Part IA

Analysis I

Year

2023
2022
2021
2020
2019
2018
2017
2016
2015
2014
2013
2012
2011
2010
2009
2008
2007
2006
2005
2004
2003
2002
2001

Paper 1, Section I

3E Analysis

Let $a \in \mathbb{R}$ and let f and g be real-valued functions defined on $\mathbb{R}$. State and prove the chain rule for $F(x) = g\big(f(x)\big)$.

Now assume that f and g are non-constant on any interval. Must the function $F(x) = g\big(f(x)\big)$ be non-differentiable at $x = a$ if

(i) f is differentiable at a and g is not differentiable at $f(a)$?

(ii) f is not differentiable at a and g is differentiable at $f(a)$?

(iii) f is not differentiable at a and g is not differentiable at $f(a)$?

Justify your answers.

Paper 1, Section I

4E Analysis

State the comparison test. Prove that if $\sum_{n=0}^{\infty} a_n z_0^n$ converges and $|z_1| < |z_0|$, then $\sum_{n=0}^{\infty} a_n z_1^n$ converges absolutely.

Define the *radius of convergence* of a complex power series. [You do not need to show that the radius of convergence is well-defined.]

If $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R_1 and $\sum_{n=0}^{\infty} b_n z^n$ has radius of convergence R_2, show that the radius of convergence R of the series $\sum_{n=0}^{\infty} a_n b_n z^n$ satisfies $R \geqslant R_1 R_2$.

Paper 1, Section II

9E Analysis

(a) Let $x_1 > 0$ and define a sequence (x_n) by

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{1}{x_{n-1}}\right) \text{ for } n > 1.$$

Prove that $\lim_{n\to\infty} x_n = 1$.

Show that if a real sequence (x_n) satisfies

$$0 \leqslant x_{m+n} \leqslant x_m + x_n \quad \text{for all } m, n = 1, 2, \ldots,$$

then the sequence (x_n/n) is (i) bounded and (ii) convergent.

(b) Suppose that a series $\sum_{n=1}^{\infty} a_n$ of real numbers converges but not absolutely. Let

$$P_n = \sum_{i=1}^{n} \big(|a_i| + a_i\big), \qquad N_n = \sum_{i=1}^{n} \big(|a_i| - a_i\big).$$

Show that $\lim_{n\to\infty} P_n/N_n = 1$.

State the alternating series test. Let (b_n) be a sequence of positive real numbers such that

$$\lim_{n\to\infty} n\left(\frac{b_n}{b_{n+1}} - 1\right) = p,$$

where p is a positive real number. Show that the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges.

Paper 1, Section II

10E Analysis

State and prove the intermediate value theorem.

Give, with justification, an example of a function $\phi : [a, \infty) \to \mathbb{R}$ such that, for any $b > a$, ϕ takes on $[a, b]$ every value between $\phi(a)$ and $\phi(b)$ but ϕ is not continuous on $[a, b]$.

If a function $f : [a, b] \to \mathbb{R}$ is monotone on $[a, b]$ and takes every value between $f(a)$ and $f(b)$, show that f is continuous on $[a, b]$.

Let $g : (a, b) \to \mathbb{R}$ be a continuous function and suppose that there are sequences $x_n \to a$ and $y_n \to a$ as $n \to \infty$ such that $g(x_n) \to l$ and $g(y_n) \to L$ with $l < L$. Show that for each $\lambda \in [l, L]$ there is a sequence $z_n \to a$ such that $g(z_n) \to \lambda$.

Paper 1, Section II
11E Analysis

(a) State the mean value theorem. Deduce that

$$\frac{a-b}{a} < \log\frac{a}{b} < \frac{a-b}{b} \quad \text{for } 0 < b < a.$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be an n-times differentiable function, where $n > 0$. Show that for each $a \in \mathbb{R}$ and $h > 0$ there exists $b \in (a, a+nh)$ such that

$$\frac{1}{h^n}\Delta_h^n f(a) = f^{(n)}(b),$$

where $\Delta_h^{k+1} f(x) = \Delta_h^1\big(\Delta_h^k f(x)\big)$ and $\Delta_h^1 f(x) = f(x+h) - f(x)$.

(c) Let $I \subset \mathbb{R}$ be an open (non-empty) interval and $a \in I$. Suppose that a function $\varphi : I \to \mathbb{R}$ has a finite limit at a and $\lim_{x\to a}\varphi(x) = \varphi(a) + 1$. Can φ be the derivative of some differentiable function f on I? Justify your answer.

Paper 1, Section II
12E Analysis

Define the *upper* and *lower integral* of a function on $[a, b]$ and what it means for a function to be *(Riemann) integrable* on $[a, b]$.

(a) Let $\lfloor y \rfloor = \max\{i \in \mathbb{Z} : i \leqslant y\}$. Show that the function

$$u(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{if } x \neq 0, \qquad u(0) = 0,$$

is integrable on $[0, 1]$. [You may assume that every continuous function on a closed bounded interval is integrable.]

(b) Let $f : [A, B] \to \mathbb{R}$ be a continuous function and $A < a < x < B$. Prove that

$$\lim_{h\to 0}\frac{1}{h}\int_a^x \big(f(t+h) - f(t)\big)dt = f(x) - f(a).$$

[Any version of the fundamental theorem of calculus from the course can be assumed if accurately stated.]

(c) Show that if a function $g : [a, b] \to \mathbb{R}$ is integrable, then there exists a sequence of continuous functions $\varphi_n : [a, b] \to \mathbb{R}$ such that $\int_\alpha^\beta g(x)dx = \lim_{n\to\infty}\int_\alpha^\beta \varphi_n(x)dx$ for any subinterval $[\alpha, \beta] \subseteq [a, b]$.

Paper 1, Section I
3D Analysis I

State the alternating series test. Deduce that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges. Is this series absolutely convergent? Justify your answer.

Find a divergent series which has the same terms $\frac{(-1)^n}{\sqrt{n}}$ taken in a different order. You should justify the divergence.

[*You may use the comparison test, provided that you accurately state it.*]

Paper 1, Section I
4D Analysis I

Let $a \in \mathbb{R}$ and let f and g be continuous real-valued functions defined on $\mathbb{R}$ which are not identically zero on any interval containing a.

Must the function $F(x) = f(x) + g(x)$ be non-differentiable at $a \in \mathbb{R}$ if (a) f is differentiable at a and g is not differentiable at a; (b) both f and g are not differentiable at a?

Must the function $G(x) = f(x)g(x)$ be non-differentiable at $a \in \mathbb{R}$ if (a) f is differentiable at a and g is not differentiable at a; (b) both f and g are not differentiable at a?

Justify your answers.

Paper 1, Section II
9D Analysis I

(a) Let a_n be a sequence of real numbers. Show that if a_n converges, the sequence $\frac{1}{n}\sum_{k=1}^{n} a_k$ also converges and $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = \lim_{n\to\infty} a_n$.

If $\frac{1}{n}\sum_{k=1}^{n} a_k$ converges, must a_n converge too? Justify your answer.

(b) Let x_n be a sequence of real numbers with $x_n > 0$ for all n. By considering the sequence $\log x_n$, or otherwise, show that if x_n converges then $\lim_{n\to\infty} \sqrt[n]{x_1 x_2 \ldots x_n} = \lim_{n\to\infty} x_n$. You may assume that exp and log are continuous functions.

Deduce that if the sequence $\dfrac{x_n}{x_{n-1}}$ converges, then $\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \dfrac{x_n}{x_{n-1}}$.

(c) What is a *Cauchy sequence*? State the general principle of convergence for real sequences.

Let a_n be a decreasing sequence of positive real numbers and suppose that the series $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} n a_n = 0$.

Paper 1, Section II
10D Analysis I

Prove that every continuous real-valued function on a closed bounded interval is bounded and attains its bounds. [The Bolzano–Weierstrass theorem can be assumed provided it is accurately stated.]

Give an example of a continuous function $\phi : (0,1) \to \mathbb{R}$ that is bounded but does not attain its bounds and an example of a function $\psi : [0,1] \to \mathbb{R}$ that is not bounded on any interval $[a,b]$ such that $0 \leqslant a < b \leqslant 1$. Justify your examples.

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove that the functions

$$m(x) = \inf_{a \leqslant \xi \leqslant x} f(\xi) \qquad \text{and} \qquad M(x) = \sup_{a \leqslant \xi \leqslant x} f(\xi)$$

are also continuous on $[a,b]$.

Let a function $g : (0,\infty) \to \mathbb{R}$ be continuous and bounded. Show that for every $T > 0$ there exists a sequence x_n such that $x_n \to \infty$ and

$$\lim_{n\to\infty} (g(x_n + T) - g(x_n)) = 0.$$

[The intermediate value theorem can be assumed.]

Paper 1, Section II
11D Analysis I

In this question $a < b$ are real numbers.

(a) State and prove Rolle's theorem. State and prove the mean value theorem.

(b) Prove that if a continuous function $f : [a,b] \to \mathbb{R}$ is differentiable on (a,b) and is not a linear function, then $f'(\xi) > \dfrac{f(b) - f(a)}{b - a}$ for some ξ with $a < \xi < b$.

(c) Let $f : [a,b] \to \mathbb{R}$ be a continuous function and let f be differentiable on (a,b). Must there exist, for every $\xi \in (a,b)$, two points x_1, x_2 with $a \leqslant x_1 < \xi < x_2 \leqslant b$ such that $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi)$? Give a proof or counterexample as appropriate.

(d) Let functions f and g be continuous on $[a,b]$ and differentiable on (a,b) with $g(a) \neq g(b)$ and suppose that $f'(x)$ and $g'(x)$ never vanish for the same value of x. By considering $\lambda f + \mu g + \nu$ for suitable real constants λ, μ, ν, or otherwise, prove that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)} \qquad \text{for some } \xi \text{ with } a < \xi < b.$$

Give an example to show that the condition that $f'(x)$ and $g'(x)$ never vanish for the same x cannot be omitted.

Paper 1, Section II
12D Analysis I

Let $f : [0,1] \to \mathbb{R}$ be a monotone function.

Show that for all dissections $\mathcal{D}$ and $\mathcal{D}'$ of $[0,1]$ one has $L_{\mathcal{D}}(f) \leqslant U_{\mathcal{D}'}(f)$, where $L_{\mathcal{D}}(f)$ and $U_{\mathcal{D}'}(f)$ are the lower and upper sums of f for the respective dissections. Show further that for each $\varepsilon > 0$ there is a dissection $\mathcal{D}$ such that $U_{\mathcal{D}}(f) - L_{\mathcal{D}}(f) < \varepsilon$. Deduce that f is integrable.

Show that

$$\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f\big(\frac{k}{n}\big) \right| < \frac{|f(1) - f(0)|}{n}$$

for all positive integers n.

Let a function F be continuous on some open interval containing $[0,1]$ and have a continuous derivative F' on $[0,1]$. Denote

$$\Delta_n = \int_0^1 F(x)dx - \frac{1}{n}\sum_{k=1}^{n} F\big(\frac{k}{n}\big).$$

Stating clearly any results from the course that you require, show that

$$\lim_{n\to\infty} n\Delta_n = (F(0) - F(1))/2.$$

[*Hint: it might be helpful to consider* $\int_{(k-1)/n}^{k/n} \big(F(x) - F(\frac{k}{n})\big)dx$.]

Paper 1, Section I
3F Analysis I

State and prove the alternating series test. Hence show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges. Show also that

$$\frac{7}{12} \leqslant \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \leqslant \frac{47}{60}.$$

Paper 1, Section I
4F Analysis I

State and prove the Bolzano–Weierstrass theorem.

Consider a bounded sequence (x_n). Prove that if every convergent subsequence of (x_n) converges to the same limit L then (x_n) converges to L.

Paper 1, Section II
9F Analysis I

(a) State the intermediate value theorem. Show that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous bijection and $x_1 < x_2 < x_3$ then either $f(x_1) < f(x_2) < f(x_3)$ or $f(x_1) > f(x_2) > f(x_3)$. Deduce that f is either strictly increasing or strictly decreasing.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. Which of the following statements are true, and which can be false? Give a proof or counterexample as appropriate.

(i) If f and g are continuous then $f \circ g$ is continuous.

(ii) If g is strictly increasing and $f \circ g$ is continuous then f is continuous.

(iii) If f is continuous and a bijection then f^{-1} is continuous.

(iv) If f is differentiable and a bijection then f^{-1} is differentiable.

Paper 1, Section II
10F Analysis I

Let $f : [a, b] \to \mathbb{R}$ be a continuous function.

(a) Let $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$. If $g : [a, b] \to \mathbb{R}$ is a positive continuous function, prove that

$$m \int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant M \int_a^b g(x)dx$$

directly from the definition of the Riemann integral.

(b) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function. Show that

$$\int_0^{1/\sqrt{n}} nf(x)e^{-nx}dx \to f(0)$$

as $n \to \infty$, and deduce that

$$\int_0^1 nf(x)e^{-nx}dx \to f(0)$$

as $n \to \infty$.

Paper 1, Section II
11F Analysis I

Let $f : \mathbb{R} \to \mathbb{R}$ be n-times differentiable, for some $n > 0$.

(a) State and prove Taylor's theorem for f, with the Lagrange form of the remainder. [You may assume Rolle's theorem.]

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function such that $f(0) = 1$ and $f'(0) = 0$, and satisfying the differential equation $f''(x) = -f(x)$. Prove carefully that

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} .$$

Paper 1, Section II

12F Analysis I

(a) Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with $a_n \in \mathbb{C}$. Show that there exists $R \in [0, \infty]$ (called the *radius of convergence*) such that the series is absolutely convergent when $|z| < R$ but is divergent when $|z| > R$.

Suppose that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ is $R = 2$. For a fixed positive integer k, find the radii of convergence of the following series. [You may assume that $\lim_{n\to\infty} |a_n|^{1/n}$ exists.]

(i) $\displaystyle\sum_{n=0}^{\infty} a_n^k z^n$.

(ii) $\displaystyle\sum_{n=0}^{\infty} a_n z^{kn}$.

(iii) $\displaystyle\sum_{n=0}^{\infty} a_n z^{n^2}$.

(b) Suppose that there exist values of z for which $\sum_{n=0}^{\infty} b_n e^{nz}$ converges and values for which it diverges. Show that there exists a real number S such that $\sum_{n=0}^{\infty} b_n e^{nz}$ diverges whenever $\mathrm{Re}(z) > S$ and converges whenever $\mathrm{Re}(z) < S$.

Determine the set of values of z for which

$$\sum_{n=0}^{\infty} \frac{2^n e^{inz}}{(n+1)^2}$$

converges.

Paper 1, Section I

3E Analysis I

(a) Let f be continuous in $[a, b]$, and let g be strictly monotonic in $[\alpha, \beta]$, with a continuous derivative there, and suppose that $a = g(\alpha)$ and $b = g(\beta)$. Prove that

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(u))g'(u)du\,.$$

[Any version of the fundamental theorem of calculus may be used providing it is quoted correctly.]

(b) Justifying carefully the steps in your argument, show that the improper Riemann integral

$$\int_0^{e^{-1}} \frac{dx}{x(\log \frac{1}{x})^\theta}$$

converges for $\theta > 1$, and evaluate it.

Paper 1, Section II

9D Analysis I

(a) State Rolle's theorem. Show that if $f : \mathbb{R} \to \mathbb{R}$ is $N + 1$ times differentiable and $x \in \mathbb{R}$ then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(N)}(0)}{N!}x^N + \frac{f^{(N+1)}(\theta x)}{(N+1)!}x^{N+1}\,,$$

for some $0 < \theta < 1$. Hence, or otherwise, show that if $f'(x) = 0$ for all $x \in \mathbb{R}$ then f is constant.

(b) Let $s : \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that

$$s'(x) = c(x), \quad c'(x) = -s(x), \quad s(0) = 0 \quad \text{and} \quad c(0) = 1.$$

Prove that

(i) $s(x)c(a - x) + c(x)s(a - x)$ is independent of x,

(ii) $s(x + y) = s(x)c(y) + c(x)s(y)$,

(iii) $s(x)^2 + c(x)^2 = 1$.

Show that $c(1) > 0$ and $c(2) < 0$. Deduce there exists $1 < k < 2$ such that $s(2k) = c(k) = 0$ and $s(x + 4k) = s(x)$.

Paper 1, Section II

10F Analysis I

(a) Let (x_n) be a bounded sequence of real numbers. Show that (x_n) has a convergent subsequence.

(b) Let (z_n) be a bounded sequence of complex numbers. For each $n \geqslant 1$, write $z_n = x_n + iy_n$. Show that (z_n) has a subsequence (z_{n_j}) such that (x_{n_j}) converges. Hence, or otherwise, show that (z_n) has a convergent subsequence.

(c) Write $\mathbb{N} = \{1, 2, 3, \ldots\}$ for the set of positive integers. Let M be a positive real number, and for each $i \in \mathbb{N}$, let $X^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, \ldots)$ be a sequence of real numbers with $|x_j^{(i)}| \leqslant M$ for all i, $j \in \mathbb{N}$. By induction on i or otherwise, show that there exist sequences $N^{(i)} = (n_1^{(i)}, n_2^{(i)}, n_3^{(i)}, \ldots)$ of positive integers with the following properties:

- for all $i \in \mathbb{N}$, the sequence $N^{(i)}$ is strictly increasing;
- for all $i \in \mathbb{N}$, $N^{(i+1)}$ is a subsequence of $N^{(i)}$; and
- for all $k \in \mathbb{N}$ and all $i \in \mathbb{N}$ with $1 \leqslant i \leqslant k$, the sequence

$$(x^{(i)}_{n_1^{(k)}}, x^{(i)}_{n_2^{(k)}}, x^{(i)}_{n_3^{(k)}}, \ldots)$$

  converges.

Hence, or otherwise, show that there exists a strictly increasing sequence (m_j) of positive integers such that for all $i \in \mathbb{N}$ the sequence $(x^{(i)}_{m_1}, x^{(i)}_{m_2}, x^{(i)}_{m_3}, \ldots)$ converges.

Paper 1, Section I

3E Analysis I

State the Bolzano-Weierstrass theorem.

Let (a_n) be a sequence of non-zero real numbers. Which of the following conditions is sufficient to ensure that $(1/a_n)$ converges? Give a proof or counter-example as appropriate.

(i) $a_n \to \ell$ for some real number ℓ.

(ii) $a_n \to \ell$ for some non-zero real number ℓ.

(iii) (a_n) has no convergent subsequence.

Paper 1, Section I

4F Analysis I

Let $\sum_{n=1}^{\infty} a_n x^n$ be a real power series that diverges for at least one value of x. Show that there exists a non-negative real number R such that $\sum_{n=1}^{\infty} a_n x^n$ converges absolutely whenever $|x| < R$ and diverges whenever $|x| > R$.

Find, with justification, such a number R for each of the following real power series:

(i) $\sum_{n=1}^{\infty} \frac{x^n}{3^n}$;

(ii) $\sum_{n=1}^{\infty} x^n \left(1 + \frac{1}{n}\right)^n$.

Paper 1, Section II

9D Analysis I

Let $g : \mathbb{R} \to \mathbb{R}$ be a function that is continuous at at least one point $z \in \mathbb{R}$. Suppose further that g satisfies

$$g(x + y) = g(x) + g(y)$$

for all $x, y \in \mathbb{R}$. Show that g is continuous on $\mathbb{R}$.

Show that there exists a constant c such that $g(x) = cx$ for all $x \in \mathbb{R}$.

Suppose that $h : \mathbb{R} \to \mathbb{R}$ is a continuous function defined on $\mathbb{R}$ and that h satisfies the equation

$$h(x + y) = h(x)h(y)$$

for all $x, y \in \mathbb{R}$. Show that h is either identically zero or everywhere positive. What is the general form for h?

Paper 1, Section II

10D Analysis I

State and prove the Intermediate Value Theorem.

State the Mean Value Theorem.

Suppose that the function g is differentiable everywhere in some open interval containing $[a, b]$, and that $g'(a) < k < g'(b)$. By considering the functions h and f defined by

$$h(x) = \frac{g(x) - g(a)}{x - a} \quad (a < x \leqslant b), \quad h(a) = g'(a)$$

and

$$f(x) = \frac{g(b) - g(x)}{b - x} \quad (a \leqslant x < b), \quad f(b) = g'(b),$$

or otherwise, show that there is a subinterval $[\alpha, \beta] \subseteq [a, b]$ such that

$$\frac{g(\beta) - g(\alpha)}{\beta - \alpha} = k.$$

Deduce that there exists $c \in (a, b)$ with $g'(c) = k$.

Paper 1, Section II

11E Analysis I

Let (a_n) and (b_n) be sequences of positive real numbers. Let $s_n = \sum_{i=1}^{n} a_i$.

(a) Show that if $\sum a_n$ and $\sum b_n$ converge then so does $\sum \left(a_n^2 + b_n^2\right)^{1/2}$.

(b) Show that if $\sum a_n$ converges then $\sum \sqrt{a_n a_{n+1}}$ converges. Is the converse true?

(c) Show that if $\sum a_n$ diverges then $\sum \frac{a_n}{s_n}$ diverges. Is the converse true?

[*For part (c), it may help to show that for any $N \in \mathbb{N}$ there exist $m \geqslant n \geqslant N$ with*

$$\frac{a_{n+1}}{s_{n+1}} + \frac{a_{n+2}}{s_{n+2}} + \ldots + \frac{a_m}{s_m} \geqslant \frac{1}{2}.]$$

Paper 1, Section II

12F Analysis I

Let $f\colon [0,1] \to \mathbb{R}$ be a bounded function. Define the upper and lower integrals of f. What does it mean to say that f is *Riemann integrable*? If f is Riemann integrable, what is the *Riemann integral* $\int_0^1 f(x)\,dx$?

Which of the following functions $f\colon [0,1] \to \mathbb{R}$ are Riemann integrable? For those that are Riemann integrable, find $\int_0^1 f(x)\,dx$. Justify your answers.

(i) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$;

(ii) $f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$,

where $A = \{x \in [0,1] : x \text{ has a base-3 expansion containing a } 1\}$;

[Hint: You may find it helpful to note, for example, that $\frac{2}{3} \in A$ as one of the base–3 expansions of $\frac{2}{3}$ is 0.1222... .]

(iii) $f(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$,

where $B = \{x \in [0,1] : x \text{ has a base–3 expansion containing infinitely many 1s}\}$.

Paper 1, Section I

3E Analysis I

Prove that an increasing sequence in $\mathbb{R}$ that is bounded above converges.

Let $f\colon \mathbb{R} \to (0, \infty)$ be a decreasing function. Let $x_1 = 1$ and $x_{n+1} = x_n + f(x_n)$. Prove that $x_n \to \infty$ as $n \to \infty$.

Paper 1, Section I

4D Analysis I

Define the *radius of convergence* R of a complex power series $\sum a_n z^n$. Prove that $\sum a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$.

If $|a_n| \leqslant |b_n|$ for all n does it follow that the radius of convergence of $\sum a_n z^n$ is at least that of $\sum b_n z^n$? Justify your answer.

Paper 1, Section II

9F Analysis I

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and let $x \in \mathbb{R}$. Define what it means for f to be *continuous* at x. Show that f is continuous at x if and only if $f(x_n) \to f(x)$ for every sequence (x_n) with $x_n \to x$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant polynomial. Show that its image $\{f(x) : x \in \mathbb{R}\}$ is either the real line $\mathbb{R}$, the interval $[a, \infty)$ for some $a \in \mathbb{R}$, or the interval $(-\infty, a]$ for some $a \in \mathbb{R}$.

(c) Let $\alpha > 1$, let $f : (0, \infty) \to \mathbb{R}$ be continuous, and assume that $f(x) = f(x^\alpha)$ holds for all $x > 0$. Show that f must be constant.

Is this also true when the condition that f be continuous is dropped?

Paper 1, Section II
10F Analysis

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$. Show that f is continuous at x_0.

(b) State the Mean Value Theorem. Prove the following inequalities:

$$|\cos(e^{-x}) - \cos(e^{-y})| \leqslant |x - y| \quad \text{for } x, y \geqslant 0$$

and

$$\log(1 + x) \leqslant \frac{x}{\sqrt{1 + x}} \quad \text{for } x \geqslant 0.$$

(c) Determine at which points the following functions from $\mathbb{R}$ to $\mathbb{R}$ are differentiable, and find their derivatives at the points at which they are differentiable:

$$f(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases} \qquad g(x) = \cos(|x|), \qquad h(x) = x|x|.$$

(d) Determine the points at which the following function from $\mathbb{R}$ to $\mathbb{R}$ is continuous:

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \text{ or } x = 0 \\ 1/q & \text{if } x = p/q \text{ where } p \in \mathbb{Z} \setminus \{0\} \text{ and } q \in \mathbb{N} \text{ are relatively prime.} \end{cases}$$

Paper 1, Section II

11E Analysis I

State and prove the Comparison Test for real series.

Assume $0 \leqslant x_n < 1$ for all $n \in \mathbb{N}$. Show that if $\sum x_n$ converges, then so do $\sum x_n^2$ and $\sum \frac{x_n}{1-x_n}$. In each case, does the converse hold? Justify your answers.

Let (x_n) be a decreasing sequence of positive reals. Show that if $\sum x_n$ converges, then $nx_n \to 0$ as $n \to \infty$. Does the converse hold? If $\sum x_n$ converges, must it be the case that $(n \log n)x_n \to 0$ as $n \to \infty$? Justify your answers.

Paper 1, Section II

12D Analysis I

(a) Let $q_1, q_2, \ldots$ be a fixed enumeration of the rationals in $[0,1]$. For positive reals $a_1, a_2, \ldots$, define a function f from $[0,1]$ to $\mathbb{R}$ by setting $f(q_n) = a_n$ for each n and $f(x) = 0$ for x irrational. Prove that if $a_n \to 0$ then f is Riemann integrable. If $a_n \not\to 0$, can f be Riemann integrable? Justify your answer.

(b) State and prove the Fundamental Theorem of Calculus.

Let f be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, and set $g(x) = f'(x)$ for $0 \leqslant x \leqslant 1$. Must g be Riemann integrable on $[0,1]$?

Paper 1, Section I

3F Analysis I

Given an increasing sequence of non-negative real numbers $(a_n)_{n=1}^{\infty}$, let

$$s_n = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Prove that if $s_n \to x$ as $n \to \infty$ for some $x \in \mathbb{R}$ then also $a_n \to x$ as $n \to \infty$.

Paper 1, Section II

11F Analysis I

(a) Let $(x_n)_{n=1}^{\infty}$ be a non-negative and decreasing sequence of real numbers. Prove that $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k x_{2^k}$ converges.

(b) For $s \in \mathbb{R}$, prove that $\sum_{n=1}^{\infty} n^{-s}$ converges if and only if $s > 1$.

(c) For any $k \in \mathbb{N}$, prove that

$$\lim_{n\to\infty} 2^{-n} n^k = 0.$$

(d) The sequence $(a_n)_{n=0}^{\infty}$ is defined by $a_0 = 1$ and $a_{n+1} = 2^{a_n}$ for $n \geqslant 0$. For any $k \in \mathbb{N}$, prove that

$$\lim_{n\to\infty} \frac{2^{n^k}}{a_n} = 0.$$

Paper 1, Section I

4E Analysis I

Show that if the power series $\sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{C})$ converges for some fixed $z = z_0$, then it converges absolutely for every z satisfying $|z| < |z_0|$.

Define the *radius of convergence* of a power series.

Give an example of $v \in \mathbb{C}$ and an example of $w \in \mathbb{C}$ such that $|v| = |w| = 1$, $\displaystyle\sum_{n=1}^{\infty} \frac{v^n}{n}$ converges and $\displaystyle\sum_{n=1}^{\infty} \frac{w^n}{n}$ diverges. [You may assume results about standard series without proof.] Use this to find the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$.

Paper 1, Section II

9D Analysis I

(a) State the Intermediate Value Theorem.

(b) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *differentiable* at a point $a \in \mathbb{R}$. If f is differentiable everywhere on $\mathbb{R}$, must f' be continuous everywhere? Justify your answer.

State the Mean Value Theorem.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable everywhere. Let $a, b \in \mathbb{R}$ with $a < b$.
If $f'(a) \leqslant y \leqslant f'(b)$, prove that there exists $c \in [a, b]$ such that $f'(c) = y$. [*Hint: consider the function g defined by*

$$g(x) = \frac{f(x) - f(a)}{x - a}$$

if $x \neq a$ and $g(a) = f'(a)$.]

If additionally $f(a) \leqslant 0 \leqslant f(b)$, deduce that there exists $d \in [a, b]$ such that $f'(d) + f(d) = y$.

Paper 1, Section II

10D Analysis I

Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : (a, b) \to \mathbb{R}$.

(a) Define what it means for f to be *continuous* at $y_0 \in (a, b)$.

f is said to have a *local minimum* at $c \in (a, b)$ if there is some $\varepsilon > 0$ such that $f(c) \leqslant f(x)$ whenever $x \in (a, b)$ and $|x - c| < \varepsilon$.

If f has a local minimum at $c \in (a, b)$ and f is differentiable at c, show that $f'(c) = 0$.

(b) f is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda)f(y)$$

for every $x, y \in (a, b)$ and $\lambda \in [0, 1]$. If f is convex, $r \in \mathbb{R}$ and $\big[y_0 - |r|\,,\, y_0 + |r|\big] \subset (a, b)$, prove that

$$(1 + \lambda)f(y_0) - \lambda f(y_0 - r) \leqslant f(y_0 + \lambda r) \leqslant (1 - \lambda)f(y_0) + \lambda f(y_0 + r)$$

for every $\lambda \in [0, 1]$.

Deduce that if f is convex then f is continuous.

If f is convex and has a local minimum at $c \in (a, b)$, prove that f has a global minimum at c, i.e., that $f(x) \geqslant f(c)$ for every $x \in (a, b)$. [*Hint: argue by contradiction.*] Must f be differentiable at c? Justify your answer.

Paper 1, Section II

12E Analysis I

Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function defined on the closed, bounded interval $[a,b]$ of $\mathbb{R}$. Suppose that for every $\varepsilon > 0$ there is a dissection $\mathcal{D}$ of $[a,b]$ such that $S_{\mathcal{D}}(f) - s_{\mathcal{D}}(f) < \varepsilon$, where $s_{\mathcal{D}}(f)$ and $S_{\mathcal{D}}(f)$ denote the lower and upper Riemann sums of f for the dissection $\mathcal{D}$. Deduce that f is Riemann integrable. [You may assume without proof that $s_{\mathcal{D}}(f) \leqslant S_{\mathcal{D}'}(f)$ for all dissections $\mathcal{D}$ and $\mathcal{D}'$ of $[a,b]$.]

Prove that if $f\colon [a,b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.

Let $g\colon (0,1] \to \mathbb{R}$ be a bounded continuous function. Show that for any $\lambda \in \mathbb{R}$, the function $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} g(x) & \text{if } 0 < x \leqslant 1 \ , \\ \lambda & \text{if } x = 0 \ , \end{cases}$$

is Riemann integrable.

Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable function with one-sided derivatives at the endpoints. Suppose that the derivative f' is (bounded and) Riemann integrable. Show that

$$\int_a^b f'(x)\,dx = f(b) - f(a) \ .$$

[*You may use the Mean Value Theorem without proof.*]

Paper 1, Section I

3D Analysis I

What does it mean to say that a sequence of real numbers (x_n) converges to x? Suppose that (x_n) converges to x. Show that the sequence (y_n) given by

$$y_n = \frac{1}{n}\sum_{i=1}^{n} x_i$$

also converges to x.

Paper 1, Section I

4F Analysis I

Let a_n be the number of pairs of integers $(x, y) \in \mathbb{Z}^2$ such that $x^2 + y^2 \leqslant n^2$. What is the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$? [You may use the comparison test, provided you state it clearly.]

Paper 1, Section II

9E Analysis I

State the Bolzano–Weierstrass theorem. Use it to show that a continuous function $f : [a, b] \to \mathbb{R}$ attains a global maximum; that is, there is a real number $c \in [a, b]$ such that $f(c) \geqslant f(x)$ for all $x \in [a, b]$.

A function f is said to attain a local maximum at $c \in \mathbb{R}$ if there is some $\varepsilon > 0$ such that $f(c) \geqslant f(x)$ whenever $|x - c| < \varepsilon$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable, and that $f''(x) < 0$ for all $x \in \mathbb{R}$. Show that there is at most one $c \in \mathbb{R}$ at which f attains a local maximum.

If there is a constant $K < 0$ such that $f''(x) < K$ for all $x \in \mathbb{R}$, show that f attains a global maximum. [*Hint: if $g'(x) < 0$ for all $x \in \mathbb{R}$, then g is decreasing.*]

Must $f : \mathbb{R} \to \mathbb{R}$ attain a global maximum if we merely require $f''(x) < 0$ for all $x \in \mathbb{R}$? Justify your answer.

Paper 1, Section II

10E Analysis I

Let $f : \mathbb{R} \to \mathbb{R}$. We say that $x \in \mathbb{R}$ is a real root of f if $f(x) = 0$. Show that if f is differentiable and has k distinct real roots, then f' has at least $k-1$ real roots. [Rolle's theorem may be used, provided you state it clearly.]

Let $p(x) = \sum_{i=1}^{n} a_i x^{d_i}$ be a polynomial in x, where all $a_i \neq 0$ and $d_{i+1} > d_i$. (In other words, the a_i are the nonzero coefficients of the polynomial, arranged in order of increasing power of x.) The *number of sign changes* in the coefficients of p is the number of i for which $a_i a_{i+1} < 0$. For example, the polynomial $x^5 - x^3 - x^2 + 1$ has 2 sign changes. Show by induction on n that the number of positive real roots of p is less than or equal to the number of sign changes in its coefficients.

Paper 1, Section II

11D Analysis I

If (x_n) and (y_n) are sequences converging to x and y respectively, show that the sequence $(x_n + y_n)$ converges to $x + y$.

If $x_n \neq 0$ for all n and $x \neq 0$, show that the sequence $\left(\dfrac{1}{x_n}\right)$ converges to $\dfrac{1}{x}$.

(a) Find $\lim\limits_{n\to\infty} \left(\sqrt{n^2+n} - n\right)$.

(b) Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}}$ converges.

Justify your answers.

Paper 1, Section II

12F Analysis I

Let $f : [0,1] \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leqslant |x - y|$ for all $x, y \in [0,1]$.

Show that f is continuous and that for all $\varepsilon > 0$, there exists a piecewise constant function g such that

$$\sup_{x\in[0,1]} |f(x) - g(x)| \leqslant \varepsilon.$$

For all integers $n \geqslant 1$, let $u_n = \int_0^1 f(t)\cos(nt)dt$. Show that the sequence (u_n) converges to 0.

Paper 1, Section I

3F Analysis I

Find the following limits:

(a) $\lim_{x\to 0} \frac{\sin x}{x}$

(b) $\lim_{x\to 0} (1+x)^{1/x}$

(c) $\lim_{x\to\infty} \frac{(1+x)^{\frac{x}{1+x}} \cos^4 x}{e^x}$

Carefully justify your answers.

[You may use standard results provided that they are clearly stated.]

Paper 1, Section I

4E Analysis I

Let $\sum_{n\geqslant 0} a_n z^n$ be a complex power series. State carefully what it means for the power series to have *radius of convergence* R, with $0 \leqslant R \leqslant \infty$.

Find the radius of convergence of $\sum_{n\geqslant 0} p(n) z^n$, where $p(n)$ is a fixed polynomial in n with coefficients in $\mathbb{C}$.

Paper 1, Section II

9F Analysis I

Let $(a_n), (b_n)$ be sequences of real numbers. Let $S_n = \sum_{j=1}^n a_j$ and set $S_0 = 0$. Show that for any $1 \leqslant m \leqslant n$ we have

$$\sum_{j=m}^{n} a_j b_j = S_n b_n - S_{m-1} b_m + \sum_{j=m}^{n-1} S_j (b_j - b_{j+1}).$$

Suppose that the series $\sum_{n\geqslant 1} a_n$ converges and that (b_n) is bounded and monotonic. Does $\sum_{n\geqslant 1} a_n b_n$ converge?

Assume again that $\sum_{n\geqslant 1} a_n$ converges. Does $\sum_{n\geqslant 1} n^{1/n} a_n$ converge?

Justify your answers.

[You may use the fact that a sequence of real numbers converges if and only if it is a Cauchy sequence.]

Paper 1, Section II

10D Analysis I

(a) For real numbers a, b such that $a < b$, let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Prove that f is bounded on $[a,b]$, and that f attains its supremum and infimum on $[a,b]$.

(b) For $x \in \mathbb{R}$, define

$$g(x) = \begin{cases} |x|^{\frac{1}{2}} \sin(1/\sin x), & x \neq n\pi \\ 0, & x = n\pi \end{cases} \qquad (n \in \mathbb{Z}).$$

Find the set of points $x \in \mathbb{R}$ at which $g(x)$ is continuous.

Does g attain its supremum on $[0, \pi]$?

Does g attain its supremum on $[\pi, 3\pi/2]$?

Justify your answers.

Paper 1, Section II

11D Analysis I

(i) State and prove the intermediate value theorem.

(ii) Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. The chord joining the points $\big(\alpha, f(\alpha)\big)$ and $\big(\beta, f(\beta)\big)$ of the curve $y = f(x)$ is said to be *horizontal* if $f(\alpha) = f(\beta)$. Suppose that the chord joining the points $\big(0, f(0)\big)$ and $\big(1, f(1)\big)$ is horizontal. By considering the function g defined on $[0, \frac{1}{2}]$ by

$$g(x) = f(x + \tfrac{1}{2}) - f(x),$$

or otherwise, show that the curve $y = f(x)$ has a horizontal chord of length $\frac{1}{2}$ in $[0,1]$. Show, more generally, that it has a horizontal chord of length $\frac{1}{n}$ for each positive integer n.

Paper 1, Section II

12E Analysis I

Let $f\colon [0,1] \to \mathbb{R}$ be a bounded function, and let $\mathcal{D}_n$ denote the dissection $0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1$ of $[0,1]$. Prove that f is Riemann integrable if and only if the difference between the upper and lower sums of f with respect to the dissection $\mathcal{D}_n$ tends to zero as n tends to infinity.

Suppose that f is Riemann integrable and $g\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Prove that $g \circ f$ is Riemann integrable.

[You may use the mean value theorem provided that it is clearly stated.]

Paper 1, Section I

3D Analysis I

Show that every sequence of real numbers contains a monotone subsequence.

Paper 1, Section I

4F Analysis I

Find the radius of convergence of the following power series:

$$\text{(i)} \sum_{n\geqslant 1} \frac{n!}{n^n} z^n; \qquad \text{(ii)} \sum_{n\geqslant 1} n^n z^{n!}.$$

Paper 1, Section II

9D Analysis I

(a) Show that for all $x \in \mathbb{R}$,

$$\lim_{k\to\infty} 3^k \sin(x/3^k) = x,$$

stating carefully what properties of sin you are using.

Show that the series $\sum_{n\geqslant 1} 2^n \sin(x/3^n)$ converges absolutely for all $x \in \mathbb{R}$.

(b) Let $(a_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers tending to zero. Show that for $\theta \in \mathbb{R}$, θ not a multiple of 2π, the series

$$\sum_{n\geqslant 1} a_n e^{in\theta}$$

converges.

Hence, or otherwise, show that $\sum_{n\geqslant 1} \frac{\sin(n\theta)}{n}$ converges for all $\theta \in \mathbb{R}$.

Paper 1, Section II

10E Analysis I

(i) State the Mean Value Theorem. Use it to show that if $f\colon (a,b) \to \mathbb{R}$ is a differentiable function whose derivative is identically zero, then f is constant.

(ii) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and $\alpha > 0$ a real number such that for all x, $y \in \mathbb{R}$,

$$|f(x) - f(y)| \leqslant |x - y|^{\alpha}.$$

Show that f is continuous. Show moreover that if $\alpha > 1$ then f is constant.

(iii) Let $f\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). Assume also that the right derivative of f at a exists; that is, the limit

$$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$$

exists. Show that for any $\epsilon > 0$ there exists $x \in (a,b)$ satisfying

$$\left| \frac{f(x) - f(a)}{x - a} - f'(x) \right| < \epsilon.$$

[You should not assume that f' is continuous.]

Paper 1, Section II

11E Analysis I

(i) Prove Taylor's Theorem for a function $f\colon \mathbb{R} \to \mathbb{R}$ differentiable n times, in the following form: for every $x \in \mathbb{R}$ there exists θ with $0 < \theta < 1$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n)}(\theta x)}{n!} x^n.$$

[You may assume Rolle's Theorem and the Mean Value Theorem; other results should be proved.]

(ii) The function $f\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable, and satisfies the differential equation $f'' - f = 0$ with $f(0) = A$, $f'(0) = B$. Show that f is infinitely differentiable. Write down its Taylor series at the origin, and prove that it converges to f at every point. Hence or otherwise show that for any a, $h \in \mathbb{R}$, the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} h^k$$

converges to $f(a+h)$.

Paper 1, Section II

12F Analysis I

Define what it means for a function $f\colon [0,1] \to \mathbb{R}$ to be (Riemann) integrable. Prove that f is integrable whenever it is

(a) continuous,

(b) monotonic.

Let $\{q_k : k \in \mathbb{N}\}$ be an enumeration of all rational numbers in $[0,1)$. Define a function $f\colon [0,1] \to \mathbb{R}$ by $f(0) = 0$,

$$f(x) = \sum_{k \in Q(x)} 2^{-k}, \quad x \in (0,1],$$

where

$$Q(x) = \{k \in \mathbb{N} : q_k \in [0,x)\}.$$

Show that f has a point of discontinuity in every interval $I \subset [0,1]$.

Is f integrable? [Justify your answer.]

Paper 1, Section I
3D Analysis I

Show that $\exp(x) \geqslant 1 + x$ for $x \geqslant 0$.

Let (a_j) be a sequence of positive real numbers. Show that for every n,

$$\sum_1^n a_j \leqslant \prod_1^n (1 + a_j) \leqslant \exp\left(\sum_1^n a_j\right).$$

Deduce that $\prod_1^n (1 + a_j)$ tends to a limit as $n \to \infty$ if and only if $\sum_1^n a_j$ does.

Paper 1, Section I
4F Analysis I

(a) Suppose $b_n \geqslant b_{n+1} \geqslant 0$ for $n \geqslant 1$ and $b_n \to 0$. Show that $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

(b) Does the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converge or diverge? Explain your answer.

Paper 1, Section II
9D Analysis I

(a) Determine the radius of convergence of each of the following power series:

$$\sum_{n \geqslant 1} \frac{x^n}{n!}, \qquad \sum_{n \geqslant 1} n! x^n, \qquad \sum_{n \geqslant 1} (n!)^2 x^{n^2}.$$

(b) State Taylor's theorem.

Show that

$$(1 + x)^{1/2} = 1 + \sum_{n \geqslant 1} c_n x^n,$$

for all $x \in (0, 1)$, where

$$c_n = \frac{\frac{1}{2}(\frac{1}{2} - 1) \ldots (\frac{1}{2} - n + 1)}{n!}.$$

Paper 1, Section II
10E Analysis I

(a) Let $f\colon [a,b] \to \mathbb{R}$. Suppose that for every sequence (x_n) in $[a,b]$ with limit $y \in [a,b]$, the sequence $(f(x_n))$ converges to $f(y)$. Show that f is continuous at y.

(b) State the Intermediate Value Theorem.

Let $f\colon [a,b] \to \mathbb{R}$ be a function with $f(a) = c < f(b) = d$. We say f is *injective* if for all x, $y \in [a,b]$ with $x \neq y$, we have $f(x) \neq f(y)$. We say f is *strictly increasing* if for all x, y with $x < y$, we have $f(x) < f(y)$.

(i) Suppose f is strictly increasing. Show that it is injective, and that if $f(x) < f(y)$ then $x < y$.

(ii) Suppose f is continuous and injective. Show that if $a < x < b$ then $c < f(x) < d$. Deduce that f is strictly increasing.

(iii) Suppose f is strictly increasing, and that for every $y \in [c,d]$ there exists $x \in [a,b]$ with $f(x) = y$. Show that f is continuous at b. Deduce that f is continuous on $[a,b]$.

Paper 1, Section II
11E Analysis I

(i) State (without proof) Rolle's Theorem.

(ii) State and prove the Mean Value Theorem.

(iii) Let f, $g\colon [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$. Show that there exists $\xi \in (a,b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Deduce that if moreover $f(a) = g(a) = 0$, and the limit

$$\ell = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists, then

$$\frac{f(x)}{g(x)} \to \ell \quad \text{as } x \to a.$$

(iv) Deduce that if $f\colon \mathbb{R} \to \mathbb{R}$ is twice differentiable then for any $a \in \mathbb{R}$

$$f''(a) = \lim_{h \to 0} \frac{f(a+h)+f(a-h)-2f(a)}{h^2}.$$

Paper 1, Section II

12F Analysis I

Fix a closed interval $[a, b]$. For a bounded function f on $[a, b]$ and a dissection $\mathcal{D}$ of $[a, b]$, how are the lower sum $s(f, \mathcal{D})$ and upper sum $S(f, \mathcal{D})$ defined? Show that $s(f, \mathcal{D}) \leqslant S(f, \mathcal{D})$.

Suppose $\mathcal{D}'$ is a dissection of $[a, b]$ such that $\mathcal{D} \subseteq \mathcal{D}'$. Show that

$$s(f, \mathcal{D}) \leqslant s(f, \mathcal{D}') \text{ and } S(f, \mathcal{D}') \leqslant S(f, \mathcal{D}) .$$

By using the above inequalities or otherwise, show that if $\mathcal{D}_1$ and $\mathcal{D}_2$ are two dissections of $[a, b]$ then

$$s(f, \mathcal{D}_1) \leqslant S(f, \mathcal{D}_2) .$$

For a function f and dissection $\mathcal{D} = \{x_0, \ldots, x_n\}$ let

$$p(f, \mathcal{D}) = \prod_{k=1}^{n} \left[1 + (x_k - x_{k-1}) \inf_{x \in [x_{k-1}, x_k]} f(x) \right] .$$

If f is non-negative and Riemann integrable, show that

$$p(f, \mathcal{D}) \leqslant e^{\int_a^b f(x) dx} .$$

[You may use without proof the inequality $e^t \geqslant t + 1$ for all t.]

Paper 1, Section I

3E Analysis I

What does it mean to say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous* at $x_0 \in \mathbb{R}$?

Give an example of a continuous function $f\colon (0,1] \to \mathbb{R}$ which is bounded but attains neither its upper bound nor its lower bound.

The function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and non-negative, and satisfies $f(x) \to 0$ as $x \to \infty$ and $f(x) \to 0$ as $x \to -\infty$. Show that f is bounded above and attains its upper bound.

[Standard results about continuous functions on closed bounded intervals may be used without proof if clearly stated.]

Paper 1, Section I

4F Analysis I

Let $f, g : [0,1] \to \mathbb{R}$ be continuous functions with $g(x) \geqslant 0$ for $x \in [0,1]$. Show that

$$\int_0^1 f(x)g(x)\,dx \leqslant M \int_0^1 g(x)\,dx\,,$$

where $M = \sup\{|f(x)| : x \in [0,1]\}$.

Prove there exists $\alpha \in [0,1]$ such that

$$\int_0^1 f(x)g(x)\,dx = f(\alpha) \int_0^1 g(x)\,dx\,.$$

[Standard results about continuous functions and their integrals may be used without proof, if clearly stated.]

Paper 1, Section II

9E Analysis I

(a) What does it mean to say that the sequence (x_n) of real numbers *converges* to $\ell \in \mathbb{R}$?

Suppose that $(y_n^{(1)})$, $(y_n^{(2)})$,..., $(y_n^{(k)})$ are sequences of real numbers converging to the same limit ℓ. Let (x_n) be a sequence such that for every n,

$$x_n \in \{y_n^{(1)}, y_n^{(2)}, \ldots, y_n^{(k)}\}\,.$$

Show that (x_n) also converges to ℓ.

Find a collection of sequences $(y_n^{(j)})$, $j = 1, 2, \ldots$ such that for every j, $(y_n^{(j)}) \to \ell$ but the sequence (x_n) defined by $x_n = y_n^{(n)}$ diverges.

(b) Let a, b be real numbers with $0 < a < b$. Sequences (a_n), (b_n) are defined by $a_1 = a$, $b_1 = b$ and

$$\text{for all } n \geqslant 1, \quad a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}\,.$$

Show that (a_n) and (b_n) converge to the same limit.

Paper 1, Section II

10D Analysis I

Let (a_n) be a sequence of reals.

(i) Show that if the sequence $(a_{n+1} - a_n)$ is convergent then so is the sequence $\left(\frac{a_n}{n}\right)$.

(ii) Give an example to show the sequence $\left(\frac{a_n}{n}\right)$ being convergent does not imply that the sequence $(a_{n+1} - a_n)$ is convergent.

(iii) If $a_{n+k} - a_n \to 0$ as $n \to \infty$ for each positive integer k, does it follow that (a_n) is convergent? Justify your answer.

(iv) If $a_{n+f(n)} - a_n \to 0$ as $n \to \infty$ for every function f from the positive integers to the positive integers, does it follow that (a_n) is convergent? Justify your answer.

Paper 1, Section II

11D Analysis I

Let f be a continuous function from $(0,1)$ to $(0,1)$ such that $f(x) < x$ for every $0 < x < 1$. We write f^n for the n-fold composition of f with itself (so for example $f^2(x) = f(f(x))$).

(i) Prove that for every $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$.

(ii) Must it be the case that for every $\epsilon > 0$ there exists n with the property that $f^n(x) < \epsilon$ for all $0 < x < 1$? Justify your answer.

Now suppose that we remove the condition that f be continuous.

(iii) Give an example to show that it need not be the case that for every $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$.

(iv) Must it be the case that for *some* $0 < x < 1$ we have $f^n(x) \to 0$ as $n \to \infty$? Justify your answer.

Paper 1, Section II

12F Analysis I

(a) (i) State the ratio test for the convergence of a real series with positive terms.

(ii) Define the radius of convergence of a real power series $\sum_{n=0}^{\infty} a_n x^n$.

(iii) Prove that the real power series $f(x) = \sum_n a_n x^n$ and $g(x) = \sum_n (n+1) a_{n+1} x^n$ have equal radii of convergence.

(iv) State the relationship between $f(x)$ and $g(x)$ within their interval of convergence.

(b) (i) Prove that the real series

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad g(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

have radius of convergence ∞.

(ii) Show that they are differentiable on the real line $\mathbb{R}$, with $f' = -g$ and $g' = f$, and deduce that $f(x)^2 + g(x)^2 = 1$.

[You may use, without proof, general theorems about differentiating within the interval of convergence, provided that you give a clear statement of any such theorem.]

Paper 1, Section I
3F Analysis I

(a) State, without proof, the Bolzano–Weierstrass Theorem.

(b) Give an example of a sequence that does not have a convergent subsequence.

(c) Give an example of an unbounded sequence having a convergent subsequence.

(d) Let $a_n = 1 + (-1)^{\lfloor n/2 \rfloor}\big(1 + 1/n\big)$, where $\lfloor x \rfloor$ denotes the integer part of x. Find all values c such that the sequence $\{a_n\}$ has a subsequence converging to c. For each such value, provide a subsequence converging to it.

Paper 1, Section I
4D Analysis I

Find the radius of convergence of each of the following power series.

(i) $\displaystyle\sum_{n \geqslant 1} n^2 z^n$

(ii) $\displaystyle\sum_{n \geqslant 1} n^{n^{1/3}} z^n$

Paper 1, Section II
9F Analysis I

(a) State, without proof, the ratio test for the series $\displaystyle\sum_{n \geqslant 1} a_n$, where $a_n > 0$. Give examples, without proof, to show that, when $a_{n+1} < a_n$ and $a_{n+1}/a_n \to 1$, the series may converge or diverge.

(b) Prove that $\displaystyle\sum_{k=1}^{n-1} \frac{1}{k} \geqslant \log\, n$.

(c) Now suppose that $a_n > 0$ and that, for n large enough, $\dfrac{a_{n+1}}{a_n} \leqslant 1 - \dfrac{c}{n}$ where $c > 1$. Prove that the series $\displaystyle\sum_{n \geqslant 1} a_n$ converges.

[*You may find it helpful to prove the inequality* $\log\,(1 - x) < -x$ *for* $0 < x < 1$.]

Paper 1, Section II

10E Analysis I

State and prove the Intermediate Value Theorem.

A *fixed point* of a function $f : X \to X$ is an $x \in X$ with $f(x) = x$. Prove that every continuous function $f : [0,1] \to [0,1]$ has a fixed point.

Answer the following questions with justification.

(i) Does every continuous function $f : (0,1) \to (0,1)$ have a fixed point?

(ii) Does every continuous function $f : \mathbb{R} \to \mathbb{R}$ have a fixed point?

(iii) Does every function $f : [0,1] \to [0,1]$ (not necessarily continuous) have a fixed point?

(iv) Let $f : [0,1] \to [0,1]$ be a continuous function with $f(0) = 1$ and $f(1) = 0$. Can f have exactly two fixed points?

Paper 1, Section II

11E Analysis I

For each of the following two functions $f : \mathbb{R} \to \mathbb{R}$, determine the set of points at which f is continuous, and also the set of points at which f is differentiable.

$$\text{(i) } f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases},$$

$$\text{(ii) } f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

By modifying the function in (i), or otherwise, find a function (not necessarily continuous) $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable at 0 and nowhere else.

Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that f is not differentiable at the points $1/2, 1/3, 1/4, \ldots$, but is differentiable at all other points.

Paper 1, Section II

12D Analysis I

State and prove the Fundamental Theorem of Calculus.

Let $f : [0,1] \to \mathbb{R}$ be integrable, and set $F(x) = \int_0^x f(t)\,\mathrm{d}t$ for $0 < x < 1$. Must F be differentiable?

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and set $g(x) = f'(x)$ for $0 \leqslant x \leqslant 1$. Must the Riemann integral of g from 0 to 1 exist?

Paper 1, Section I
3D Analysis I

Let $\sum_{n\geqslant 0} a_n z^n$ be a complex power series. State carefully what it means for the power series to have radius of convergence R, with $R \in [0, \infty]$.

Suppose the power series has radius of convergence R, with $0 < R < \infty$. Show that the sequence $|a_n z^n|$ is unbounded if $|z| > R$.

Find the radius of convergence of $\sum_{n\geqslant 1} z^n/n^3$.

Paper 1, Section I
4E Analysis I

Find the limit of each of the following sequences; justify your answers.

(i)
$$\frac{1+2+\ldots+n}{n^2}\,;$$

(ii)
$$\sqrt[n]{n}\,;$$

(iii)
$$(a^n+b^n)^{1/n} \qquad \text{with} \quad 0 < a \leqslant b\,.$$

Paper 1, Section II
9E Analysis I

Determine whether the following series converge or diverge. Any tests that you use should be carefully stated.

(a)
$$\sum_{n\geqslant 1} \frac{n!}{n^n}\ ;$$

(b)
$$\sum_{n\geqslant 1} \frac{1}{n+(\log n)^2}\ ;$$

(c)
$$\sum_{n\geqslant 1} \frac{(-1)^n}{1+\sqrt{n}}\ ;$$

(d)
$$\sum_{n\geqslant 1} \frac{(-1)^n}{n\,(2+(-1)^n)}\ .$$

Paper 1, Section II
10F Analysis I

(a) State and prove Taylor's theorem with the remainder in Lagrange's form.

(b) Suppose that $e:\ \mathbb{R}\to\mathbb{R}$ is a differentiable function such that $e(0)=1$ and $e'(x)=e(x)$ for all $x\in\mathbb{R}$. Use the result of (a) to prove that

$$e(x)=\sum_{n\geqslant 0}\frac{x^n}{n!}\quad \text{for all}\quad x\in\mathbb{R}.$$

[*No property of the exponential function may be assumed.*]

Paper 1, Section II
11D Analysis I

Define what it means for a bounded function $f : [a, \infty) \to \mathbb{R}$ to be Riemann integrable.

Show that a monotonic function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, where $-\infty < a < b < \infty$.

Prove that if $f : [1, \infty) \to \mathbb{R}$ is a decreasing function with $f(x) \to 0$ as $x \to \infty$, then $\sum_{n \geqslant 1} f(n)$ and $\int_1^\infty f(x)\, dx$ either both diverge or both converge.

Hence determine, for $\alpha \in \mathbb{R}$, when $\sum_{n \geqslant 1} n^\alpha$ converges.

Paper 1, Section II
12F Analysis I

(a) Let $n \geqslant 1$ and f be a function $\mathbb{R} \to \mathbb{R}$. Define carefully what it means for f to be n times differentiable at a point $x_0 \in \mathbb{R}$.

Set $\operatorname{sign}(x) = \begin{cases} x/|x|, & x \neq 0, \\ 0, & x = 0. \end{cases}$

Consider the function $f(x)$ on the real line, with $f(0) = 0$ and

$$f(x) = x^2 \operatorname{sign}(x) \left| \cos \frac{\pi}{x} \right|, \quad x \neq 0.$$

(b) Is $f(x)$ differentiable at $x = 0$?

(c) Show that $f(x)$ has points of non-differentiability in any neighbourhood of $x = 0$.

(d) Prove that, in any finite interval I, the derivative $f'(x)$, at the points $x \in I$ where it exists, is bounded: $|f'(x)| \leqslant C$ where C depends on I.

Paper 1, Section I

3F Analysis I

Determine the limits as $n \to \infty$ of the following sequences:

(a) $a_n = n - \sqrt{n^2 - n}$;

(b) $b_n = \cos^2\left(\pi\sqrt{n^2+n}\right)$.

Paper 1, Section I

4E Analysis I

Let $a_0, a_1, a_2, \ldots$ be a sequence of complex numbers. Prove that there exists $R \in [0, \infty]$ such that the power series $\sum_{n=0}^{\infty} a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$.

Give an example of a power series $\sum_{n=0}^{\infty} a_n z^n$ that diverges if $z = \pm 1$ and converges if $z = \pm \mathrm{i}$.

Paper 1, Section II

9F Analysis I

For each of the following series, determine for which real numbers x it diverges, for which it converges, and for which it converges absolutely. Justify your answers briefly.

(a) $$\sum_{n \geqslant 1} \frac{3 + \left(\sin x\right)^n}{n} \left(\sin x\right)^n,$$

(b) $$\sum_{n \geqslant 1} \left|\sin x\right|^n \frac{(-1)^n}{\sqrt{n}},$$

(c) $$\sum_{n \geqslant 1} \underbrace{\sin\left(0.99\ \sin\left(0.99 \ldots \sin\left(0.99\ x\right) \ldots\right)\right)}_{n \text{ times}}.$$

Paper 1, Section II

10D Analysis I

State and prove the intermediate value theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $P = (a, b)$ be a point of the plane $\mathbb{R}^2$. Show that the set of distances from points $(x, f(x))$ on the graph of f to the point P is an interval $[A, \infty)$ for some value $A \geqslant 0$.

Paper 1, Section II

11D Analysis I

State and prove Rolle's theorem.

Let f and g be two continuous, real-valued functions on a closed, bounded interval $[a, b]$ that are differentiable on the open interval (a, b). By considering the determinant

$$\phi(x) = \begin{vmatrix} 1 & 1 & 0 \\ f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \end{vmatrix} = g(x)\,(f(b) - f(a)) - f(x)\,(g(b) - g(a)) \ ,$$

or otherwise, show that there is a point $c \in (a, b)$ with

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)) \ .$$

Suppose that $f, g : (0, \infty) \to \mathbb{R}$ are differentiable functions with $f(x) \to 0$ and $g(x) \to 0$ as $x \to 0$. Prove carefully that if the limit $\lim\limits_{x \to 0} \dfrac{f'(x)}{g'(x)} = \ell$ exists and is finite, then the limit $\lim\limits_{x \to 0} \dfrac{f(x)}{g(x)}$ also exists and equals ℓ.

Paper 1, Section II

12E Analysis I

(a) What does it mean for a function $f : [a, b] \to \mathbb{R}$ to be *Riemann integrable*?

(b) Let $f : [0, 1] \to \mathbb{R}$ be a bounded function. Suppose that for every $\delta > 0$ there is a sequence

$$0 \leqslant a_1 < b_1 \leqslant a_2 < b_2 \leqslant \ldots \leqslant a_n < b_n \leqslant 1$$

such that for each i the function f is Riemann integrable on the closed interval $[a_i, b_i]$, and such that $\sum_{i=1}^{n}(b_i - a_i) \geqslant 1 - \delta$. Prove that f is Riemann integrable on $[0, 1]$.

(c) Let $f : [0, 1] \to \mathbb{R}$ be defined as follows. We set $f(x) = 1$ if x has an infinite decimal expansion that consists of 2s and 7s only, and otherwise we set $f(x) = 0$. Prove that f is Riemann integrable and determine $\int_0^1 f(x)\,\mathrm{d}x$.

1/I/3F **Analysis I**

State the ratio test for the convergence of a series.

Find all real numbers x such that the series

$$\sum_{n=1}^{\infty} \frac{x^n - 1}{n}$$

converges.

1/I/4E **Analysis I**

Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable, and for $0 \leqslant x \leqslant 1$ set $F(x) = \int_0^x f(t)\,dt$.

Assuming that f is continuous, prove that for every $0 < x < 1$ the function F is differentiable at x, with $F'(x) = f(x)$.

If we do not assume that f is continuous, must it still be true that F is differentiable at every $0 < x < 1$? Justify your answer.

1/II/9F **Analysis I**

Investigate the convergence of the series

(i) $$\sum_{n=2}^{\infty} \frac{1}{n^p (\log n)^q}$$

(ii) $$\sum_{n=3}^{\infty} \frac{1}{n\,(\log\log n)^r}$$

for positive real values of p, q and r.

[*You may assume that for any positive real value of* α, $\log n < n^\alpha$ *for* n *sufficiently large. You may assume standard tests for convergence, provided that they are clearly stated.*]

1/II/10D **Analysis I**

(a) State and prove the intermediate value theorem.

(b) An *interval* is a subset I of $\mathbb{R}$ with the property that if x and y belong to I and $x < z < y$ then z also belongs to I. Prove that if I is an interval and f is a continuous function from I to $\mathbb{R}$ then $f(I)$ is an interval.

(c) For each of the following three pairs (I, J) of intervals, either exhibit a continuous function f from I to $\mathbb{R}$ such that $f(I) = J$ or explain briefly why no such continuous function exists:

(i) $I = [0,1]\,, \quad J = [0,\infty)\,;$

(ii) $I = (0,1]\,, \quad J = [0,\infty)\,;$

(iii) $I = (0,1]\,, \quad J = (-\infty,\infty)\,.$

1/II/11D **Analysis I**

(a) Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ and suppose that both f and g are differentiable at the real number x. Prove that the product fg is also differentiable at x.

(b) Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let $g(x) = x^2 f(x)$ for every x. Prove that g is differentiable at x if and only if either $x = 0$ or f is differentiable at x.

(c) Now let f be any continuous function from $\mathbb{R}$ to $\mathbb{R}$ and let $g(x) = f(x)^2$ for every x. Prove that g is differentiable at x if and only if at least one of the following two possibilities occurs:

(i) f is differentiable at x;

(ii) $f(x) = 0$ and

$$\frac{f(x+h)}{|h|^{1/2}} \longrightarrow 0 \quad \text{as} \quad h \to 0\,.$$

1/II/12E **Analysis I**

Let $\sum_{n=0}^{\infty} a_n z^n$ be a complex power series. Prove that there exists an $R \in [0, \infty]$ such that the series converges for every z with $|z| < R$ and diverges for every z with $|z| > R$.

Find the value of R for each of the following power series:

(i) $$\sum_{n=1}^{\infty} \frac{1}{n^2} z^n ;$$

(ii) $$\sum_{n=0}^{\infty} z^{n!} .$$

In each case, determine at which points on the circle $|z| = R$ the series converges.

1/I/3F **Analysis**

Prove that, for positive real numbers a and b,

$$2\sqrt{ab} \leqslant a + b\,.$$

For positive real numbers $a_1, a_2, \ldots$, prove that the convergence of

$$\sum_{n=1}^{\infty} a_n$$

implies the convergence of

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}\,.$$

1/I/4D **Analysis**

Let $\sum_{n=0}^{\infty} a_n z^n$ be a complex power series. Show that there exists $R \in [0, \infty]$ such that $\sum_{n=0}^{\infty} a_n z^n$ converges whenever $|z| < R$ and diverges whenever $|z| > R$.

Find the value of R for the power series

$$\sum_{n=1}^{\infty} \frac{z^n}{n}\,.$$

1/II/9F **Analysis**

Let $a_1 = \sqrt{2}$, and consider the sequence of positive real numbers defined by

$$a_{n+1} = \sqrt{2 + \sqrt{a_n}}\,, \qquad n = 1, 2, 3, \ldots\,.$$

Show that $a_n \leqslant 2$ for all n. Prove that the sequence $a_1, a_2, \ldots$ converges to a limit.

Suppose instead that $a_1 = 4$. Prove that again the sequence $a_1, a_2, \ldots$ converges to a limit.

Prove that the limits obtained in the two cases are equal.

1/II/10E **Analysis**

State and prove the Mean Value Theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that, for every $x \in \mathbb{R}$, $f''(x)$ exists and is non-negative.

(i) Show that if $x \leq y$ then $f'(x) \leq f'(y)$.

(ii) Let $\lambda \in (0,1)$ and $a < b$. Show that there exist x and y such that

$$f\big(\lambda a + (1-\lambda)b\big) = f(a) + (1-\lambda)(b-a)f'(x) = f(b) - \lambda(b-a)f'(y)$$

and that

$$f\big(\lambda a + (1-\lambda)b\big) \leq \lambda f(a) + (1-\lambda)f(b)\,.$$

1/II/11E **Analysis**

Let $a < b$ be real numbers, and let $f : [a,b] \to \mathbb{R}$ be continuous. Show that f is bounded on $[a,b]$, and that there exist $c, d \in [a,b]$ such that for all $x \in [a,b]$, $f(c) \leq f(x) \leq f(d)$.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$\lim_{x\to+\infty} g(x) = \lim_{x\to-\infty} g(x) = 0\,.$$

Show that g is bounded. Show also that, if a and c are real numbers with $0 < c \leq g(a)$, then there exists $x \in \mathbb{R}$ with $g(x) = c$.

1/II/12D **Analysis**

Explain carefully what it means to say that a bounded function $f : [0,1] \to \mathbb{R}$ is *Riemann integrable.*

Prove that every continuous function $f : [0,1] \to \mathbb{R}$ is Riemann integrable.

For each of the following functions from $[0,1]$ to $\mathbb{R}$, determine with proof whether or not it is Riemann integrable:

(i) the function $f(x) = x \sin \dfrac{1}{x}$ for $x \neq 0$, with $f(0) = 0$;

(ii) the function $g(x) = \sin \dfrac{1}{x}$ for $x \neq 0$, with $g(0) = 0$.

1/I/3F **Analysis**

Let $a_n \in \mathbb{R}$ for $n \geqslant 1$. What does it mean to say that the infinite series $\sum_n a_n$ converges to some value A? Let $s_n = a_1 + \cdots + a_n$ for all $n \geqslant 1$. Show that if $\sum_n a_n$ converges to some value A, then the sequence whose n-th term is

$$(s_1 + \cdots + s_n)/n$$

converges to some value $\tilde{A}$ as $n \to \infty$. Is it always true that $A = \tilde{A}$? Give an example where $(s_1 + \cdots + s_n)/n$ converges but $\sum_n a_n$ does not.

1/I/4D **Analysis**

Let $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ be power series in the complex plane with radii of convergence R and S respectively. Show that if $R \neq S$ then $\sum_{n=0}^{\infty}(a_n + b_n)z^n$ has radius of convergence $\min(R, S)$. [*Any results on absolute convergence that you use should be clearly stated.*]

1/II/9E **Analysis**

State and prove the Intermediate Value Theorem.

Suppose that the function f is differentiable everywhere in some open interval containing $[a, b]$, and that $f'(a) < k < f'(b)$. By considering the functions g and h defined by

$$g(x) = \frac{f(x) - f(a)}{x - a} \quad (a < x \leqslant b)\ , \quad g(a) = f'(a)$$

and

$$h(x) = \frac{f(b) - f(x)}{b - x} \quad (a \leqslant x < b)\ , \quad h(b) = f'(b),$$

or otherwise, show that there is a subinterval $[a', b'] \subseteq [a, b]$ such that

$$\frac{f(b') - f(a')}{b' - a'} = k.$$

Deduce that there exists $c \in (a, b)$ with $f'(c) = k$. [*You may assume the Mean Value Theorem.*]

1/II/10E **Analysis**

Prove that if the function f is infinitely differentiable on an interval (r, s) containing a, then for any $x \in (r, s)$ and any positive integer n we may expand $f(x)$ in the form

$$f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + R_n(f, a, x),$$

where the remainder term $R_n(f, a, x)$ should be specified explicitly in terms of $f^{(n+1)}$.

Let $p(t)$ be a nonzero polynomial in t, and let f be the real function defined by

$$f(x) = p\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) \quad (x \neq 0)\ , \quad f(0) = 0.$$

Show that f is differentiable everywhere and that

$$f'(x) = q\left(\frac{1}{x}\right)\exp\left(-\frac{1}{x^2}\right) \quad (x \neq 0)\ , \quad f'(0) = 0,$$

where $q(t) = 2t^3p(t) - t^2p'(t)$. Deduce that f is infinitely differentiable, but that there exist arbitrarily small values of x for which the remainder term $R_n(f, 0, x)$ in the Taylor expansion of f about 0 does not tend to 0 as $n \to \infty$.

1/II/11F **Analysis**

Consider a sequence $(a_n)_{n\geqslant 1}$ of real numbers. What does it mean to say that $a_n \to a \in \mathbb{R}$ as $n \to \infty$? What does it mean to say that $a_n \to \infty$ as $n \to \infty$? What does it mean to say that $a_n \to -\infty$ as $n \to \infty$? Show that for every sequence of real numbers there exists a subsequence which converges to a value in $\mathbb{R} \cup \{\infty, -\infty\}$. [*You may use the Bolzano–Weierstrass theorem provided it is clearly stated.*]

Give an example of a bounded sequence $(a_n)_{n\geqslant 1}$ which is not convergent, but for which

$$a_{n+1} - a_n \to 0 \quad \text{as} \quad n \to \infty.$$

1/II/12D **Analysis**

Let f_1 and f_2 be Riemann integrable functions on $[a, b]$. Show that $f_1 + f_2$ is Riemann integrable.

Let f be a Riemann integrable function on $[a, b]$ and set $f^+(x) = \max(f(x), 0)$. Show that f^+ and $|f|$ are Riemann integrable.

Let f be a function on $[a, b]$ such that $|f|$ is Riemann integrable. Is it true that f is Riemann integrable? Justify your answer.

Show that if f_1 and f_2 are Riemann integrable on $[a, b]$, then so is $\max(f_1, f_2)$. Suppose now $f_1, f_2, \ldots$ is a sequence of Riemann integrable functions on $[a, b]$ and $f(x) = \sup_n f_n(x)$; is it true that f is Riemann integrable? Justify your answer.

1/I/3F **Analysis**

Define the *supremum* or *least upper bound* of a non-empty set of real numbers.

Let A denote a non-empty set of real numbers which has a supremum but no maximum. Show that for every $\epsilon > 0$ there are infinitely many elements of A contained in the open interval

$$(\sup A - \epsilon\,,\ \sup A).$$

Give an example of a non-empty set of real numbers which has a supremum *and* maximum and for which the above conclusion does not hold.

1/I/4D **Analysis**

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series in the complex plane with radius of convergence R. Show that $|a_n z^n|$ is unbounded in n for any z with $|z| > R$. State clearly any results on absolute convergence that are used.

For every $R \in [0, \infty]$, show that there exists a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence R.

1/II/9F **Analysis**

Examine each of the following series and determine whether or not they converge. Give reasons in each case.

(*i*)
$$\sum_{n=1}^{\infty} \frac{1}{n^2}\,,$$

(*ii*)
$$\sum_{n=1}^{\infty} \frac{1}{n^2 + (-1)^{n+1}\, 2n + 1}\,,$$

(*iii*)
$$\sum_{n=1}^{\infty} \frac{n^3 + (-1)^n\, 8n^2 + 1}{n^4 + (-1)^{n+1}\, n^2}\,,$$

(*iv*)
$$\sum_{n=1}^{\infty} \frac{n^3}{e^{e^n}}\,.$$

1/II/10D **Analysis**

Explain what it means for a bounded function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable.

Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function. Show that for each $x \in (0, \infty)$, there exists a unique point $g(x) \in (0, x)$ such that

$$\frac{1}{x} \int_0^x f(t)\, dt = f(g(x)).$$

Find $g(x)$ if $f(x) = e^{-x}$.

Suppose now that f is differentiable and $f'(x) < 0$ for all $x \in (0, \infty)$. Prove that g is differentiable at all $x \in (0, \infty)$ and $g'(x) > 0$ for all $x \in (0, \infty)$, stating clearly any results on the inverse of f you use.

1/II/11E **Analysis**

Prove that if f is a continuous function on the interval $[a, b]$ with $f(a) < 0 < f(b)$ then $f(c) = 0$ for some $c \in (a, b)$.

Let g be a continuous function on $[0, 1]$ satisfying $g(0) = g(1)$. By considering the function $f(x) = g(x + \frac{1}{2}) - g(x)$ on $[0, \frac{1}{2}]$, show that $g(c + \frac{1}{2}) = g(c)$ for some $c \in [0, \frac{1}{2}]$. Show, more generally, that for any positive integer n there exists a point $c_n \in [0, \frac{n-1}{n}]$ for which $g(c_n + \frac{1}{n}) = g(c_n)$.

1/II/12E **Analysis**

State and prove Rolle's Theorem.

Prove that if the real polynomial p of degree n has all its roots real (though not necessarily distinct), then so does its derivative p'. Give an example of a cubic polynomial p for which the converse fails.

1/I/3D **Analysis**

Define the *supremum* or *least upper bound* of a non-empty set of real numbers.

State the Least Upper Bound Axiom for the real numbers.

Starting from the Least Upper Bound Axiom, show that if (a_n) is a bounded monotonic sequence of real numbers, then it converges.

1/I/4E **Analysis**

Let $f(x) = (1+x)^{1/2}$ for $x \in (-1,1)$. Show by induction or otherwise that for every integer $r \geq 1$,

$$f^{(r)}(x) = (-1)^{r-1}\frac{(2r-2)!}{2^{2r-1}(r-1)!}(1+x)^{\frac{1}{2}-r}.$$

Evaluate the series

$$\sum_{r=1}^{\infty}(-1)^{r-1}\frac{(2r-2)!}{8^r r!(r-1)!}\,.$$

[You may use Taylor's Theorem in the form

$$f(x) = f(0) + \sum_{r=1}^{n}\frac{f^{(r)}(0)}{r!}x^r + \int_0^x \frac{(x-t)^n f^{(n+1)}(t)}{n!}\,dt$$

without proof.]

1/II/9D **Analysis**

i) State Rolle's theorem.

Let $f, g : [a,b] \to \mathbb{R}$ be continuous functions which are differentiable on (a,b).

ii) Prove that for some $c \in (a,b)$,

$$(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c).$$

iii) Suppose that $f(a) = g(a) = 0$, and that $\lim_{x\to a+}\dfrac{f'(x)}{g'(x)}$ exists and is equal to L. Prove that $\lim_{x\to a+}\dfrac{f(x)}{g(x)}$ exists and is also equal to L.

[You may assume there exists a $\delta > 0$ such that, for all $x \in (a, a+\delta)$, $g'(x) \neq 0$ and $g(x) \neq 0$.]

iv) Evaluate $\lim_{x\to 0}\dfrac{\log\cos x}{x^2}$.

1/II/10E **Analysis**

Define, for an integer $n \geq 0$,

$$I_n = \int_0^{\pi/2} \sin^n x \, dx.$$

Show that for every $n \geq 2$, $nI_n = (n-1)I_{n-2}$, and deduce that

$$I_{2n} = \frac{(2n)!}{(2^n n!)^2} \frac{\pi}{2} \quad \text{and} \quad I_{2n+1} = \frac{(2^n n!)^2}{(2n+1)!}.$$

Show that $0 < I_n < I_{n-1}$, and that

$$\frac{2n}{2n+1} < \frac{I_{2n+1}}{I_{2n}} < 1.$$

Hence prove that

$$\lim_{n \to \infty} \frac{2^{4n+1}(n!)^4}{(2n+1)(2n)!^2} = \pi.$$

1/II/11F **Analysis**

Let f be defined on $\mathbb{R}$, and assume that there exists at least one point $x_0 \in \mathbb{R}$ at which f is continuous. Suppose also that, for every $x, y \in \mathbb{R}$, f satisfies the equation

$$f(x+y) = f(x) + f(y).$$

Show that f is continuous on $\mathbb{R}$.

Show that there exists a constant c such that $f(x) = cx$ for all $x \in \mathbb{R}$.

Suppose that g is a continuous function defined on $\mathbb{R}$ and that, for every $x, y \in \mathbb{R}$, g satisfies the equation

$$g(x+y) \;=\; g(x)g(y).$$

Show that if g is not identically zero, then g is everywhere positive. Find the general form of g.

1/II/12F **Analysis**

(i) Show that if $a_n > 0$, $b_n > 0$ and

$$\frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$$

for all $n \geqslant 1$, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) Let

$$c_n = \binom{2n}{n} 4^{-n}.$$

By considering $\log c_n$, or otherwise, show that $c_n \to 0$ as $n \to \infty$.

[*Hint*: $\log(1-x) \leqslant -x$ for $x \in (0,1)$.]

(iii) Determine the convergence or otherwise of

$$\sum_{n=1}^{\infty} \binom{2n}{n} x^n$$

for (a) $x = \frac{1}{4}$, (b) $x = -\frac{1}{4}$.

1/I/3B **Analysis**

Define what it means for a function of a real variable to be *differentiable* at $x \in \mathbb{R}$.

Prove that if a function is differentiable at $x \in \mathbb{R}$, then it is continuous there.

Show directly from the definition that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at 0 with derivative 0.

Show that the derivative $f'(x)$ is not continuous at 0.

1/I/4C **Analysis**

Explain what is meant by the *radius of convergence* of a power series.

Find the radius of convergence R of each of the following power series:

$$\text{(i)} \quad \sum_{n=1}^{\infty} n^{-2} z^n, \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \left(n + \frac{1}{2^n} \right) z^n .$$

In each case, determine whether the series converges on the circle $|z| = R$.

1/II/9F **Analysis**

Prove the Axiom of Archimedes.

Let x be a real number in $[0, 1]$, and let m, n be positive integers. Show that the limit

$$\lim_{m \to \infty} \left[\lim_{n \to \infty} \cos^{2n} (m! \pi x) \right]$$

exists, and that its value depends on whether x is rational or irrational.

[*You may assume standard properties of the cosine function provided they are clearly stated.*]

1/II/10F **Analysis**

State without proof the *Integral Comparison Test* for the convergence of a series $\sum_{n=1}^{\infty} a_n$ of non-negative terms.

Determine for which positive real numbers α the series $\sum_{n=1}^{\infty} n^{-\alpha}$ converges.

In each of the following cases determine whether the series is convergent or divergent:

(i) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \log n}$,

(ii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n \log\, n)\, (\log\log\, n)^2}$,

(iii) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^{(1+1/n)} \log n}$.

1/II/11B **Analysis**

Let $f : [a, b] \to \mathbb{R}$ be continuous. Define the *integral* $\int_a^b f(x)dx$. (You are not asked to prove existence.)

Suppose that m, M are real numbers such that $m \leqslant f(x) \leqslant M$ for all $x \in [a, b]$. Stating clearly any properties of the integral that you require, show that

$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)\,.$$

The function $g : [a, b] \to \mathbb{R}$ is continuous and non-negative. Show that

$$m \int_a^b g(x)dx \leqslant \int_a^b f(x)g(x)dx \leqslant M \int_a^b g(x)dx\,.$$

Now let f be continuous on $[0, 1]$. By suitable choice of g show that

$$\lim_{n\to\infty} \int_0^{1/\sqrt{n}} nf(x)e^{-nx}dx = f(0)\ ,$$

and by making an appropriate change of variable, or otherwise, show that

$$\lim_{n\to\infty} \int_0^1 nf(x)e^{-nx}dx = f(0)\,.$$

1/II/12C **Analysis**

State carefully the formula for integration by parts for functions of a real variable.

Let $f : (-1,1) \to \mathbb{R}$ be infinitely differentiable. Prove that for all $n \geqslant 1$ and all $t \in (-1,1)$,

$$f(t) = f(0) + f'(0)t + \frac{1}{2!}f''(0)t^2 + \ldots + \frac{1}{(n-1)!}f^{(n-1)}(0)t^{n-1} + \frac{1}{(n-1)!}\int_0^t f^{(n)}(x)(t-x)^{n-1}\,dx.$$

By considering the function $f(x) = \log(1-x)$ at $x = 1/2$, or otherwise, prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$$

converges to $\log 2$.

1/I/3C **Analysis I**

Suppose $a_n \in \mathbb{R}$ for $n \geqslant 1$ and $a \in \mathbb{R}$. What does it mean to say that $a_n \to a$ as $n \to \infty$? What does it mean to say that $a_n \to \infty$ as $n \to \infty$?

Show that, if $a_n \neq 0$ for all n and $a_n \to \infty$ as $n \to \infty$, then $1/a_n \to 0$ as $n \to \infty$. Is the converse true? Give a proof or a counter example.

Show that, if $a_n \neq 0$ for all n and $a_n \to a$ with $a \neq 0$, then $1/a_n \to 1/a$ as $n \to \infty$.

1/I/4C **Analysis I**

Show that any bounded sequence of real numbers has a convergent subsequence.

Give an example of a sequence of real numbers with no convergent subsequence.

Give an example of an unbounded sequence of real numbers with a convergent subsequence.

1/II/9C **Analysis I**

State some version of the fundamental axiom of analysis. State the alternating series test and prove it from the fundamental axiom.

In each of the following cases state whether $\sum_{n=1}^{\infty} a_n$ converges or diverges and prove your result. You may use any test for convergence provided you state it correctly.

(i) $a_n = (-1)^n (\log(n+1))^{-1}$.

(ii) $a_{2n} = (2n)^{-2}$, $a_{2n-1} = -n^{-2}$.

(iii) $a_{3n-2} = -(2n-1)^{-1}$, $a_{3n-1} = (4n-1)^{-1}$, $a_{3n} = (4n)^{-1}$.

(iv) $a_{2^n+r} = (-1)^n (2^n + r)^{-1}$ for $0 \leqslant r \leqslant 2^n - 1$, $n \geqslant 0$.

1/II/10C **Analysis I**

Show that a continuous real-valued function on a closed bounded interval is bounded and attains its bounds.

Write down examples of the following functions (no proof is required).

(i) A continuous function $f_1 : (0,1) \to \mathbb{R}$ which is not bounded.

(ii) A continuous function $f_2 : (0,1) \to \mathbb{R}$ which is bounded but does not attain its bounds.

(iii) A bounded function $f_3 : [0,1] \to \mathbb{R}$ which is not continuous.

(iv) A function $f_4 : [0,1] \to \mathbb{R}$ which is not bounded on any interval $[a,b]$ with $0 \leqslant a < b \leqslant 1$.

[*Hint: Consider first how to define f_4 on the rationals.*]

1/II/11C **Analysis I**

State the mean value theorem and deduce it from Rolle's theorem.

Use the mean value theorem to show that, if $h : \mathbb{R} \to \mathbb{R}$ is differentiable with $h'(x) = 0$ for all x, then h is constant.

By considering the derivative of the function g given by $g(x) = e^{-ax} f(x)$, find all the solutions of the differential equation $f'(x) = af(x)$ where $f : \mathbb{R} \to \mathbb{R}$ is differentiable and a is a fixed real number.

Show that, if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(x) = \int_0^x f(t)\, dt$$

is differentiable with $F'(x) = f(x)$.

Find the solution of the equation

$$g(x) = A + \int_0^x g(t)\, dt$$

where $g : \mathbb{R} \to \mathbb{R}$ is differentiable and A is a real number. You should explain why the solution is unique.

1/II/12C **Analysis I**

Prove Taylor's theorem with some form of remainder.

An infinitely differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfies the differential equation

$$f^{(3)}(x) = f(x)$$

and the conditions $f(0) = 1$, $f'(0) = f''(0) = 0$. If $R > 0$ and j is a positive integer, explain why we can find an M_j such that

$$|f^{(j)}(x)| \leqslant M_j$$

for all x with $|x| \leqslant R$. Explain why we can find an M such that

$$|f^{(j)}(x)| \leqslant M$$

for all x with $|x| \leqslant R$ and all $j \geqslant 0$.

Use your form of Taylor's theorem to show that

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} \ .$$

1/I/3D **Analysis I**

What does it mean to say that $u_n \to l$ as $n \to \infty$?

Show that, if $u_n \to l$ and $v_n \to k$, then $u_n v_n \to lk$ as $n \to \infty$.

If further $u_n \neq 0$ for all n and $l \neq 0$, show that $1/u_n \to 1/l$ as $n \to \infty$.

Give an example to show that the non-vanishing of u_n for all n need not imply the non-vanishing of l.

1/I/4D **Analysis I**

Starting from the theorem that any continuous function on a closed and bounded interval attains a maximum value, prove Rolle's Theorem. Deduce the Mean Value Theorem.

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(t) > 0$ for all t show that f is a strictly increasing function.

Conversely, if f is strictly increasing, is $f'(t) > 0$ for all t ?

1/II/9D **Analysis I**

(i) If $a_0, a_1, \ldots$ are complex numbers show that if, for some $w \in \mathbb{C}, w \neq 0$, the set $\{|a_n w^n| : n \geq 0\}$ is bounded and $|z| < |w|$, then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely. Use this result to define the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$.

(ii) If $|a_n|^{1/n} \to R$ as $n \to \infty$ $(0 < R < \infty)$ show that $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence equal to $1/R$.

(iii) Give examples of power series with radii of convergence 1 such that (a) the series converges at all points of the circle of convergence, (b) diverges at all points of the circle of convergence, and (c) neither of these occurs.

1/II/10D **Analysis I**

Suppose that f is a continuous real-valued function on $[a, b]$ with $f(a) < f(b)$. If $f(a) < v < f(b)$ show that there exists c with $a < c < b$ and $f(c) = v$.

Deduce that if f is a continuous function from the closed bounded interval $[a, b]$ to itself, there exists at least one fixed point, i.e., a number d belonging to $[a, b]$ with $f(d) = d$. Does this fixed point property remain true if f is a continuous function defined (i) on the open interval (a, b) and (ii) on $\mathbb{R}$? Justify your answers.

1/II/11D **Analysis I**

(i) Show that if $g : \mathbb{R} \to \mathbb{R}$ is twice continuously differentiable then, given $\epsilon > 0$, we can find some constant L and $\delta(\epsilon) > 0$ such that

$$|g(t) - g(\alpha) - g'(\alpha)\,(t - \alpha)| \;\leq\; L|t - \alpha|^2$$

for all $|t - \alpha| < \delta(\epsilon)$.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable on $[a, b]$ (with one-sided derivatives at the end points), let f' and f'' be strictly positive functions and let $f(a) < 0 < f(b)$.

If $F(t) = t - (f(t)/f'(t))$ and a sequence $\{x_n\}$ is defined by $b = x_0, x_n = F(x_{n-1})$ $(n > 0)$, show that $x_0, x_1, x_2, \ldots$ is a decreasing sequence of points in $[a, b]$ and hence has limit α. What is $f(\alpha)$? Using part (i) or otherwise estimate the rate of convergence of x_n to α, i.e., the behaviour of the absolute value of $(x_n - \alpha)$ for large values of n.

1/II/12D **Analysis I**

Explain what it means for a function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable on $[a, b]$, and give an example of a bounded function that is not Riemann integrable.

Show each of the following statements is true for continuous functions f, but false for general Riemann integrable functions f.

(i) If $f : [a, b] \to \mathbb{R}$ is such that $f(t) \geq 0$ for all t in $[a, b]$ and $\int_a^b f(t)\,dt = 0$, then $f(t) = 0$ for all t in $[a, b]$.

(ii) $\int_a^t f(x)\,dx$ is differentiable and $\frac{d}{dt}\int_a^t f(x)\,dx = f(t)$.