modes.

1/I/5E   **Fluid Dynamics**

Explain how a streamfunction ψ can be used to represent in Cartesian Coordinates an incompressible flow in two dimensions. Show that the streamlines are given by $\psi = \text{const.}$

Consider the two-dimensional incompressible flow

$$\mathbf{u}(x, y, t) = (x + \sin t, -y).$$

(a) Find the streamfunction, and hence the streamlines at $t = \dfrac{\pi}{2}$.

(b) Find the path of a fluid particle released at $t = 0$ from $(x_0, 1)$. For what value of x_0 does the particle *not* tend to infinity?

1/II/17E   **Fluid Dynamics**

State Bernoulli's expression for the pressure in an unsteady potential flow with conservative force $-\nabla\chi$.

A spherical bubble in an incompressible liquid of density ρ has radius $R(t)$. If the pressure far from the bubble is p_∞ and inside the bubble is p_b, show that

$$p_b - p_\infty = \rho\left(\frac{3}{2}\dot{R}^2 + R\ddot{R}\right).$$

Calculate the kinetic energy $K(t)$ in the flow outside the bubble, and hence show that

$$\dot{K} = (p_b - p_\infty)\dot{V},$$

where $V(t)$ is the volume of the bubble.

If $p_b(t) = p_\infty V_0/V$, show that

$$K = K_0 + p_\infty\left(V_0\ln\frac{V}{V_0} - V + V_0\right),$$

where $K = K_0$ when $V = V_0$.

Part IB   2005

2/I/8E   **Fluid Dynamics**

For a steady flow of an incompressible fluid of density ρ, show that

$$\mathbf{u} \times \boldsymbol{\omega} = \nabla H \,,$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity and H is to be found. Deduce that H is constant along streamlines.

Now consider a flow in the xy-plane described by a streamfunction $\psi(x, y)$. Evaluate $\mathbf{u} \times \boldsymbol{\omega}$ and deduce from $H = H(\psi)$ that

$$\frac{dH}{d\psi} + \omega = 0.$$

3/II/18E   **Fluid Dynamics**

Consider the velocity potential in plane polar coordinates

$$\phi(r, \theta) = U \left(r + \frac{a^2}{r} \right) \cos\theta + \frac{\kappa\theta}{2\pi}.$$

Find the velocity field and show that it corresponds to flow past a cylinder $r = a$ with circulation κ and uniform flow U at large distances.

Find the distribution of pressure p over the surface of the cylinder. Hence find the x and y components of the force on the cylinder

$$(F_x, F_y) = \int (\cos\theta, \sin\theta) pa \, d\theta.$$

4/II/18E   **Fluid Dynamics**

A fluid of density ρ_1 occupies the region $z > 0$ and a second fluid of density ρ_2 occupies the region $z < 0$. State the equations and boundary conditions that are satisfied by the corresponding velocity potentials ϕ_1 and ϕ_2 and pressures p_1 and p_2 when the system is perturbed so that the interface is at $z = \zeta(x, t)$ and the motion is irrotational.

Seek a set of linearised equations and boundary conditions when the disturbances are proportional to $e^{i(kx - \omega t)}$, and derive the dispersion relation

$$\omega^2 = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} gk,$$

where g is the gravitational acceleration.

1/I/9C   **Fluid Dynamics**

From the general mass-conservation equation, show that the velocity field $\mathbf{u}(\mathbf{x})$ of an incompressible fluid is solenoidal, i.e. that $\nabla \cdot \mathbf{u} = 0$.

Verify that the two-dimensional flow

$$\mathbf{u} = \left(\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2} \right)$$

is solenoidal and find a streamfunction $\psi(x, y)$ such that $\mathbf{u} = (\partial\psi/\partial y, -\partial\psi/\partial x)$.

1/II/20C   **Fluid Dynamics**

A layer of water of depth h flows along a wide channel with uniform velocity $(U, 0)$, in Cartesian coordinates (x, y), with x measured downstream. The bottom of the channel is at $y = -h$, and the free surface of the water is at $y = 0$. Waves are generated on the free surface so that it has the new position $y = \eta(x, t) = a\, e^{i(\omega t - kx)}$.

Write down the equation and the full nonlinear boundary conditions for the velocity potential ϕ (for the perturbation velocity) and the motion of the free surface.

By linearizing these equations about the state of uniform flow, show that

$$\frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} = \frac{\partial\phi}{\partial y}, \qquad \frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x} + g\eta = 0 \qquad \text{on} \quad y = 0,$$

$$\frac{\partial\phi}{\partial y} = 0 \qquad\qquad\qquad \text{on} \quad y = -h,$$

where g is the acceleration due to gravity.

Hence, determine the dispersion relation for small-amplitude surface waves

$$(\omega - kU)^2 = gk\tanh kh.$$

3/I/10C   **Fluid Dynamics**

State Bernoulli's equation for unsteady motion of an irrotational, incompressible, inviscid fluid subject to a conservative body force $-\nabla\chi$.

A long vertical U-tube of uniform cross section contains an inviscid, incompressible fluid whose surface, in equilibrium, is at height h above the base. Derive the equation

$$h\frac{d^2\zeta}{dt^2} + g\zeta = 0$$

governing the displacement ζ of the surface on one side of the U-tube, where t is time and g is the acceleration due to gravity.

3/II/21C   **Fluid Dynamics**

Use separation of variables to determine the irrotational, incompressible flow

$$\mathbf{u} = U\frac{a^3}{r^3}\left(\cos\theta\,\mathbf{e}_r + \tfrac{1}{2}\sin\theta\,\mathbf{e}_\theta\right)$$

around a solid sphere of radius a translating at velocity U along the direction $\theta = 0$ in spherical polar coordinates r and θ.

Show that the total kinetic energy of the fluid is

$$K = \tfrac{1}{4}M_f U^2,$$

where M_f is the mass of fluid displaced by the sphere.

A heavy sphere of mass M is released from rest in an inviscid fluid. Determine its speed after it has fallen through a distance h in terms of M, M_f, g and h.

4/I/8C   **Fluid Dynamics**

Write down the vorticity equation for the unsteady flow of an incompressible, inviscid fluid with no body forces acting.

Show that the flow field

$$\mathbf{u} = (-x,\, x\,\omega(t),\, z - 1)$$

has uniform vorticity of magnitude $\omega(t) = \omega_0 e^t$ for some constant ω_0.

4/II/18C   **Fluid Dynamics**

Use Euler's equation to derive the momentum integral

$$\int_S (pn_i + \rho n_j u_j u_i) \, dS = 0$$

for the steady flow $\mathbf{u} = (u_1, u_2, u_3)$ and pressure p of an inviscid, incompressible fluid of density ρ, where S is a closed surface with normal $\mathbf{n}$.

A cylindrical jet of water of area A and speed u impinges axisymmetrically on a stationary sphere of radius a and is deflected into a conical sheet of vertex angle α as shown. Gravity is being ignored.
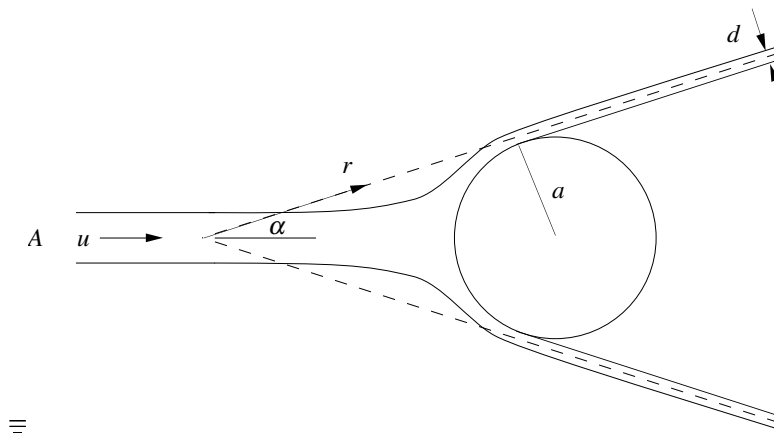
Use a suitable form of Bernoulli's equation to determine the speed of the water in the conical sheet, being careful to state how the equation is being applied.

Use conservation of mass to show that the width $d(r)$ of the sheet far from the point of impact is given by

$$d = \frac{A}{2\pi r \sin \alpha},$$

where r is the distance along the sheet measured from the vertex of the cone.

Finally, use the momentum integral to determine the net force on the sphere in terms of ρ, u, A and α.

1/I/6C   **Fluid Dynamics**

An unsteady fluid flow has velocity field given in Cartesian coordinates (x, y, z) by $\mathbf{u} = (1, xt, 0)$, where t denotes time. Dye is released into the fluid from the origin continuously. Find the position at time t of the dye particle that was released at time s and hence show that the dye streak lies along the curve

$$y = \tfrac{1}{2}tx^2 - \tfrac{1}{6}x^3.$$

1/II/15C   **Fluid Dynamics**

Starting from the Euler equations for incompressible, inviscid flow

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla p, \qquad \nabla \cdot \mathbf{u} = 0,$$

derive the vorticity equation governing the evolution of the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

Consider the flow

$$\mathbf{u} = \beta(-x, -y, 2z) + \Omega(t)(-y, x, 0),$$

in Cartesian coordinates (x, y, z), where t is time and β is a constant. Compute the vorticity and show that it evolves in time according to

$$\boldsymbol{\omega} = \omega_0 e^{2\beta t}\mathbf{k},$$

where ω_0 is the initial magnitude of the vorticity and $\mathbf{k}$ is a unit vector in the z-direction.

Show that the material curve $C(t)$ that takes the form

$$x^2 + y^2 = 1 \quad \text{and} \quad z = 1$$

at $t = 0$ is given later by

$$x^2 + y^2 = a^2(t) \quad \text{and} \quad z = \frac{1}{a^2(t)},$$

where the function $a(t)$ is to be determined.

Calculate the circulation of $\mathbf{u}$ around C and state how this illustrates Kelvin's circulation theorem.

3/I/8C   **Fluid Dynamics**

Show that the velocity field

$$\mathbf{u} = \mathbf{U} + \frac{\mathbf{\Gamma} \times \mathbf{r}}{2\pi r^2},$$

where $\mathbf{U} = (U,0,0)$, $\mathbf{\Gamma} = (0,0,\Gamma)$ and $\mathbf{r} = (x,y,0)$ in Cartesian coordinates (x,y,z), represents the combination of a uniform flow and the flow due to a line vortex. Define and evaluate the circulation of the vortex.

Show that

$$\oint_{C_R} (\mathbf{u} \cdot \mathbf{n})\mathbf{u}\, dl \to \tfrac{1}{2}\mathbf{\Gamma} \times \mathbf{U} \qquad \text{as} \qquad R \to \infty,$$

where C_R is a circle $x^2 + y^2 = R^2$, $z = \text{const}$. Explain how this result is related to the lift force on a two-dimensional aerofoil or other obstacle.

3/II/18C   **Fluid Dynamics**

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid in the absence of gravity.

Water of density ρ is driven through a tube of length L and internal radius a by the pressure exerted by a spherical, water-filled balloon of radius $R(t)$ attached to one end of the tube. The balloon maintains the pressure of the water entering the tube at $2\gamma/R$ in excess of atmospheric pressure, where γ is a constant. It may be assumed that the water exits the tube at atmospheric pressure. Show that

$$R^3 \ddot{R} + 2R^2 \dot{R}^2 = -\frac{\gamma a^2}{2\rho L}. \tag{$\dagger$}$$

Solve equation ($\dagger$), by multiplying through by $2R\dot{R}$ or otherwise, to obtain

$$t = R_0^2 \left(\frac{2\rho L}{\gamma a^2}\right)^{1/2} \left[\frac{\pi}{4} - \frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right],$$

where $\theta = \sin^{-1}(R/R_0)$ and R_0 is the initial radius of the balloon. Hence find the time when $R = 0$.

4/I/7C   **Fluid Dynamics**

Inviscid fluid issues vertically downwards at speed u_0 from a circular tube of radius a. The fluid falls onto a horizontal plate a distance H below the end of the tube, where it spreads out axisymmetrically.

Show that while the fluid is falling freely it has speed

$$u = u_0 \left[1 + \frac{2g}{u_0^2}(H - z) \right]^{1/2},$$

and occupies a circular jet of radius

$$R = a \left[1 + \frac{2g}{u_0^2}(H - z) \right]^{-1/4},$$

where z is the height above the plate and g is the acceleration due to gravity.

Show further that along the plate, at radial distances $r \gg a$ (i.e. far from the falling jet), where the fluid is flowing almost horizontally, it does so as a film of height $h(r)$, where

$$\frac{a^4}{4r^2h^2} = 1 + \frac{2g}{u_0^2}(H - h).$$

4/II/16C   **Fluid Dynamics**

Define the terms *irrotational flow* and *incompressible flow*. The two-dimensional flow of an incompressible fluid is given in terms of a streamfunction $\psi(x, y)$ as

$$\mathbf{u} = (u, v) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)$$

in Cartesian coordinates (x, y). Show that the line integral

$$\int_{\mathbf{x_1}}^{\mathbf{x_2}} \mathbf{u} \cdot \mathbf{n} \, dl = \psi(\mathbf{x_2}) - \psi(\mathbf{x_1})$$

along any path joining the points $\mathbf{x_1}$ and $\mathbf{x_2}$, where $\mathbf{n}$ is the unit normal to the path. Describe how this result is related to the concept of mass conservation.

Inviscid, incompressible fluid is contained in the semi-infinite channel $x > 0$, $0 < y < 1$, which has rigid walls at $x = 0$ and at $y = 0$, 1, apart from a small opening at the origin through which the fluid is withdrawn with volume flux m per unit distance in the third dimension. Show that the streamfunction for irrotational flow in the channel can be chosen (up to an additive constant) to satisfy the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

and boundary conditions

$$
\begin{aligned}
\psi &= 0 && \text{on } y = 0,\ x > 0, \\
\psi &= -m && \text{on } x = 0,\ 0 < y < 1, \\
\psi &= -m && \text{on } y = 1,\ x > 0, \\
\psi &\to -my && \text{as } x \to \infty,
\end{aligned}
$$

if it is assumed that the flow at infinity is uniform. Solve the boundary-value problem above using separation of variables to obtain

$$\psi = -my + \frac{2m}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \, \sin n\pi y \, e^{-n\pi x}.$$

1/I/6C   **Fluid Dynamics**

A fluid flow has velocity given in Cartesian co-ordinates as $\mathbf{u} = (kty, 0, 0)$ where k is a constant and t is time. Show that the flow is incompressible. Find a stream function and determine an equation for the streamlines at time t.

At $t = 0$ the points along the straight line segment $x = 0$, $0 \leqslant y \leqslant a$, $z = 0$ are marked with dye. Show that at any later time the marked points continue to form a segment of a straight line. Determine the length of this line segment at time t and the angle that it makes with the x-axis.

1/II/15C   **Fluid Dynamics**

State the unsteady form of Bernoulli's theorem.

A spherical bubble having radius R_0 at time $t = 0$ is located with its centre at the origin in unbounded fluid. The fluid is inviscid, has constant density ρ and is everywhere at rest at $t = 0$. The pressure at large distances from the bubble has the constant value p_∞, and the pressure inside the bubble has the constant value $p_\infty - \triangle p$. In consequence the bubble starts to collapse so that its radius at time t is $R(t)$. Find the velocity everywhere in the fluid in terms of $R(t)$ at time t and, assuming that surface tension is negligible, show that R satisfies the equation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = -\frac{\triangle p}{\rho}\ .$$

Find the total kinetic energy of the fluid in terms of $R(t)$ at time t. Hence or otherwise obtain a first integral of the above equation.

3/I/8C   **Fluid Dynamics**

State and prove Kelvin's circulation theorem.

Consider a planar flow in the unbounded region outside a cylinder for which the vorticity vanishes everywhere at time $t = 0$. What may be deduced about the circulation around closed loops in the fluid at time t:

(i) that do not enclose the cylinder;

(ii) that enclose the cylinder?

Give a brief justification for your answer in each case.

Part IB

3/II/18C   **Fluid Dynamics**

Use Euler's equation to derive Bernoulli's theorem for the steady flow of an inviscid fluid of uniform density ρ in the absence of body forces.

Such a fluid flows steadily through a long cylindrical elastic tube having circular cross-section. The variable z measures distance downstream along the axis of the tube. The tube wall has thickness $h(z)$, so that if the external radius of the tube is $r(z)$, its internal radius is $r(z) - h(z)$, where $h(z) \geqslant 0$ is a given slowly-varying function that tends to zero as $z \to \pm\infty$. The elastic tube wall exerts a pressure $p(z)$ on the fluid given as

$$p(z) = p_0 + k[r(z) - R],$$

where p_0, k and R are positive constants. Far upstream, r has the constant value R, the fluid pressure has the constant value p_0, and the fluid velocity u has the constant value V. Assume that gravity is negligible and that $h(z)$ varies sufficiently slowly that the velocity may be taken as uniform across the tube at each value of z. Use mass conservation and Bernoulli's theorem to show that $u(z)$ satisfies

$$\frac{h}{R} = 1 - \left(\frac{V}{u}\right)^{1/2} + \frac{1}{4}\lambda\left[1 - \left(\frac{u}{V}\right)^2\right], \quad \text{where} \quad \lambda = \frac{2\rho V^2}{kR} \; .$$

Sketch a graph of h/R against u/V. Show that if $h(z)$ exceeds a critical value $h_c(\lambda)$, no such flow is possible and find $h_c(\lambda)$.

Show that if $h < h_c(\lambda)$ everywhere, then for given h the equation has two positive solutions for u. Explain how the given value of λ determines which solution should be chosen.

4/I/7C   **Fluid Dynamics**

If $\mathbf{u}$ is given in Cartesian co-ordinates as $\mathbf{u} = (-\Omega y, \Omega x, 0)$, with Ω a constant, verify that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla(-\tfrac{1}{2}\mathbf{u}^2) \; .$$

When incompressible fluid is placed in a stationary cylindrical container of radius a with its axis vertical, the depth of the fluid is h. Assuming that the free surface does not reach the bottom of the container, use cylindrical polar co-ordinates to find the equation of the free surface when the fluid and the container rotate steadily about this axis with angular velocity Ω.

Deduce the angular velocity at which the free surface first touches the bottom of the container.

4/II/16C   **Fluid Dynamics**

Use Euler's equation to show that in a planar flow of an inviscid fluid the vorticity $\boldsymbol{\omega}$ satisfies

$$\frac{D\boldsymbol{\omega}}{Dt} = 0 \ .$$

Write down the velocity field associated with a point vortex of strength κ in unbounded fluid.

Consider now the flow generated in unbounded fluid by two point vortices of strengths κ_1 and κ_2 at $\mathbf{x}_1(t) = (x_1, y_1)$ and $\mathbf{x}_2(t) = (x_2, y_2)$, respectively. Show that in the subsequent motion the quantity

$$\mathbf{q} = \kappa_1 \mathbf{x}_1 + \kappa_2 \mathbf{x}_2$$

remains constant. Show also that the separation of the vortices, $|\mathbf{x}_2 - \mathbf{x}_1|$, remains constant.

Suppose finally that $\kappa_1 = \kappa_2$ and that the vortices are placed at time $t = 0$ at positions $(a, 0)$ and $(-a, 0)$. What are the positions of the vortices at time t?

Part IB

1/I/6G   Fluid Dynamics

Determine the pressure at a depth z below the surface of a static fluid of density ρ subject to gravity g. A rigid body having volume V is fully submerged in such a fluid. Calculate the buoyancy force on the body.

An iceberg of uniform density ρ_I is observed to float with volume V_I protruding above a large static expanse of seawater of density ρ_w. What is the total volume of the iceberg?

1/II/15G   Fluid Dynamics

A fluid motion has velocity potential $\phi(x, y, t)$ given by

$$\phi = \epsilon y \cos(x - t)$$

where ϵ is a constant. Find the corresponding velocity field $\mathbf{u}(x, y, t)$. Determine $\nabla \cdot \mathbf{u}$.

The *time-average* of a quantity $\psi(x, y, t)$ is defined as $\frac{1}{2\pi} \int_0^{2\pi} \psi(x, y, t)\, dt$. Show that the time-average of this velocity field at every point (x, y) is zero.

Write down an expression for the fluid acceleration and find the time-average acceleration at (x, y).

Suppose now that $|\epsilon| \ll 1$. The material particle at $(0, 0)$ at time $t = 0$ is marked with dye. Write down equations for its subsequent motion and verify that its position (x, y) at time $t > 0$ is given (correct to terms of order ϵ^2) as

$$x = \epsilon^2(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t),$$
$$y = \epsilon \sin t \ .$$

Deduce the time-average velocity of the dyed particle correct to this order.

3/I/8G   Fluid Dynamics

Inviscid incompressible fluid occupies the region $y > 0$, which is bounded by a rigid barrier along $y = 0$. At time $t = 0$, a line vortex of strength κ is placed at position (a, b). By considering the flow due to an image vortex at $(a, -b)$, or otherwise, determine the velocity potential in the fluid.

Derive the position of the original vortex at time $t > 0$.

3/II/18G   **Fluid Dynamics**

State the form of Bernoulli's theorem appropriate for an unsteady irrotational motion of an inviscid incompressible fluid.

A circular cylinder of radius a is immersed in unbounded inviscid fluid of uniform density ρ. The cylinder moves in a prescribed direction perpendicular to its axis, with speed U. Use cylindrical polar coordinates, with the direction $\theta = 0$ parallel to the direction of the motion, to find the velocity potential in the fluid.

If U depends on time t and gravity is negligible, determine the pressure field in the fluid at time t. Deduce the fluid force per unit length on the cylinder.

[In cylindrical polar coordinates, $\nabla\phi = \dfrac{\partial\phi}{\partial r}\mathbf{e}_r + \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\mathbf{e}_\theta$.]

4/I/7G   **Fluid Dynamics**

Starting from the Euler equation, derive the *vorticity equation* for the motion of an inviscid incompressible fluid under a conservative body force, and give a physical interpretation of each term in the equation. Deduce that in a flow field of the form $\mathbf{u} = (u(x, y, t), v(x, y, t), 0)$ the vorticity of a material particle is conserved.

Find the vorticity for such a flow in terms of the stream function ψ. Deduce that if the flow is steady, there must be a function f such that

$$\nabla^2\psi = f(\psi)\ .$$

4/II/16G   **Fluid Dynamics**

A long straight canal has rectangular cross-section with a horizontal bottom and width $w(x)$ that varies slowly with distance x downstream. Far upstream, w has a constant value W, the water depth has a constant value H, and the velocity has a constant value U. Assuming that the water velocity is steady and uniform across the channel, use mass conservation and Bernoulli's theorem, which should be stated carefully, to show that the water depth $h(x)$ satisfies

$$\left(\frac{W}{w}\right)^2 = \left(1 + \frac{2}{F}\right)\left(\frac{h}{H}\right)^2 - \frac{2}{F}\left(\frac{h}{H}\right)^3 \ \text{ where }\ F = \frac{U^2}{gH}\ .$$

Deduce that for a given value of F, a flow of this kind can exist only if $w(x)$ is everywhere greater than or equal to a critical value w_c, which is to be determined in terms of F.

Suppose that $w(x) > w_c$ everywhere. At locations where the channel width exceeds W, determine graphically, or otherwise, under what circumstances the water depth exceeds H.