Topics in Representation Theory

Here are a few exercises to get you started.

1. Let $X \trianglelefteq_r R$. Show that RX is an ideal of R which is nilpotent if X is.

2. Assume R has no non-zero nilpotent ideals and X is a minimal right ideal of R. Show (i) & (iv). Discuss how far (ii) & (iii) are true.

   (i) $X^2 = X$;

   (ii) ~~if~~ $\exists$ x, y ~~are non-zero elements of~~ in X ~~then~~ with $xy \neq 0$;

   (iii) ~~if~~ $\exists$ $0 \neq x \in X$ then $xX = X$;

   (iv) $X = eR$ for some idempotent e.

[Hint for (ii). Consider $X \cap Y$, where Y is the set $\{r \mid r \in R, xr = 0\}$.

3. Let e be an idempotent in R. Prove

   (i) $eR \cap Re = eRe$ and this is a ring;

   (ii) $eRe \cong \operatorname{Hom}_R(eR, eR)$;

   (iii) when is eRe a division ring?

4. Let R be an Artinian ring and M a non-zero Artinian module. Show that M is finitely generated.

[Hint. Assume every proper submodule is finitely generated. Show you may assume first that $J(R) = 0$, then that R is simple. Conclude by Wedderburn II.]

II

Here are a few more, to keep you going

1. Prove the Krull-Schmidt Theorem for finitely generated R-modules, where R is a finite ring. (In our course R will be $\mathbb{Z}/p^r\mathbb{Z}\,G$ with p prime, G finite.)

2. Prove (i) $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/(m,n)\mathbb{Z}$;

(ii) for a commutative ring S and ideal X of S prove

$M \otimes_S S/X \cong M/MX$ for any S-module M.

3. Let M and V be finitely generated kG-modules (k a field, G finite). Write M^* for the dual space $\mathrm{Hom}_k(M, k)$. Show first that

$M^* \otimes_k V \cong \mathrm{Hom}_k(M, V)$ as k-spaces, via $\theta \otimes v \longmapsto (\mu \mapsto (\mu\theta)v)$

and then that is is a kG-map.

4. Let H be a subgroup of a free group F. Look up what a 'Schreier' transversal to the cosets of H in F is and the Nielsen-Schreier theorem that gives bases for H in terms of a basis of F and a Schreier transversal. Then do question 5 below.

5. (a) Let $F_1 = \langle x, y, z \rangle$ be free of rank 3 and let $\pi_1 : F_1 \longrightarrow A_3$ be given by $x \mapsto (123)$, $y \mapsto 1$, $z \mapsto (132)$. Let $R_1 = \mathrm{Ker}\, \pi_1$. Using $\{1, x, z\}$ as a Schreier transversal, show that R_1 is free of rank 7 and work out a basis.

5 ctd.

(b) Let $F = \langle a, b \rangle$ be free of rank 2 and let $\pi: F \to S_2$ be given by $a \mapsto (12)$, $b \mapsto 1$. Let $R = \operatorname{Ker} \pi$. Show that R is freely generated by b, a^2, aba^{-1}.

(c) With F as in (b), let $\varpi : F \to S_3$ be given by $a \mapsto (12)$ $b \mapsto (123)$. Show that $\operatorname{Ker} \varpi$ equals $\langle {a^2}^F, {b^3}^F, {baba^{-1}}^F \rangle$.

6. Let A be a finite Abelian group. View A as a $\mathbb{Z}$-module and let p be prime. Show that $A \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is the Sylow p-subgroup of A.

7. Let G be a group and M a $\mathbb{Z}G$-module. A map $d: G \to M$ is a <u>derivation</u> if and only if $(xy)d = (xd)y + yd$ for all x, y in G. The derivations form an additive group $\operatorname{Der}(G, M)$. Define $d^+ : \mathfrak{g} \to M$ by $(x-1)d^+ = xd$. Show $d \to d^+$ is an isomorphism from $\operatorname{Der}(G, M)$ onto $\operatorname{Hom}_{\mathbb{Z}G}(\mathfrak{g}, M)$.

III A few more.

1. Let $E = G \ltimes M$ be the split extension of M by G, where M is a $\mathbb{Z}G$-module. E is defined by

$$(g, \mu)(g', \mu') = (gg', \mu g' + \mu').$$

Let $d : G \to M$ and define $s : G \longrightarrow E$ by $gs = (g, gd)$

Show s is a homomorphism if and only if $d \in \mathrm{Der}(G, M)$.

2. Let F be a free group with basis X. Prove that the augmentation ideal $\mathfrak{f}$ of $\mathbb{Z}F$ is a free $\mathbb{Z}F$-module with basis $X - 1 = \{x - 1 \mid x \in X\}$. [Hint: You must show that any map from $X - 1$ to any $\mathbb{Z}F$-module M extends uniquely to a $\mathbb{Z}F$-hom $\mathfrak{f} \longrightarrow M$. In other words (cf. Sheet II) that any map $\phi : X \longrightarrow M$ extends uniquely to a derivation of F to M. Define $s : X \longrightarrow F \ltimes M$ in the obvious way and use question 1.]

3. Let S be commutative and G finite. Let M be an SG-lattice. Let $M^* = \mathrm{Hom}_S(S, M)$. For $\mu \in M$ define μ^{**} by $\theta\mu^{**} = \mu\theta$ $(\theta \in M^*)$. Show that $\mu \mapsto \mu^{**}$ is an SG-map. By using the projectivity of M as an S-module and the fact that $S^{**} \cong S$, prove M and M^{**} are isomorphic SG-lattices.

4. Let S be commutative and G finite. Consider the map $SG \to (SG)^*$ determined by $g \to \varphi$ where $x\varphi = \delta_{gx}$ (Kronecker delta), and show it is an isomorphism of SG-modules. Deduce that if M is any SG-projective then M^* is also projective.

5. Let M, N be SG-lattices and P a projective SG-lattice.

Suppose $0 \to P \xrightarrow{i} M \xrightarrow{t} N \to 0$ is exact. Use the S-projectivity of N to show $M^* \xrightarrow{i^*} P^*$ $(\theta \to \theta i)$ is an epimorphism. From the exactness $0 \to N^* \xrightarrow{t^*} M^* \xrightarrow{i^*} P^*$ deduce $M^* \underset{SG}{\cong} P^* \oplus N^*$. Deduce further that $M \underset{SG}{\cong} P \oplus N$.

6. Let S, T be commutative rings. Let M be an S-module, P a T-module and N an (S, T)-bimodule. Then $M \otimes_S N$ is naturally a T-module, $N \otimes_T P$ an S-module, and $(M \otimes_S N) \otimes_T P \cong M \otimes_S (N \otimes_T P)$.

7. Let $\alpha : V^* \otimes_k V \to \mathrm{Hom}_k(V, V)$ be such that $(\theta \otimes v)\alpha :$ $w \to \theta(v).w$. Let $\beta : V^* \otimes_k V \to k$ be such that $(\theta \otimes v)\beta = \theta(v)$. For $h \in \mathrm{Hom}_k(V, V)$, consider $\beta\alpha^{-1}h$ and show it equals trace h. If V_1, V_2 are k-spaces and $h_i \in \mathrm{Hom}_k(V_i, V_i)$ show trace $(h_1 \otimes h_2) =$ trace h_1. trace h_2.

Examples IV.

1. If $\bar{R}$ is a relation module, then so is $\bar{R} \oplus \mathbb{Z}G$.

(Method. Let $1 \to R \to F \xrightarrow{\alpha} G \to 1$ be exact with F free on $x_1, \ldots, x_n$. Let F_1 be free on $x_1, \ldots, x_n, x$. Let $\pi : F_1 \twoheadrightarrow F$ be defined by $\pi_F = id_F$, $x\pi = 1$. Let $R_1 = \ker \pi\alpha$. Let $X = \ker \pi$.

Show $F_1 = F [X$ is the split extension of X by F and $R_1 = R [X$.

Then $\bar{R}_1 = \bar{R} \oplus X/_{X'[R,X]}$. The elements of F form a Schreier transversal to the cosets of X. Show X is free on the x^f $(f \in F)$, and hence $\bar{X} \cong \mathbb{Z}F$. Then conclude.)

2. If $d(G) = 1$ show that $pr(G) = 0$.

3. ~~Let~~ Prove that if $\mathfrak{g}$ is the augmentation ideal of $\mathbb{Z}G$ then $\mathfrak{g}/_{\mathfrak{g}^2} \cong G/_{G'}$.

4. Let $G = H [A$ be a split extension of a simple $\mathbb{Z}H$-module A by H. If $d \in Der(H, A)$, show that the set of elements in G of the form (h, hd) $(h \in H)$ is a subgroup H_d, with $G = H_d A$ and $H_d \cap A = 1$.

Show that the number of subgroups of G that are complementary to A is $|Der(H, A)|$.

5. For a group X, let $\varphi_n(X)$ be the number of ordered n-tuples $(x_1, x_2, \ldots, x_n)$ such that $X = \langle x_1, \ldots, x_n \rangle$.

Let G be a finite soluble group and A a minimal normal subgroup of G.

(i) Show that $d(G) \leq 1 + d(G/A)$;

(ii) Let $B_1, \ldots, B_n$ be elements of G/A such that $G/A = \langle B_1, \ldots, B_n \rangle$. Show that if $b_1 \in B_1, \ldots, b_n \in B_n$ and if $H = \langle b_1, b_2, \ldots, b_n \rangle$ then $G = HA$ and either $G = H$ or else $H \cap A = 1$. Let c be the number of complements of A in G. Show that

$$\varphi_n(G/A)\left(|A|^n - c\right) = \varphi_n(G).$$

(iii) Show that if $d(G) = 1 + d(G/A)$ then

$$|A|^{d(G/A)} = |\mathrm{Der}(G/A, A)|.$$

6. Suppose $L_1, \ldots, L_n$ are different normal subgroups of H. Show that if each H/L_i is simple and non-Abelian then $H/L_1 \cap \cdots \cap L_n \cong H/L_1 \times H/L_2 \times \cdots \times H/L_n$.

7. Let $N \triangleleft G$ and suppose $d(G/N) \leq r$. Suppose $G/N = \langle B_1, \dots, B_r \rangle$. There are $|N|^r$ r-tuples $\underset{\sim}{b} = (b_1, \dots, b_r)$ with $b_i \in B_i$, $1 \leq i \leq r$. Show that for a subgroup S of G then either $NS < G$ and no $\underset{\sim}{b}$ lies in S^r or else $NS = G$ and there are $|S \cap N|^r$ vectors $\underset{\sim}{b}$ in S^r. Deduce that the number s of vectors $\underset{\sim}{b}$ such that $\langle b_1, \dots, b_r \rangle = G$ is independent of the choice of generating system $B_1, \dots, B_r$ of r elements of G/N. Deduce further that if $r \geq d(G)$ then $s > 0$.