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The zeta function of $\mathfrak{sl}_2$ and resolution of singularities

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1. *Introduction*

Let L be a ring additively isomorphic to $\mathbb{Z}^d$. The zeta function of L is defined to be

$$\zeta_L(s) = \sum_{H \leqslant L} |L : H|^{-s},$$

where the sum is taken over all subalgebras H of finite index in L. This zeta function has a natural Euler product decomposition:

$$\zeta_L(s) = \prod_{p \text{ prime}} \zeta_{L\otimes\mathbb{Z}_p}(s).$$

These functions were introduced in a paper of Grunewald, Segal and Smith [**5**] where the local factors $\zeta_{L\otimes\mathbb{Z}_p}(s)$ were shown to always be rational functions in p^{-s}. The proof depends on representing the local zeta function as a definable p-adic integral and then appealing to a general result of Denef's [**1**] about the rationality of such integrals. The proof of Denef relies on Macintyre's Quantifier Elimination for $\mathbb{Q}_p$ [**8**] followed by techniques developed by Igusa [**6**] which employ resolution of singularities.

Essentially two explicit examples are calculated in [**5**], the integrals and method of evaluation at that stage being mainly of theoretical importance:

(1) if $L = \mathbb{Z}^d$ then

$$\zeta_L(s) = \zeta(s)\cdots\zeta(s-d+1);$$

(2) if $L = \begin{pmatrix} 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \\ 0 & 0 & 0 \end{pmatrix} = H(\mathbb{Z})$, the discrete Heisenberg algebra, then

$$\zeta_L(s) = \zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)\zeta(3s-3)^{-1}.$$

(There are more extensive examples in [**5**] of zeta functions counting only ideals in nilpotent Lie algebras.)

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In a recent paper ([**2**]) the following expression is derived for another 3-dimensional Lie algebra:

(3) if $L = \mathfrak{sl}_2(\mathbb{Z})$ then there exists a rational function $P_2(Y)$ (at that time not yet calculated) such that

$$\zeta_L(s) = P_2(2^{-s}) \cdot \zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-1)\zeta(3s-1)^{-1}.$$

The proof of this expression in [**2**] depends on some hard calculation made by Ilani [**7**] for the case $\zeta_{p(L\otimes\mathbb{Z}_p)}(s)$, p odd, together with a result which explains the connection between $\zeta_L(s)$ and $\zeta_{pL}(s)$. Ilani had two methods of proof for his calculations: one relying on small dimension and connections between two and three generator subalgebras; the other a direct calculation of the associated p-adic integral which relies on a complicated case analysis and a computer. The calculation is too complicated to provide any details in his paper. Neither method of calculation is easy.

The purpose of the current paper is to provide a relatively straightforward evaluation of the integral associated to $\mathfrak{sl}_2(\mathbb{Z}_p)$. The method we shall employ can also be adapted to complete the missing calculation of $p = 2$. It also has the potential to be implemented in higher dimensional Lie algebras.

The paper also serves to demonstrate that recent work [**3**] of du Sautoy and Grunewald on which the calculation is based is more than just of theoretical importance. In that paper an explicit expression is derived for the rational functions expressing the local factors valid for almost all primes p. The method of proof breaks up into a number of stages:

(1) Elimination of quantifiers is performed by hand without recourse to Macintyre's result. This depends on just solving linear equations. The integrals are reduced to *cone integrals* defined as follows: let $f_0, g_0, \dots, f_l, g_l \in \mathbb{Z}[X_1, \dots, X_n]$ then the cone integral associated with these polynomials $\mathscr{D} = \{f_0, g_0, \dots, f_l, g_l\}$ is defined to be:

$$Z_{\mathscr{D},p}(s) = \int_{V_p} |f_0(s)|^s |g_0(s)| d\mu,$$

where $d\mu$ is the additive Haar measure on $\mathbb{Z}_p^n$ and

$$V_p = \left\{\mathbf{x} \in \mathbb{Z}_p^n \colon v(f_i(\mathbf{x})) \leqslant v(g_i(\mathbf{x})) \text{ for } i = 1, \dots, l\right\}.$$

(2) A resolution of singularities $h\colon Y \to \mathbb{A}^n$ is made of the polynomial $F(\mathbf{X}) = f_0 \cdot g_0 \cdots f_l \cdot g_l$. Let E_i ($i \in T$, T finite) denote the irreducible components corresponding to $\left(h^{-1}(D)\right)_{\text{red}}$ where $D = \operatorname{Spec}(\mathbb{Q}[\mathbf{X}]/(F))$. A resolution of singularities means that the E_i are non-singular varieties intersecting with normal crossings. For those primes for which the resolution has good reduction, the integral reduces to:

(a) a calculation of the number of points mod$p$ on various configurations of the varieties E_i : for each $I \subset T$ we need to calculate

$$c_p(I) = \operatorname{card}\{a \in Y(\mathbb{F}_p) \colon a \in E_i(\mathbb{F}_p) \quad \text{if and only if } i \in I\};$$

(b) for each I, a calculation of a geometric progression over lattice points lying in a cone C defined by linear inequalities depending on the numerical data of the resolution:

$$P_I(s) = \sum_{(n_1,\dots,n_m)\in\Lambda} p^{(B_1-A_1 s)n_1} \cdots p^{(B_m-A_m s)n_m},$$

where

$$\Lambda = \left\{ (n_1, \ldots, n_m) \in \mathbb{N}^m : \sum_{j=1}^{m} a_{ij} n_j \leqslant \sum_{j=1}^{m} b_{ij} n_j, \quad \text{where } i = 1, \ldots, l \right\}.$$

These geometric progressions can be calculated by decomposing the cone C into open simplicial cones with fundamental regions of volume 1.

The explicit expression for $\zeta_{L\otimes\mathbb{Z}_p}(s)$ for primes p for which the resolution has good reduction is then

$$\zeta_{L\otimes\mathbb{Z}_p}(s) = \sum_{I\subset T} c_I(p) P_I(s).$$

This explicit expression is put to theoretical use in proving in [**3**] that the global zeta function $\zeta_L(s)$ has meromorphic continuation beyond its region of convergence. It also isolates the dependence on p of these local rational functions. The only part of this calculation which depends on p is the calculation of the coefficients $c_p(I)$.

However it is also a practical procedure which can be implemented in examples. We demonstrate in this paper how this can be done for the case of $\mathfrak{sl}_2(\mathbb{Z})$. This is the first such implementation of this theoretical method and we hope that the details provided here will be a catalyst for other such implementations in higher dimensional Lie algebras.

In the case of $\mathfrak{sl}_2(\mathbb{Z})$ there is one prime for which our resolution has bad reduction, namely $p = 2$. However we show that it is still possible to adapt the procedure above to calculate this missing case. Essentially, we have to count points on varieties modulo higher powers of 2.

This gives us then a proof of the following:

Theorem 1. (1) *If $p > 2$ then*

$$\zeta_{\mathfrak{sl}_2(\mathbb{Z}_p)}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-1)\zeta_p(3s-1)^{-1}.$$

(2) *If $p = 2$ then*

$$\zeta_{\mathfrak{sl}_2(\mathbb{Z}_2)}(s) = \zeta_2(s)\zeta_2(s-1)\zeta_2(2s-2)\zeta_2(2s-1)(1 + 6\cdot 2^{-2s} - 8\cdot 2^{-3s}).$$

In [**2**], the exact location of the pole of $\zeta_{\mathfrak{sl}_2(\mathbb{Z})}(s)$ was unknown since the local factor at $p = 2$ had not been calculated. Having done this, we can now improve corollary 3·6 of [**2**]:

Corollary 2. *$\zeta_{\mathfrak{sl}_2(\mathbb{Z})}(s)$ converges on $\Re(s) > 2$ and has a simple pole at $s = 2$. Hence*

$$a_1(\mathfrak{sl}_2(\mathbb{Z})) + \cdots + a_n(\mathfrak{sl}_2(\mathbb{Z})) \sim c\cdot n^2$$

where

$$c = \frac{20}{31}\cdot\frac{\zeta(2)^2\zeta(3)}{\zeta(5)}.$$

The evaluation of the integral in the case of primes with good reduction (i.e. $p > 2$) translates formally into a calculation of the associated motivic zeta function, a claim made in [**4**]:

COROLLARY 3. *Let k be a characteristic zero field. Denote by $\mathscr{X}_t^{\leqslant}$ the space of $k[[t]]$-subalgebras of $\mathfrak{sl}_2(k[[t]])$ of finite codimension. Then*

$$P_{\mathfrak{sl}_2(k[[t]]),\mathscr{X}_t^{\leqslant}}(\mathbf{L}^{-s}) = (1-\mathbf{L}^{-s})^{-1}(1-\mathbf{L}^{1-s})^{-1}(1-\mathbf{L}^{2-2s})^{-1}(1-\mathbf{L}^{1-2s})^{-1}(1-\mathbf{L}^{1-3s}),$$

where $\mathbf{L}$ is the Lefschetz motive and $P_{\mathfrak{sl}_2(k[[t]]),\mathscr{X}_t^{\leqslant}}(\mathbf{L}^{-s})$ is the motivic zeta function defined in [**4**] *as the power series*

$$P_{L,\mathscr{X}}(T) = \sum_{n=0}^{\infty}[A_n(\mathscr{X})]T^n,$$

where the coefficient $[A_n(\mathscr{X})]$ is the element of the Grothendieck ring defined by the subalgebras in $\mathscr{X}$ of codimension n.

2. *Cone integrals*

The cone integral representing $\zeta_{\mathfrak{sl}_2(\mathbb{Z}_p)}(s)$ is as follows (see [**2**])

$$\zeta_{\mathfrak{sl}_2(\mathbb{Z}_p)}(s) = (1-p^{-1})^{-3}\int_W |a|^{s-1}\,|x|^{s-2}\,|z|^{s-3}\,|d\mu|\,,$$

where

$$W = \left\{ \begin{pmatrix} a & c & b \\ 0 & x & y \\ 0 & 0 & z \end{pmatrix} \in \mathrm{Tr}_3(R)\colon \begin{array}{l} v(x) \leqslant v(4cy) \\ v(x) \leqslant v(4cz) \\ v(xz) \leqslant v(ax^2+4bxy-4cy^2) \end{array} \right\}.$$

It is interesting to compare this with the cone integral associated to the Heisenberg Lie algebra $H(\mathbb{Z}_p)$ which takes the following form:

$$\zeta_{H(\mathbb{Z}_p)}(s) = (1-p^{-1})^{-3}\int_V |a_1|^{s-1}\,|b_2|^{s-2}\,|c_3|^{s-3}\,|d\mu|\,,$$

where

$$V = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{pmatrix} \in \mathrm{Tr}_3(R)\colon v(c_3) \leqslant v(a_1b_2) \right\}.$$

The point of the resolution of singularities of the cone data is to break the integral up into pieces on which the polynomials become monomial. Once the polynomials are monomial, to know the valuation of the polynomial it suffices to know just the valuation of the individual variables since $v(X_1^{a_1}\dots X_d^{a_d}) = a_1v(X_1)+\cdots+a_dv(X_d)$. It was shown in [**3**] why this then reduces the integral to a calculation of a geometric progression over lattice points representing the possible valuations of the variables lying in a cone C defined by linear inequalities coming from the cone conditions.

In the case of the Heisenberg group, we see that already the polynomial $F = a_1b_2c_3$ is a union of non-singular varieties with normal crossings, hence there is no need for any resolution of singularities. The integral reduces immediately to a calculation of the following geometric series:

$$\sum_{A+B\geqslant C} p^{-As}p^{B(1-s)}p^{C(2-s)}.$$

An analysis of this sum by decomposing the associated cone into open simplicial cones with fundamental regions of volume 1 can be found in [**4**]. Or else a direct calculation leads to the expression recorded in the introduction.

Returning to the integral for $\mathfrak{sl}_2(\mathbb{Z}_p)$, the trouble of course lies in the fact that in the third cone condition defining W, knowledge of the valuations of the individual variables does not tell us about $v(ax^2 + 4bxy - 4cy^2)$. The polynomial

$$F = (ax^2 + 4bxy - 4cy^2)acxyz,$$

which is the reduced form of the polynomial associated to the cone data of $\mathfrak{sl}_2(\mathbb{Z}_p)$, has singularities (where we mean singularities not coming from normal crossings).

We begin by understanding the singularities of F. We then show that three blow-ups can be performed so that $\mathbb{Z}_p^6$ is transformed into a space on which F is non-singular. We then show how these blow-ups translate into judicious choices of change of variables in the integral representing $\zeta_{\mathfrak{sl}_2(\mathbb{Z}_p)}(s)$ reducing the integral to the calculation essentially of three integrals of the type encountered in the Heisenberg algebra. These can be evaluated in a straightforward manner.

The philosophy is that understanding the singularities of the associated polynomial guides us to an efficient way to decompose the integral which avoids the complicated case analysis that required Ilani's calculation to rely on a computer to complete.

3. *Singularities and blow-ups*

We begin by calculating the singularities of the one non-singular irreducible component, namely $ax^2 + 4bxy - 4cy^2$. The partial derivatives give us the following description of the singular set:

$$\begin{aligned} x^2 &= 0,\\ 4xy &= 0,\\ -4y^2 &= 0,\\ 2ax + 4by &= 0,\\ 4bx - 8cy &= 0. \end{aligned}$$

Therefore the singular set is the four-dimensional subspace $x = y = 0$.

3·1. *Blow-up at $x = y = 0$*

We start by blowing up over this subspace. The blowing up of $\mathbb{A}^6$ at $x = y = 0$ is defined by the equation $xy' = x'y$ inside $\mathbb{A}^6 \times \mathbb{P}^1$. It looks like $\mathbb{A}^6$ except that every point on $x = y = 0$ has been replaced by a $\mathbb{P}^1$.

We obtain the total inverse image of $ax^2 + 4bxy - 4cy^2 = 0$ by considering the equations $ax^2 + 4bxy - 4cy^2$ and $xy' = x'y$ inside $\mathbb{A}^6 \times \mathbb{P}^1$. Now $\mathbb{P}^1$ is covered by the open set $x' \neq 0$ and $y' \neq 0$ which we consider separately.

If $x' \neq 0$ then we can set $x' = 1$ and use y' as an affine parameter. Then we have the equations in $\mathbb{A}^7$

$$\begin{aligned} ax^2 + 4bxy - 4cy^2 &= 0,\\ xy' &= y. \end{aligned}$$

Substituting we get

$$x^2(a + 4by' - 4c(y')^2) = 0,$$

which gives two non-singular irreducible components with normal crossings.

Similarly on the other chart we get

$$ax^2 + 4bxy - 4cy^2 = 0,$$
$$x = x'y.$$

Substituting:

$$y^2(a(x')^2 + 4bx' - 4c) = 0,$$

which also gives two non-singular irreducible components with normal crossings.

We break the space $\mathbb{Z}_p^6 = \left\{(x, y, z, a, b, c) \in \mathbb{Z}_p^6\right\}$ into two disjoint pieces

$$U_1 = \left\{(x, y, z, a, b, c) \in \mathbb{Z}_p^6 \colon v(x) \leqslant v(y)\right\},$$
$$U_2 = \left\{(x, y, z, a, b, c) \in \mathbb{Z}_p^6 \colon v(x) > v(y)\right\}.$$

Denote by $h_{xy} \colon B_{xy} \to \mathbb{A}^6$ the blow-up we are considering where B_{xy} is the subset of $\mathbb{A}^6 \times \mathbb{P}^1$ defined by the equation $xy' = x'y$. Let $\theta_y \colon \; B_{xy}|_{x' \neq 0} \to \mathbb{A}^6$ be the affine chart defined by $(x, y, z, a, b, c, x', y') \to (x, z, a, b, c, y')$ and θ_x the corresponding chart on $y' \neq 0$. Then $h_{xy}^{-1}(U_1) \subset B_{xy}|_{x' \neq 0}$ and $\theta_y \circ h_{xy}^{-1}(U_1) = \mathbb{Z}_p^6 = \left\{(x, z, a, b, c, y') \in \mathbb{Z}_p^6\right\}$ and F on this chart is given by

$$x^2(a + 4by' - 4c(y')^2)acx^2y'z = 0. \tag{3·1}$$

Whilst $h_{xy}^{-1}(U_2) \subset B_{xy}|_{y' \neq 0}$ and

$$\theta_x \circ h_{xy}^{-1}(U_2) = \mathbb{Z}_p^5 \times p\mathbb{Z}_p = \left\{(y, z, a, b, c, x') \in \mathbb{Z}_p^6 \colon x' \in p\mathbb{Z}_p\right\}$$

and F on this chart is given by

$$y^2(a(x')^2 + 4bx' - 4c)acx'y^2z = 0. \tag{3·2}$$

Although each individual irreducible component of the transform of F in each of these charts is non-singular, we have singularities coming from non-normal crossings. We have to perform two further blow-ups, one on each chart.

3·2. *Blow-up at $y' = a = 0$*

Consider first $x^2(a + 4by' - 4c(y')^2)acx^2y'z = 0$ on $\mathbb{Z}_p^6$. Then $f_1 = a + 4by' - 4c\,(y')^2$ and $f_2 = a$ have non-normal crossings with singular set $y' = b = 0$ by consideration of the matrix of partial derivatives:

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial a} & \dfrac{\partial f_1}{\partial b} & \dfrac{\partial f_1}{\partial c} & \dfrac{\partial f_1}{\partial y'} \\ \dfrac{\partial f_2}{\partial a} & \dfrac{\partial f_2}{\partial b} & \dfrac{\partial f_2}{\partial c} & \dfrac{\partial f_2}{\partial y'} \end{pmatrix} = \begin{pmatrix} 1 & 4y' & -4\,(y')^2 & 4b - 8cy' \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Here we do a blow-up on $y' = a = 0$ to effect our desingularization: break $\mathbb{Z}_p^6 = \left\{(x, z, a, b, c, y') \in \mathbb{Z}_p^6\right\}$ into two pieces

$$V_1 = \left\{(x, z, a, b, c, y') \in \mathbb{Z}_p^6 \colon v(y') \leqslant v(a)\right\},$$
$$V_2 = \left\{(x, z, a, b, c, y') \in \mathbb{Z}_p^6 \colon v(y') > v(a)\right\}.$$

Then on $\theta_a \circ h_{y'a}^{-1}(V_1) = \mathbb{Z}_p^6 = \left\{(x, z, a', b, c, y') \in \mathbb{Z}_p^6\right\}$, F is given by

$$x^2y'\left(a' + 4b - 4cy'\right)a'y'cx^2y'z = 0.$$

Because b is now free we get that the irreducible components corresponding to the transform of F have normal crossings.

On $\theta_{y'} \circ h_{y'a}^{-1}(V_2) = \mathbb{Z}_p^5 \times p\mathbb{Z}_p = \left\{(x, z, a', b, c, y'') \in \mathbb{Z}_p^6 : y'' \in p\mathbb{Z}_p\right\}$, F is given by

$$x^2 a(1 + 4by'' - 4ca(y'')^2)acx^2y''az = 0.$$

But this variety on $\mathbb{Z}_p^5 \times p\mathbb{Z}_p$ is the same as the variety $x^2a^2cx^2y''az = 0$ since $y'' \in p\mathbb{Z}_p$ which means $(1 + 4by'' - 4ca(y'')^2) \neq 0$. Hence we have a variety again with normal crossings.

3·3. *Blow-up at* $x' = c = 0$

Consider next $y^2(a(x')^2 + 4bx' - 4c)acx'y^2z = 0$ on $\{(y, z, a, b, c, x') \in \mathbb{Z}_p^6 : x' \in p\mathbb{Z}_p\}$. Here $f_1 = (a(x')^2 + 4bx' - 4c)$ and $f_2 = c$ have non-normal crossings with singular set $b = x' = 0$ by consideration of the matrix of partial derivatives:

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial a} & \dfrac{\partial f_1}{\partial b} & \dfrac{\partial f_1}{\partial c} & \dfrac{\partial f_1}{\partial x'} \\[2ex] \dfrac{\partial f_2}{\partial a} & \dfrac{\partial f_2}{\partial b} & \dfrac{\partial f_2}{\partial c} & \dfrac{\partial f_2}{\partial x'} \end{pmatrix} = \begin{pmatrix} (x')^2 & 4x' & -4 & 2ax' + 4b \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This time we blow-up at $x' = c = 0$. Break $\left\{(y, z, a, b, c, x') \in \mathbb{Z}_p^6 : x' \in p\mathbb{Z}_p\right\}$ into two pieces:

$$W_1 = \left\{(y, z, a, b, c, x') \in \mathbb{Z}_p^6 : x' \in p\mathbb{Z}_p, v(x') \leqslant v(c)\right\},$$
$$W_2 = \left\{(y, z, a, b, c, x') \in \mathbb{Z}_p^6 : x' \in p\mathbb{Z}_p, v(x') > v(c)\right\}.$$

Then on $\theta_c \circ h_{x'c}^{-1}(W_1) = \mathbb{Z}_p^4 \times p\mathbb{Z}_p^2 = \left\{(y, z, a, b, c', x') \in \mathbb{Z}_p^6 : c', x' \in p\mathbb{Z}_p\right\}$, F is given by

$$y^2x'\left(ax' + 4b - 4c'\right)ac'\left(x'\right)^2y^2z = 0.$$

Because b is now free we get that the irreducible components corresponding to the transform of F have normal crossings.

On $\theta_{x'} \circ h_{x'c}^{-1}(W_2) = \mathbb{Z}_p^5 \times p\mathbb{Z}_p = \left\{(y, z, a, b, c, x'') \in \mathbb{Z}_p^6 : x'' \in p\mathbb{Z}_p\right\}$, F is given by

$$y^2c(a(x'')^2 + 4bx'' - 4)ac^2x''y^2z = 0.$$

Provided we avoid the prime $p = 2$, this variety on $\mathbb{Z}_p^5 \times p\mathbb{Z}_p$ is the same as the variety $y^2cac^2x''y^2z = 0$ since $x'' \in p\mathbb{Z}_p$ which means $(a(x'')^2 + 4bx'' - 4) \neq 0$. Hence we have a variety again with normal crossings.

The combined resolution of singularities given by these three blow-ups has bad reduction at the prime $p = 2$. However, we shall see that even for $p = 2$ these blow-ups transform the integral into pieces that we can still evaluate.

Having shown how to desingularize the variety defined by the cone data of $\mathfrak{sl}_2$, let us now implement this resolution to calculate an explicit expression for the associated zeta function. As we shall see, these three blow-ups correspond to a judicious choice of change of variable in the integrals. The following can therefore be understand with no understanding of the underlying algebraic geometry which motivated the choices made.

4. *Summing lattice points in cones*

Before working on the integral for $\mathfrak{sl}_2(\mathbb{Z}_p)$ itself, we will evaluate the following, which will be useful later. It is basically a variation of the Heisenberg integral:

For $n_1, n_2, m_4, m_5 \in \mathbb{N}$ put

$$\begin{aligned} J(\mathbf{w};\mathbf{v};n_1,n_2,m_4,m_5) &= (1-p^{-1})^{-5} \int_{\substack{v(w_1)\leqslant v(w_2w_3)+n_1 \\ v(w_3)\leqslant v(w_1w_4^{m_4}w_5^{m_5})+n_2}} |w_1|^{v_1-1}\dots|w_5|^{v_5-1}d\nu, \\ &= \sum_{\substack{W_1\leqslant W_2+W_3+n_1 \\ W_3\leqslant W_1+m_4W_4+m_5W_5+n_2}} p^{-W_1v_1}\dots p^{-W_5v_5}. \end{aligned}$$

This sum breaks into two cases:

(i) $W_1 \leqslant W_3$: let $W_3' = W_3 - W_1$.

$$\begin{aligned} J_{(i)} &= \sum_{W_3'\leqslant m_4W_4+m_5W_5+n_2} p^{-W_1v_1}p^{-W_2v_2}p^{-(W_1+W_3')v_3}p^{-W_4v_4}p^{-W_5v_5} \\ &= \frac{1}{(1-p^{-(v_1+v_3)})\prod_{i=2}^{5}(1-p^{-v_i})} \\ &\quad -\frac{p^{-(n_2+1)v_3}}{(1-p^{-(v_1+v_3)})(1-p^{-v_2})(1-p^{-v_3})(1-p^{-(v_3m_4+v_4)})(1-p^{-(v_3m_5+v_5)})}. \end{aligned}$$

(ii) $W_1 > W_3$: let $W_1' = W_1 - W_3$.

$$\begin{aligned} J_{(\mathrm{ii})} &= \sum_{1\leqslant W_1'\leqslant W_2+n_1} p^{-(W_1'+W_3)v_1}p^{-W_2v_2}\dots p^{-W_5v_5} \\ &= \frac{\sum_{W_2} p^{-v_1}(1-p^{-(W_2+n_1)v_1})p^{-W_2v_2}}{(1-p^{-v_1})(1-p^{-(v_1+v_3)})(1-p^{-v_4})(1-p^{-v_5})}, \\ &= \frac{1}{(1-p^{-v_1})(1-p^{-(v_1+v_3)})(1-p^{-v_4})(1-p^{-v_5})}\left(\frac{p^{-v_1}}{(1-p^{-v_2})}-\frac{p^{-(n_1+1)v_1}}{(1-p^{-(v_1+v_2)})}\right). \end{aligned}$$

Hence

$$J_{(i)} + J_{(ii)} = \frac{Q(p)}{(1-p^{-(v_1+v_2)})(1-p^{-(v_1+v_3)})(1-p^{-(v_3m_4+v_4)})(1-p^{-(v_3m_5+v_5)})\prod_{i=1}^{5}(1-p^{-v_i})},$$

where

$$Q(p) = \left(\begin{array}{c} (1-p^{-v_1})(1-p^{-(v_1+v_2)})(1-p^{-(v_3m_4+v_4)})(1-p^{-(v_3m_5+v_5)}) \\ -p^{-(n_2+1)v_3}(1-p^{-v_1})(1-p^{-(v_1+v_2)})(1-p^{-v_4})(1-p^{-v_5}) \\ +p^{-v_1}(1-p^{-(v_1+v_2)})(1-p^{-v_3})(1-p^{-(v_3m_4+v_4)})(1-p^{-(v_3m_5+v_5)}) \\ -p^{-(n_1+1)v_1}(1-p^{-v_2})(1-p^{-v_3})(1-p^{-(v_3m_4+v_4)})(1-p^{-(v_3m_5+v_5)}). \end{array}\right).$$

We record two particular cases that will be important in the calculation for $p > 2$. The general form will be useful later in the case of $p = 2$.

(i) $J(\mathbf{w};\mathbf{v};0,0,1,0)$ has the following form

$$\frac{(1-p^{-(v_1+v_2+v_3+v_4)})}{(1-p^{-(v_1+v_2)})(1-p^{-(v_1+v_3)})(1-p^{-(v_3+v_4)})(1-p^{-v_2})(1-p^{-v_4})(1-p^{-v_5})}, \qquad (4{\cdot}1)$$

(ii) $J(\mathbf{w};\mathbf{v};0,0,1,1)$ has the following form

$$\frac{(1-p^{-(v_1+v_2+v_3+v_4)}-p^{-(v_1+v_2+v_3+v_5)}-p^{-(v_3+v_4+v_5)}+p^{-(v_1+v_2+v_3+v_4+v_5)}+p^{-(v_1+v_2+2v_3+v_4+v_5)})}{(1-p^{-(v_1+v_2)})(1-p^{-(v_1+v_3)})(1-p^{-(v_3+v_4)})(1-p^{-(v_3+v_5)})(1-p^{-v_2})(1-p^{-v_4})(1-p^{-v_5})}. \tag{4·2}$$

5. *Odd p*

With this result noted, we will now proceed to find the zeta function for $\mathfrak{sl}_2(\mathbf{Z}_p)$ by evaluating the integral:

$$I = \int_{\substack{v(x)\leqslant v(cy)\\ v(x)\leqslant v(cz)\\ v(xz)\leqslant v(ax^2+4bxy-4cy^2)}} |a|^{s-1}\,|x|^{s-2}\,|z|^{s-3}d\nu.$$

The problems here are caused by the quadratic term in the third condition, but these can be resolved by means of breaking the region of integration into various sections and performing an appropriate change of variables in each case, using the resolution of singularities explained in the previous section as guidance.

Case 1: $v(x)\leqslant v(y)$. Let $y\to xy'$, so that $y'=y/x\in\mathbf{Z}_p$.

$$\begin{aligned} I_1 &= \int_{\substack{(x)\;\leqslant\; v(cxy')\\ v(x)\;\leqslant\; v(cz)\\ v(xz)\;\leqslant\; v(ax^2+4bx^2y'-4cx^2y'^2)}} |a|^{s-1}\,|x|^{s-2}\,|z|^{s-3}|x|d\nu \\ &= \int_{\substack{v(x)\;\leqslant\; v(cz)\\ v(z)\;\leqslant\; v(x)+v(a+4by'-4cy'^2)}} |a|^{s-1}\,|x|^{s-1}\,|z|^{s-3}d\nu. \end{aligned}$$

The quadratic term may be resolved further.

Case 1*a*: $v(y')\leqslant v(a)$. Let $a\to a'y'$, so that $a'=a/y'\in\mathbf{Z}_p$.

$$\begin{aligned} I_{1a} &= \int_{\substack{v(x)\;\leqslant\; v(cz)\\ v(z)\;\leqslant\; v(x)+v(a'y'+4by'-4cy'^2)}} |a'y'|^{s-1}\,|x|^{s-1}\,|z|^{s-3}|y'|d\nu \\ &= \int_{\substack{v(x)\;\leqslant\; v(cz)\\ v(z)\;\leqslant\; v(x)+v(b')+v(y')}} |a'|^{s-1}\,|x|^{s-1}\,|y'|^{s}\,|z|^{s-3}d\nu, \end{aligned}$$

where $b'=a'+4b-4cy'$.

We can now apply the calculation (4·2) of the previous section with

$$\begin{aligned} \mathbf{w} &= (x,c,z,b',y'),\\ \mathbf{v} &= (s,1,(s-2),1,(s+1)), \end{aligned}$$

$$I_{1a}=\frac{(1-p^{-1})^4(1-p^{-2s}-p^{-3s}-p^{-2s}+p^{-(3s+1)}+p^{-(4s-1)})}{(1-p^{-s})(1-p^{-(s+1)})^2(1-p^{-(2s-2)})(1-p^{-(s-1)})(1-p^{-(2s-1)})}.$$

Case 1*b*: $v(y') > v(a)$. Let $y' \to ay''$, so that $y'' = y'/a \in p\mathbf{Z}_p$.

$$I_{1b} = \int_{\substack{v(x) \leqslant v(cz) \\ v(z) \leqslant v(x)+v(a+4bay''-4ca^2y''^2) \\ v(y'') \geqslant 1}} |a|^{s-1}\,|x|^{s-1}\,|z|^{s-3}|a|d\nu$$

$$= \int_{\substack{v(x) \leqslant v(cz) \\ v(z) \leqslant v(x)+v(a) \\ v(y'') \geqslant 1}} |a|^{s}\,|x|^{s-1}\,|z|^{s-3}d\nu,$$

as $v(1 + 4by'' - 4acy''^2) = 0$.

We apply the calculation of $J(\mathbf{w};\mathbf{v};0,0,1,0)$ in (4·1) with

$$\mathbf{w} = (x, c, z, a, b),$$
$$\mathbf{v} = (s, 1, s-2, s+1, 1)$$

to get

$$I_{1b} = \frac{(1-p^{-1})^3p^{-1}(1-p^{-3s})}{(1-p^{-(s+1)})^2(1-p^{-(2s-1)})(1-p^{-(2s-2)})}.$$

Case 2: $v(x) > v(y)$. Let $x \to x'y$, so that $x' = x/y \in p\mathbf{Z}_p$.

$$I_2 = \int_{\substack{v(x'y) \leqslant v(cy) \\ v(x'y) \leqslant v(cz) \\ v(x'yz) \leqslant v(ax'^2y^2+4bx'y^2-4cy^2) \\ v(x') \geqslant 1}} |a|^{s-1}\,|x'y|^{s-2}\,|z|^{s-3}|y|d\nu$$

$$= \int_{\substack{v(x') \leqslant v(c) \\ v(x'y) \leqslant v(cz) \\ v(x'z) \leqslant v(y)+v(ax'^2+4bx'-4c) \\ v(x') \geqslant 1}} |a|^{s-1}\,|x'|^{s-2}\,|y|^{s-1}|z|^{s-3}d\nu.$$

Case 2*a*: $v(x') \leqslant v(c)$. Let $c \to c'x'$, so that $c' = c/x' \in \mathbf{Z}_p$.

$$I_{2a} = \int_{\substack{v(x') \leqslant v(c'x') \\ v(x'y) \leqslant v(c'x'z) \\ v(x'z) \leqslant v(y)+v(ax'^2+4bx'-4c'x') \\ v(x') \geqslant 1}} |a|^{s-1}\,|x'|^{s-2}\,|y|^{s-1}\,|z|^{s-3}|x'|d\nu,$$

$$= \int_{\substack{v(y) \leqslant v(c'z) \\ v(z) \leqslant v(y)+v(b') = v(yb') \\ v(x') \geqslant 1}} |a|^{s-1}\,|x'|^{s-1}\,|y|^{s-1}\,|z|^{s-3}d\nu,$$

where we set $b' = ax' + 4b - 4c'$.

Similarly to Case 1*b*, we apply the calculation of $J(\mathbf{w};\mathbf{v};0,0,1,0)$ in (4·1) with

$$\mathbf{w} = (y, c', z, b', a),$$
$$\mathbf{v} = (s, 1, s-2, 1, s)$$

so that

$$I_{2a} = \frac{(1-p^{-1})^4 p^{-s}(1-p^{-2s})}{(1-p^{-s})^2(1-p^{-(s+1)})(1-p^{-(s-1)})(1-p^{-(2s-2)})}$$
$$= \frac{(1-p^{-1})^4 p^{-s}(1+p^{-s})}{(1-p^{-s})(1-p^{-(s+1)})(1-p^{-(s-1)})(1-p^{-(2s-2)})}.$$

Case 2*b*: $v(x') > v(c)$. Let $x' \to cx''$, so that $x'' = x'/c \in p\mathbf{Z}_p$.

$$I_{2b} = \int_{\substack{v(x''c) \leqslant v(c) \\ v(x''cy) \leqslant v(cz) \\ v(x''cz) \leqslant v(y)+v(ac^2x''^2+4bcx''-4c) \\ v(x'') \geqslant 1}} |a|^{s-1}\,|x''c|^{s-2}\,|y|^{s-1}\,|z|^{s-3}|c|d\nu.$$

Here, the first condition gives $v(x'') \leqslant 0$. This, combined with the final condition, $v(x'') \geqslant 1$, gives that

$$I_{2b} = 0.$$

So,

$$\begin{aligned} I &= I_{1a} + I_{1b} + I_{2a} \\ &= \frac{(1-p^{-1})^3}{(1-p^{-s})(1-p^{-(s+1)})^2(1-p^{-(s-1)})(1-p^{-(2s-1)})(1-p^{-(2s-2)})} \times K, \end{aligned}$$

where

$$\begin{aligned} K &= (1-p^{-1})(1-2p^{2s}-p^{-3s}+p^{-(3s+1)}+p^{-(4s-1)}) \\ &\quad + p^{-1}(1-p^{-3s})(1-p^{-s})(1-p^{-(s-1)}) \\ &\quad +(1-p^{-1})p^{-s}(1+p^{-s})(1-p^{-(s+1)})(1-p^{-(2s-1)}) \\ &= (1-p^{-(s+1)})^2(1-p^{-(3s-1)}). \end{aligned}$$

And so, finally, we have

$$\begin{aligned} \zeta_{sl_2(\mathbf{Z}_p)}(s) &= (1-p^{-1})^{-3} I \\ &= \frac{\zeta_p(s)\zeta_p(s-1)\zeta_p(2s-1)\zeta_p(2s-2)}{\zeta_p(3s-1)}. \end{aligned}$$

This confirms Theorem 1(1).

6. $p = 2$

We will use p throughout, rather than 2, as this will avoid confusion with coefficients. However, when working out some expressions, we will set $p = 2$ and $X = p^{-s}$, as this will make the calculations easier, as will be seen.

To find the zeta function for $\mathfrak{sl}_2(\mathbb{Z}_2)$ again we need to evaluate the integral:

$$I = \int_{\substack{v(x) \leqslant v(4cy) \\ v(x) \leqslant v(4cz) \\ v(xz) \leqslant v(ax^2+4bxy-4cy^2)}} |a|^{s-1}|x|^{s-2}|z|^{s-3}d\nu.$$

As with the calculations for $p \neq 2$, we break the region of integration into various sections and perform appropriate changes of variables in each case.

Case 1: $v(x) \leqslant v(y)$. Let $y \to xy'$, so that $y' = y/x \in \mathbb{Z}_p$.

$$
\begin{aligned}
I_1 &= \int\limits_{\substack{v(x) \leqslant v(4cxy') \\ v(x) \leqslant v(4cz) \\ v(xz) \leqslant v(ax^2+4bx^2y'-4cx^2y'^2)}} |a|^{s-1}|x|^{s-2}|z|^{s-3}|x|d\nu \\
&= \int\limits_{\substack{v(x) \leqslant v(4cz) \\ v(z) \leqslant v(x)+v(a+4by'-4cy'^2)}} |a|^{s-1}\,|x|^{s-1}\,|z|^{s-3}d\nu.
\end{aligned}
$$

The quadratic term may be resolved further.

Case 1a: $v(y') \leqslant v(a)$. Let $a \to a'y'$, so that $a' = a/y' \in \mathbb{Z}_p$.

$$
\begin{aligned}
I_{1a} &= \int\limits_{\substack{v(x) \leqslant v(4cz) \\ v(z) \leqslant v(x)+v(a'y'+4by'-4cy'^2)}} |a'y'|^{s-1}\,|x|^{s-1}\,|z|^{s-3}|y'|d\nu \\
&= \int\limits_{\substack{v(x) \leqslant v(4cz) \\ v(z) \leqslant v(xy')+v(b')}} |a'|^{s-1}\,|x|^{s-1}\,|y'|^{s}\,|z|^{s-3}d\nu,
\end{aligned}
$$

where $b' = a' + 4b - 4cy'$.

This splits into two parts, depending on the valuation of a'.

Case 1a(i): $v(a') \leqslant 1$. Let $a'' = a' + 4b - 4cy'$, so $v(a'') = v(a') \leqslant 1$.

$$
\begin{aligned}
I_{1a(\mathrm{i})} &= \int\limits_{\substack{v(x) \leqslant v(4cz) \\ v(z) \leqslant v(xy'a'') \\ v(a'') \leqslant 1}} |a''|^{s-1}\,|x|^{s-1}\,|y'|^{s}\,|z|^{s-3}d\nu \\
&= \left(J(\mathbf{w};\mathbf{v};2,0,1,0)(1-p^{-1}) + J(\mathbf{w};\mathbf{v};2,1,1,0)(1-p^{-1})p^{-s}\right)(1-p^{-1})^5,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{w} &= (x, c, z, y', b), \\
\mathbf{v} &= (s, 1, s-2, s+1, 1).
\end{aligned}
$$

Hence

$$
I_{1a(\mathrm{i})} = \frac{\left(1 - \frac{7}{4}X^2 - \frac{5}{2}X^3 + X^4 + \frac{1}{2}X^5\right)}{16(1-\frac{1}{2}X)^2(1-4X^2)(1-2X^2)}.
$$

Case 1a(ii): $v(a') > 1$. Let $b' = \frac{1}{4}(a' + 4b - 4cy') \in \mathbb{Z}_p$.

$$\begin{aligned} I_{1a(\mathrm{ii})} &= \int\limits_{\substack{v(x) \leqslant v(4cz) \\ v(z) \leqslant v(4xy'b') \\ v(a') > 1}} |a'|^{s-1}\,|x|^{s-1}\,|y'|^{s}\,|z|^{s-3} d\nu \\ &= \frac{p^{-2s}\left(1-p^{-1}\right)}{(1-p^{-s})} J(\mathbf{w};\mathbf{v};2,2,1,1)(1-p^{-1})^5, \end{aligned}$$

where

$$\begin{aligned} \mathbf{w} &= (x, c, z, y', b'), \\ \mathbf{v} &= (s, 1, s-2, s+1, 1). \end{aligned}$$

Then

$$I_{1a(\mathrm{ii})} = \frac{X^2\left(1 + \frac{5}{2}X + \frac{7}{2}X^2 - 10X^3 + 5X^4 + 2X^5\right)}{16(1-X)(1-\frac{1}{2}X)^2(1-4X^2)(1-2X^2)(1-2X)}.$$

Now, $I_{1a} = I_{1a(\mathrm{i})} + I_{1a(\mathrm{ii})}$, which gives:

$$I_{1a} = \frac{1 - \frac{3}{2}X + \frac{7}{2}X^2 - 15X^3 + 28X^4 - 16X^5 + 4X^6}{16(1-X)(1-\frac{1}{2}X)^2(1-2X)(1-2X^2)(1-4X^2)}.$$

Case 1b: $v(y') > v(a)$. Let $y' \to ay''$, so that $y'' = y'/a \in p\mathbb{Z}_p$.

$$\begin{aligned} I_{1b} &= \int\limits_{\substack{v(x) \leqslant v(4cz) \\ v(z) \leqslant v(x)+v(a+4bay''-4ca^2y''^2) \\ v(y'') \geqslant 1}} |a|^{s-1}\,|x|^{s-1}\,|z|^{s-3}|a| d\nu \\ &= \int\limits_{\substack{v(x) \leqslant v(4cz) \\ v(z) \leqslant v(xa) \\ v(y'') \geqslant 1}} |a|^{s}\,|x|^{s-1}\,|z|^{s-3} d\nu, \end{aligned}$$

as $v(1 + 4by'' - 4acy''^2) = 0$. Hence

$$\begin{aligned} I_{1b} &= p^{-1}(1-p^{-1})^5 J(\mathbf{w};\mathbf{v};2,0,1,0) \\ &= \frac{(1 + \frac{1}{2}X + \frac{1}{2}X^2 - 2X^3 - X^4)}{16(1-\frac{1}{2}X)^2(1-4X^2)(1-2X^2)}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{w} &= (x, c, z, a, b), \\ \mathbf{v} &= (s, 1, s-2, s+1, 1). \end{aligned}$$

Case 2: $v(x) > v(y)$. Let $x \to x'y$, so that $x' = x/y \in p\mathbb{Z}_p$.

$$\begin{aligned} I_2 &= \int_{\substack{v(x'y) \leqslant v(4cy) \\ v(x'y) \leqslant v(4cz) \\ v(x'yz) \leqslant v(ax'^2y^2+4bx'y^2-4cy^2) \\ v(x') \geqslant 1}} |a|^{s-1}\,|x'y|^{s-2}\,|z|^{s-3}|y|d\nu \\ &= \int_{\substack{v(x') \leqslant v(4c) \\ v(x'y) \leqslant v(4cz) \\ v(x'z) \leqslant v(y)+v(ax'^2+4bx'-4c) \\ v(x') \geqslant 1}} |a|^{s-1}\,|x'|^{s-2}\,|y|^{s-1}|z|^{s-3}d\nu. \end{aligned}$$

Case 2*a*: $v(x') \leqslant v(c)$. Let $c \to c'x'$, so that $c' = c/x' \in \mathbb{Z}_p$.

$$\begin{aligned} I_{2a} &= \int_{\substack{v(x') \leqslant v(4c'x') \\ v(x'y) \leqslant v(4c'x'z) \\ v(x'z) \leqslant v(y)+v(ax'^2+4bx'-4c'x') \\ v(x') \geqslant 1}} |a|^{s-1}\,|x'|^{s-2}\,|y|^{s-1}\,|z|^{s-3}|x'|d\nu \\ &= \int_{\substack{v(y) \leqslant v(4c'z) \\ v(z) \leqslant v(y)+v(ax'+4b-4c') \\ v(x') \geqslant 1}} |a|^{s-1}\,|x'|^{s-1}\,|y|^{s-1}\,|z|^{s-3}d\nu. \end{aligned}$$

This splits into three cases, depending on the valuations of a and x'.

Case 2*a*(i): $v(a) = 0, v(x') = 1$. So $v(ax' + 4b - 4c') = 1$

$$\begin{aligned} I_{2a(\mathrm{i})} &= \int_{\substack{v(y) \leqslant v(4c'z) \\ v(z) \leqslant v(y)+1 \\ v(a)=0,\ v(x')=1}} |a|^{s-1}\,|x'|^{s-1}\,|y|^{s-1}\,|z|^{s-3}d\nu \\ &= (1-p^{-1})^2p^{-s}(1-p^{-1})^5J(\mathbf{w};\mathbf{v};2,1,0,0) \\ &= \frac{X(1+\frac{9}{2}X-\frac{3}{2}X^2)}{16(1-\frac{1}{2}X)(1-4X^2)}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{w} &= (y, c', z, b, b), \\ \mathbf{v} &= (s, 1, s-2, 1, 1). \end{aligned}$$

(Note that the repetition of b contributes nothing since $m_4 = m_5 = 0$.)

Case 2a(ii): $(v(a)=0, v(x')>1)$ or $(v(a)\geqslant 1, v(x')\geqslant 1)$. Let $b'=\frac{1}{4}(ax'+4b-4c')\in\mathbb{Z}_p$.

$$\begin{aligned}
I_{2a(\mathrm{ii})} &= \int_{\substack{v(y)\,\leqslant\, v(4c'z)\\ v(z)\,\leqslant\, v(y)+v(b')+2\\ (v(a)=0,v(x')>1)\text{ or }(v(a)\geqslant 1,v(x')\geqslant 1)}} |a|^{s-1.1}\,|x'|^{s-1.1}\,|y|^{s-1.1}\,|z|^{s-3}d\nu\\
&= \left((1-p^{-1})\frac{(1-p^{-1})p^{-2s}}{(1-p^{-s})}+\frac{(1-p^{-1})p^{-s}}{(1-p^{-s})}\frac{(1-p^{-1})p^{-s}}{(1-p^{-s})}\right)(1-p^{-1})^5 J(\mathbf{w};\mathbf{v};2,2,1,0)\\
&= \frac{X^2(1+\frac{5}{2}X+\frac{11}{2}X^2-5X^3)}{8(1-X)^2(1-2X)(1-4X^2)},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{w} &= (y,c',z,b',D),\\
\mathbf{v} &= (s,1,s-2,1,1)
\end{aligned}$$

(the introduction of the dummy variable D has no effect since $m_5=0$).

So combining these cases we get:

$$I_{2a} = I_{2a(\mathrm{i})} + I_{2a(\mathrm{ii})} = \frac{X(1+\frac{5}{2}X-\frac{21}{2}X^2+35X^3-32X^4+8X^5)}{16(1-\frac{1}{2}X)(1-X)^2(1-2X)(1-4X^2)}.$$

Case 2b: $v(x') > v(c)$. Let $x'\to cx''$, so that $x''=x'/c\in p\mathbb{Z}_p$.

$$\begin{aligned}
I_{2b} &= \int_{\substack{v(x''c)\,\leqslant\, v(4c)\\ v(x''cy)\,\leqslant\, v(4cz)\\ v(x''cz)\,\leqslant\, v(y)+v(ac^2x''^2+4bcx''-4c)\\ v(x'')\,\geqslant\, 1}} |a|^{s-1}\,|x''c|^{s-2}\,|y|^{s-1}\,|z|^{s-3}|c|d\nu\\
&= \int_{\substack{1\,\leqslant\, v(x'')\,\leqslant\, 2\\ v(x''y)\,\leqslant\, v(4z)\\ v(x''z)\,\leqslant\, v(y)+v(acx''^2+4bx''-4)}} |a|^{s-1}\,|c|^{s-1}\,|x''|^{s-2}\,|y|^{s-1}\,|z|^{s-3}d\nu.
\end{aligned}$$

This case is quite fiddly, and breaks into various parts, as follows.

Case 2b(i): $v(x'')=2$. Then $v(acx''^2+4bx''-4)=v(4)=2$

$$\begin{aligned}
I_{2b(\mathrm{i})} &= \int_{\substack{v(x'')\,=\,2\\ v(y)\,\leqslant\, v(z)\\ v(z)\,\leqslant\, v(y)}} |a|^{s-1}\,|c|^{s-1}\,|x''|^{s-2}\,|y|^{s-1}\,|z|^{s-3}d\nu\\
&= \frac{2X^2}{16(1-X)^2(1-4X^2)}.
\end{aligned}$$

Case 2b(ii): $v(x'') = 1, v(a) = 0, v(c) = 0$. Let $b' = \frac{1}{8}(acx''^2 + 4bx'' - 4) \in \mathbb{Z}_p$.

$$\begin{aligned}
I_{2b(\mathrm{ii})} &= \int\limits_{\substack{v(y) \leqslant v(2z) \\ v(z) \leqslant v(y)+v(4b') \\ v(x'') = 1,\ v(a) = 0,\ v(c) = 0}} |a|^{s-1}\,|c|^{s-1}\,|x''|^{s-2}\,|y|^{s-1}\,|z|^{s-3} d\nu \\
&= (1-p^{-1})^3 p^{-s+1}(1-p^{-1})^5 J(\mathbf{w};\mathbf{v};2,1,0,0), \\
&= \frac{X(1+3X+6X^2)}{16(1-2X)(1-4X^2)},
\end{aligned}$$

with

$$\begin{aligned}
\mathbf{w} &= (z, b', y, D, D), \\
\mathbf{v} &= (s-2, 1, s, 1, 1),
\end{aligned}$$

where D is a dummy variable which has no effect since $m_4 = m_5 = 0$.

Case 2b(iii): $v(x'') = 1, (v(a), v(c)) \neq (0,0)$. Then $v(acx''^2 + 4bx'' - 4) = 2$.

$$I_{2b(\mathrm{iii})} = \int\limits_{\substack{v(y) \leqslant v(2z) \\ v(z) \leqslant v(y)+1 \\ v(x'') = 1,\ (v(a),v(c)) \neq (0,0)}} |a|^{s-1}\,|c|^{s-1}\,|x''|^{s-2}\,|y|^{s-1}\,|z|^{s-3} d\nu.$$

This does not satisfy the conditions of the general integral J, but it is not hard to work out directly.

$$\begin{aligned}
I_{2b(\mathrm{iii})} &= \frac{(1-p^{-1})^5 p^{-(s-1)}\left(2p^{-s} - p^{-2s}\right)}{(1-p^{-s})^2} \times \sum_{\substack{Y \leqslant Z+1 \\ Z \leqslant Y+1}} p^{-Ys} p^{-Z(s-2)} \\
&= \frac{X^2(2-X)(1+5X)}{16(1-X)^2(1-4X^2)}.
\end{aligned}$$

And so,

$$\begin{aligned}
I_{2b} &= I_{2b(\mathrm{i})} + I_{2b(\mathrm{ii})} + I_{2b(\mathrm{iii})} \\
&= \frac{X(1+5X+2X^2-32X^3+16X^4)}{16(1-X)^2(1-2X)(1-4X^2)}.
\end{aligned}$$

Finally, then, we have the result

$$\begin{aligned}
I &= I_{1a} + I_{1b} + I_{2a} + I_{2b} \\
&= \frac{1+6X^2-8X^3}{8(1-X)(1-2X)(1-2X^2)(1-4X^2)} \\
&= \frac{(1-p^{-1})^3(1+3p^{-(2s-1)} - p^{-(3s-3)})}{(1-p^{-s})(1-p^{-(s-1)})(1-p^{-(2s-1)})(1-p^{-(2s-2)})}.
\end{aligned}$$

And so,

$$\zeta_{\mathfrak{sl}_2(\mathbb{Z}_2)} = (1 + 3\cdot 2^{-(2s-1)} - 2^{-(3s-3)})\,\zeta_2(s)\,\zeta_2(s-1)\,\zeta_2(2s-1)\,\zeta_2(2s-2).$$

This proves Theorem 1(2).

There are three good tests for the accuracy of such a calculation. Firstly, the expression after it is put over a common denominator reduces to an unexpectedly compact form, something not guaranteed when looking at the expressions for each of the pieces $I_{1a}, I_{1b}, I_{2a}, I_{2b}$. Secondly, that the first few terms of the power series agree with computer calculations for low-index subalgebras. Thirdly, that someone else has independently got the same answer via a different method. Our calculation passes all these tests, the third provided by Juliette While, a student of Dan Segal, at about the same time as our calculation was completed.

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