Topics in Set Theory

Lectured by O. Kolman

Michaelmas Term 2012

Axiomatics. The formal axiomatic system of ordinary set theory (ZFC). Models of set theory. Absoluteness. Simple independence results. Transfinite recursion. Ranks. Reflection principles. Constructibility. [4]

Infinitary combinatorics. Cofinality. Stationary sets. Fodor's lemma. Solovay's theorem. Cardinal exponentiation. Beth and Gimel functions. Generalized Continuum Hypothesis. Singular Cardinals Hypothesis. Prediction principles (diamonds, squares, black boxes). Partial orders. Aronszajn and Suslin trees. Martin's Axiom. Suslin's Hypothesis. [6]

Forcing. Generic extensions. The forcing theorems. Examples. Adding reals; collapsing cardinals. Introduction to iterated forcing. Internal forcing axioms. Proper forcing. [4]

Large cardinals. Introduction to large cardinals. Ultrapowers. Scott's theorem. [2]

Partition relations and possible cofinality theory. Partition relations. Model-theoretic methods. Ramsey's theorem; Erdős-Rado theorem. Kunen's theorem. Walks on ordinals. Todorcevic's theorem. Introduction to pcf theory. [4]

Applications. Selection from algebra, analysis, geometry, and topology. [4]

Please let me know of corrections: glt1000@cam.ac.uk

Last updated: Tue 29^{th} Aug, 2017

Pre-requisite Mathematics

Logic and Set Theory is essential.

Literature

Basic material
† Drake, F. R., Singh, D., Intermediate Set Theory, John Wiley, Chichester, 1996.
Eklof, P. C., Mekler, A. H., Almost Free Modules, rev. ed., North-Holland, Amsterdam, 2002.
Halbeisen, L., Combinatorial Set Theory With a Gentle Introduction to Forcing, Springer, Berlin, 2012.
Kanamori, A., The Higher Infinite, 2nd ed., Springer, Berlin, 2009.
† Kunen, K., Set Theory, reprint, Studies in Logic, 34, College Publications, London, 2011.

Advanced topics
Burke, M. R., Magidor, M., Shelahs pcf theory and its applications, Ann. Pure Appl. Logic 50 (1990), 207–254.
Kanamori, A., Foreman, M., Handbook of Set Theory, Springer, Berlin, 2012.
† Shelah, S., Proper and Improper Forcing, 2nd ed., Springer, Berlin, 1998. Chapters 1 and 2.
Shelah, S. Cardinal Arithmetic, Oxford University Press, New York, 1994.
Todorcevic, S., Combinatorial dichotomies in set theory, Bull. Symbolic Logic 17 (2011), 1–72.

Lecture 1

Introduction

The **Continuum Hypothesis** (CH) asserts that if X is any uncountable subset of the real numbers $\mathbb{R}$, then there is a bijection from X onto $\mathbb{R}$. It is an answer to Cantor's Continuum question: for which ordinal α is $2^{\aleph_0} = \aleph_\alpha$?

This is Hilbert's First Problem (1900). However, there are *other* answers. Indeed, the response to Cantor's question is among the most striking and surprising results of 20$^{\text{th}}$ Century mathematics.

Stated loosely, CH is *independent* of the principles of ordinary mathematics. More precisely, if ZFC is *consistent* then ZFC does not prove CH and ZFC does not prove $\neg$CH (the negation of CH, namely $2^{\aleph_0} > \aleph_1$).

One of the aims of this course is to provide an introduction to the ideas and methods developed since Gödel (1938) and Cohen (1963) to handle this and other insoluble problems.

Independence results appear *everywhere*, not just in set theory. Let us consider some examples (to correct any naive misconceptions/prejudices, etc).

Example 1. Combinatorics and the order type of the real line.

Let $\mathbb{T} = \langle T, \leqslant_T \rangle$ be a **tree**, i.e.

(1) $\leqslant_T$ is a partial order on T ;

(2) $\{y \in T : y <_T x\}$ is well ordered.

A **chain** in $\mathbb{T}$ is a set $C \subset T$ such that C is totally ordered: $x \leqslant y$ or $y \leqslant x$ for all $x, y \in C$. An **antichain** in $\mathbb{T}$ is a set $A \subset T$ such that $(\forall x \neq y \in A)(x \nleqslant y \wedge y \nleqslant x)$.

Suppose κ is an infinite cardinal. A **κ-Suslin tree** is a tree $\mathbb{T} = \langle T, \leqslant_T \rangle$ such that

(1) $|T| = \kappa$

(2) every chain and every antichain in $\mathbb{T}$ has cardinality less than κ.

Proposition. There are no $\aleph_0$-Suslin trees.

Proof. Let $\mathbb{T}$ be an infinite tree with no infinite antichains. We show that $\mathbb{T}$ has an infinite chain. Note that since $|T| = \aleph_0$, there exists $x_0 \in T$ such that $\{y \in T : y > x_0\}$ is infinite: for if each x had only finitely many descendants then we could inductively construct an infinite antichain.

Now continue by induction to define $\{x_n : n \in \omega\}$, as follows: given $x_n \in T$ such that $|\{y \in T : y > x_n\}| = \aleph_0$, pick $x_{n+1} > x_n$ such that $|\{y \in T : y > x_{n+1}\}| = \aleph_0$.

Then $\{x_n : n \in \omega\}$ is an infinite chain in $\mathbb{T}$. So $\mathbb{T}$ is not $\aleph_0$-Suslin. □

Question. What about $\aleph_1$-Suslin trees?

Could we prove (as in the case of $\aleph_0$-Suslin trees) that there are none?

We shall see that this question cannot be answered on the basis of ZFC. The assertion that there are *no* $\aleph_1$-Suslin trees is called **Suslin's Hypothesis**.

Example 2. Commutative Algebra.

Definition. An infinite abelian group A is **free** if $A \cong \bigoplus_{i \in I} \mathbb{Z}$ for some set I, where $\mathbb{Z}$ is the infinite cyclic group under addition.

Definition. An abelian group G is called a **Whitehead group** (W**-group**) if when we write $G = F/K$ with F and K free, every homomorphism $f : K \to \mathbb{Z}$ extends to a homomorphism from F to $\mathbb{Z}$.

For example, a free abelian group is a W-group. Every countable W-group is free (harder).

Question. Is every W-group free?

It emerges that the existence of an uncountable non-free W-group is independent of ZFC.

Example 3. Complex Analysis.

Let $\mathcal{H}(\mathbb{C})$ be the family of entire holomorphic functions.

Definition. A family $\mathcal{F} \subset \mathcal{H}(\mathbb{C})$ is called **orbit-countable** if for all $n \in \mathbb{C}$, we have $|\{f(w_0) : f \in \mathcal{F}\}| \leqslant \aleph_0$.

Every countable family $\mathcal{F} \subset \mathcal{H}(\mathbb{C})$ is orbit-countable.

Question. If a family $\mathcal{F} \subset \mathcal{H}(\mathbb{C})$ is orbit-countable, is $\mathcal{F}$ countable?

The answer (due to Erdős) is that the existence of an uncountable orbit-countable family is equivalent to CH.

Example 4. Euclidean Geometry (Sierpinski; Kuratowski; Davies)

Consider the assertions about the Euclidean plane and space:

(P1) Euclidean 3-dimensional space can be decomposed into 3 sets E_i ($i = 1, 2, 3$) such that each line parallel to the coordinate axis OX_i intersects E_i in only a finite number of points.

(Q1) The Euclidean plane can be decomposed into 3 sets E_i ($i = 1, 2, 3$) such that for some 3 directions v_i in the plane, each line in the direction v_i intersects E_i in only a finite number of points.

Theorem (Sierpinski et al). (P1) $\iff$ (Q1) $\iff$ $2^{\aleph_0} = \aleph_1$.

So, for example, the rudimentary geometric assertion (Q1) about partitioning the Euclidean plane can neither be proved nor disproved from the principles of ordinary mathematics.

Example 5. Commutative Algebra (Łoś).

Suppose $\varphi : \prod_{i \in I} \mathbb{Z} \to \mathbb{Z}$ is a homomorphism of groups.

Question. If $\varphi|_{\bigoplus_{i \in I} \mathbb{Z}} \equiv 0$, is $\varphi \equiv 0$?

Theorem (Specker). If $I = \mathbb{N}$, and $\varphi|_{\bigoplus_{i \in I} \mathbb{Z}} \equiv 0$, then $\varphi \equiv 0$.

Theorem (Łoś). The following are equivalent:

(1) There exists a non-zero group homomorphism $\varphi : \prod_{i\in I}\mathbb{Z} \to \mathbb{Z}$ such that $\varphi|_{\bigoplus_{i\in I}\mathbb{Z}} \equiv 0$.

(2) The cardinal $|I|$ is a measurable cardinal.

Measurable cardinals are examples of large cardinals. They are sets that are so big that their existence cannot be proved from the principles of ordinary mathematics. Yet, they determine whether there exist non-trivial homomorphisms of the Cartesian power $\prod_{i\in I}\mathbb{Z}$ fulfilling (1).

Definition. An uncountable cardinal κ is **measurable** iff there is a κ-complete non-principal ultrafilter on κ, i.e. there is a set $U \subset \mathcal{P}(\kappa)$ such that

(1) U is a filter on κ (i.e., $A, B \in U$ implies $A \cap B \in U$; $\emptyset \notin U$; and if $A \in U$ and $A \subset B \subset \kappa$, then $B \in U$)

(2) for all $A \subset \kappa$, either $A \in U$ or $\kappa \setminus A \in U$

(3) for all $\alpha < \kappa$, $\{\alpha\} \notin U$

(4) if $\lambda < \kappa$ and $A_\alpha \in U$ for all $\alpha < \lambda$, then $\bigcap_{\alpha<\lambda} A_\alpha \in U$.

Logical questions. Although these five examples are not explicitly set-theoretic in character, the solutions are all independent of the principles of ordinary set theory. Why? What is the source of the complexity that gives rise to set-theoretic independence? What sorts of properties and classes of structures are immune to independence phenomena?

Lecture 2

Chapter 1. Axiomatics

In this lecture, we review the mathematical logic required for independence results in set theory, specify the vocabulary, the language and the axioms of the first-order theory ZFC. We then describe the architecture of independence proofs, and introduce some useful models of (subtheories of) ZFC. As a pay-off, we shall be able to show:

(1) any proof of the existence of the set of real numbers in first-order set theory *must* necessarily use the power set axiom.

(2) the first-order theory ZFC is not finitely axiomatisable

(3) the existence of a strongly inaccessible cardinal cannot be proved from ZFC

What does (3) mean?

Definition. A cardinal κ is **strongly inaccessible** iff

1. κ is an uncountable regular cardinal, and
2. κ is a strong limit cardinal.

Definition. A cardinal κ is **regular** iff κ *cannot* be written as a union of less than κ many sets of size less than κ.

For example, $\aleph_0$ is regular – it cannot be written as a finite union of finite sets. Also, $\aleph_1$ is regular – countable unions of countable sets are countable, under AC. But $\aleph_\omega$ is *not* regular, since $\aleph_\omega = \bigcup_{n<\omega}\aleph_n$. (Non-regular infinite cardinals are called **singular**.)

Definition. A cardinal κ is a **strong limit** iff $(\forall\lambda < \kappa)(2^\lambda < \kappa)$.

For example, $\aleph_0$ is a strong limit.

We can manufacture strong limits using the $\beth$ function (bet function). We define it by transfinite recursion:

$$\beth_0 = \aleph_0$$
$$\beth_{\alpha+1} = 2^{\beth_\alpha}$$
$$\beth_\delta = \bigcup_{\alpha<\delta} \beth_\alpha \quad \text{when } \delta \text{ is a limit ordinal}$$

So $\beth_\omega$ is a strong limit cardinal: for if $\lambda < \beth_\omega$ then $\lambda \leqslant \beth_n$ for some $n \in \omega$, and so $2^\lambda \leqslant 2^{\beth_n} = \beth_{n+1} < \beth_\omega$. However, $\beth_\omega$ is not regular.

The question is then natural: can one prove there exist strongly inaccessible cardinals?

Let us turn to the technical definitions from mathematical logic.

Definition. A **vocabulary** is a family τ of symbols that includes relation symbols, function symbols and constant symbols.

The **vocabulary of set theory** has one binary relation symbol $\in$.

If τ is a vocabulary, then the first-order language $L(\tau)$ or L_τ is the family of terms and formulas formed from τ using the syntactic rules of first-order logic.

For definiteness, we specify that the **logical vocabulary** has the symbols:

$$\text{variables } v_0, v_1, v_2, \ldots, \quad \text{and} \quad),\ (,\ =,\ \wedge,\ \neg,\ \exists$$

But of course we shall use the customary meta-linguistic abbreviations $\vee$, $\rightarrow$, $\forall$, and other conventions too. We write $L(\in)$ or $L_\in$ for the language of set theory.

Suppose τ is a vocabulary. A **τ-theory** is a family T of τ-sentences (i.e., sentences in $L(\tau)$).

For a theory T and a sentence φ (in $L(\tau)$), we write $T \vdash \varphi$ to mean that φ is deducible (provable) from T, i.e. there is a formal first-order deduction of φ from T.

For families $S, T \subset L(\tau)$, we write $T \vdash S$ to mean $T \vdash \varphi$ for all $\varphi \in S$.

A theory $T \subset L(\tau)$ is **consistent** iff for some τ-sentence φ, $T \nvdash \varphi$ (i.e., T does not prove φ); equivalently, for every $\varphi \in L(\tau)$, we have $T \nvdash \varphi \wedge \neg\varphi$.

A theory T is **complete** iff T is maximal consistent, i.e. if $T \subset T'$ and T' is consistent then $T = T'$. Equivalently, T is complete iff for every $\varphi \in L(\tau)$, either $\varphi \in T$ or $\neg\varphi \in T$.

With this review of syntax over, we can define the theory ZFC precisely. The axioms are the transcriptions into $L(\in)$ of the formal principles of naive (or semi-axiomatic) set theory.

The axioms of ZFC are:

(A0) Set Existence. $\exists x(x = x)$

(A1) Extensionality. $\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b$

Notes. 1. We are using the meta-language to write this axiom.

2. We tacitly adopt the **generality** interpretation of free variables: that free variables a and b are understood to be universally quantified.

(A2) Null set. $\exists x \forall y\ \neg(y \in x)$

(A3) Pair set. $\exists x \forall y (y \in x \leftrightarrow y = a \vee y = b)$

(A4) Union. $\exists x \forall y (y \in x \leftrightarrow \exists z (y \in z \wedge z \in a))$

(A5) Power set. $\exists x \forall y (y \in x \leftrightarrow y \subset a)$, where $y \subset a$ abbreviates $\forall z (z \in y \to z \in a)$

$(A6)_\varphi$ Separation (restricted comprehension, subset). For each φ in $L(\in)$ in which x does not appear,

$$\exists x \forall y (y \in x \leftrightarrow y \in a \wedge \varphi(y))$$

(i.e., $x = \{y \in a : \varphi(y)\}$ is a set)

$(A7)_\psi$ Replacement. For each $\psi(z, y)$ in $L(\in)$ in which x does not appear,

$$\forall z \forall u \forall v \big(\psi(z,u) \wedge \psi(z,v) \to u = v\big) \longrightarrow \exists x \forall y \big(y \in x \leftrightarrow \exists z (z \in a \wedge \psi(z,y))\big)$$

(i.e., $x = \{F(z) : z \in a\}$ is a set, where $F(z) = y \iff \psi(z,y)$.)

(A8) Foundation. $\exists x (x \in a) \to \exists x (x \in a \wedge x \cap a = \emptyset)$

(A9) Infinity. $\exists w (\emptyset \in w \wedge \forall x (x \in w \to S(x) \in w))$, where $S(x) = x \cup \{x\}$

(A10) Choice.

$$\forall x \forall y \big(x \in a \wedge y \in a \wedge x \neq y \to x \cap y = \emptyset \wedge \forall x (x \in a \to x \neq \emptyset)\big)$$
$$\longrightarrow \exists c \forall x (x \in a \to \exists u (c \cap x = \{u\}))$$

Lecture 3 **Definition.** The first-order theories ZF and ZFC are those with axioms A0–A9 and A0–A10 respectively.

Comments. Why these choices?

1. Why axiomatise?
   - reduce vagueness of naive set theory and avoid paradoxes
   - provide rigorous concept of provability (and unprovability)
   - discover new set-theoretic principles
2. Why this *first-order* axiomatisation?
   - the vocabulary $\{\in\}$ is economised and suffices to express everything pertinent
   - first-order logic is sound, complete and compact
   - infinitary logics are not compact (infinitary logic allows infinite conjunctions and infinite strings of quantifiers)
   - second-order and higher order logics are generally not compact and the quantifiers range over subsets of domains
3. Why *these* axioms?
   - see the Shoenfield handout (or Part II *Logic & Set Theory*)

The semantics and model theory of set theory

We discuss the concepts from first-order model theory that we need for the analysis of ZF and ZFC.

Definition. Let $\varphi(x)$ be a formula in $L(\in)$. We refer to $\{x : \varphi(x)\}$ as a **class**. (Classes need not be sets: e.g., V is the class $\{x : x = x\}$.)

We may allow parameters $x_1, \ldots, x_n \in V$. E.g., $M = \{x : \varphi(x, x_1, \ldots, x_n)\}$.

Definition. Suppose τ is a vocabulary. A **τ-structure** is a pair $\mathbb{M} = (M, \tau^M)$ where M is a non-empty class, and τ^M is a family of relations, operations and constants on M which interpret the symbols of τ:

- if $R \in \tau$ is an n-ary relation symbol then $R^M \subset M^n$
- if $f(x_1, \ldots, x_n) \in \tau$, then $f^M : M^n \to M$ (and we allow $dom(f) \subset M^n$)
- if $c \in \tau$ is a constant symbol, then $c^M \in M$

In the case of $L(\in)$, a τ-structure is simply a pair $(M, \in^M)$ where $\in^M \subset M \times M$.

Definition.

1. Suppose $\varphi \in L(\in)$ is a sentence. We write $\mathbb{M} \models \varphi$ to mean φ is true in $\mathbb{M}$.
2. If free variables of the formula ψ are included in $x_1, \ldots, x_n$ and $a_1, \ldots, a_n \in M$, we write $\mathbb{M} \models \varphi[a_1, \ldots, a_n]$ to mean $a_1, \ldots, a_n$ satisfy $\varphi(x_1, \ldots, x_n)$ in $\mathbb{M}$ (when $x_1, \ldots, x_n$ are assigned the values $a_1, \ldots, a_n$)
3. If T is a τ-theory, we write $\mathbb{M} \models T$ to mean $\mathbb{M} \models \varphi$ for every $\varphi \in T$, and say that $\mathbb{M}$ is a **model** of T.

   We write $\mathrm{Mod}(T)$ for the class of models of T.
4. We say a τ-theory T is **finitely axiomatisable** if there exists a finite family T_0 of τ-sentences such that $\mathrm{Mod}(T) = \mathrm{Mod}(T_0)$.

The fundamental results from first-order logic we need are the following.

Theorem (Completeness). Suppose T is a τ-theory and φ is a τ-sentence.

(1) $T \models \varphi$ (i.e., every model of T is a model of φ) iff $T \vdash \varphi$.

(2) T is consistent iff T has a model.

Theorem (Compactness). Suppose T is a τ-theory. Then T has a model iff every finite subfamily of T has a model (i.e., T is **finitely satisfiable**).

Let us define some important examples of structures for $L(\in)$.

Definition. We define by transfinite recursion on $\alpha \in \mathbf{Ord}$ the set V_α as follows:

(1) $V_0 = \emptyset$

(2) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$, the power set of V_α

(3) $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$, when α is a limit ordinal

Definition. The class $\{V_\alpha : \alpha \in \mathbf{Ord}\}$ is called the **von Neumann hierarchy**.

Exercise. V_ω is a model of ZFC$\setminus\{$INFINITY$\}$.

Proposition.

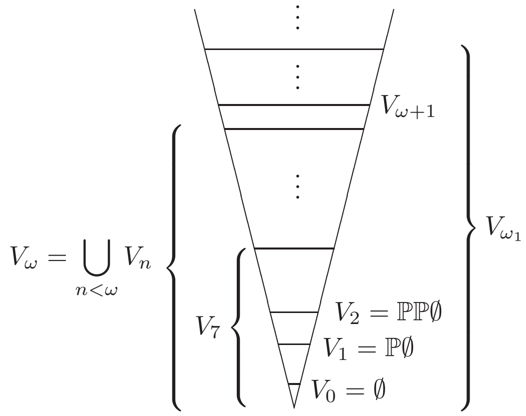
1. If $\alpha < \beta$ then $V_\alpha \subset V_\beta$.
2. V_α is transitive: if $x \in y \in V_\alpha$ then $x \in V_\alpha$ (i.e., $x \in V_\alpha \implies x \subset V_\alpha$).

(Hint: use transfinite induction on α.)

Definition. The class WF of **well-founded sets** is defined by the formula $WF(x)$

$$(\exists \alpha \in \mathbf{Ord})(x \in V_\alpha)$$

Picture.



Definition. Let λ be an infinite cardinal. We define $H(\lambda) = \{x : |\mathrm{trcl}(x)| < \lambda\}$, where $\mathrm{trcl}(x)$ is the transitive closure of x.

$H(\lambda)$, also written H_λ, is the family of sets of cardinality hereditarily less than λ.

Lecture 4

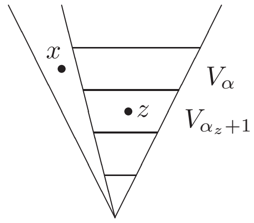
Reflection Principles and Non-finite Axiomatizability of ZF and ZFC

We motivate the Reflection Principle with a simple proposition.

Proposition. ZF $\vdash \forall x\, WF(x)$, i.e. ZF $\vdash \forall x \exists \alpha \in \mathbf{Ord}\ (x \in V_\alpha)$.

Proof. Let $x \in V$ and suppose $x \notin WF$.

Consider $u = \{y \in \mathrm{trcl}(\{x\}) : y \notin WF\}$. Then $u \neq \emptyset$ (as $x \in U$). By Foundation, u has an $\in$-minimal element y.

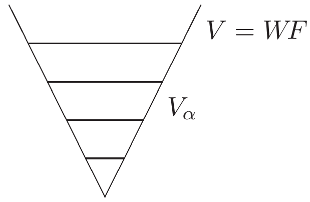


For all $z \in y$, we have $z \in WF$ (by $\in$-minimality of y), so there exists α_z minimal such that $z \in V_{\alpha_z+1}$.

$\{\alpha_z : z \in y\}$ is a set, by Replacement. So $\sup\{\alpha_z + 1 : z \in y\} \in$ **Ord**, and for all $z \in y$, we have $z \in V_\alpha$. So $y \subset V_\alpha$, and hence $y \in V_{\alpha+1} = \mathcal{P}(V_\alpha)$, contradicting $y \notin WF$.

So x must belong to WF. So we have shown that $V = WF$. $\square$

Thus our picture (under Foundation) simplifies to:



Comments.

1. Note, we do *not* attempt to show $\mathrm{ZF} \vdash \forall x WF(x)$ by constructing a formal derivation from the axioms. We argue informally in V (a class), and then observe that the argument could be recast as a formal deduction.
2. We can restate the Proposition as follows. Let ZF^0 be ZF without Foundation. By the Deduction Theorem of first-order logic, the Proposition is exactly $\mathrm{ZF}^0 \vdash$ Foundation $\rightarrow (V = WF)$.

Exercise. Show that $\mathrm{ZF}^0 \vdash (V = WF) \rightarrow$ Foundation.

Returning to the picture of V, it is natural to ask: how similar are V and V_α? We consider which first-order properties of V are "reflected" in V_α, for large α.

There are two aspects to this problem.

1. For φ an axiom of ZFC or first-order property in $L(\in)$, and $\alpha \in \mathbf{Ord}$, is V_α a model of φ?
2. For which axioms φ of ZFC can one prove there exist (arbitrarily large) $\alpha \in \mathbf{Ord}$ such that $V_\alpha \models \varphi$.

There is a technical difficulty to overcome: what does it mean to write $V \models \varphi$, or $A \models \varphi$ when A is a proper class?

Recall by Tarski's Theorem on indefinability of truth that we cannot define (in ZFC) $V \models \varphi$. We deal with this by *relativising formulas* to a class.

Definition. Suppose $A = \{x : A(x)\}$ is a class where $A(x)$ is a formula in $L(\in)$ with free variable x.

For each formula φ in $L(\in)$ we define φ^A, the relativisation of φ to A, as follows:

- if φ is $v_i \in v_j$ or $v_i = v_j$, then φ^A is $v_i \in v_j$ or $v_i = v_j$
- if φ is $\varphi_1 \wedge \varphi_2$ or $\neg\varphi_3$, then φ^A is $\varphi_1^A \wedge \varphi_2^A$ or $\neg(\varphi_3^A)$
- if φ is $\exists y \psi(y)$, then φ^A is $\exists y(y \in A \wedge \psi^A(y))$ – more formally, φ^A is $\exists y(A(y) \wedge \psi^A(y))$

Exercise. Write out φ^A for axioms (A0)–(A5).

Intuitively, $\mathrm{ZF} \vdash \varphi^A$ means $A \models \varphi$, i.e. A is a model of φ when the bound quantifiers are relativised to the class A.

Definition. Let A be a class. We say $\{A_\alpha : \alpha \in \mathbf{Ord}\}$ is a **cumulative hierarchy** for A if

(1) A_α is a set

(2) $\alpha < \beta$ implies $A_\alpha \subset A_\beta$

(3) $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$ if δ is a limit ordinal

(4) $A = \bigcup_{\alpha \in \mathbf{Ord}} A_\alpha$

Theorem (Reflection Principle). Suppose $A = \bigcup_{\alpha \in \mathbf{Ord}} A_\alpha$ is a cumulative hierarchy. Suppose $\varphi(y_1, \ldots, y_n)$ is a formula in $L(\in)$.

Then there exist unboundedly many $\alpha \in \mathbf{Ord}$ such that:

$$\forall x_1, \ldots, x_n \in A_\alpha, \quad \varphi(x_1, \ldots, x_n)^A \longleftrightarrow \varphi(x_1, \ldots, x_n)^{A_\alpha}$$

Proof. The idea is to "close under Skolem-like functions".

Write $\bar{y}$ for $(y_1, \ldots, y_n)$. List all of the subformulas of $\varphi(\bar{y})$ as $\psi_1, \ldots, \psi_k$. Wlog all existential subformulas (i.e. of the form $\exists u(\ldots)$) appear at the beginning of the list, and if $\varphi(\bar{y})$ is $\exists x \psi(x, \bar{y})$ then ψ_1 is $\varphi(\bar{y})$. Let $j_i + 1$ be the number of free variables in ψ_i.

The "Skolem-like" functions. For $y_1, \ldots, y_{j_i} \in A$, define $f_i : A^{j_i} \to \mathbf{Ord}$ by

$$f_i(y_1, \ldots, y_{j_i}) = \begin{cases} \text{the least ordinal } \alpha \text{ such that} \\ \quad \text{if } \psi_i(x, y_1, \ldots, y_{j_i})^A \text{ for some } x \in A \\ \quad \text{then } \psi_i(x, y_1, \ldots, y_{j_i})^A \text{ for some } x \in A_\alpha \\ \\ 0 \text{ otherwise} \end{cases}$$

For any γ, $f_i\left[A_\gamma^{j_i}\right]$ is a set by Replacement, so there is an ordinal $\beta_i(\gamma)$ greater than all ordinals in $f_i\left[A_\gamma^{j_i}\right]$.

Let $\beta(\gamma) = \max\{\beta_i(\gamma) : i \leqslant k\}$.

Now, given $\alpha \in \mathbf{Ord}$, define $\langle \gamma_n : n < \omega \rangle$ by

$$\gamma_0 = \alpha, \quad \gamma_{n+1} = \beta(\gamma_n), \quad \text{and} \quad \gamma_\omega = \sup\{\gamma_n : n \in \omega\}$$

Claim. γ_ω is as required. I.e., for all $y_1, \ldots, y_n \in A_{\gamma_\omega}$, we have

$$\varphi(\bar{y})^A \longleftrightarrow \varphi(\bar{y})^{A_{\gamma_\omega}}$$

This we prove by induction on complexity of φ.

Lecture 5 The claim is trivial if $\varphi(\bar{y})$ is atomic. If $\varphi(\bar{y})$ is $\varphi_1(\bar{y}) \wedge \varphi_2(\bar{y})$ or $\neg\varphi_3(\bar{y})$, then the claim follows by induction hypothesis on $\varphi_1, \varphi_2, \varphi_3$.

The interesting case is when the formula $\varphi(\bar{y})$ is $\exists x \psi_i(x, \bar{y})$.

If $\varphi(\bar{y})^{A_\gamma}$ then there is $x \in A_{\gamma_\omega}$ such that $\psi_i(x, \bar{y})^{A_{\gamma_\omega}}$. So by induction hypothesis applied to $\psi_i(x, \bar{y})$ we obtain $\psi_i(x, \bar{y})^A$, i.e. $\exists x \psi_i(x, \bar{y})^A$, i.e. $\varphi(\bar{y})^A$.

Conversely, if $\varphi(\bar{y})^A$ and $\bar{y} \in A_{\gamma_\omega}^{j_i}$, then there is an $m < \omega$ with $y_1, \ldots, y_{j_i} \in A_{\gamma_m}$. What do we know about $\xi = f_i(\bar{y})$? Well, $\xi \leqslant \gamma_{m+1}$.

We are given that there is $x \in A$ such that $\psi_i(x, \bar{y})^A$, so by construction there is $x \in A_{f_i(\bar{y})}$ such that $\psi_i(x, \bar{y})^A$. But $A_{f_i(\bar{y})} \subset A_{\gamma_\omega}$, so by induction hypothesis we obtain $\psi_i(x, \bar{y})^{A_{\gamma_\omega}}$, or in other words $\varphi(\bar{y})^{A_{\gamma_\omega}}$, as required. $\square$

Corollary 1. Suppose $\varphi(\bar{x})$ is a formula in $L(\in)$.

Then $\mathrm{ZF} \vdash \forall \beta \, \exists \alpha > \beta \, \forall y_1, \ldots, y_n \in V_\alpha \; \varphi(\bar{y})^{V_\alpha} \longleftrightarrow \varphi(\bar{y})$.

Proof. By the Proposition, ZF $\vdash V = WF$, so $\{V_\alpha : \alpha \in \mathbf{Ord}\}$ is a cumulative hierarchy for V. Now apply the Reflection Principle with $A_\alpha = V_\alpha$ and $A = V$. □

Corollary 2. The theories ZF and ZFC are not finitely axiomatisable.

Proof. By contradiction and Gödel's Second Incompleteness Theorem.

Suppose T_0 is a finite set of sentences in $L(\in)$ and $\mathrm{Mod}(ZFC) = \mathrm{Mod}(T_0)$.

Let φ be $\bigwedge T_0$, the conjunction of T_0. Now, by the Reflection Principle applied to φ using $\{V_\alpha : \alpha \in \mathbf{Ord}\}$, there exists $\alpha \in \mathbf{Ord}$ such that $\varphi^{V_\alpha} \leftrightarrow \varphi$.

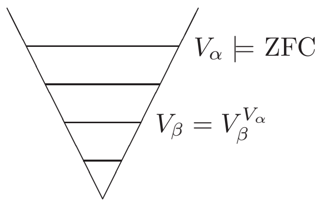
ZFC $\vdash \varphi$, so $ZFC \vdash \varphi^{V_\alpha}$, so $V_\alpha \models \varphi$, and so $V_\alpha \models \psi$ for every $\psi \in$ ZFC.

In other words, we have proved in ZFC that there exists a set (namely V_α) which is a model of ZFC. That is to say, we have proved the consistency of ZFC just using the axioms of ZFC. This contradicts Gödel's Second Incompleteness Theorem. □

We might prefer a proof that does not appeal to Gödel. Here is the idea.

For the V_α in the above proof, "$V_\alpha \models$ ZFC" – $(*)$

Let α be minimal with this property $(*)$.



If $\left\{V_\beta^{V_\alpha} : \beta < \alpha\right\}$ were a cumulative hierarchy for V_α, then in V_α we could apply the Reflection Principle to $\varphi = \bigwedge T_0$ to get a new ordinal $\gamma < \alpha$ such that $\varphi^{V_\gamma^{V_\alpha}} \leftrightarrow \varphi^{V_\alpha}$.

But $\varphi^{V_\alpha} \leftrightarrow \varphi$, so $\varphi^{V_\gamma^{V_\alpha}} \leftrightarrow \varphi$. Since "$V_\alpha \models$ ZFC", $V_\gamma^{V_\alpha}$ "must be" V_γ.

So $\gamma < \alpha$ and V_γ reflects φ, contradicting the minimality of α.

This is a vaguely expressed argument, but it can be made rigorous. The issue concerns how or whether defined terms change their meanings as one moves from one model to another. This will lead us to study the concept of **absoluteness**.

However, some further comment on the strength of the Reflection Principle is necessary.

Theorem. Let LM (standing for "Levy-Montague") be the axioms of ZFC without the Replacement axiom scheme, but with all instances of the Reflection Principle (one for each formula $\varphi(\bar{y})$).

Then, for every formula $\varphi(\bar{y})$, LM $\vdash \mathrm{Replacement}_{\varphi(\bar{y})}$.

Proof. The idea is to use the axiom of Separation combined with $\mathrm{RP}_{\varphi(\bar{y})}$ to find an appropriate witness for Replacement.

Details will be in an exercise. □

Cumulative hierarchies are very useful as a source of new axioms of set theory.

Definition. Suppose A is a set. Then the collection $\mathrm{Def}(A)$ is the family of sets $y \subset A$ such that for some formula $\varphi(x, y_1, \ldots, y_n)$ in $L(\in)$ and $a_1, \ldots, a_n \in A$, we have $y = \{x : \varphi(x, a_1, \ldots, a_n)^A\}$.

$\mathrm{Def}(A)$ is the collection of **definable** subsets of A.

We define the cumulative hierarchy of **constructible** sets as follows.

1. $L_0 = \emptyset$
2. $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$
3. $L_\delta = \bigcup_{\alpha<\delta} L_\alpha$ when δ is a limit ordinal.

The class $L = \bigcup_{\alpha\in\mathbf{Ord}} L_\alpha$ is the universe of constructible sets. The **Axiom of Constructibility** is the assertion that $V = L$, i.e. $\forall x \exists \alpha \in \mathbf{Ord}(x \in L_\alpha)$.

The Axiom of Constructibility is independent of ZFC. It is a very powerful statement about the regularity of the universe V. Although the study of the class L is not the goal of this course, we note that L is a transitive class model of ZFC+GCH. It is the smallest transitive model of ZFC containing all of the ordinals.

Lecture 6 The intuition behind the definition of the class L is that instead of the “maximally fat” power set operation, one puts into the power set of a set x only those subsets of x that necessarily must be there, i.e. the ones that have names (in the sense of definability).

It is instructive to begin the study of L by comparing the basic properties of V_α and L_α.

Theorem (Basic properties of V_α).

(1) $\alpha \leqslant \beta \to V_\alpha \subset V_\beta$

(2) $\alpha < \beta \to V_\alpha \in V_\beta$

(3) V_α is transitive: $x \in V_\alpha \to x \subset V_\alpha$

(4) $V_\alpha = \{x \in WF : \mathrm{rank}(x) < \alpha\}$, where $\mathrm{rank}(x) = \min\{\beta \in \mathbf{Ord} : x \in V_{\beta+1}\}$

(5) $y \in x \to \mathrm{rank}(y) < \mathrm{rank}(x)$

(6) $\mathrm{rank}(\alpha) = \alpha$ for all $\alpha \in \mathbf{Ord}$

(7) $\mathbf{Ord} \cap V_\alpha = \alpha \quad \big(= \{\beta : \beta < \alpha\}\big)$

(8) $\forall n$, $|V_n| < \aleph_0$, and $|V_\omega| = \aleph_0$, and, with (AC), $|V_{\omega+\alpha}| = \beth_\alpha$ for all $\alpha \in \mathbf{Ord}$

Proof. All are done by transfinite induction on $\alpha \in \mathbf{Ord}$. (Exercise.) □

Theorem (Basic properties of L_α).

(0) L_α is a set and $L_\alpha \subset V_\alpha$

(1) L_α is transitive

(2) $|L_n| < \aleph_0$, $L_n = V_n$ for $n < \omega$, and $L_\omega = V_\omega$

(3) $|L_\alpha| = |\alpha|$ for $\alpha \geqslant \omega$

(4) $\alpha \leqslant \beta \to L_\alpha \subset L_\beta$

(5) $\alpha < \beta \to L_\alpha \in L_\beta$ (use the formula $x = x$), and $L_\alpha \subsetneq L_\beta$

(6) $\alpha \in L_{\alpha+1}$ and $\alpha \notin L_\alpha$

Proof. In most cases, these are like the proofs for V_α using transfinite induction. (Exercise.) □

Comment. For most $\alpha \in \mathbf{Ord}$, we have $L_\alpha \subsetneq V_\alpha$, by (8). Thus in L, we have greater control over the power set operation.

These basic properties of V_α and L_α will be useful when we come to establish relative consistency results.

We define next the concepts of **relative consistency** and **inner models**.

Definition. Let φ be a sentence in $L(\in)$ and T a theory. We say that φ is **consistent relative to** T if the consistency of T implies the consistency of $T \cup \{\varphi\}$.

Analogously, we can define the concept of S being consistent relative to T for another theory S.

There is a very useful criterion for relative consistency.

Theorem. Let T be a theory in $L(\in)$ (or some expansion of $L(\in)$), and φ be a sentence. Suppose $A(x)$ is a formula in $L(\in)$ with exactly the free variable x. If

(1) $T \vdash \exists x A(x)$

(2) for every axiom σ in T, $T \vdash \sigma^A$

(3) $T \vdash \varphi^A$

then φ is consistent relative to T.

Proof. Let us prove the contrapositive. So $T \cup \{\varphi\}$ is inconsistent. Then there is a finite subset $\{\sigma_1, \ldots, \sigma_n\} \subset T$ such that $S_0 = \{\sigma_1, \ldots, \sigma_n, \varphi\}$ is inconsistent. Thus S_0 has no model.

Hence, $\models \sigma_1 \wedge \ldots \wedge \sigma_n \to \neg\varphi$.

Recall from Logic, if ψ is any sentence, then $\models \psi$ implies $\models \exists x A(x) \to \psi^A$ – $(*)$

So by $(*)$, $\models \exists x A(x) \to (\sigma_1 \wedge \ldots \wedge \sigma_n \to \neg\varphi)^A$.

That is, $\models \exists x A(x) \to (\sigma_1^A \wedge \ldots \wedge \sigma_n^A \to \neg\varphi^A)$.

Now by (1) and (2), $T \vdash \neg\varphi^A$. But by (3), $T \vdash \varphi^A$. Thus $T \vdash \neg\varphi^A \wedge \varphi^A$, so T is inconsistent. □

Definition. A class A is called an **inner model** of a theory T (with respect to a theory S) if

(1) A is transitive

(2) for every σ in T, we have $S \vdash \sigma^A$

The previous theorem helps us find inner models. We now have several useful classes (V_α, L_α, H_λ, WF, L) for the application of the criterion to obtain relative consistency results.

An example is the following theorem of Gödel: L is an inner model of ZFC+$V = L$+GCH.

Lecture 7

Chapter 2. Infinitary Combinatorics

Definition. A cardinal κ is a **successor** cardinal if $\kappa = \lambda^+$ for some cardinal λ.

A cardinal κ is a **limit** cardinal if κ is not a successor cardinal.

Remark. *Every* infinite cardinal is a *limit ordinal*.

Definition. Suppose that $\mathbb{P} = (P, \leqslant)$ is a partial order. A set $A \subset P$ is a **cover** of $\mathbb{P}$ if

$$(\forall p \in P)(\exists q \in A)(p \leqslant q)$$

The **cofinality** of $\mathbb{P}$ is the least cardinal λ for which there exists a cover of $\mathbb{P}$ of size λ:

$$\mathrm{cf}(\mathbb{P}) = \min\{|A| : A \subset \mathbb{P} \text{ is a cover of } \mathbb{P}\}$$

Notes.

(1) $\mathrm{cf}(\mathbb{P}) \leqslant |\mathbb{P}|$

(2) $\mathrm{cf}(\alpha) \leqslant \alpha$ for every ordinal α

(3) $\mathrm{cf}(\alpha + 1) = 1$ (since $\{\alpha\}$ is a cover)

(4) $\mathrm{cf}(\mathcal{P}(X), \subset) = 1$

Lemma 1. Suppose $\alpha \in \mathbf{Ord}$. Then there exists a function $f : \mathrm{cf}(\alpha) \to \alpha$ such that

(1) $\mathrm{range}(f)$ is a cover of α

(2) f is strictly increasing: $\xi < \zeta \to f(\xi) < f(\zeta)$

Proof. Let A be a cover of α of size $\mathrm{cf}(\alpha)$, say $A = \{\alpha_\zeta : \zeta < \mathrm{cf}(\alpha)\}$. Define $f : \mathrm{cf}(\alpha) \to \alpha$ by

$$f(\zeta) = \max\left\{\alpha_\zeta,\ \sup\{f(\xi) + 1 : \xi < \zeta\}\right\}$$

Why is f well-defined for all $\zeta \in \mathrm{cf}(\alpha)$? By minimality of $|A|$, f is well-defined. Then (1) and (2) are immediate. □

Definition. We say that $f : \alpha \to \beta$ is a **cofinal** map (or that f maps α into β **cofinally**) if $\mathrm{range}(f)$ is unbounded in β.

Proposition 2. The following are equivalent:

(1) $\mathrm{cf}(\beta) = \alpha$

(2) α is the least ordinal such that there is a cofinal map $f : \alpha \to \beta$

Proof.

(1) $\Rightarrow$ (2). Let A be a cover of cardinality $\mathrm{cf}(\beta) = \alpha$. So α is a cardinal. By Lemma 1, there is an increasing function $f : \alpha \to \beta$ such that $\mathrm{range}(f)$ is a cover of β. So $\mathrm{range}(f)$ is unbounded in β and is therefore cofinal.

α is the least ordinal with this property. For if $\delta < \alpha$ and $g : \delta \to \beta$, then $|\mathrm{range}(g)| \leqslant |\delta| \leqslant \delta < \alpha = \mathrm{cf}(\beta)$, so $\mathrm{range}(g)$ is not a cover and hence is not cofinal.

Thus α is the least cardinal such that there is $f : \alpha \to \beta$ cofinally.

(2) $\Rightarrow$ (1). Let $f : \alpha \to \beta$ be cofinal, and α is minimal, so $|\alpha| = \alpha$. Then $A = \mathrm{range}(f)$ is a cover of β, so $\mathrm{cf}(\beta) \leqslant |A| \leqslant |\alpha| = \mathrm{cf}(\beta)$. □

Definition. A cardinal κ is **regular** if there is no family $\{A_\alpha : \alpha < \lambda\}$ such that $\kappa = \bigcup_{\alpha<\lambda} A_\alpha$, with $|A_\alpha| < \kappa$ and $\lambda < \kappa$.

If κ is not regular, we say that κ is **singular**.

Examples. $\aleph_0$ is regular, as is $\aleph_1$. But $\aleph_\omega$ is singular, since $\aleph_\omega = \bigcup_{n<\omega} \aleph_n$. And $\beth_\omega$ is also singular.

Proposition 3. A cardinal κ is singular iff $\mathrm{cf}(\kappa) < \kappa$.

Proof.

($\Leftarrow$) Suppose $\lambda = \mathrm{cf}(\kappa)$, and $\lambda < \kappa$ and $A \subset \kappa$ is a cover of size λ, say $A = \{A_\zeta : \zeta < \lambda\}$.

Let $A_\zeta = \{\beta < \kappa : \beta < \alpha_\zeta + 1\} = \alpha_\zeta + 1$.

Then $|A_\zeta| = |\alpha_\zeta + 1| = |\alpha_\zeta| < \kappa$, and $\lambda < \kappa$, and $\kappa = \bigcup_{\zeta<\lambda} A_\zeta$. (Take any $\beta \leqslant \kappa$, then $\beta \in A_\zeta$ for some ζ since A is a conver of κ.)

So κ is singular.

($\Rightarrow$) Suppose κ is singular, and let λ be minimal such that $\kappa = \bigcup_{\zeta<\lambda} A_\zeta$ with $|A_\zeta| < \kappa$.

Case 1: some A_α is a cover of κ. Then $\mathrm{cf}(\kappa) \leqslant |A_\alpha| < \kappa$, and we are done.

Case 2: otherwise. So no A_ζ covers κ. Thus for all $\zeta < \lambda$, there is α_ζ such that $\forall \beta \in A_\zeta$, $\beta < \alpha_\zeta$. Let $A = \{\alpha_\zeta : \zeta < \lambda\}$. Then A is a cover of κ, and $|A| \leqslant \lambda < \kappa$, so $\mathrm{cf}(\kappa) < \kappa$. □

Corollary 4. $\mathrm{cf}(\kappa^+) = \kappa^+$ for every infinite cardinal κ.

Proof. Let $\lambda = \mathrm{cf}(\kappa^+)$. Write $\kappa = \bigcup_{\alpha<\lambda} A_\alpha$, with $|A_\alpha| < \kappa^+$.

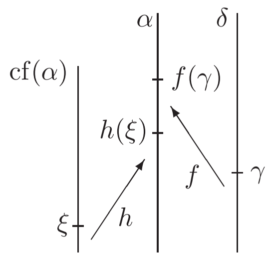
Then $\kappa^+ = \left|\bigcup_{\alpha<\lambda} A_\alpha\right| = \lambda \cdot \sup\{|A_\alpha| : \alpha < \lambda\} = \max\left\{\lambda,\ \sup\{|A_\alpha| : \alpha < \lambda\}\right\}$.

But $|A_\alpha| \leqslant \kappa$, so $\kappa^+ = \max\{\lambda, \kappa\} = \lambda = \mathrm{cf}(\kappa^+)$. □

Lecture 8

Lemma 5. Suppose δ is a limit ordinal, and $f : \delta \to \alpha$ is strictly increasing and cofinal. Then $\mathrm{cf}(\delta) = \mathrm{cf}(\alpha)$.

Proof. First, $\mathrm{cf}(\alpha) \leqslant \mathrm{cf}(\delta)$. Let $g : \mathrm{cf}(\delta) \to \delta$ be as in Lemma 1. Now, $\mathrm{range}(f \circ g)$ is a cover of α. So $\mathrm{cf}(\alpha) \leqslant \mathrm{cf}(\delta)$.



Second, $\mathrm{cf}(\delta) \leqslant \mathrm{cf}(\alpha)$. Let $h : \mathrm{cf}(\alpha) \to \alpha$ be cofinal, and let $k(\xi)$ be the least $\gamma < \delta$ such that $f(\gamma) > h(\xi)$.

Claim: $k : \mathrm{cf}(\alpha) \to \delta$ is cofinal.

Why? If $i < \delta$ then $f(i) < \alpha$. There exists ξ with $f(i) \leqslant h(\xi)$, and $k(\xi) = \min\{\gamma : f(\gamma) > h(\xi)\}$, and so $i < k(\xi) = \gamma$. (If $\gamma < i$ then $f(\gamma) < f(i)$ and we get a contradiction.)

So $\mathrm{cf}(\delta) \leqslant \mathrm{cf}(\alpha)$. □

Corollary 6. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$.

Proof. By Lemma 1, there is a strictly increasing cofinal map $f : \delta = \mathrm{cf}(\alpha) \to \alpha$. By Lemma 5, since $\mathrm{cf}(\alpha)$ is a limit ordinal (since $\mathrm{cf}(\alpha)$ is a cardinal), it follows that $\mathrm{cf}(\delta) = \mathrm{cf}(\alpha)$, i.e. $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$. □

Thus $\mathrm{cf}(\alpha)$ is always a *regular* cardinal.

Corollary 7. $\aleph_0$ is regular.

Proof. Any cover is infinite.

Corollary 8. If δ is a limit ordinal, then $\mathrm{cf}(\aleph_\delta) = \mathrm{cf}(\delta)$.

Proof. By Lemma 1, there is a strictly increasing cofinal map $f : \mathrm{cf}(\delta) \to \delta$. Define $g : \mathrm{cf}(\delta) \to \aleph_\delta$ by $g(\xi) = \aleph_{f(\xi)}$.

So $\mathrm{cf}(\aleph_\delta) = \mathrm{cf}(\delta)$, by Lemma 5, because g is strictly increasing and cofinal. □

Exercise. Prove that for all $\alpha \in \mathbf{Ord}$, we have $\alpha \leqslant \aleph_\alpha$.

Corollary 9. If $\aleph_\delta$ is an uncountable regular limit ordinal, then $\aleph_\delta = \delta$.

Proof. We have

$$\begin{array}{rcll} \aleph_\delta & = & \mathrm{cf}(\aleph_\delta) & \text{by regularity} \\ & = & \mathrm{cf}(\delta) & \text{by Corollary 8} \\ & \leqslant & \delta & \\ & \leqslant & \aleph_\delta & \text{by the exercise} \end{array}$$

So $\aleph_\delta = \delta$. □

Observation. The converse is false. I.e., there exists a singular δ with $\aleph_\delta = \delta$.

For example, let $\alpha_0 = \omega$ and $\alpha_{n+1} = \aleph_{\alpha_n}$, and $\alpha = \sup_{n<\omega} \alpha_n$.

Then $\aleph_\alpha = \alpha$, but $\mathrm{cf}(\aleph_\alpha) = \omega$.

Cofinality enables us to improve Cantor's Theorem, that $2^\kappa > \kappa$.

Definition. Suppose that $\{X_i : i \in I\}$ are pairwise disjoint sets, with $|X_i| = \lambda_i$. We define the (cardinal) sum

$$\sum_{i \in I} \lambda_i = \left| \bigcup_{i \in I} X_i \right|$$

and we define the (cardinal) product

$$\prod_{i \in I} \lambda_i = \left| \left\{ f : f \text{ is a function from } I \text{ into } \bigcup X, \text{ with } f(i) \in X_i \forall i \in I \right\} \right|$$

Theorem 10 (König). Suppose $\lambda_\alpha < \kappa_\alpha$, for all $\alpha < \delta$. Then $\sum_{\alpha<\delta} \lambda_\alpha < \prod_{\alpha<\delta} \kappa_\alpha$.

Proof. Clearly, $\sum_{\alpha<\delta} \lambda_\alpha \leqslant \prod_{\alpha<\delta} \kappa_\alpha$. (Exercise: use the natural inclusion map.)

Note $\sum_{\alpha<\delta} \lambda_\alpha = \left| \bigcup_{\alpha<\delta} \{\alpha\} \times \lambda_\alpha \right|$. Suppose that $G : \bigcup_{\alpha<\delta} \{\alpha\} \times \lambda_\alpha \to \prod_{\alpha<\delta} \kappa_\alpha$.

We show that G is *not* a surjection.

Define $h \in \prod_{\alpha<\delta} \kappa_\alpha$ as follows. For $\alpha < \delta$,

$$h(\alpha) = \min \big\{ \xi : \xi \in \kappa_\alpha \setminus \underbrace{\{\pi_\alpha G(\alpha,\zeta) : \zeta < \lambda_\alpha\}}_{\text{has cardinality } \leqslant \lambda_\alpha} \big\}$$

where π_α is the projection of $\prod_{\beta<\delta} \kappa_\beta$ onto κ_α.

h is well-defined, since $\lambda_\alpha < \kappa_\alpha$. Now it is evident that $h \notin \text{range}(G)$ – by construction, $h(\alpha) \neq \pi_\alpha G(\alpha,\xi)$ for all $\xi < \lambda_\alpha$. □

Corollary 11. If $\lambda \geqslant \text{cf}(\kappa)$, then $\kappa^\lambda > \kappa$.

Proof. Write $\kappa = \sum_{i<\text{cf}(\kappa)} \lambda_i$, with $\lambda_i < \kappa$. Then by Theorem 10,

$$\kappa = \sum_{i<\text{cf}(\kappa)} \lambda_i < \prod_{i<\text{cf}(\kappa)} \kappa = \kappa^{\text{cf}(\kappa)} \leqslant \kappa^\lambda$$

□

Corollary 12. If $\kappa \geqslant \aleph_0$, then $\text{cf}(2^\kappa) > \kappa$.

Proof. Let $\lambda = \text{cf}(2^\kappa)$. By Corollary 11, $(2^\kappa)^\lambda > 2^\kappa$.

But $(2^\kappa)^\lambda = 2^{\kappa\lambda} = 2^{\max\{\kappa,\lambda\}}$. So $2^{\max\{\kappa,\lambda\}} > 2^\kappa$.

So $\max\{\kappa,\lambda\} > \kappa$, and so $\lambda = \text{cf}(2^\kappa) > \kappa$. □

Lecture 9 We know from Corollaries 11 and 12 that $\forall\kappa\ \exists\lambda^*\ \aleph_0 \leqslant \lambda^* \leqslant \text{cf}(\kappa)$ and $\kappa^{\lambda^*} > \kappa$.

Three natural questions occur:

(1) What is the least such λ^*?

(2) For the least λ^* above,

   (a) how large is κ^{λ^*} relative to κ?

   (b) how large is $\kappa^{\text{cf}(\kappa)}$ relative to κ?

(3) How are 2^κ and $\kappa^{\text{cf}(\kappa)}$ related (if at all)?

Key examples to think about:

(1) What is $\aleph_\omega^{\aleph_0}$, given $2^{\aleph_n}$, $n < \omega$?

(2) What is $\aleph_{\omega_1}^{\aleph_1}$, given $2^{\aleph_\alpha}$, $\alpha < \omega_1$?

Answers were given by Shelah and Silver.

In this lecture, we examine some hypotheses which bear on these questions.

Definition.

(1) The **Generalised Continuum Hypothesis** (GCH) is the statement:

$$(\forall \kappa \geqslant \aleph_0)(2^\kappa = \kappa^+)$$

(2) The **Continuum Hypothesis** (CH) is the statement:

$$2^{\aleph_0} = \aleph_1$$

(3) The **weak Generalised Continuum Hypothesis** (wGCH) is the statement:

$$(\forall \kappa \geqslant \aleph_0)(2^\kappa < 2^{\kappa^+})$$

(4) The **weak Continuum Hypothesis** (wCH) is the statement:

$$2^{\aleph_0} < 2^{\aleph_1}$$

Remark. GCH $\to$ wGCH, and CH $\to$ wCH.

We look at how GCH impacts the operation of cardinal exponentiation and infinite products of cardinals.

Notation. We write ${}^X Y = \{f : f \text{ is a function from } X \text{ into } Y\}$.

Proposition 13. Let $\kappa \geqslant \aleph_0$. Then κ is regular iff $(*)$ $(\forall \lambda < \kappa)\big({}^\lambda \kappa = \bigcup_{\alpha<\kappa} {}^\lambda \alpha\big)$

Proof.

($\Rightarrow$) Suppose $\lambda < \kappa = \mathrm{cf}(\kappa)$. Note key observation: if $f \in {}^\lambda \kappa$, then range(f) is bounded in κ, and so there is $\alpha < \kappa$ such that $f \in {}^\lambda \alpha$.

So ${}^\lambda \kappa \subset \bigcup_{\alpha<\kappa} {}^\lambda \alpha$ and clearly $\bigcup_{\alpha<\kappa} {}^\lambda \alpha \subset {}^\lambda \kappa$. Thus $(*)$ holds.

($\Leftarrow$) Suppose $(*)$ holds. If $\lambda < \kappa$ and $f : \lambda \to \kappa$ then there is $\alpha < \kappa$ with $f \in {}^\lambda \alpha$ by $(*)$. So f is not cofinal in κ. Thus $\lambda < \mathrm{cf}(\kappa)$ for all $\lambda < \kappa$.

So $\kappa \leqslant \mathrm{cf}(\kappa) \leqslant \kappa$. I.e., κ is regular. □

Lemma 14. For all $\kappa \geqslant \aleph_0$, if $\lambda < \mathrm{cf}(\kappa)$ and $(\forall \mu < \kappa)(2^\mu \leqslant \kappa)$, then $\kappa^\lambda = \kappa$.

Proof. If $\lambda < \mathrm{cf}(\kappa)$, then

$$\begin{aligned} \kappa^\lambda &= |{}^\lambda \kappa| = \Big|\bigcup_{\alpha<\kappa} {}^\lambda \alpha\Big| \\ &\overset{(*)}{=} \Big|\sum_{\alpha<\kappa} {}^\lambda \alpha\Big| = \sum_{\alpha<\kappa} |{}^\lambda \alpha| = \sum_{\alpha<\kappa} |\alpha|^\lambda \\ &\leqslant \sum_{\alpha<\kappa} \big(2^{|\alpha|}\big)^\lambda = \sum_{\alpha<\kappa} 2^{|\alpha|\lambda} \leqslant \sum_{\alpha<\kappa} \kappa = \kappa\kappa = \kappa \leqslant \kappa^\lambda \end{aligned}$$

(Note that $(*)$ uses the key observation from the proof of Proposition 13.) □

Thus GCH answer all of the opening question about λ^*, $\kappa^{\mathrm{cf}(\kappa)}$ and 2^κ. GCH implies that $\lambda^* = \mathrm{cf}(\kappa)$, and $\kappa^{\lambda^*} = \kappa^{\mathrm{cf}(\kappa)} = \kappa^+$.

Definition. For cardinals λ, κ, we define the **weak power** of κ to λ to be

$$\kappa^{<\lambda} = \sum_{\mu<\lambda} \kappa^{\mu}$$

Lemma 15. For all $\kappa \geqslant \aleph_0$,

(1) if $\kappa = \mathrm{cf}(\kappa)$ and $(\forall \mu < \kappa)(2^{\mu} \leqslant \kappa)$, then $\kappa^{<\kappa} = \kappa$

(2) Assume GCH. Then $\kappa = \mathrm{cf}(\kappa)$ iff $\kappa^{<\kappa} = \kappa$.

Proof.

(1) By Lemma 14, for all $\lambda < \kappa = \mathrm{cf}(\kappa)$, we have $\kappa^{\lambda} = \kappa$.

So $\kappa^{<\kappa} = \sum_{\lambda<\kappa} \kappa^{\lambda} = \sum_{\lambda<\kappa} \kappa = \kappa\kappa = \kappa$.

(2) Note GCH implies that $(\forall \lambda < \kappa)\big(2^{\lambda} = \lambda^{+} \leqslant \kappa\big)$.

($\Rightarrow$) This is part (1).

($\Leftarrow$) If $\lambda = \mathrm{cf}(\kappa) < \kappa$, then by König's Theorem, $\kappa^{\lambda} > \kappa$, and so $\kappa < \kappa^{<\kappa}$. □

Lemma 16. For all $\kappa \geqslant \aleph_0$,

(1) $\kappa^{\kappa} \geqslant \kappa^{<\kappa} \geqslant 2^{<\kappa} \geqslant \kappa$

(2) If $\kappa = \lambda^{+}$, then $\kappa^{<\kappa} = 2^{<\kappa} = 2^{\lambda}$

Proof.

(1) This is trivial.

(2) Note $2^{<\kappa} = 2^{\lambda}$ and $\kappa^{<\kappa} = \kappa^{\lambda}$. Hence $2^{\lambda} = 2^{<\kappa} \leqslant \kappa^{<\kappa} = \kappa^{\lambda} \leqslant (2^{\lambda})^{\lambda} = 2^{\lambda}$. □

Corollary 17. GCH holds iff $(\forall \kappa \geqslant \aleph_0)(2^{<\kappa} = \kappa)$.

Lemma 18. If κ is a limit cardinal, then $2^{\kappa} = (2^{<\kappa})^{\mathrm{cf}(\kappa)}$.

Proof. Write $\kappa = \sum_{i<\mathrm{cf}(\kappa)} \lambda_i$, with $\lambda_i < \kappa$. Then

$$2^{\kappa} = 2^{\sum \lambda_i} = \prod_{i<\mathrm{cf}(\kappa)} 2^{\lambda_i} \leqslant \prod_{i<\mathrm{cf}(\kappa)} 2^{<\kappa} = (2^{<\kappa})^{\mathrm{cf}(\kappa)} \leqslant (2^{\kappa})^{\mathrm{cf}(\kappa)} = 2^{\kappa}$$

□

Lecture 10 **Theorem 19 (Bukovský-Hechler).** Let $\kappa > \mathrm{cf}(\kappa)$ be such that

$$\exists \lambda_0 \ \forall \lambda \ \lambda_0 \leqslant \lambda < \kappa \to 2^{\lambda} = 2^{\lambda_0} \quad (*)$$

Then $2^{\kappa} = 2^{\lambda_0}$.

Proof. Since $\mathrm{cf}(\kappa) < \kappa$, by $(*)$ and wlog, $\mathrm{cf}(\kappa) \leqslant \lambda_0$.

Now, by Lemma 18, $2^{\kappa} = (2^{<\kappa})^{\mathrm{cf}(\kappa)} = (2^{\lambda_0})^{\mathrm{cf}(\kappa)} = 2^{\lambda_0}$. □

Definition. The **Gimel function** $\gimel(\kappa)$ is defined on the class of infinite cardinals by

$$\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)}.$$

Why is $\gimel(\kappa)$ important in current research in cardinal exponentiation?

Note. If $\kappa, \lambda \geqslant \aleph_0$, then $\mathrm{cf}(\kappa^\lambda) > \lambda$. (Exercise: use König again.) In particular, $\gimel(\kappa) > \kappa$.

Definition. The **Gimel Hypothesis** (Solovay, 1974) states: for every *singular* cardinal κ,

$$\gimel(\kappa) = \max\left\{2^{\mathrm{cf}(\kappa)}, \kappa^+\right\}$$

Remarks.

(1) GCH implies the Gimel Hypothesis. (Exercise.)

(2) We proved in ZFC that $\gimel(\kappa) \geqslant 2^{\mathrm{cf}(\kappa)}$ and $\gimel(\kappa) > \kappa$. The Gimel Hypothesis asserts that $\gimel(\kappa)$ is the least ZFC-permitted value (relative to the other values of $\gimel(\lambda)$ for regular λ).

For example, under CH, the Gimel Hypothesis says $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$.

(3) Why only for singular cardinals? We know the for regular cardinals κ, we have $\gimel(\kappa) = 2^\kappa$, and it follows from work of Easton (1964, 1970) that for any "reasonably-defined" cardinal-valued function F defined on regular cardinals such that

(i) $\kappa < \lambda \to F(\kappa) \leqslant F(\lambda)$
(ii) $\mathrm{cf}(F(\kappa)) > \kappa$

it is consistent with ZFC and the Gimel Hypothesis that for all regular cardinals κ we have $2^\kappa = F(\kappa)$.

So, in particular, the Gimel Hypothesis does *not* imply GCH.

Failures of the Gimel Hypothesis are an active subject of research. An equivalent statement is the following.

Definition. The **Singular Cardinals Hypothesis** (SCH) states: for every singular cardinal κ,

$$\text{if } 2^{\mathrm{cf}(\kappa)} < \kappa, \text{ then } \gimel(\kappa) = \kappa^+$$

Exercise. Show that the SCH is equivalent to:

$$\text{for all } \textit{regular} \text{ cardinals } \kappa, \lambda, \text{ if } 2^\lambda < \kappa, \text{ then } \kappa^\lambda = \kappa$$

Corollary 20 (Bukovský). The continuum function $\kappa \mapsto 2^\kappa$ can be defined in terms of the Gimel function. In other words, if one knows $\gimel$, then one knows 2^κ for all κ.

Proof.

(a) If $\kappa = \mathrm{cf}(\kappa)$, then $2^\kappa = \gimel(\kappa)$.

(b) If κ is a limit cardinal and the continuum function is eventually constant below κ (as in Theorem 19), then (check)

$$2^\kappa = 2^{<\kappa}\gimel(\kappa)$$

(c) If κ is a limit cardinal and the continuum function is not eventually constant below κ, then

$$2^\kappa = \gimel(2^{<\kappa})$$

because $\mathrm{cf}(2^\kappa) = \mathrm{cf}(\kappa)$. (Check.)

Thus given $\beth(\kappa)$ we can compute 2^κ.

To close this section on cardinal exponentiation, we note that the following theorems of Silver and Shelah illustrate how different exponentiation of singular cardinals is.

Theorem.

(1) (Silver, 1974) If $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$

(2) (Shelah, $\leqslant$1994) $pp(\aleph_\omega) < \aleph_{\omega_4}$

In simplistic terms, $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4} + (2^{\aleph_0})^+$.

So, for example, if $2^{\aleph_0} < \aleph_\omega$, then $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$.

Note: $pp(\kappa)$ is defined in terms of cofinalities of products of regular cardinals.

We leave exponentiation and turn to the study of prediction principles and families of large subsets of limit ordinals.

Definition. Suppose that γ is a (limit) ordinal. We say that $C \subset \gamma$ is a **club** of γ (or in γ) if

(1) C is closed in γ – i.e., if $\delta \subset C$ has $\sup \delta \in \gamma$ then $\sup \delta \in C$

(2) C is unbounded in γ – i.e., $\forall \alpha < \gamma\ \exists \beta \in C\ \alpha < \beta < \gamma$.

("Club" is from "*cl*osed *unb*ounded".)

Lecture 11 **Remarks.**

(1) If γ is a successor ordinal, then clubs are not very interesting, since e.g. if $\gamma = \delta + 1$, then $\{\delta\}$ is a club.

(2) $C \subset \gamma$ is closed iff C contains all of its limit points: δ is a **limit point** if $(\forall \alpha < \delta)$ $(\exists \beta \in C)$ $(\alpha < \beta < \delta)$ (where γ is a limit ordinal)

Examples.

(1) γ is closed in γ.

(2) If $\mathrm{cf}(\gamma) > \aleph_0$, then $\lim(\gamma) = \{\delta < \gamma : \delta \text{ is a limit ordinal}\}$ is club in γ.

(3) If $C \subset \gamma$ is unbounded in γ, then the set C^* of limit points of C is club in γ.

(4) If $\lambda > \aleph_0$ and $\mathrm{cf}(\lambda) = \aleph_0$, then any cofinal ω-sequence in λ is club in λ.

(5) If λ is a limit cardinal, then $\{\kappa < \lambda : \kappa \text{ is a cardinal}\}$ is club in λ.

(6) If λ is a limit cardinal, then $\{\kappa^+ : \langle\lambda\rangle\}$ is not club in λ (it is not closed)

Remark. The concept of a club of γ is most interesting when γ is an ordinal of uncountable cofinality.

Remark. If C_i is closed in γ, then $\bigcap_{i \in I} C_i$ is also closed in γ.

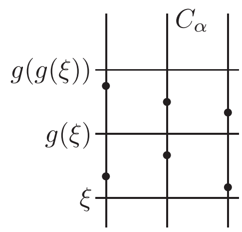
However, arbitrary intersections of clubs will not in general be club. Suppose that $\kappa = \mathrm{cf}(\lambda)$ and $f : \kappa \to \lambda$ is cofinal in λ. Consider $C_\alpha = \{\beta < \lambda : \beta \geqslant f(\alpha)\} = [f(\alpha), \lambda)$. Then C_α is club in λ for all $\alpha < \kappa$, but $\bigcap_{\alpha < \kappa} C_\alpha = \emptyset$.

Proposition 23. Suppose $\mathrm{cf}(\gamma) > \aleph_0$, and $\kappa < \mathrm{cf}(\gamma)$.

Then $\bigcap_{\alpha<\kappa} C_\alpha$ is club in γ whenever C_α is club in γ for all $\alpha < \kappa$.

Proof. Let $C = \bigcap_{\alpha<\kappa} C_\alpha$. For later, wlog, $\alpha < \beta \to C_\alpha \supset C_\beta$.

Clearly, C is closed. We show that C is unbounded.



Let $\xi < \gamma$ be given. Let $f_\alpha(\xi) = \min\{C_\alpha \setminus (\xi + 1)\}$.
(I.e., remove ξ and its predecessors.)

Let $g(\xi) = \sup\{f_\alpha(\xi) : \alpha < \kappa\} < \gamma$ so $g : \gamma \to \gamma$. (Recall $\kappa < \mathrm{cf}(\gamma)$.)

Define $g_0(\xi) = \xi$, and $g_{n+1}(\xi) = g(g_n(\xi))$, and $g_\omega(\xi) = \sup_{n<\omega} g_n(\xi)$.

It follows that $\xi < g_\omega(\xi) < \gamma$, and $g_\omega(\xi) \in C$. □

Corollary 24. Suppose $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ and $\kappa < \lambda$.

If C_α is club in λ for all $\alpha < \kappa$ then $\bigcap_{\alpha<\kappa} C_\alpha$ is club in λ.

Definition. Suppose γ is a limit ordinal. A set $S \subset \gamma$ is **stationary** in γ if $S \cap C \neq \emptyset$ for every club C of γ.

Examples. Let $\kappa = \mathrm{cf}(\kappa) \ < \ \lambda = \mathrm{cf}(\lambda)$, with $\lambda > \aleph_0$.

Let $S^\lambda_\kappa = \{\delta \in \lambda : \mathrm{cf}(\delta) = \kappa\}$. Then S^λ_κ is stationary in λ.

Proof. Recall f is continuous if $f(\delta) = \sup\{f(\alpha) : \alpha < \delta\}$.

Given a club C, define $f : \lambda \to C$, strictly increasing and continuous. Note $f(\kappa) \in C$ and $f(\kappa) \in S^\lambda_\kappa$, so $S^\lambda_\kappa \cap C \neq \emptyset$. □

Example. If $\lambda = \aleph_1$ and $\kappa = \aleph_0$, then $S^{\aleph_1}_{\aleph_0}$ is actually a club in $\aleph_1$, since $S^{\aleph_1}_{\aleph_0} = \lim(\aleph_1)$.

However, in general, this is false. S^λ_κ does not contain a club of λ since any club must contain elements of *every* cofinality $< \lambda$.

Proposition 25.

(1) If C is club and S is stationary in λ then $C \cap S$ is stationary in λ.
(2) If $\kappa < \mathrm{cf}(\lambda)$ and $\bigcup\{X_\alpha : \alpha < \kappa\}$ is stationary in λ, then some X_α is stationary in λ.

Proof.

1. By Proposition 23.
2. If $C_\alpha \cap X_\alpha = \emptyset$ for all $\alpha < \kappa$, then $C = \bigcap_{\alpha<\kappa} C_\alpha$ is club by Proposition 23, but $C \cap \bigcup\{X_\alpha : \alpha < \kappa\} = \emptyset$, contradiction. □

Definition. Suppose $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ and $X_\alpha \subset \lambda$ for $\alpha < \lambda$. We define the **diagonal intersection** to be the set

$$\mathop{\Delta}_{\alpha<\lambda} X_\alpha = \{\xi < \lambda : (\forall \alpha < \xi)(\xi \in X_\alpha)\}$$

Proposition 26. Suppose that $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ and each C_α is club in λ. Then $C = \mathop{\Delta}_{\alpha<\lambda} C_\alpha$ is club in λ.

Lecture 12

Proof. Clearly C is closed. To see that C is unbounded in λ, define $g(\xi)$ for $\xi < \lambda$ as follows:

$$g(\xi) = \min\Bigg(\underbrace{\bigcap_{\alpha<\xi} C_\alpha}_{\text{club by Prop 23}} \setminus (\xi+1)\Bigg)$$

By Proposition 23, g is well-defined, and $\xi < g(\xi) < \lambda$.

Let $g_0(\xi) = \xi$, and $g_{n+1} = g(g_n(\xi))$, and $g_\omega(\xi) = \sup\limits_{n<\omega} g_n(\xi)$.

Claim. $g_\omega(\xi) \in C$.

Why? Suppose $\alpha < g_\omega(\xi)$. Then $\exists n\ (\forall m \geqslant n)\ (\alpha < g_n(\xi) \in C_\alpha)$, and we know that C_α is closed, so $g_\omega(\xi) \in C_\alpha$.

Thus C is club in λ. □

Theorem 27 (Pressing Down Lemma / Fodor's Lemma). Suppose $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ and $S \subset \lambda$ is stationary in λ. If $f : S \to \lambda$ is **regressive**, i.e. $\forall \delta \in S\ f(\delta) < \delta$, then there is some stationary $S' \subset S$ such that $f|_{S'}$ is constant. I.e., $\exists \alpha < \lambda$ such that $f^{-1}(\{\alpha\})$ is stationary.

Proof. Otherwise, there exists a club C_α such that $C_\alpha \cap f^{-1}(\{\alpha\}) = \emptyset$. Then, by Proposition 26, $C = \mathop{\Delta}_{\alpha<\lambda} C_\alpha$ is club.

In particular, it follows that $C \cap S \neq \emptyset$. Pick $\xi \in C \cap S$.

If $\alpha < \xi$, then $\xi \in C_\alpha$. So $f(\xi) \neq \alpha$. So $f(\xi) \geqslant \xi$ – (1).

But $\xi \in S$, and so $f(\xi) < \xi$ – (2), contradicting (1). □

Theorem 28 (Ulam). Suppose $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. Then there exists a family of λ *disjoint* stationary subsets of λ.

Proof. There are two cases.

Case 1. λ is a limit cardinal.

Consider $\{S^\lambda_\kappa : \kappa = \mathrm{cf}(\kappa) < \lambda\}$. Recall: $S^\lambda_\kappa = \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$

This is as required.

Case 2. $\lambda = \kappa^+$.

Let $\{A_{\alpha\xi} : \alpha < \lambda,\ \xi < \kappa\}$ be an Ulam (λ,κ)-matrix on λ, i.e.

(1) $A_{\alpha\xi} \cap A_{\beta\xi} = \emptyset$ if $\alpha < \beta < \lambda$

(2) $\left| \lambda \setminus \bigcup_{\xi<\kappa} A_{\alpha\xi} \right| \leqslant \kappa$.

We can find $A_{\alpha\xi}$ with these properties as in Exercise Sheet 1:

$\forall \alpha \in [\kappa, \kappa^+)$, let $f_\alpha : \kappa \to \alpha$ be a surjection, and $A_{\alpha\xi} = \{\beta < \lambda : f_\beta(\xi) = \alpha\}$.

Note, by (2), $\bigcup_{\xi<\kappa} A_{\alpha\xi}$ is stationary (by Proposition 25).

By Proposition 25 again, for some $\xi_\alpha < \kappa$, $A_{\alpha\xi_\alpha}$ is stationary in λ.

So there exists ξ^* such that $E = \{\alpha : \xi_\alpha = \xi^*\}$ is stationary.

Thus $\{A_{\alpha\xi^*} : \alpha \in E\}$ is the required family. □

Corollary 29. Let $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. There are 2^λ stationary subsets of λ.

Proof. For the family $\{B_\alpha : \alpha < \lambda\}$ of stationary subsets in Theorem 28 and non-empty $X \subset \lambda$, take $F_\lambda = \bigcup_{\alpha \in X} B_\alpha$. □

Prediction Principles

Intuitively, a set-theoretic prediction principle is a list of guesses (or approximations) of subsets of a cardinal. In this sense, CH is a prediction principle:

$$\mathrm{CH} \longleftrightarrow \exists \langle X_\alpha : \alpha < \omega_1 \rangle \; \forall Y \subset \omega \; \exists \alpha < \omega_1 \; Y = X_\alpha$$

AC_λ is also a prediction principle:

$$\mathrm{AC}_\lambda \longrightarrow \exists \langle X_\alpha : \alpha < 2^\lambda \rangle \; \forall Y \subset \lambda \; \exists \alpha < 2^\lambda \; Y = X_\alpha$$

The syntactic form of these statements leads one to consider stronger statements.

Definition. Let S be a stationary subset of $\lambda = \mathrm{cf}(\lambda) > \aleph_0$.

The **diamond** on S, denoted $\diamondsuit_S$, is the statement: $\exists \langle A_\alpha : \alpha \in S \rangle$ such that

(1) $A_\alpha \subset \alpha$

(2) for all $X \subset \lambda$, the set $\{\alpha \in S : X \cap \alpha = A_\alpha\}$ is stationary in λ

If $S = \lambda$, we write $\diamondsuit_\lambda$ for $\diamondsuit_S$, and we write $\diamondsuit$ for $\diamondsuit_{\aleph_1}$.

Lecture 13 **Remarks.**

(1) $S \subset S'$ implies $\diamondsuit_S \to \diamondsuit_{S'}$. (Recall: for *stationary* sets.)

(2) $\diamondsuit_{\lambda^+} \to 2^\lambda = \lambda^+$, so $\diamondsuit \to$ CH.

(3) $\diamondsuit_S$ is equivalent to: $\exists \langle f_a : \alpha \in S \rangle$ such that

  (i) $f_\alpha : \alpha \to \alpha$

  (ii) for all $f : \lambda \to \lambda$, the set $\{\alpha \in S : f|_\alpha = f_\alpha\}$ is stationary.

Proof.

(1) is trivial.

(2) We show $\mathbb{P}(\lambda) \subset \{A_\alpha : \alpha \in \lambda^+\}$.
If $X \subset \lambda$, then there is $\beta < \lambda^+$ with $X \subset \beta$. The set $E = \{\alpha < \lambda^+ : X \cap \alpha = A_\alpha\}$ is stationary. Pick $\alpha \in E$ with $\alpha > \beta$. Now $X = X \cap \alpha = A_\alpha$.

(3) will be on Exercise Sheet 3.

Remark. $V = L$ implies $\diamondsuit_S$, but we will show the relative consistency of $\diamondsuit$ by forcing.

One of the most striking and important applications of $\diamondsuit$ is to prove the existence of a Suslin tree.

Theorem 30. $\diamondsuit$ implies that there exists a Suslin tree.

It will be convenient first to recall some definitions relating to trees.

Definitions.

(1) A **tree** is a partial order $\mathbb{T} = \langle T, \leqslant_T \rangle$ such that for all $x \in T$, the set $\widehat{x} = \{y \in T : y <_T x\}$ is well-ordered.

(2) The order-type of $\widehat{x}$ under $<_T$ is called the **height** of x in $\mathbb{T}$, denoted $\mathrm{ht}_{\mathbb{T}}(x)$.

(3) If $\alpha \in \mathbf{Ord}$, the α^{th} **level** of $\mathbb{T}$ is the set $T_\alpha = \mathrm{Lev}_\alpha(\mathbb{T}) = \{x \in T : \mathrm{ht}_{\mathbb{T}}(x) = \alpha\}$.

We write $\mathbb{T}|_\alpha$ for the partial order $\mathbb{T}$ restricted to the set $T|_\alpha$, where $T|_\alpha = \bigcup_{\beta<\alpha} T|_\beta$.

(4) A **branch** of $\mathbb{T}$ is a linearly ordered subset b such that $x \in b \wedge y <_T x \rightarrow y \in b$.

We say that b is an α**-branch** if b has order-type α.

(5) A branch b is **maximal** if b is not *properly* contained in any other branch of $\mathbb{T}$. AC implies that every branch can be extended to a maximal branch.

Remark. $\widehat{x}$ is a branch. If x has no successors in $\mathbb{T}$, then $\widehat{x} \cup \{x\}$ is maximal.

(6) An **antichain** in $\mathbb{T}$ is a subset $A \subset T$ such that no two elements of A are comparable in $<_T$. An antichain A is maximal if it is not properly contained in any other antichain of $\mathbb{T}$.

AC implies that every antichain can be extended to a maximal antichain.

Remark. If $T_\alpha \neq \emptyset$, then T_α is a maximal antichain.

(7) Let δ be an ordinal and λ a cardinal. A tree $\mathbb{T}$ is a (δ, λ)**-tree** if

(i) $\forall \alpha < \delta$, $T_\alpha \neq \emptyset$
(ii) $T_\delta = \emptyset$
(iii) $\forall \alpha < \delta$, $|T_\alpha| < \lambda$

So a (δ, λ)-tree has "height" δ and "width" less than λ.

(8) $\mathbb{T}$ has **unique limits** if whenever δ is a limit ordinal, if we have $x, y \in T_\delta$ with $\widehat{x} = \widehat{y}$, then $x = y$.

(9) A (δ, λ)-tree $\mathbb{T}$ is **normal** if

   (i) $\mathbb{T}$ has unique limits

   (ii) $|T_0| = 1$

   (iii) If $\alpha, \alpha+1 < \delta$, $x \in T_\alpha$, then there are distinct $y_1, y_2 \in T_{\alpha+1}$ such that $x <_T y_1$ and $x <_T y_2$

   (iv) If $\alpha < \beta < \delta$ and $x \in T_\alpha$, then there is $y \in T_\beta$ such that $x <_T y$

(10) Let λ be an infinite cardinal. A **λ-tree** is a normal (λ, λ)-tree.

(11) A **Suslin tree** is an $\aleph_1$-tree with *no* uncountable antichains.

Lecture 14 We now prove Theorem 30, that $\diamond \to \neg$(Suslin's Hypothesis).

Proof. Assume $\diamond$ and let $\langle A_\alpha : \alpha < \omega_1 \rangle$ be a $\diamond$-sequence. By transfinite recursion, we construct a Suslin tree $\mathbb{T} = \langle T, <_T \rangle$, with $T = \omega_1$, such that

(1) $\mathbb{T} = \bigcup_{\alpha < \omega_1} \mathbb{T}|_\alpha$

(2) $\mathbb{T}|_\alpha$ is a (normal) $(\alpha, \aleph_1)$-tree

The elements of T_ω are the finite ordinals, and for infinite α, the elements of T_α will be ordinals from the set $\{\xi : \omega\alpha \leqslant \xi < \omega\alpha + \omega\}$.

How do we go?

(0) $T_0 = \{0\}$.

(1) If $n \in \omega$ and $\mathbb{T}|_{n+1}$ is defined, define $\mathbb{T}|_{n+2}$ by taking the elements of T_n in turn and, for each element, picking the next two unused finite ordinals to be its successors in T_{n+1}.

(2) If $\alpha \geqslant \omega$ and $\mathbb{T}|_{\alpha+1}$ is defined, define $\mathbb{T}|_{\alpha+2}$ by using the ordinals in $\{\xi : \omega\alpha \leqslant \xi < \omega\alpha + \omega\}$ to provide each element in T_α with two successors in $T_{\alpha+1}$.

   This is possible, since $|T_\alpha| \leqslant \aleph_0$.

(3) If $\alpha \geqslant \omega$ and α is a limit ordinal, and $\mathbb{T}|_\alpha$ is defined, for each $x \in T|_\alpha$, pick an α-branch b_x with $x \in b_x$, subject to the condition: if A_α is a maximal antichain in $\mathbb{T}|_\alpha$, then $b_x \cap A_\alpha \neq \emptyset$. (Remember that A_α is from the $\diamond$-sequence – one of the 'guesses'.)

   Why is this possible?

   There exists $a \in A_\alpha$ such that x and a are comparable: $x \leqslant_T a$ or $a <_T x$. If $x \leqslant_T a$ then pick b_x to be some α-branch extending $\widehat{a} \cup \{a\}$, and if $a <_T x$ then pick b_x to be an α-branch extending $\widehat{x} \cup \{x\}$.

   If A_α is not a maximal antichain in $\mathbb{T}|_\alpha$, pick b_x to be any α-branch containing x.

   For each $\alpha \in T|_\alpha$, add a 1-point extension from $\{\xi : \omega\alpha \leqslant \xi < \omega\alpha + \omega\}$ to the α-branch b_x. This is possible since $\left|T|_\alpha\right| \leqslant \aleph_0$.

Let $\mathbb{T} = \bigcup_{\alpha < \omega_1} \mathbb{T}|_\alpha$. Then $\mathbb{T}$ is an $\aleph_1$-tree. This is clear.

It remains to show that $\mathbb{T}$ is Suslin, i.e. $\mathbb{T}$ has no uncountable antichains. It's enough to show that every maximal antichain of $\mathbb{T}$ is countable.

Suppose that $X \subset \omega_1$, and X is a maximal antichain in $\mathbb{T}$. Consider

$$C = \{\alpha \in \omega_1 : \omega\alpha = \alpha \text{ and } X \cap \alpha \text{ is a maximal antichain in } \mathbb{T}|_\alpha\}$$

Claim. C is a club.

Proof of claim. It's easy to see that $C_1 = \{\alpha < \omega_1 : \omega\alpha = \alpha\}$ is a club, and check that $C_2 = \{\alpha \in \lim(\omega_1) : X \cap \alpha \text{ is a maximal antichain in } \mathbb{T}|_\alpha\}$ is a club.

Why is C_2 club? Easily C_2 is closed: $\alpha_n \in C_2 \to \alpha_n \to \alpha$.

C_2 is unbounded. Let $\beta_0 < \omega_1$. We find $\beta \in C_2$ with $\beta > \beta_0$. Given β_n, define β_{n+1} to be the least limit ordinal $\gamma > \beta_n$ such that every element of $\mathbb{T}|_{\beta_n}$ is comparable to some element $X \cap \gamma$. Let $\beta = \sup\limits_{n<\omega} \beta_n$. So $\beta \in C_2$, and so $C = C_1 \cap C_2$ is a club.

So now by $\diamondsuit$, the set $S = \{\alpha < \omega_1 : X \cap \alpha = A_\alpha\}$ is stationary in ω_1, and so $C \cap S \neq \emptyset$.

Pick $\alpha \in C \cap S$. Then $X \cap \alpha = A_\alpha$, and so A_α is a maximal antichain in $\mathbb{T}|_\alpha$.

By construction of T_α (and every T_β, $\beta > \alpha$), every element of T_α lies above some element of $A_\alpha = X \cap \alpha$ (i.e., is comparable with some element in $X \cap \alpha$).

So $X \cap \alpha$ is a maximal antichain in $\mathbb{T}$.

So $X \cap \alpha = X$, and so $|X| = |X \cap \alpha| \leqslant |\alpha| \leqslant \aleph_0$, since $\alpha < \omega_1$.

Thus X is countable. $\square$

It is natural to ask (and István Juhász did) whether this is optimal. Could one replace $\diamondsuit$ by a weaker principle? In this connection, we mention the **club principle**, $\clubsuit$.

Definition. $\clubsuit$ asserts that there exists $\langle C_\delta : \delta \in \lim(\omega_1)\rangle$ such that

(1) $\sup C_\delta = \delta$
(2) for every uncountable $X \subset \omega_1$, the set $\{\delta \in \lim(\omega_1) : C_\delta \subset X\}$ is stationary in $\aleph_1$.

Remarks.

(1) $\diamondsuit \to \clubsuit$
(2) Juhász's Question: does $\clubsuit$ imply the existence of a Suslin tree?

Lecture 15 **Remark.** (Regarding $\diamondsuit \to \neq$SH.)

Some model theory allows one a less ad hoc proof that the set C in the proof is a club. Recall that X is a maximal antichain in $\mathbb{T}$. Consider the τ-structure $\mathbb{T}_X = \langle T, \leqslant_T, X\rangle$.

The set $\{\alpha \in \lim(\omega_1) : \mathbb{T}_X|_\alpha \preceq \mathbb{T}_X\}$ is club by question 5(iv) on Exercise Sheet 3. This is the set C in the proof.

We shall return to $\diamondsuit_\lambda$ later. We now introduce a new axiom MA (Martin's Axiom) and use it to show that there are no Suslin trees.

Definition. Suppose $\mathbb{P} = \langle P, \leqslant_P \rangle$ is a partial order. Wlog, $\mathbb{P}$ has a least element O_P.

(1) Two elements $p, q \in P$ are **compatible** if $(\exists r \in P)(p \leqslant r \wedge q \leqslant r)$. Otherwise p, q are **incompatible**.

(2) A subset $G \subset P$ is **directed** if $(\forall p \in G)(\forall q \in G)(\exists r \in G)(p \leqslant r \wedge q \leqslant r)$.

(3) A subset $D \subset \mathbb{P}$ is **dense** in $\mathbb{P}$ if $(\forall p \in P)(\exists d \in D)(p \leqslant d)$.

(4) Let κ be an infinite cardinal. We say that $\mathbb{P}$ satisfies the **κ-chain condition** if every antichain in $\mathbb{P}$ has cardinality less than κ.

(5) If $\kappa = \aleph_1$, then the $\aleph_1$-chain condition is called the **countable chain condition** (CCC).

Caveat. Some sources reverse the order in the above (and subsequent) definitions. E.g., they say p, q are compatible if $(\exists r \in P)(r \leqslant p \wedge r \leqslant q)$.

Definition (cont.)

(6) Suppose $\mathcal{D}$ is a family of dense sets in $\mathbb{P}$, and $G \subset P$ is directed. We say that G is **$\mathcal{D}$-generic** (or **generic relative to** $\mathcal{D}$) if $\bigwedge_{D \in \mathcal{D}} G \cap D \neq \emptyset$

Proposition 31. Suppose $\mathcal{D} = \{D_n : n < \omega\}$ is a countable family of dense sets in the partial order $\mathbb{P}$. Then there exists a $\mathcal{D}$-generic set $G \subset P$.

Proof. By induction, we define $\langle d_n : n < \omega \rangle$.

$D_0 \neq \emptyset$, so pick $d_0 \in D_0$. Given $d_n \in D_n$, since D_{n+1} is dense in $\mathbb{P}$, there exists $d_{n+1} \in D_{n+1}$ with $d_n \leqslant d_{n+1}$.

Now $G = \{d_n : n < \omega\}$ is a $\mathcal{D}$-generic set, as required. □

In general, we cannot "improve" Proposition 31 to uncountable families of dense sets. Uncountable antichains can wreak havoc. However, if we require that $\mathbb{P}$ has the CCC then there is a relatively consistent statement.

Definition. Let κ be an infinite cardinal. The statement MA_κ asserts: for every CCC partial order $\mathbb{P}$ and every family $\mathcal{D} = \{D_\alpha : a < \kappa\}$ of dense subsets in $\mathbb{P}$, there exists a $\mathcal{D}$-generic set $G \subset P$ in $\mathbb{P}$.

Martin's Axiom is the statement: $\displaystyle\bigwedge_{\aleph_0 \leqslant \kappa < 2^{\aleph_0}} \mathrm{MA}_\kappa$.

Remark. CH $\to$ MA.

So for this reason, we generally tacitly assume $\neg$CH when we apply MA.

Proposition 32.

(1) $\mathrm{ZFC} \vdash \mathrm{MA}_{\aleph_0}$

(2) $\mathrm{MA}_\kappa \to 2^{\aleph_0} > \kappa$

(3) $\mathrm{ZFC} \vdash \neg\mathrm{MA}_{2^{\aleph_0}}$.

Proof.

(1) is immediate from Proposition 31.

(2) We must find a poset $\mathbb{P}$ and some dense sets.

Let $\langle f_\alpha \in {}^\omega 2 : \alpha < \kappa \rangle$. We aim to find $g \in {}^\omega 2$ with $\bigwedge_{\alpha<\kappa} g \neq f_\alpha$.

We pick $\mathbb{P}$ to approximate g. Let P be the set of finite functions from ω into $\{0,1\}$:

$$P = \left\{ f : \mathrm{dom}(f) \subset \omega,\ f : \mathrm{dom}(f) \to \{0,1\},\ |\mathrm{dom}(f)| < \aleph_0 \right\}$$

and define $f \leqslant_P h$ iff $h|_{\mathrm{dom}(f)} = f$, i.e. h extends f.

$\mathbb{P} = (P, \leqslant_P)$ is CCC since $|P| = \aleph_0$.

Let $E_n = \{p \in P : n \in \mathrm{dom}(p)\}$. Easily, E_n is dense in $\mathbb{P}$.

For $\alpha < \kappa$, let

$$D_\alpha = \left\{ p \in P : \exists n \in \mathrm{dom}(p),\ f_\alpha(n) \neq p(n) \right\} .$$

Then D_α is dense in $\mathbb{P}$.

Now let $\mathcal{D} = \{D_\alpha, E_n : \alpha < \kappa, n < \omega\}$. By MA_κ, there exists a $\mathcal{D}$-generic set G.

Let $g = \bigcup G$. Note: g is a function since G is directed.

We have $\mathrm{dom}(g) = \omega$ since $G \cap E_n \neq \emptyset$, and $\mathrm{range}(g) \subset \{0,1\}$.

And, for all $\alpha < \kappa$, we have $g \neq f_\alpha$ since $G \cap D_\alpha \neq \emptyset$.

Thus $g \in {}^\omega 2$ and $g \neq f_\alpha$ for all $\alpha < \kappa$. So $2^{\aleph_0} > \kappa$.

(3) is immediate from (2).

Lecture 16 The axioms $\mathrm{MA}_{\aleph_1}$ and MA say that the universe contains generic sets (for families of dense sets in CCC posets, provided the families are of size $< 2^{\aleph_0}$). We now prove:

Theorem 33. $\mathrm{MA}_{\aleph_1}$ implies that there are no Suslin trees.

Lemma. If $\mathbb{T}$ is a Suslin tree, then every chain in $\mathbb{T}$ is countable.

Proof. By contradiction. Suppose C is an uncountable chain in $\mathbb{T}$. Wlog, C is maximal. So $C \cap T_\alpha \neq \emptyset$ for all $\alpha < \omega_1$. So there exists $c_\alpha \in C \cap T_\alpha$, and by normality (iii) of a Suslin tree, there exists $a_{\alpha+1} \in T_{\alpha+1} \setminus C$ such that $\{a_{\alpha+1}, c_{\alpha+1}\}$ is an antichain in $T_{\alpha+1}$.

Consider $A = \{a_{\alpha+1} : \alpha < \omega_1\}$. Clearly, A is an antichain, and $|A| = \aleph_1$, contradiction. □

"Why can one not simply define inductively an uncountable chain in a Suslin tree by using normality to grab a chain member at each next level?"

This will not work, since at limit levels δ, there is no uniform single choice for an element c_δ to continue the chain at the limit level above all the lower elements c_α, $\alpha < \delta$. By normality, one can certainly ascend one-by-one, but not above a boatload.

"What is the cofinality $\mathrm{cf}(\mathbb{T})$ of a Suslin tree $\mathbb{T}$?"

If C is a cover of a Suslin tree $\mathbb{T}$, then $T = \bigcup\{\widehat{c} : c \in C\} \cup C$; so if C were countable, then some $\widehat{c}$ would have cardinality $\aleph_1$ and be an uncountable chain in $\mathbb{T}$. This is impossible since Suslin trees have no uncountable chains by the Lemma. Hence there is no countable cover of $\mathbb{T}$. Since $\mathbb{T}$ has cardinality $\aleph_1$, any cover has cardinality at least, and hence exactly, $\aleph_1$. It follows that $\mathrm{cf}(\mathbb{T}) = \aleph_1$.

Back to Theorem 33.

Proof. Towards a contradiction, suppose that $\mathbb{T}$ is a Suslin tree. Then $\mathbb{T}$ is a CCC partial order.

Let $D_\alpha = \{y \in T : \mathrm{ht}_{\mathbb{T}}(y) > \alpha\}$ for $\alpha < \omega_1$. Clearly, $D_\alpha \neq \emptyset$ and D_α is dense in $\mathbb{T}$.

By $\mathrm{MA}_{\aleph_1}$, there is a $\mathcal{D}$-generic set G for $\mathcal{D} = \{D_\alpha : \alpha < \omega_1\}$. Clearly, G is a chain in $\mathbb{T}$ since G is directed, and $|G| = \aleph_1$, contradicting the lemma. □

$\diamond$ and MA have many applications across ordinary mathematics. Here is a more recent short application of MA to abelian group theory.

Example (sketch). Let A be an infinite abelian group. We write $A^* = \mathrm{Hom}(A, \mathbb{Z})$ for the dual of A. We say that A is **free** if $A \cong \oplus_{i\in I}\mathbb{Z}$ for some set I.

Let κ be an infinite cardinal. We say that A is κ-free if every subgroup of A of cardinality $< \kappa$ is free.

For example, $\mathbb{Z}^\omega$ is $\aleph_1$-free.

The **Trivial Dual Conjecture for** κ states: there exists a κ-free group A with $A^* = 0$.

This is denoted TDU_κ.

Remarks.

(1) $\mathrm{TDU}_{\aleph_0}$ is true: e.g., $\mathbb{Q} = (\mathbb{Q}, +)$.

(2) What about $\mathrm{TDU}_{\aleph_1}$?
$\mathrm{TDU}_{\aleph_1}$ is true, but it's a decent theorem due to Eda (1989).

(3) What about $\mathrm{TDU}_{\aleph_2}$?

Theorem. MA implies that if A is $\aleph_2$-free and $|A| < 2^{\aleph_0}$, then A is separable.

Definition. An abelian group A is **separable** if every finite subset of A is contained in a free direct summand of A.

Equivalently, any pure cyclic subgroup of A is a free direct summand of A.

$B \leqslant A$ is **pure** in A if: $\forall b \in B$, $\forall n \in \omega$, if $A \models \exists x\,(nx = b)$, then $B \models \exists x\,(nx = b)$

Proof. Let $h \in A$. Suppose that $\langle h\rangle$ is pure cyclic. It is enough to show that $\langle h\rangle$ is a free abelian summand. Let $\eta(h) = 1$. We use MA to extend η to a homomorphism $\Phi \in \mathrm{Hom}(A, \mathbb{Z})$.

Let $\mathbb{P} = (P, \leqslant_P)$ be the set

$$\{\varphi : \varphi \in \mathrm{Hom}(A_\varphi, \mathbb{Z}), \varphi(h) = 1, A_\varphi \text{ is a pure finitely-generated subgroup of } A\}$$

Then $\varphi \leqslant \psi$ if ψ extends φ.

Some facts which require proof:

- $\mathbb{P}$ is a CCC partial order
- For $a \in A$, let $D_a = \{\varphi \in P : a \in A_\varphi\}$

Then D_α is dense in $\mathbb{P}$.

By MA, there exists a $\mathcal{D}$-generic set G since $|\mathcal{D}| < 2^{\aleph_0} = |A|$.

Let $\Phi = \bigcup G$. Then $\Phi \in \mathrm{Hom}(A, \mathbb{Z})$ and by the First Isomorphism Theorem, $A/\ker\Phi \cong \mathbb{Z}$. So $\ker\Phi$ is a direct summand of A. This completes the proof (modulo some algebra). □

Open question. What is the status of $\mathrm{TDU}_{\aleph_\omega}$?

Lecture 17

Chapter 3. Forcing

In this chapter, we present **forcing**, a method for constructing extensions of models of (subtheories of) ZFC. We then use the technique to prove the independence of the Continuum Hypothesis, $\diamondsuit$, $\clubsuit$, and some other combinatorial principles, but also $V = L$.

There are several approaches to the presentation of forcing. We shall follow the presentation of Shelah in *Proper and Improper Forcing*, chapter 1.

We start from the assumption that ZFC is consistent, and has a countable model $\mathbb{M}$. Wlog, we shall also assume:

(1) $\mathbb{M}$ is a standard $\in$-model: if $\mathbb{M} = (M, E^{\mathbb{M}})$ then $E^{\mathbb{M}} = \in|_{M\times M}$. In other words, the membership predicate of $\mathbb{M}$ is *real* membership.

(2) The universe of $\mathbb{M}$ is a transitive set: $x \in y \in M \to x \in M$.

These additional assumptions come "for free" and they simplify the presentation. (We can justify (1) and (2) either using the fact that L, the universe of constructible sets, is a standard $\in$-model and use the Downward Löwenheim-Skolem theorem to obtain a countable elementary submodel; alternatively, use the Reflection Principle to cut down to a V_α for an arbitrary large finite fragment of ZFC, and take a countable elementary submodel of V_α; finally, use the Mostowski Collapse to obtain an isomorphic *transitive* model.)

So throughout this presentation, we shall assume that $\mathbb{M}$ is a countable transitive model satisfying (1) and (2). We call $\mathbb{M}$ a CTM.

We wish to extend $\mathbb{M}$.

Definition. A **forcing** is a partial order $\mathbb{P} = (P, \leqslant_P)$. We shall, wlog, assume that forcings have minimal elements. It is also not necessary to assume that $\mathbb{P}$ is antisymmetric. In some sources, this is called a **pre-partial order** or **quasi-order**.

The elements of a forcing are called (**forcing**) **conditions**.

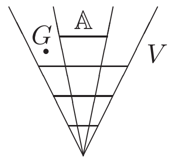
The extensions we construct are defined using forcings and generic filters in V. These forcings and generic sets control the truth values of sentences in the extensions.

Definition. Suppose $\mathbb{A}$ is a model of ZFC (or a fragment of ZFC), and $\mathbb{P} \in \mathbb{A}$ (so P and $\leqslant_P$ belong to $\mathbb{A}$). We say that $G \subset P$ is a **filter** in $\mathbb{P}$ if

(1) G is directed: $(\forall x, y \in G)\,(\exists z \in G)\,(x \leqslant z \wedge y \leqslant z)$

(2) G is **downward-closed**: $x \leqslant y \in G \rightarrow x \in G$.

Note. The definition does *not* require $G \in V$ to belong to $\mathbb{A}$.



Definition. We say a subset $D \subset P$ is **dense open** in $\mathbb{P}$ if

(1) D is dense in $\mathbb{P}$: $p \in P \rightarrow (\exists d \in D)(p \leqslant d)$

(2) D is open (upward-closed): $(\forall p, q \in P)(q \leqslant p \wedge q \in D \rightarrow p \in D)$

We say that a filter G in $\mathbb{P}$ is **generic** (**over** $\mathbb{A}$) if $G \cap D \neq \emptyset$ for every dense open set $D \subset P$ in $\mathbb{A}$.

Our aim is to build an extension $\mathbb{M}[G]$ for a given CTM $\mathbb{M}$ and a generic filter G over $\mathbb{M}$ with the properties:

(1) $\mathbb{M} \subset \mathbb{M}[G]$, i.e. $\mathbb{M}$ is a substructure of $\mathbb{M}[G]$

(2) $\mathbb{M} \cap \mathbf{Ord} = \mathbb{M}[G] \cap \mathbf{Ord}$

(3) $G \in \mathbb{M}[G]$

(4) $\mathbb{M}[G]$ is the smallest CTM such that (1), (2), (3) hold.

How will it work?

In intuitive terms, if one thinks of the conditions $p \in P$ as potential information about G, then $\{p : p \in G\}$ should be contradiction-free, and this amounts to saying that G should be directed.

If $q \leqslant p \in G$, this amounts to saying that stronger information (p) implies weaker information (q) about G, so G should be downward-closed.

How about genericity?

The genericity condition on G amounts to saying that G is generic in the sense of "non-specific" or random, i.e. G contains consistent general information.

Are there any generic sets over a CTM? And where?

Proposition 1. Suppose $\mathbb{M}$ is a CTM, and $\mathbb{P} \in \mathbb{M}$ is a forcing. Then there exists a generic filter G ($\in V$) on $\mathbb{P}$ over $\mathbb{M}$. In fact, for any $p \in P$, there is a generic G over $\mathbb{M}$ such that $p \in G$.

Lecture 18 **Proof.** (See Proposition 31 in Chapter 2.)

In V, $\mathbb{M}$ is countable, and so there is a list $\{D_n : n \in \mathbb{N}\}$ of all the dense open sets in $\mathbb{P}$

that belong to $\mathbb{M}$. Now, inductively, pick $p_{n+1} \geqslant p_n$ with $p_{n+1} \in D_{n+1}$ (using denseness of D_{n+1}) and $p_1 = p$. Let $G = \{q \in P : \exists n \in \omega : q \leqslant p_n\}$.

G is a generic filter over $\mathbb{M}$, as required. $\square$

Note. This argument happens in V, not in $\mathbb{M}$.

In general, $G \notin M$.

Proposition 2. Suppose $\mathbb{P}$ is **separative**, i.e. $\forall p \in P$, $\exists q, r \in P$, with $p \leqslant q$, $p \leqslant r$ and $q \perp r$. (Every condition has two incompatible extensions.)

Then $G \notin M$, for any generic filter G in $\mathbb{P}$ over $\mathbb{M}$.

Proof. By contradiction. Suppose $G \in \mathbb{M}$. Then the set $D = P \setminus G$ belongs to $\mathbb{M}$. But $D \neq \emptyset$ and D is dense open, so $G \cap D \neq \emptyset$. Contradiction. $\square$

Now let us try to motivate the construction of the forcing extension.

First, note that we *do* need to consider a *new* sort of model-theoretic extension. Why? Two general reasons:

(1) ZFC and its subtheories are not model-complete.

A theory is **model-complete** if whenever $\mathbb{A} \subset \mathbb{B}$ (substructure) then $\mathbb{A} \preceq \mathbb{B}$ (elementary substructure), i.e. its submodels are elementary submodels. For example, $V_\omega \subset V_{\omega_1}$, but $V_\omega \npreceq V_{\omega_1}$.

Recall. A substructure $\mathbb{A} \subset \mathbb{B}$ is an **elementary substructure** of $\mathbb{B}$ if for every formula $\varphi(x_1, \ldots, x_n)$ and $a_1, \ldots, a_n \in A$, we have $\mathbb{A} \models \varphi[a_1, \ldots, a_n] \Rightarrow \mathbb{B} \models \varphi[a_1, \ldots, a_n]$.

(2) The CTM $\mathbb{M}$ might (for all we know) be a model also of $V = L$. Then if a submodel $\mathbb{M}^-$ of $\mathbb{M}$ contains all of the ordinals of $\mathbb{M}$ then $\mathbb{M}^- = \mathbb{M}$, because everything in $\mathbb{M}^-$ is constructible.

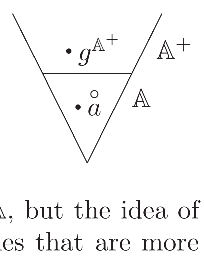
The other way to extend $\mathbb{M}$ is to find an elementary extension $\mathbb{M}^+$. But this does not help to prove independence, because $Th(\mathbb{M}) = Th(\mathbb{M}^+)$ if $\mathbb{M} \preceq \mathbb{M}^+$, so e.g. if $\mathbb{M} \models$ CH then $\mathbb{M}^+ \models$ CH.

Recall. If $\mathbb{A}$ is a τ-structure, the **diagram** of $\mathbb{A}$ is the collection of atomic sentences and negations of atomic sentences in the language τ_A (τ with constants $\mathring{a}$ for each $a \in A$) which are true in $\mathbb{A}$.

The **elementary diagram** of $\mathbb{A}$ is $\{\varphi : \mathbb{A} \models \varphi, \varphi$ is a sentence in the vocabulary of $\tau_A\}$.

In model-theoretic terms, we are looking for a concept intermediate between the diagram of $\mathbb{M}$ and the elementary diagram of $\mathbb{M}$. Where should we go?

One way to construct an extension of a model $\mathbb{A}$ is to add new constants $\mathring{a}$ (for $a \in A$) and a witness constant g, and observe that the elementary diagram with the sentences $\mathring{a} \neq g$ for each $a \in A$ is finitely satisfiable, hence by the Compactness Theorem has a model $\mathbb{A}^+$, and $g^{\mathbb{A}^+} \notin A$.



Of course, this is too much, since $\mathbb{A}^+$ is already an elementary extension of $\mathbb{A}$, but the idea of using names or new constants is a useful strategy. We therefore design names that are more complicated sets in $\mathbb{M}$, defined by transfinite recursion for elements of $\mathbb{M}$ and $\mathbb{P}$, and whose interpretations will depend upon the generic set G. In general, the referents of these names *cannot* be computed or identified in $\mathbb{M}$, since $G \notin \mathbb{M}$ (usually).

Definition. Suppose $\mathbb{P} \in \mathbb{M}$ is a forcing. We define by induction on $\alpha \in \mathbf{Ord}$ the **$\mathbb{P}$-names** (or **names**) **of rank** $\leqslant \alpha$ as follows.

A set τ is a $\mathbb{P}$-name of rank $\leqslant \alpha$ if $\tau = \{(p_i, \tau_i) : i < i_0\}$, where $p_i \in P$, τ_i is a $\mathbb{P}$-name of rank $\leqslant \beta_i < \alpha$, and $i_0 \in \mathbf{Ord}$.

A **$\mathbb{P}$-name** is a $\mathbb{P}$-name of rank $\leqslant \alpha$ for some $\alpha \in \mathbf{Ord}$.

We write $\mathbb{M}^{\mathbb{P}}$ for the collection of $\mathbb{P}$-names that are elements of $\mathbb{M}$.

We let the **(name) rank** of τ, $\mathrm{rk}_n(\tau) = \alpha$ if τ is a $\mathbb{P}$-name of rank $\leqslant \alpha$, but not of rank $\leqslant \beta$ for any $\beta < \alpha$.

We define for $a \in M$, by induction on $\mathrm{rk}(a)$, a $\mathbb{P}$-name $\dot{a} = \{(p, \dot{b}) : p \in P, b \in a\}$.

Note that $\dot{a}$ *is* a $\mathbb{P}$-name. (Check.)

Examples of $\mathbb{P}$-names.

(1) rank $\leqslant 0$: $\emptyset$
(2) rank $\leqslant 1$: $\emptyset$, $\{(p_i, \emptyset) : i < i_0\}$.

Lecture 19 **Definition (cont.)** We define the $\mathbb{P}$-name $\Gamma = \{(p, \dot{p}) : p \in \mathbb{P}\}$.

We define the **revised (name) rank** $\mathrm{rk}_r(\tau) = \begin{cases} 0 \text{ if } \tau = \dot{a} \text{ for some } a \in \mathbb{M} \\ \bigcup\{\mathrm{rk}_r(\sigma) + 1 : (p, \sigma) \in \tau \text{ for some } p \in \mathbb{P}\} \end{cases}$

Notation. $V^{\mathbb{P}}$ is the class of all $\mathbb{P}$-names. We use $\underset{\sim}{f}$, $\underset{\sim}{\tau}$, $\underset{\sim}{a}$ for $\mathbb{P}$-names that are not of the form $\dot{a}$ for $a \in \mathbb{M}$.

When no confusion arises, we lapse into using a instead of $\dot{a}$.

This completes the definitions of the names.

We turn next to define the values or interpretations of the names. This definition by transfinite recursion is given in V (not in $\mathbb{M}$), since in general $G \notin \mathbb{M}$.

Definition. Suppose $\mathbb{P} \in \mathbb{M}$ is a forcing and $G \subset P$ is a generic filter over $\mathbb{M}$. We define by induction on $\alpha \in \mathbf{Ord}$.

(1) If τ is a $\mathbb{P}$-name of rank $\leqslant \alpha$, say $\tau = \{(p_i, \tau_i) : p_i \in P, i < i_0\}$ where τ_i is a $\mathbb{P}$-name of rank $\leqslant \beta_i$, $\beta_i < \alpha$, then the **interpretation** (or **value**) of τ relative to G is

$$\tau[G] = \{\tau_i[G] : p_i \in G, i < i_0\}$$

(2) $\mathbb{M}[G] = \{\tau[G] : \tau \in \mathbb{M} \text{ and } \tau \text{ is a } \mathbb{P}\text{-name}\}$.

Note: $\tau[G]$ is also written τ_G, $val_G(\tau)$, $val(\tau, G)$ and other variants.

Write $\in^{\mathbb{M}[G]} = \in_{\mathbb{M}[G] \times \mathbb{M}[G]}$.

In this course, $\mathbb{M}$ is called the **ground model**, and $\mathbb{M}[G]$ the **generic extension** of $\mathbb{M}$ relative to G (by G).

Finally, we define the **forcing relation** $\Vdash_{\mathbb{P}}$ (or $\Vdash$) which connects truth in generic extensions with definable sets in $\mathbb{M}$ and the combinatorial properties of $\mathbb{P}$.

Definition. Suppose $\mathbb{P} \in \mathbb{M}$ is a forcing, $\tau_1, \ldots, \tau_n$ are $\mathbb{P}$-names, and $\varphi(x_1, \ldots, x_n)$ is a formula in the vocabulary of set theory (possibly with a unary predicate symbol $M(x)$ for $\mathbb{M}$).

We say that a condition $p \in \mathbb{P}$ **forces** $\varphi(\tau_1, \ldots, \tau_n)$, denoted $p \Vdash_{\mathbb{P}} \varphi(\tau_1, \ldots, \tau_n)$, if

$$\mathbb{M}[G] \vdash \varphi\Big[\tau_1[G], \ldots, \tau_n[G]\Big]$$

for every generic filter G in $\mathbb{P}$ over $\mathbb{M}$ such that $p \in G$.

Remark. The definition of $\Vdash_{\mathbb{P}}$ is given in V, apparently. However, we shall prove that the binary predicate $\Vdash_{\mathbb{P}}$ is *definable in* $\mathbb{M}$.

This is a critical fact about forcing.

We can now state precisely the Forcing Theorem.

Theorem 3. Suppose $\mathbb{M}$ is a CTM, and $\mathbb{P} \in \mathbb{M}$ is a forcing. Then for every generic filter G in $\mathbb{P}$ over $\mathbb{M}$, there exists a CTM $\mathbb{M}[G]$ (defined as above) such that

(1) $\mathbb{M}[G]$ is an extension of $\mathbb{M}$. $\mathbb{M} \subset \mathbb{M}[G]$, $G \in \mathbb{M}[G]$.

(2) $\mathbf{Ord} \cap \mathbb{M} = \mathbf{Ord} \cap \mathbb{M}[G]$, i.e. $\mathbb{M}$ and $\mathbb{M}[G]$ have the *same ordinals*.

(3) For every sentence φ in the vocabulary of ZFC (possibly with a unary predicate symbol $M(x)$),
$$\mathbb{M}[G] \vdash \varphi \text{ iff } (\exists p \in G)(p \Vdash_{\mathbb{P}} \varphi)$$

(4) The predicate $\Vdash_{\mathbb{P}}$ is definable in $\mathbb{M}$.

(5) $\mathbb{M}[G]$ is minimal with the above properties: if $\mathbb{M}^+$ is any transitive model of ZFC with $\mathbb{M} \subset \mathbb{M}^+$, $G \in \mathbb{M}^+$, then $\mathbb{M}[G] \subset \mathbb{M}^+$.

It will take some preparatory work to prove $\mathbb{M}[G]$ has properties (1)–(5).

Let us consider some examples to see how this theorem helps us to establish independence results.

Example 1. Adding a Cohen real.

Let $\mathbb{P} = \Big\{f : f : \mathrm{dom}(f) \subset \omega \to \{0,1\}, |\mathrm{dom}(f)| < \aleph_0\Big\}$, $\leqslant_{\mathbb{P}}$ is extension of functions, i.e. $f \leqslant g \Leftrightarrow g|_{\mathrm{dom}(f)} = f$.

So we have $\mathbb{P} \in \mathbb{M}$. In $\mathbb{M}$, the sets $D_n = \{f \in P : n \in \mathrm{dom}(f)\}$ are dense open and definable in $\mathbb{M}$, so belong to $\mathbb{M}$. So $G \cap D_n \neq \emptyset$.

What do we know about $g = \bigcup G$? g is a function from ω to $\{0,1\}$.

Lecture 20

If G is generic, then $g = \bigcup G$ is a "real", i.e. $g : \omega \to \{0,1\}$. And since $\mathbb{P}$ is separative, $G \notin \mathbb{M}$ and so $g \notin \mathbb{M}$. So g is a real which is "new".

If we could add enough new reals, then CH would fail in the generic extension $\mathbb{M}[G]$.

Example 2. Let $\mathbb{P}_\lambda = \{f : f \text{ is a finite function from } \lambda \times \omega \text{ into } \{0,1\}\}$, where $\lambda \in \mathbb{M}$, and $f \leqslant g \Leftrightarrow g|_{\mathrm{dom}(f)=f}$.

If G is generic over $\mathbb{M}$, define $g_\alpha : \omega \to \{0,1\}$ by $g_\alpha(n) = \bigcup G(\alpha, n)$. Then $\langle g_\alpha : \alpha < \lambda^{\mathbb{M}}\rangle$ is a sequence of new reals.

However, this would not be quite enough to "blow up" the continuum to size λ in $\mathbb{M}[G]$, because maybe $\lambda^{\mathbb{M}}$ could be small in $\mathbb{M}[G]$. We would need to ensure that cardinals do not "collapse".

Example 3. $\mathbb{P}_\diamond = \{\bar{a}_\alpha = \langle A_i : i < \alpha\rangle : \alpha < \omega, A_i \subset \bar{\imath}\ \forall i < \alpha\}$, with $\bar{a}_\alpha \leqslant \bar{a}_\beta \Leftrightarrow \bar{a}_\alpha$ is an initial segment of $\bar{a}_\beta$.

A $\diamond$-sequence in $\mathbb{M}[G]$ will be $\bar{A} = \bigcup\{\bar{a} : \bar{a} \in G\}$, where G is a generic filter over $\mathbb{M}$.

Example 4. $\mathbb{P} = \{f : f \text{ is a function from a countable (in } \mathbb{M}) \text{ ordinal into } \mathcal{P}(\omega)\}$, with $f \leqslant g \Leftrightarrow f \subset g$.

A generic filter G gives rise to a map $g = \bigcup G : \omega_1^{\mathbb{M}} \to \mathcal{P}(\omega)^{\mathbb{M}}$.

To make sure that $\mathbb{M}[G]$ is a model of CH, we must be sure that $\omega_1^{\mathbb{M}} = \omega_1^{\mathbb{M}[G]}$ and $\mathcal{P}(\omega)^{\mathbb{M}} = \mathcal{P}(\omega)^{\mathbb{M}[G]}$.

To check that $\mathbb{M}[G]$ is an extension of $\mathbb{M}$ and that, e.g. in the examples above, $\omega_1^{\mathbb{M}} = \omega_1^{\mathbb{M}[G]}$ and $\mathcal{P}(\omega)^{\mathbb{M}} = \mathcal{P}(\omega)^{\mathbb{M}[G]}$, we need to show that for transitive models, the simplest set-theoretic concepts are invariant under extension. The concept of invariance of a property or a term is important in its own right. We study it briefly in a more general setting.

Absoluteness

Definition. Suppose $\varphi(x_1, \ldots, x_n)$ is a formula in the vocabulary of ZFC (or some expansion), and $\mathbb{A} \subset \mathbb{B}$ are classes. We say:

(1) $\varphi(x_1, \ldots, x_n)$ is **absolute between $\mathbb{A}$ and $\mathbb{B}$** if

$$\forall x_1, \ldots, x_n \in A \ \left(\varphi(x_1, \ldots, x_n)^{\mathbb{A}} \leftrightarrow \varphi(x_1, \ldots, x_n)^{\mathbb{B}}\right)$$

(2) A term t is **absolute between $\mathbb{A}$ and $\mathbb{B}$** if the formula $x = t$ is absolute between $\mathbb{A}$ and $\mathbb{B}$.

(3) If φ or t is absolute between $\mathbb{A}$ and V, then φ or t is **absolute for $\mathbb{A}$**.

(4) If φ or t is absolute for any *transitive* class $\mathbb{A}$, then φ or t is **absolute**.

Why is absoluteness useful? What concepts and properties are absolute?

Example. Suppose $\mathbb{A}$ is a class, $x \in \mathbb{A}$, and $V \models (\exists y)(y \in x)$. A priori, y need not belong to $\mathbb{A}$. However, if $\mathbb{A}$ is transitive, then $y \in \mathbb{A}$. So for transitive $\mathbb{A}$, $\exists y(y \in x) \leftrightarrow \left(\exists y(y \in x)\right)^{\mathbb{A}}$.

This example suggests that atomic formulas and formulas all of whose quantifiers are restricted will have the property $(\exists y \in x)\varphi \leftrightarrow \left((\exists y \in x)\varphi\right)^{\mathbb{A}}$.

I.e., properties expressed by formulas with restricted quantifiers are absolute, leading to the following definition of Δ_0-formulas.

Definition. We define that class of $\Sigma_0 = \Pi_0 = \Delta_0$ formulas as follows:

(1) If φ is atomic, then φ is Σ_0.

(2) If $\varphi_1, \varphi_2, \varphi_3$ are Σ_0, then $\varphi_1 \to \varphi_2$ and $\neg\varphi_3$ are Σ_0. (And $\varphi_1 \leftrightarrow \varphi_2$, $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$ are also Σ_0 by conventional abbreviations.)

(3) If φ is Σ_0, then $\exists x(x \in y \wedge \varphi)$ is Σ_0. (We abbreviate $\exists x(x \in y \wedge \varphi)$ as $\exists x \in y\, \varphi$.) Also, $\forall x(x \in y \to \varphi)$ is Σ_0 (abbreviated as $(\forall x \in y)\varphi$).

(See handout for a list of Δ_0-formulas.)

Definition. Suppose T is a theory. We say that φ is Σ_0^T (T-provably Σ_0) if for some Σ_0-formula ψ, we have $T \vdash \varphi \leftrightarrow \psi$.

If T is ZF or stronger, we omit T.

Lemma. Suppose $\varphi(x_1, \ldots, x_n)$ is Σ_0 (or provably equivalent to a Σ_0-formula), and $\mathbb{A}$ is a transitive class.

Then $\forall x_1, \ldots, x_n \in \mathbb{A}\ \varphi(x_1, \ldots, x_n) \leftrightarrow \varphi(x_1, \ldots, x_n)^{\mathbb{A}}$.

I.e., $\varphi(x_1, \ldots, x_n)$ is absolute.

Proof. Straightforward induction on the complexity of $\varphi(x_1, \ldots, x_n)$. □

With these matters clarified, we turn to the proof of the Forcing Theorem.

Lemma 4. Let G be a generic filter in $\mathbb{P}$ over $\mathbb{M}$. Then

(1) For all $a \in \mathbb{M}$, we have $\dot{a}[G] = a$, and $\Gamma[G] = G$.

So $\mathbb{M} \subset \mathbb{M}[G]$, and $G \in \mathbb{M}[G]$.

(2) $\mathbb{M}[G]$ is transitive.

(3) If τ is a $\mathbb{P}$-name in $\mathbb{M}$, then $\mathrm{rk}_r(\tau) \leqslant \mathrm{rk}_n(\tau)$, and $\mathrm{rank}(\tau[G]) \leqslant \mathrm{rk}_n(\tau)$.

So $\mathbf{Ord} \cap \mathbb{M} = \mathbf{Ord} \cap \mathbb{M}[G]$.

(4) $\mathbb{M}[G]$ is minimal: if $\mathbb{M}^+$ is a transitive model of ZFC with $\mathbb{M} \subset \mathbb{M}^+$ and $G \in \mathbb{M}^+$, then $\mathbb{M}[G] \subset \mathbb{M}^+$.

Lecture 21 First, a remark on absoluteness of transfinite recursion.

(1) If $s(y, z)$ is absolute and ZFC $\vdash t(\alpha) = s(t|_\alpha, \alpha)$, then $y = t(\alpha)$ is absolute.

(2) If s_0 and $s_1(y)$ are absolute terms and ZFC $\vdash t(0) = s_0 \wedge t(n+1) = s(t(n))$, then $y = t(n)$ is absolute.

Proof of Lemma 4.

(1) Induction on $\mathbb{P}$-name rank $\leqslant \alpha$.

$x \in \dot{a}[G] \leftrightarrow x = \dot{b}[G]$ for some $b \in a$ with $(p, \dot{b}) \in \dot{a}$.

By induction, this $\leftrightarrow x = b$ for some $b \in a$. So $x \in a$ and $\dot{a}[G] = a$.

Then $\Gamma[G] = \{(p, \dot{p}) : p \in P\}[G] = \{\dot{p}[G] : p \in G\} = \{p : p \in G\} = G$.

(2) $x \in \tau[G] \in \mathbb{M}[G] \to x = \tau_i[G]$ for some $\mathbb{P}$-name τ_i, etc, which $\to x \in \mathbb{M}[G]$.

(3) The claims about rank are proved by induction (exercise).

We show that $\mathbf{Ord} \cap \mathbb{M} = \mathbf{Ord} \cap \mathbb{M}[G]$. Clearly, $\mathbf{Ord} \cap \mathbb{M} \subset \mathbf{Ord} \cap \mathbb{M}[G]$ by (1).

If $\beta \in \mathbf{Ord} \cap \mathbb{M}[G]$, say $\beta = \underset{\sim}{\tau}[G]$ for some $\mathbb{P}$-name $\tau \in \mathbb{M}^{\mathbb{P}}$.

Then $\beta = \mathrm{rank}(\beta) = \mathrm{rank}(\underset{\sim}{\tau}[G]) \leqslant \mathrm{rk}_n(\tau) \in \mathbf{Ord} \cap \mathbb{M}$, since ranks are defined by transfinite recursion, so are absolute.

So $\beta \leqslant \alpha \in \mathbf{Ord} \cap \mathbb{M}$, so $\beta \in \mathbb{M}$, since $\mathbb{M}$ is transitive and $\alpha \in \mathbf{Ord}$.

Thus $\mathbf{Ord} \cap \mathbb{M} = \mathbf{Ord} \cap \mathbb{M}[G]$.

(4) For $\tau \in \mathbb{M}^{\mathbb{P}}$, we see that $\tau \in \mathbb{M}^+$, $G \in \mathbb{M}^+$.

So $\tau[G] = \tau[G]^{\mathbb{M}^+} \in \mathbb{M}^+$, so $\mathbb{M}[G] \subset \mathbb{M}^+$. □

We start proving $\mathbb{M}[G]$ satisfies the axioms of ZFC. We get some very easily.

Proposition 5. With the usual assumptions on $\mathbb{M}, \mathbb{P}, G$, we have that $\mathbb{M}[G]$ satisfies the following.

(0) Set existence
(1) Extensionality
(2) Null set
(3) Foundation
(4) Pair set
(5) Union
(6) Infinity

Proof.

(0) $\emptyset \neq \mathbb{M} \subset \mathbb{M}[G]$.

(1) $\mathbb{M}[G]$ is transitive

(2), (3), (6) absoluteness of terms involved in the axioms.

(4) Let $x = \sigma[G]$, $y = \tau[G] \in \mathbb{M}[G]$, with $\sigma, \tau \in \mathbb{M}^{\mathbb{P}}$.

Define a $\mathbb{P}$-name $\mathrm{upair}(\sigma, \tau) \in \mathbb{M}^{\mathbb{P}}$, by $\mathrm{upair}(\sigma, \tau) = \{\langle 0_{\mathbb{P}}, \sigma\rangle, \langle 0_{\mathbb{P}}, \tau\rangle\}$.

It's easy now to check $\mathrm{upair}(\sigma, \tau)[G] = \{\sigma[G], \tau[G]\} = \{x, y\}$.

(5) Suppose that $x = \tau[G] \in \mathbb{M}[G]$. Two ways to check union.
(i) manufacture a $\mathbb{P}$-name $u \in \mathbb{M}^{\mathbb{P}}$ such that $u[G] = \bigcup x$
(ii) "on credit". Find a $\mathbb{P}$-name $\rho \in \mathbb{M}^{\mathbb{P}}$ such that $\bigcup x \subset \rho[G]$ and then appeal to Separation. (But this assumes that we have checked that Separation holds in $\mathbb{M}[G]$.)

We indicate (1). We want $(\bigcup x)[G] = \{z[G] : \exists u[G] \in x[G], z[G] \in u[G]\}$.

Reflecting suggests the following $\mathbb{P}$-name as a candidate to give $\bigcup x$ in $\mathbb{M}[G]$.

$$u = \{(r, \rho) : \exists p, q \in P, \exists \sigma \in \mathbb{M}^{\mathbb{P}}, \langle p, \sigma\rangle \in \tau \wedge \langle q, \rho\rangle \in \sigma \wedge p \leqslant r \wedge q \leqslant r\}$$

This is a $\mathbb{P}$-name in $\mathbb{M}$, and it is easy enough to check that $u[G] = \bigcup x = \bigcup(\tau[G])$.

If one does (2), take $\rho = \bigcup \mathrm{range}(\tau)$. We have $\mathrm{range}(\tau) = \{\sigma : \exists p \in P, \langle p, \sigma\rangle \in \tau\}$.

ρ is a $\mathbb{P}$-name in $\mathbb{M}$, and $\bigcup x \subset \rho[G]$. Now apply Separation. □

It remains to verify Power set, Replacement, Separation and Choice in $\mathbb{M}[G]$. These axioms will require us to produce more complicated $\mathbb{P}$-names in $\mathbb{M}$. Hence we must treat the definability of $\Vdash$ in $\mathbb{M}$.

Lecture 22 To verify these remaining axioms, we shall assume proven the following two clauses of the Forcing Theorem.

(3) $\mathbb{M}[G] \models \varphi[\tau_1[G], \ldots, \tau_n[G]]$ iff $(\exists p \in G)(p \Vdash \varphi(\tau_1, \ldots, \tau_n))$.

(4) The relation $\Vdash_{\mathbb{P}}$ is definable in $\mathbb{M}$; more precisely, given $\varphi(x_1, \ldots, x_n)$, there is a formula $\varphi^*(x_1, \ldots, x_n, x, y)$ which is absolute in $\mathbb{M}$ such that for all $a_1, \ldots, a_n \in \mathbb{M}$,

$$p \Vdash \varphi(\dot{a}_1, \ldots, \dot{a}_n) \text{ iff } M \models \varphi^*(a_1, \ldots, a_n, p, \mathbb{P})$$

Notes.

(3) is sometimes called the Truth Lemma.

(4) is sometimes called the Definability Lemma.

We shall prove (3) and (4) in an appendix (they are proved by induction).

Proposition $5\frac{1}{2}$ The axioms of Separation, Power set, Replacement and Choice are all satisfied in $\mathbb{M}[G]$.

Proof.

(1) Separation.

We'll write $\sigma_G, \tau_G, \ldots$ instead of $\sigma[G], \tau[G], \ldots$, when more convenient.

Suppose that $\sigma, \tau_1, \ldots, \tau_n \in \mathbb{M}^{\mathbb{P}}$ and $\varphi(x, y, x_1, \ldots, x_n)$ is a formula. We wish to show that

$$y = \{a \in \sigma_G : \mathbb{M}[G] \models \varphi[a, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}]\}$$

belongs to $\mathbb{M}[G]$.

We define an appropriate name $\rho \in \mathbb{M}^{\mathbb{P}}$.

$$\rho = \Big\{\langle p, \pi\rangle : \langle p, \pi\rangle \in P \times \mathrm{range}(\sigma), p \Vdash \pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n)\Big\}$$

We have $\rho \in V^{\mathbb{P}}$. In fact, $\rho \in \mathbb{M}^{\mathbb{P}}$ by Definability Lemma (4).

We just check that $\rho[G] = y$.

First, $\rho[G] \subset y$. Suppose $a \in \rho[G]$. Then $a = \pi[G]$, where $\langle p, \pi\rangle \in \rho$ for some $p \in G$. By definition of ρ,

$$p \Vdash \pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n)$$

Since $p \in G$,

$$\mathbb{M}[G] \models \pi_G \in \sigma_G \wedge \varphi[\pi_G, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}]$$

(by definition of $\Vdash$).

So $a = \pi[G] \in y$.

Second, $y \subset \rho[G]$. Suppose $a \in y$. Then $a \in \sigma[G]$ and $\varphi(a, \sigma_G, \tau_{1G}, \ldots, \tau_{nG})^{\mathbb{M}[G]}$.

So $a = \pi[G]$ for some $\pi \in \text{range}(\sigma)$. So $\mathbb{M}[G] \models \pi_G \in \sigma_G \wedge \varphi[\pi_G, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}]$.

Thus $(\exists p \in G) p \Vdash \pi \in \sigma \wedge \varphi(\pi, \sigma, \tau_1, \ldots, \tau_n)$ (by Truth Lemma (3)).

So $\langle p, \pi \rangle \in \rho$ and $\pi_G \in \rho_G$. So $y \subset \rho_G$.

(2) Power set.

Suppose $x \in \mathbb{M}[G]$, $x = \sigma[G]$ for some $\sigma \in \mathbb{M}^{\mathbb{P}}$.

Let $Z_\sigma = \big\{\langle q, p \rangle : (\exists p \leqslant q)(\langle p, \rho \rangle \in \sigma)\big\}$.

Let $Z'_\sigma = \{u : u \subset Z_\sigma\}^{\mathbb{M}}$. Let $\tau = P \times Z'_\sigma$.

$\tau \in \mathbb{M}^{\mathbb{P}}$. It is an exercise to check that $(Px)^{\mathbb{M}[G]} = \tau_G$.

(3) Replacement.

Suppose $\sigma[G] \in \mathbb{M}[G]$ and we wish to find a set $\tau[G] \in \mathbb{M}[G]$ such that (in $\mathbb{M}[G]$)

$$\forall z \Big(z \in \tau[G] \leftrightarrow (\exists y)\big(y \in \sigma[G] \wedge \psi(y, z)\big)\Big)$$

i.e.

$$\tau[G] = \Big\{ z : (\exists y \in \sigma[G]) \ \psi(y, z)^{\mathbb{M}[G]} \Big\}$$

We seek a suitable name.

Let $Z_\sigma = \big\{\langle q, y \rangle : (\exists r \in P)\big(\langle r, y \rangle \in \sigma \wedge r \leqslant q\big)\big\}$

Then $Z_\sigma \in \mathbb{M}^{\mathbb{P}}$.

For each $\langle q, y \rangle \in Z_\sigma$, consider all $\mathbb{P}$-names z such that $q \Vdash \psi(\dot{y}, \dot{z})$.

There are too many $\mathbb{P}$-names z for this to be a set in $\mathbb{M}$. We cut the collection down to a set in $\mathbb{M}$ by using the rank function in $\mathbb{M}$.

Let $\rho(q, y)$ be the least rank of a set $z \in \mathbb{M}$ for which $q \Vdash \psi(\dot{y}, \dot{z})$. (If no such z exists, then $\rho(q, y) = 0$.)

Now, let

$$\tau = \Big\{ \langle w, z \rangle : (\exists \langle q, y \rangle \in Z_\sigma)\big(q \Vdash \psi(\dot{y}, \dot{z}) \wedge \text{rank}(z) \leqslant \rho(q, y)\big)\Big\}^{\mathbb{M}}$$

Check that τ_G is as required. Note that $\tau \in \mathbb{M}^{\mathbb{P}}$ because $\text{rank}(\tau)$ is bounded in $\mathbb{M}$, and $\Vdash$ is definable in $\mathbb{M}$.

(4) Choice.

Suppose $x = \tau_G \in \mathbb{M}[G]$. We show that there is a function $f \in \mathbb{M}[G]$ mapping an ordinal α onto a set containing x as a subset.

Let $\langle \sigma_\beta : \beta < \alpha \rangle$ be an enumeration of $\text{range}(\tau)$.

Since $\mathbb{M} \models \mathrm{AC}$, we may assume that this enumeration belongs to $\mathbb{M}$. Let $\mathrm{opair}(u,v)$ be the ordered pair $\langle u,v\rangle = \big\{\{u,u\},\{u,v\}\big\}$.

Let $f = \big\{\langle p, \mathrm{opair}(\dot\beta, \sigma_\beta)\rangle : p \in P, \beta < \alpha\big\} \in \mathbb{M}^{\mathbb{P}}$

Then $f[G] = \big\{\langle \beta, \sigma_\beta[G]\rangle : \beta < \alpha\big\} \in \mathbb{M}[G]$, and $x \subset \mathrm{range}(f)$, $\mathrm{dom}(f) = \alpha$.

So x is well-ordered in $\mathbb{M}[G]$. □

Corollary 6. $\mathbb{M}[G] \models \mathrm{ZFC}$.

Lecture 23 **Proposition 7.** Let $\mathbb{P} \in \mathbb{M}$ be a forcing. If $p \Vdash (\exists x)(x \in \sigma \wedge \varphi(x, \tau_1, \ldots, \tau_n)$ then there exist $q \geqslant p$ and $\pi \in \mathrm{range}(\sigma)$ such that $q \Vdash \varphi(\pi, \tau_1, \ldots, \tau_n)$.

Proof. Suppose $p \in G$ for some generic filter G.

$\mathbb{M}[G] \Vdash (\exists x)\big(x \in \sigma_G \wedge \varphi(x, \tau_{1G}, \ldots, \tau_{nG})\big)$, so for some $\pi \in \mathbb{M}^{\mathbb{P}}$ and $q' \in G$, we have $(q', \pi) \in \sigma$ and $\mathbb{M}[G] \models \pi_G \in \sigma_G \wedge \varphi(\pi_G, \tau_{1G}, \ldots, \tau_{nG})\big)$,

By the Truth Lemma, there is $q \in G$ such that $q \Vdash \pi \in \sigma \wedge \varphi(\pi, \tau_1, \ldots, \tau_G)$.

Wlog, $q \geqslant p, q'$. □

We turn to the proof of the relative consistency of CH.

Definition. Let $\mathbb{P} \in \mathbb{M}$. We say

(1) $\mathbb{P}$ **preserves cardinals** if for every generic filter G in $\mathbb{P}$ over $\mathbb{M}$, for all $\beta \in \mathbf{Ord} \cap \mathbb{M}$,

$$(\beta \text{ is a cardinal})^{\mathbb{M}} \longleftrightarrow (\beta \text{ is a cardinal})^{\mathbb{M}[G]} \quad (*)$$

If a cardinal κ in $\mathbb{M}$ ceases to be a cardinal in $\mathbb{M}[G]$, we say that $\mathbb{P}$ **collapses** κ (or that κ is **collapsed** by $\mathbb{P}$).

(2) Analogously, we say that $\mathbb{P}$ **preserves cardinals** $\leqslant \lambda \in \mathbb{M}$ if $(*)$ holds for all $\beta \leqslant \lambda$, $\beta \in \mathbf{Ord} \cap \mathbb{M}$.

Note. The finite ordinals and ω are absolute, so cardinal preservation is an issue only for $\beta > \aleph_0$. Note also that if β is a cardinal in $\mathbb{M}[G]$, then β is a cardinal in $\mathbb{M}$ since $\mathbb{M} \subset \mathbb{M}[G]$.

Definition. Let $\mathbb{P} \in \mathbb{M}$. We say

(1) $\mathbb{P}$ **preserves cofinalities** if for every limit ordinal $\delta \in \mathbb{M}$,

$$\mathrm{cf}(\delta)^{\mathbb{M}} = \mathrm{cf}(\delta)^{\mathbb{M}[G]} \quad (**)$$

(2) Analogously, $\mathbb{P}$ **preserves cofinalities** $\leqslant \lambda \in \mathbb{M}$ if $(**)$ holds for $\delta \leqslant \lambda$.

Lemma 8. If $\mathbb{P}$ preserves cofinalities, then $\mathbb{P}$ preserves cardinals.

Proof. Every infinite cardinal κ is regular or a limit.

Case 1. κ is regular in $\mathbb{M}$. Then $\mathrm{cf}(\kappa)^{\mathbb{M}[G]} = \mathrm{cf}(\kappa)^{\mathbb{M}} = \kappa$, so κ is a regular cardinal in $\mathbb{M}[G]$.

Case 2. κ is a limit cardinal in $\mathbb{M}$. Then the regular (even the successor) cardinals $\lambda \in \mathbb{M}$, $\lambda < \kappa$, are unbounded in κ. By Case 1, these λ remain regular cardinals in $\mathbb{M}[G]$ and are still unbounded in κ, and so κ is a limit cardinal in $\mathbb{M}[G]$. □

Definition.

1. Suppose $\lambda \geqslant \aleph_0$. A forcing $\mathbb{P}$ is **λ-closed** (or **λ-complete**) if for any $\gamma < \lambda$, every $\leqslant_{\mathbb{P}}$-increasing sequence $\langle p_i : i < \gamma \rangle \subseteq \mathbb{P}$ has an upper bound $p \in P$. I.e., $\bigwedge_{i<\gamma} p_i \leqslant_{\mathbb{P}} p$.
2. $\mathbb{P}$ is **countably complete** if $\mathbb{P}$ is $\aleph_0$-complete.

Lemma 9. Suppose $\mathbb{P} \in \mathbb{M}$, $\lambda \in \mathbb{M}$, and $(\mathbb{P}$ is λ-closed$)^{\mathbb{M}}$. Let $\alpha < \lambda$, $B \in \mathbb{M}$, and suppose G is generic in $\mathbb{P}$ over $\mathbb{M}$.

Then $({}^{\alpha}B)^{\mathbb{M}} = ({}^{\alpha}B)^{\mathbb{M}[G]}$.

In $\mathbb{M}[G]$, there are no *new* α-sequences, i.e. if $f : \alpha \to B$, $f \in \mathbb{M}[G]$, then $f \in \mathbb{M}$.

Proof. $\mathbb{M}[G] \models f$ is a function from α into B. Say $f = \tau_G$ for some $\tau \in \mathbb{M}^{\mathbb{P}}$.

Suppose $f \notin ({}^{\alpha}B)^{\mathbb{M}} = {}^{\alpha}B \cap \mathbb{M}$ (absolute) $= K \in \mathbb{M}$.

By the Truth Lemma, there exists $p \in G$ such that $p \Vdash$ (τ is a function from $\dot{\alpha}$ into $\dot{B}$ and $\tau \notin K$).

Now we work entirely in $\mathbb{M}$ and define by transfinite recursion a $\leqslant_{\mathbb{P}}$-increasing sequence $\langle p_\beta : \beta \leqslant \alpha \rangle$ and $\langle b_\beta \in B : \beta < \alpha \rangle$ as follows:

(1) $p_0 = p$

(2) $\xi \leqslant \zeta \longrightarrow p_\xi \leqslant_{\mathbb{P}} p_\zeta$

(3) $p_{\beta+1} \Vdash \tau(\dot{\beta}) = b_\beta$ for some $b_\beta \in B$.

Why is this possible to do?

If $\delta \leqslant \alpha$ is a limit ordinal, since $\mathbb{P}$ is λ-closed and $\alpha < \lambda$, there is $p_\delta \in \mathbb{P}$ with $\bigwedge_{\xi<\delta} p_\xi \leqslant_{\mathbb{P}} p\delta$.

So (2) is satisfied.

If p_β has been defined, $p_\beta \Vdash \tau$ is a function from $\dot{\alpha}$ into $\dot{B}$. $(p = p_0 \leqslant p_\beta)$

So $p_\beta \Vdash (\exists x)(x \in \dot{B} \wedge \tau(\dot{B}) = x)$.

By Proposition 7, there exists $b_\beta \in B$ and $p_{\beta+1} \geqslant p_\beta$ such that $p_{\beta+1} \Vdash \tau(\dot{\beta}) = \dot{b_\beta}$.

This completes the definition.

In $\mathbb{M}$, consider the function $g : \alpha \to B$ given by $g(\beta) = b_\beta$. Then $g \in K$. In particular, $g \in \mathbb{M}$.

Let H be a generic filter in $\mathbb{P}$ over $\mathbb{M}$ such that $p_\alpha \in H$. Then for all $\beta < \alpha$, $p_\beta \in H$.

$\bigwedge_{\beta<\alpha} \mathbb{M}[H] \Vdash \tau_H(\beta) = g(\beta)$. So in $\mathbb{M}[H]$, $\tau_H = g \in K \quad (*)$.

But, $p = p_0 \leqslant p_\alpha \in H$, so $p \in H$. And $p \Vdash \tau \notin \dot{K}$, so $\mathbb{M}[H] \models \tau_H \notin K$.

So $\tau_H \notin K$, which contradicts $(*)$. Thus $f \in \mathbb{M}$. □

Corollary 10. Suppose (λ is a cardinal and $\mathbb{P}$ is λ-complete)$^{\mathbb{M}}$. Then $\mathbb{P}$ preserves cofinalities $\leqslant \lambda$ and cardinals $\leqslant \lambda$.

Proof. If for some cardinal $\kappa \leqslant \lambda$ we have $\mathrm{cf}(\kappa)^{\mathbb{M}[G]} < \mathrm{cf}(\kappa)^{\mathbb{M}}$, then in $\mathbb{M}[G]$, there exist $\alpha < \kappa$ and $f \in \mathbb{M}[G]$ such that $f : \alpha \to \kappa$ is strictly increasing and cofinal in κ.

By Lemma 9, $f \in \mathbb{M}$, so $\mathrm{cf}(\kappa)^{\mathbb{M}} \leqslant \alpha < \kappa$. □

Theorem 11. Con(ZFC) $\longrightarrow$ Con(ZFC+$(2^{\aleph_0} = \aleph_1)$).

I.e., there is a model of ZFC+$(2^{\aleph_0} = \aleph_1)$.

Proof. Let $\mathbb{M}$ be a CTM. We construct $\mathbb{M}[G]$ such that $\mathbb{M}[G] \models (2^{\aleph_0} = \aleph_1)$.

We wish $\mathbb{M}[G]$ to possess a surjection from ω_1 onto $\mathcal{P}(\omega)$ in $\mathbb{M}[G]$.

In $\mathbb{M}$, let $\mathbb{P} = \big\{f : \mathrm{dom}(f) = \alpha < \omega_1, \mathrm{range}(f) \subseteq \mathcal{P}(\omega)\big\}$, with $f \leqslant_{\mathbb{P}} g$ if $g|_{\mathrm{dom}(f)} = f$.

Note:

(1) $\mathbb{P}$ is $\aleph_1$-complete. So $\omega_1^{\mathbb{M}} = \omega_1^{\mathbb{M}[G]}$, by Corollary 10.

(2) $(\mathcal{P}(w))^{\mathbb{M}} = (\mathcal{P}(\omega))^{\mathbb{M}[G]}$, since $({}^{\omega}2)^{\mathbb{M}} = ({}^{\omega}2)^{\mathbb{M}[G]}$, by Lemma 9.

Let $g = \bigcup G$, where G is a generic filter in $\mathbb{P}$ over $\mathbb{M}$.

Then $E_a = \big\{f : a \in \mathrm{range}(f)\big\}$, for $\alpha \subseteq \omega$ (in $\mathbb{M}$), is a dense open set in $\mathbb{M}$.

So $G \cap E_a \neq \emptyset$.

Hence in $\mathbb{M}[G]$, g is a surjection from $\omega_1^{\mathbb{M}[G]}$ onto $(\mathcal{P}(\omega))^{\mathbb{M}[G]}$. □

Lecture 24

Lemma 12. Suppose $\mathbb{P} \in \mathbb{M}$ and ($\mathbb{P}$ is a CCC)$^{\mathbb{M}}$. Suppose $f \in \mathbb{M}[G]$ is a function from α to β, with $\alpha, \beta \in$ **Ord** and $\alpha \geqslant \omega$.

Then there exists $y \in \mathbb{M}$ with $\mathrm{range}(f) \subseteq y$ and $(|y| \leqslant |\alpha|)^{\mathbb{M}}$.

In other words, $f \in \mathbb{M}[G]$ can be "approximated" by a set in $\mathbb{M}$.

Proof. $f = \tau_G$ for some $\tau \in \mathbb{M}^{\mathbb{P}}$. $\mathbb{M}[G] \models (\tau_G : \alpha \to \beta$ is a function).

By the Truth Lemma, there is $p \in G$ with $p \Vdash (\tau : \dot{\alpha} \to \dot{\beta}$ is a function).

So $p \Vdash (\forall \delta < \dot{\alpha})(\tau(\delta) < \dot{\beta})$. And $p \in G$, so $\mathbb{M}[G] \models (\forall \delta < \alpha)(f(\delta) < \beta)$.

Say $\gamma = f(\delta)$. Thus there is $q \in G$, $p \leqslant q$, with $q \Vdash \tau(\dot{\delta}) = \dot{\gamma}$.

For each $\delta < \alpha$, define $y_\delta = \big\{\xi < \beta | (\exists r)(p \leqslant r)(r \Vdash \tau(\dot{\delta}) = \dot{\xi})\big\}$.

Note: $y_\delta \in \mathbb{M}$ by the Definability Lemma, and $\gamma \in y_\delta$ where $\gamma = f(\delta)$.

In $\mathbb{M}$, y_δ is countable. Why? Choose for each $\xi \in y_\delta$ a condition q_ξ such that $p \leqslant q_\xi$ and $q_\xi \Vdash \tau(\dot{\delta}) = \dot{\xi}$.

Then $\{q_\xi : \xi \in y_\delta\}$ is an antichain in $\mathbb{P}$. This is easy to check.

So $\{q_\xi : \xi \in y_\delta\}$ is countable since $\mathbb{P}$ has CCC in $\mathbb{M}$, and hence $(y_\delta \text{ is countable})^{\mathbb{M}}$.

Let $y = \bigcup_{\delta<\alpha} y_\delta$. Then $y \in \mathbb{M}$ since $\alpha, y_\delta \in \mathbb{M}$: $\Big(|y| = \sum_{\delta<|a} |y_\delta| = \aleph_0|\alpha| = |\alpha|\Big)^{\mathbb{M}}$.

And range$(f) \subseteq y$ since $\forall\gamma < \delta$, we have $f(\delta) \in y_\delta$. □

Lemma 13. Suppose $(\mathbb{P} \text{ is CCC})^{\mathbb{M}}$. Then $\mathbb{P}$ preserves cofinalities and cardinals.

Proof. If not, then since $\aleph_0^{\mathbb{M}} = \aleph_0^{\mathbb{M}[G]}$, the witness to failure $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ in $\mathbb{M}$ has cofinality $\alpha = \mathrm{cf}(\kappa)^{\mathbb{M}[G]} < \kappa$ in $\mathbb{M}[G]$.

So in $\mathbb{M}[G]$, there is $f : \alpha \to \kappa$, strictly increasing and cofinal in κ. Now, by Lemma 12, there is $y \in \mathbb{M}$ such that range$(f) \subseteq y$ and $(|y| = |a|)^{\mathbb{M}}$.

So y is cofinal in κ in $\mathbb{M}$. Thus $\big(\mathrm{cf}(\kappa) \leqslant |y| = |\alpha| < \kappa\big)^{\mathbb{M}}$, so $\mathrm{cf}(\kappa)^{\mathbb{M}} < \kappa$ in $\mathbb{M}$, contradiction.

$\mathbb{P}$ must therefore preserve cofinalities and hence also cardinals. □

We shall use a useful result to check CCC.

Lemma 14 (the Δ-system Lemma). Suppose $\mathcal{A}$ is an uncountable family of finite subsets of a set X.

Then there exist an *uncountable* subfamily $\mathcal{B} \subseteq \mathcal{A}$ and a set r such that for all $a \neq b \in \mathcal{B}$, we have $a \cap b = r$.

Proof. Wlog, for all $a \in \mathcal{A}$, $|a| = n$ for a fixed $n < \omega$. We prove by induction on n.

If $n = 1$, this is the Pigeonhole Principle. So suppose $n > 1$.

Case 1. If there exists x such that $x \in a$ for uncountably many $a \in \mathcal{A}$, then evict x and consider $\{a \setminus \{x\} : a \text{ as above}\}$. Apply the induction hypothesis to get $\mathcal{B}$ and add x back to each $b \in \mathcal{B}$.

Case 2. Not case 1. Then every x leaves the a's in $\mathcal{A}$ after at most countably many a's. So one can easily define a sequence $\langle a_\alpha : \alpha < \omega_1\rangle$ with $a_\alpha \in \mathcal{A}$ with $a_\alpha \cap a_\beta = \emptyset$ for $\alpha \neq \beta$. □

Definition. $\mathrm{Fn}(A, B, \kappa) = \{f : f \text{ is a function from } A \text{ to } B \text{ and } |f| < \kappa\}$.

Corollary 15. $\mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$ has the CCC.

Proof. If $\mathcal{A} \subseteq \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$ is uncountable, apply Lemma 14 to find $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{B}| = \aleph_1$, and $f \cap g = r$ for all $f \neq g \in \mathcal{B}$.

So $h = f \cup g \in \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$ and $f \leqslant h \wedge g \leqslant h$.

So $\mathcal{A}$ is not an antichain. □

Theorem 16. $\mathrm{Con(ZFC)} \longrightarrow \mathrm{Con}(\mathrm{ZFC}+(2^{\aleph_0} > \aleph_1))$.

I.e., there is a model of ZFC+$(2^{\aleph_0} > \aleph_1)$.

Proof. Let $\mathbb{M}$ be a CTM. In $\mathbb{M}$, let $\mathbb{P} = \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$ where $\lambda \geqslant \aleph_2$ in $\mathbb{M}$.

By Corollary 15, $\mathbb{P}$ has the CCC, so $\mathbb{P}$ preserves cofinalities and cardinals, and hence

$$\aleph_1^{\mathbb{M}[G]} = \aleph_1^{\mathbb{M}}$$

$$\aleph_2^{\mathbb{M}[G]} = \aleph_2^{\mathbb{M}} \leqslant \lambda$$

The following sets are dense open and belong to $\mathbb{M}$:

$$D_{\alpha,n} = \{p \in \mathbb{P} : (\alpha, n) \in \mathrm{dom}(p)\}$$

So $G \cap D_{\alpha,n} \neq \emptyset$ for all $\alpha < \lambda$, $n \in \omega$.

In $\mathbb{M}[G]$, $\bigcup G : \lambda \times \omega \to 2$. Let $g_\alpha(n) = \bigcup G(\alpha, n)$. Then $g_\alpha : \omega \to 2$, $g_\alpha \in \mathbb{M}[G]$.

Claim. In $\mathbb{M}[G]$, $\alpha < \beta \to g_\alpha \neq g_\beta$.

Why? Otherwise $\mathbb{M}[G] \models g_\alpha = g_\beta$, so there exist $\tau_\alpha, \tau_\beta \in \mathbb{M}^{\mathbb{P}}$, and $p \in G$ such that $p \Vdash \tau_\alpha = \tau_\beta$.

Pick $n < \omega$ such that $(\alpha, n), (\beta, n) \notin \mathrm{dom}(p)$.

Extend p to q, so $q|_{\mathrm{dom}(p)} = p$, with $q(\alpha, n) = 0$ and $q(\beta, n) = 1$.

Let H be generic in $\mathbb{P}$ over $\mathbb{M}$, $q \in H$.

Then $\mathbb{M}[H] \models \tau_{\alpha H}(n) = 0$, and $\mathbb{M}[H] \models \tau_{\beta H} = 1$, but $\mathbb{M}[H] \models \tau_{\alpha H}(n) = \tau_{\beta H}$, because $p \in H$ (as $p \leqslant q$).

Contradiction. Thus in $\mathbb{M}[G]$, $g_\alpha \neq g_\beta$, and so $\mathbb{M}[G] \models (2^{\aleph_0} \geqslant \lambda \geqslant \aleph_2)$. □

Corollary 17. The Continuum Hypothesis is independent of ZFC (if ZFC is consistent).

Proof. Theorems 11 and 16. □

TOPICS IN SET THEORY: Exercise Sheet 1

Department of Pure Mathematics and Mathematical Sciences, University of Cambridge

Michaelmas 2012-2013 Dr Oren Kolman

1 (i) Assume the axiom of choice.
Suppose I is a non-empty set and for each $i \in I$, λ_i is an infinite cardinal. Show $\sum_{i \in I} \lambda_i \leq \mid I \mid sup_{i \in I} \lambda_i$, where $\mid I \mid$ is the cardinality of I. [Hint: enumerate λ_i and think up a surjection from $\mid I \mid \times \lambda$ onto $\bigcup_{i \in I} \lambda_i$ where $\lambda = sup_{i \in I} \lambda_i$.]

(ii) Suppose κ is an infinite cardinal. Prove that the (cardinal) successor κ^+ of κ is a regular cardinal. [Hint: part (i).]

2 (i) Suppose $f \in \mathcal{H}(\mathbb{C})$, i.e. f is an entire holomorphic function of the single complex variable z. Let $Z(f) = \{z \in \mathbb{C} : f(z) = 0\}$ be the zero set of f. If $f \neq 0$, what is the cardinality of $Z(f)$? [Hint: the zeros of a holomorphic function have a noteworthy topological property.]

(ii) Suppose $\mathcal{F} \subseteq \mathcal{H}(\mathbb{C})$ has cardinality $\aleph_1$. What is the cardinality of the set $\bigcup_{f \neq g \in \mathcal{F}} Z(f - g)$? [Hint: Q1(i).]

(iii) Suppose $2^{\aleph_0} > \aleph_1$ and $\mathcal{F} \subseteq \mathcal{H}(\mathbb{C})$ has cardinality $\aleph_1$. By judicious selection of a $w_0 \in \mathbb{C}$, show that $\mathcal{F}$ is not orbit countable.

(iv) Deduce the observation of Erdös that if $2^{\aleph_0} > \aleph_1$, then orbit countability is equivalent to countability.

3 (i) Let ZFC^- be the first-order theory whose axioms are obtained from ZFC by omitting the axiom of infinity. Show that **(*)** $(V_\omega, \in \cap (V_\omega \times V_\omega))$ is a model of ZFC^-.

(ii) Show that every element of V_ω is finite. Deduce that the axiom of infinity cannot be proved from the other axioms of ZFC.

(iii) Can the assertion **(*)** be proved from ZFC^-? Explain.

4 (i) Consider the assertion **DODGY**: there exists a family $\{A_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ such that (i) $\omega_1 \setminus \bigcup_{n<\omega} A_{\alpha,n}$ is finite for every $\alpha < \omega_1$, and (ii) if $\alpha \neq \beta$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for all $n < \omega$. Decide whether **DODGY** is provable or not from ZFC. [Hint: it may be easier to try part (ii) first.]

(ii) Prove there exists a family $\{A_{\alpha,n} : \alpha < \omega_1, n < \omega\}$ such that (i) $\omega_1 \setminus \bigcup_{n<\omega} A_{\alpha,n}$ is countable for every $\alpha < \omega_1$, and (ii) if $\alpha \neq \beta$, then $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for all $n < \omega$. [Hint: For each ordinal $\alpha < \omega_1$, choose a surjection f_α from ω onto α (why is this possible?), and consider the set $A_{\alpha,n} = \{\xi : f_\xi(n) = \alpha\}$.]

(iii) Can you generalize the result of part (ii) to cardinals greater than $\aleph_1$? How about $\aleph_\omega$?

5 Assume that ${\aleph_1}^{\aleph_0} = \aleph_1$. Prove ${\aleph_n}^{\aleph_0} = \aleph_n$ for all $n < \omega$. [Hint: induction.]

TOPICS IN SET THEORY: Exercise Sheet 2

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge

Michaelmas 2012-2013 Dr Oren Kolman

1 (i) Suppose that x and y belong to the class WF of well-founded sets. Find bounds for the ranks of the following sets in terms of the ranks of x and y: $\cup x$, $P(x)$, $\{x\}$, $x \times y$, $x \cup y$, $x \cap y$, $\{x, y\}$, $\langle x, y\rangle$, and ${}^y x$.

(ii) Calculate the ranks of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.

2 (i) Suppose that $\mathbf{M}$ is a transitive model of ZF (or a large enough finite fragment of ZF including the Power Set Axiom) and let $x \in \mathbf{M}$. Prove that $P(x)^{\mathbf{M}} = P(x) \cap \mathbf{M}$. Deduce that the Power Set Axiom holds in $\mathbf{M}$ if and only if $\forall x \in \mathbf{M} \exists y \in \mathbf{M}(P(x) \cap \mathbf{M} \subseteq y)$.

(ii) Suppose that V_α reflects (a large enough finite fragment T of) ZFC, let $\beta < \alpha$. Prove that $V_\beta^{V_\alpha} = V_\beta$. Hence complete the second proof that neither ZF nor ZFC is finitely axiomatizable.

3 Prove the basic properties of the hierarchy $\{V_\alpha : \alpha \in \mathbf{Ord}\}$.

(i) $\alpha \leq \beta \rightarrow V_\alpha \subseteq V_\beta$.

(ii) $\alpha < \beta \rightarrow V_\alpha \in V_\beta$.

(iii) V_α is a transitive set: $x \in V_\alpha \rightarrow x \subseteq V_\alpha$.

(iv) $V_\alpha = \{x \in WF : rank(x) < \alpha\}$, where $rank(x) = min\{\beta : x \in V_\beta + 1\}$.

(v) $y \in x \rightarrow rank(y) < rank(x)$.

(vi) $rank(\alpha) = \alpha$.

(vii) $\mathbf{Ord} \cap V_\alpha = \alpha$.

(viii) For every $n \in \omega, \mid V_n \mid < \aleph_0$; $\mid V_\omega \mid = \aleph_0$;
(AC) for all $\alpha \in \mathbf{Ord}, \mid V_{\omega+\alpha} \mid = \beth_\alpha$.

4 Prove the basic properties of the constructible hierarchy $\{L_\alpha : \alpha \in \mathbf{Ord}\}$.

(i) L_α is a set and $L_\alpha \subseteq V_\alpha$.

(ii) L_α is transitive.

(iii) For every $n \in \omega$, $| L_n | < \aleph_0$, and $L_n = V_n$; $L_\omega = V_\omega$.

(iv) For every $\alpha \geq \omega$, $| L_\alpha | = | \alpha |$.

(v) $\alpha \leq \beta \to L_\alpha \subseteq L_\beta$.

(vi) $\alpha < \beta \to L_\alpha \in L_\beta$ and $L_\alpha \subsetneq L_\beta$.

(vii) $\alpha \in L_{\alpha+1}$ and $\alpha \notin L_\alpha$.

5 (i) Suppose that $\mathbf{M}$ is a transitive class. Show that $ZFC \vdash Extensionality^{\mathbf{M}}$ and $ZFC \vdash Regularity^{\mathbf{M}}$.

(ii) For a $L(\in)$-formula $\varphi(x, y)$ with free variables $x, y, y_1, \ldots, y_n$, and a variable z not free in $\varphi(x, y)$, let $(*)_{\varphi(x,y)}$ be the assertion $(\forall x \in a)(\exists! y)\varphi(x, y) \to \exists z \forall y(y \in z \leftrightarrow \exists x(x \in a \wedge \varphi(x, y)))$. The notation $\exists! w\psi$ abbreviates the formula $\exists z \forall w(\psi \leftrightarrow w = z)$, where z is the first variable different from w and not free in ψ. Show that the schema $(*)_{\varphi(x,y)}$ is an equivalent form of the schema of the Axiom of Replacement.

(iii) Prove that the Reflection Principle for $\{V_\alpha : \alpha \in Ord\}$ implies the Axioms of Replacement. [Hint: Suppose $a \in V_\alpha$; referring to part (i), reflect to a V_β, and use the Axioms of Separation to find the right candidate for the image of a.] Comment: sometimes it may prove less onerous to check the Reflection Principle instead of Replacement, e.g., when showing that the class L is an inner model of ZFC, it is equivalent to check that L is an inner model of Levy-Montague set theory LM.

6 **

(i) Consider the assertion that for every formula $\varphi(x_1, \ldots, x_n)$ in the language of set theory, for all ordinals $\omega < \alpha < \beta$, for all $a_1, \ldots, a_n \in V_\alpha$,

$$\varphi(a_1, \ldots, a_n)^{V_\alpha} \leftrightarrow \varphi(a_1, \ldots, a_n)^{V_\beta}.$$

What is your opinion? Can the assertion be proved in ZFC? Would your view change if V_α and V_β were replaced by L_α and L_β? Would your views alter if the assertion were modified to the sharpened form: for all uncountable cardinals $\kappa < \lambda$, for every formula $\varphi(x_1, \ldots, x_n)$ in the language of set theory, for all $a_1, \ldots, a_n \in V_\kappa$,

$$\varphi(a_1, \ldots, a_n)^{L_\kappa} \leftrightarrow \varphi(a_1, \ldots, a_n)^{L_\lambda}.$$

(ii) Suppose there exists an ordinal α such that V_α is a model of ZFC. Show that the least such ordinal α has cofinality ω.

(iii) Suppose κ is a strongly inaccessible cardinal. Prove that V_κ is a model of ZFC. [1]

[1] Unrelated general gossip: apparently, if the existence of inaccessible cardinals were inconsistent with ZFC, marvellous phenomena would appear - one could prove that there are no uncountable Grothendieck universes and the axiom of universes in category theory is false. (An uncountable Grothendieck universe is exactly H_κ for an inaccessible cardinal κ, and the axiom of universes asserts that every set is in such a universe.)

(iv) Deduce that the converse of part (iii) is false.

7 (i) Prove that for every infinite cardinal κ, $2^\kappa = \kappa^\kappa$.

(ii) Let $\kappa, \lambda_i (i \in I)$ be infinite cardinals. Prove:

(a) $\kappa^{\sum_{i \in I} \lambda_i} = \prod_{i \in I} \kappa^{\lambda_i}$;

(b) $(\prod_{i \in I} \lambda_i)^\kappa = \prod_{i \in I} \lambda_i^\kappa$.

TOPICS IN SET THEORY: Exercise Sheet 3

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge

Michaelmas 2012-2013 Dr Oren Kolman

1 Suppose α and β are ordinals. Prove:

(i) $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$; $\alpha \leq \beta$ implies ${\aleph_\alpha}^{\aleph_\beta} = 2^{\aleph_\beta}$.

(ii) $\prod_{0<n<\omega} n = 2^{\aleph_0}$; $\prod_{n<\omega} \aleph_n = {\aleph_\omega}^{\aleph_0}$; $\prod_{\alpha<\omega+\omega} \aleph_\alpha = {\aleph_{\omega+\omega}}^{\aleph_0}$.

(iii) $\alpha < \beta$ implies ${\aleph_{\alpha+1}}^{\aleph_\beta} = {\aleph_\alpha}^{\aleph_\beta}\, \aleph_{\alpha+1}$.

(iv) GCH implies ${\aleph_\alpha}^{\aleph_\beta} = \begin{cases} \aleph_\alpha & \aleph_\beta < cf(\aleph_\alpha); \\ \aleph_{\alpha+1} & cf(\aleph_\alpha) \leq \aleph_\beta \leq \aleph_\alpha; \\ \aleph_{\beta+1} & \aleph_\alpha \leq \aleph_\beta. \end{cases}$

(v) ${\aleph_\omega}^{\aleph_{\omega_1}} = {\aleph_\omega}^{\aleph_0}\, 2^{\aleph_{\omega_1}}$.

(vi) $\alpha < \omega_2$ implies ${\aleph_\alpha}^{\aleph_2} = {\aleph_\alpha}^{\aleph_1}\, 2^{\aleph_2}$.

2 (i) Prove that an infinite cardinal κ is a strong limit cardinal if and only if $\kappa = \beth_\delta$ for some limit ordinal δ .

(ii) (Tarski's recursion formula) Let κ be a limit cardinal and $\lambda > 0$. Let δ be a limit ordinal such that $\lambda < cf(\delta)$. Suppose that $\{\kappa_\xi < \kappa : \xi < \delta\}$ is a strictly increasing sequence of cardinals such that $\kappa = \sum_{\xi<\delta} \kappa_\xi$. Show that $\kappa^\lambda = \sum_{\xi<\delta} {\kappa_\xi}^\lambda$.

(iii) Prove that if $\delta > 0$ is a limit ordinal, then $cf(\beth_\delta) = cf(\delta)$.

3 (i) Suppose κ is a limit cardinal and $\lambda < cf(\kappa)$. Prove $\kappa^\lambda = \sum_{\alpha<\kappa} \mid \alpha \mid^\lambda$.

(ii) Suppose κ is a limit cardinal and $\lambda \geq cf(\kappa)$. Prove $\kappa^\lambda = (\sup_{\alpha<\kappa} \mid \alpha \mid^\lambda)^{cf(\kappa)}$.

(iii) Suppose $\kappa > cf(\kappa)$ is not a strong limit cardinal. Prove $\kappa^{<\kappa} = 2^{<\kappa} > \kappa$.

(iv) Suppose $\kappa > cf(\kappa)$ is a strong limit cardinal. Prove $2^{<\kappa} = \kappa$ and $\kappa^{<\kappa} = \kappa^{cf(\kappa)}$.

(v) $2^{\aleph_0} > \aleph_\omega$ implies ${\aleph_\omega}^{\aleph_0} = 2^{\aleph_0}$.

(vi) $2^{\aleph_1} = \aleph_2$ and ${\aleph_\omega}^{\aleph_0} > \aleph_{\omega_1}$ implies ${\aleph_{\omega_1}}^{\aleph_1} = {\aleph_\omega}^{\aleph_0}$.

(vii) $2^{\aleph_0} \geq \aleph_{\omega_1}$ implies $\gimel(\aleph_\omega) = 2^{\aleph_0}$ and $\gimel(\aleph_{\omega_1}) = 2^{\aleph_1}$.

(viii) $2^{\aleph_1} = \aleph_2$ implies ${\aleph_\omega}^{\aleph_0} \neq \aleph_{\omega_1}$. [HINT: Cofinalities.]

4 Prove there exists a rigid dense subset D of the linear order $(\mathbb{R}, \leq)$, i.e. D has no non-trivial order-automorphisms. [HINT: Predictively list the order-automorphisms from $\mathbb{R}$ into $\mathbb{R}$ (using the observation that these are uniquely determined by their restrictions to $\mathbb{Q}$) and construct two disjoint sets $D \supseteq \mathbb{Q}$ and B by transfinite recursion so that every potential order-automorphism of D maps some element of D into B .]

5 (i) Let $\lambda = cf(\lambda) > \aleph_0$.
Show $U = \{\delta < \lambda : \omega\delta = \delta\}$ and $V = \{\delta < \lambda : \omega^\delta = \delta\}$ are club in λ.

(ii) Suppose $B \subseteq \lambda = cf(\lambda) > \aleph_0$. Show that B is a club of λ if and only if B is the range of a continuous strictly increasing function $f : \lambda \to \lambda$.

(iii) Prove that if $\lambda > cf(\lambda) \geq \aleph_0$, then there is a club C of λ such that no member of C is a regular cardinal. [HINT: Try the range of a continuous function $f : cf(\lambda) \to \{\alpha < \lambda : \alpha > cf(\lambda)\}$.]

(iv) Suppose $\lambda = cf(\lambda) > \aleph_0$ and $f : \lambda \to \lambda$. Show that the set $D = \{\delta < \lambda : \delta$ is closed under $f\}$ is club in λ. Deduce that if F is a family of functions from λ to λ and $| F | < \lambda$, then the set $E = \{\delta < \lambda : \delta$ is closed under $F\}$ is club in λ. Hence or otherwise, show that for a τ-structure M with universe λ, where $| \tau | < \lambda$, the set $\{\delta < \lambda : M \mid \delta$ is an elementary submodel of $M\}$ is a club in λ. [HINT: Skolem functions and the Tarski-Vaught Criterion.]

6 Let ω_1 have the order topology and suppose that $f : \omega_1 \to \mathbb{R}$ is a (topologically) continuous function. Show there exists $\alpha < \omega_1$ such that $\forall \beta > \alpha,\ f(\beta) = f(\alpha)$. [HINT: for each real $\epsilon > 0$ and limit ordinal $\xi > 0$ use the inverse image of $(f(\xi) - \epsilon, f(\xi) + \epsilon)$ to define a regressive function $g_\epsilon(\xi)$, apply Fodor's Theorem and first countability of $\mathbb{R}$.]

7 Let A be a set of cardinality $\kappa = cf(\kappa) > \aleph_0$. A *$\kappa$-filtration* of A is an indexed sequence $\{A_\alpha : \alpha < \kappa\}$ such that for all $\alpha, \beta < \kappa$

(i) $| A_\alpha | < \kappa$;

(ii) $\alpha < \beta$ implies $A_\alpha \subseteq A_\beta$;

(iii) $\delta \in lim(\kappa)$ implies $A_\delta = \cup\{A_\alpha : \alpha < \delta\}$;

(iv) $A = \cup\{A_\alpha : \alpha < \kappa\}$.

(i) Suppose $\{A_\alpha : \alpha < \kappa\}$ and $\{B_\alpha : \alpha < \kappa\}$ are κ-filtrations of A. Show the set $\{\alpha \in \kappa : A_\alpha = B_\alpha\}$ is a club of κ.

(ii) Let $\{A_\alpha : \alpha < \kappa = cf(\kappa)\}$ be a κ-filtration of A. Prove there exists a club C of κ such that for all $\alpha \in C$, $| A_{\alpha^+} \setminus A_\alpha | = | \alpha^+ \setminus \alpha |$ where α^+ is the successor of α in C, i.e. $\alpha^+ = inf\{\beta \in C : \beta > \alpha\}$.

(iii) Suppose $\lambda = cf(\lambda) > \aleph_0$ and $\{A_\alpha : \alpha < \kappa\}$ is a κ-filtration of a set A of cardinality κ. Prove Fodor's Lemma: if S is a stationary subset of κ and $f : S \to A$ is a function such that for all $\alpha \in S$, $f(\alpha) \in A_\alpha$, then there exists a stationary $S' \subseteq S$ such that $f \mid S'$ is constant

8 Prove that $\diamondsuit$ implies there exists $\{Z_\alpha : \alpha < 2^{\aleph_1}\}$ such that

(i) $\forall \alpha < 2^{\aleph_1}$ Z_α is a stationary subset of $\aleph_1$;

(ii) $\forall \alpha < \beta < 2^{\aleph_1}$ $Z_\alpha \cap Z_\beta$ is countable.

9 Suppose that S is a stationary subset of $\lambda = cf(\lambda) > \aleph_0$. Prove that $\diamondsuit_S$ is equivalent to the statement: there exists $\{f_\alpha : \alpha \in S\}$ such that

(i) $\forall \alpha \in S, f_\alpha : \alpha \to \alpha$;

(ii) $\forall f : \lambda \to \lambda, \{\alpha \in S : f \mid \alpha = f_\alpha\}$ is a stationary subset of λ.

10 (i) Show that $\diamondsuit$ implies $\clubsuit$.

(ii) * Prove Devlin's Theorem: $\clubsuit + CH$ implies $\diamondsuit$. Deduce that $\diamondsuit$ is equivalent to $\clubsuit + CH$.

11 Let $\diamondsuit'_S$ denote the following statement: there exists $\{E_\alpha : \alpha \in S\}$ such that

(i) $\forall \alpha \in S, E_\alpha$ is a countable set of subsets of α;

(ii) $\forall X \subseteq \lambda, \{\alpha \in S : X \cap \alpha \in E_\alpha\}$ is a stationary subset of λ.

Prove that $\diamondsuit'_S$ and $\diamondsuit_S$ are equivalent in ZFC.

12 (i) Suppose π is a bijection from λ^+ onto $\lambda \times \lambda^+$. Show there exists a club C of λ^+ such that for all $\delta \in C$, the restriction map $\pi \mid \delta$ is a bijection from δ onto $\lambda \times \delta$.

(ii) For a cardinal λ and a set W, let $[W]^{\leq\lambda} = \{Y \subseteq W : \mid Y \mid \leq \lambda\}$. If $2^\lambda = \lambda^+$, let $\{X_\alpha : \alpha < \lambda^+\}$ be an enumeration of $[\lambda^+]^{\leq\lambda}$ and suppose $Z \subseteq \lambda^+$. Show that for some club C of λ^+, for all $\delta \in C$ there are arbitrarily large $\alpha < \delta$ such that for some $\beta < \delta, Z \cap \alpha = X_\beta$.

(iii) Suppose $cf(\delta) = \kappa > \aleph_0$ and h is a function from $dom(h) \supseteq \delta$ into κ. Prove that the following are equivalent:

(a) h is one-to-one on some club C of δ;

(b) h is strictly increasing on some club D of δ;

(c) $range(h \mid S)$ is unbounded in κ for every stationary subset $S \subseteq \delta$.

13 Open Research Problems.

(i) Juhasz's Problem. Does $\clubsuit$ imply $\neg SH$, i.e. does $\clubsuit$ imply there exists a Suslin tree?

(ii) Assume that $cf(\chi) < \chi$ and $\lambda = \chi^+ = 2^\chi$. Determine whether $\diamondsuit_{S^\lambda_{cf(\chi)}}$ is a theorem of ZFC (or even $ZFC + GCH$).

(iii) Assume that $\lambda = \lambda^{<\lambda} = 2^\mu$ is a regular limit cardinal. Determine whether $\diamondsuit_\lambda$ is a theorem of ZFC.

TOPICS IN SET THEORY: Exercise Sheet 4

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge

Michaelmas 2012-2013 Dr Oren Kolman

This final set of exercises involves, alongside basic drill, some questions that require significant extensions of the material covered in lectures. Attempt a representative selection. Several questions are optional (they presuppose some elementary non-set-theoretic information from algebra, complex analysis, model theory, or topology); some problems are difficult (marked †) or open, and they have been included, along with guidance, to illustrate the richness and flexibility of forcing.

1 Absoluteness results

All formulas and terms are in the vocabulary of ZFC unless otherwise indicated. Suppose that φ, φ_1 and φ_2 are absolute.

(i) Suppose that ψ is a formula with the same free variables as φ such that $ZF \vdash \varphi \leftrightarrow \psi$. Show that ψ is absolute.

(ii) Suppose that $\forall y \varphi_1$, $\exists z \varphi_2$ and ψ have the same free variables; suppose $ZF \vdash \forall y \varphi_1 \leftrightarrow \psi$ and $ZF \vdash \exists z \varphi_2 \leftrightarrow \psi$. Show that ψ is absolute.

(iii) Suppose that $\psi(y)$ and the term t are absolute. Show that $\psi(t)$ is absolute.

(iv) Suppose that t is absolute. Show that $x \in t$ and $t \in x$ are absolute.

(v) Suppose that $\psi(y)$ and t are absolute. Show that $\{y \in t : \psi(y)\}$ is absolute.

(vi) Prove that the following terms and predicates are absolute (in each instance, it suffices to produce a ZF-provably equivalent absolute or Δ_0-formula):

$y \subseteq x$; $z = \{x, y\}$; $z = \{x\}$; $z = \langle x, y \rangle$; $z = \bigcup x$; $z = x \cup y$; $z = x \cap y$; $z = x \setminus y$; $z = x \times y$;
f is a function; $y = dom(f)$; $y = range(f)$;
$y = \emptyset$; $y = s(x)$; $y = 1$; $y = 2$;
$y = f''x$ (where $f''x$ means the application of f to x); $y = f \mid x$;
x is transitive; $x \in Ord$ (remember that Foundation is an axiom of ZFC);
x is a limit ordinal; $x = \omega$.

(vii) Suppose that the term $s(y, z)$ is absolute, and $ZFC \vdash (\forall \alpha) t(\alpha) = s(t \mid \alpha, \alpha)$. Show that the formula $y = t(\alpha)$ is absolute. [For a formula $\varphi(x)$, $\varphi(\alpha)$ abbreviates $(\alpha \in Ord \wedge \varphi(\alpha))$.]

(viii) Prove that the following are absolute:

$\alpha + \beta$; $\alpha \cdot \beta$; α^β; $rank(x)$; y is the transitive closure of x.

(ix) Determine which of the following are *(a)* absolute, *(b)* absolute for V_κ, when κ is a strongly inaccessible cardinal, *(c)* absolute between the ground model and its generic extensions (brief explanations suffice):

$\mathbb{Z}$, $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$, x is countable, $y = P(z)$, α is a cardinal, $\aleph_1$, S is a stationary set, C is a club, $\mathbb{P}$ is a forcing, the partial order $\mathbb{P}$ is a c.c.c. forcing, $Fn(A, B, \aleph_0)$, $Fn(A, B, \aleph_1)$, $\mathbb{T}$ is a Suslin tree (†), $y = \aleph_\delta$, $\delta = \aleph_\delta$, $z = \beth_\xi$, $x \in L_\alpha$, $(\exists \alpha)(x \in L_\alpha)$, $ZFC \vdash \varphi$;

Optional

$\daleth(\lambda)$ (†), M is an R-module over the commutative ring R, $P = NP$, the real Hilbert space ℓ^2, X is a complex Banach space, Y is an inseparable topological space, the Singular Cardinals Hypothesis (†), the Riemann Hypothesis (†).

2 Generic filters and classical theorems

(i) Prove Cantor's theorem on the $\aleph_0$-categoricity of unbounded dense linear orders: if A and B are countable unbounded dense linear orders, then A and B are isomorphic. [HINT. Let $\mathbb{P}$ be the partial order of finite partial isomorphisms between A and B under extension. Show that for all $a \in A$ and $b \in B$, the sets $D_a = \{p : a \in dom(p)\}$ and $R^b = \{q : b \in range(q)\}$ are dense open in $\mathbb{P}$. Consider what properties a filter G generic relative to the family $\{D_a, R^b : a \in A, b \in B\}$ might possess.]

(ii) For a set X and a cardinal λ, let $[X]^{<\lambda} = \{Y \subseteq X : | Y | < \lambda\}$. Let $\mathfrak{I} = \{I \subseteq \omega : \sum_{n \in I} \frac{1}{n} < \infty\}$. Let $\mathbb{Q}$ be the partial order $\{I \subseteq \omega : \sum_{n \in I} \frac{1}{n} < 1\}$ where $I \leq_{\mathbb{Q}} J$ iff $I \subseteq J$. For a cardinal λ, let $HS_{<\lambda}(\mathfrak{I})$ abbreviate the statement:

$$(\forall \mathfrak{H} \in [\mathfrak{I}]^{<\lambda})(\exists I_\infty \in \mathfrak{I})(\forall I \in \mathfrak{H})(I \subseteq^* I_\infty)$$

where for $x, y \in P(\omega), x \subseteq^* y \Leftrightarrow (| x \backslash y | < \omega)$. Prove the assertion $HS_{<\aleph_1}(\mathfrak{I})$. What can be said about $HS_{<2^{\aleph_0}}(\mathfrak{I})$? [HINT. Show $D_J = \{q \in \mathbb{Q} : J \subseteq^* q\}$ is dense in $\mathfrak{I}$ for $J \in \mathfrak{I}$.]

(iii) Consider the following (rash) statement: if the forcing $\mathbb{P}$ has the $\aleph_2$-chain condition and D is a family of less than $2^{\aleph_1}$ dense open sets in $\mathbb{P}$, then there exists a D-generic filter G in $\mathbb{P}$. Is this statement provable, refutable, or independent of ZFC? (Generalization of Martin's Axiom to higher cardinals and discovery of the strongest or optimal form of the generic principle it expresses are elusive; *Martin's Maximum* (MM; see below) and the *Proper Forcing Axiom* (PFA) are the best-known candidates.)

3 Concerning forcings, anti-chains and generic sets

(i) Prove that a filter G is generic in $\mathbb{P}$ over $\mathbb{M}$ if and only if for every maximal anti-chain $A \in \mathbb{M}$ of $\mathbb{P}$ $| G \cap A | = 1$. [HINT. One direction might use AC.]

(ii) A subset $D \subseteq \mathbb{P}$ is:

(1) *pre-dense above* $p \in \mathbb{P}$ if $(\forall q \in \mathbb{P})(q \geq p \rightarrow (\exists d \in D)(d$ and q are compatible)); D is *pre-dense* if D is pre-dense above $0_{\mathbb{P}}$.

(2) *dense above* $p \in \mathbb{P}$ if $(\forall q \in \mathbb{P})(q \geq p \rightarrow (\exists d \in D)(d \geq q))$. (So D is dense in $\mathbb{P}$ if D is dense above $0_{\mathbb{P}}$.)

Suppose that E is pre-dense in $\mathbb{P}$ and G is generic in $\mathbb{P}$ over $\mathbb{M}$. Show that $G \cap E \neq \emptyset$.

Suppose that E is pre-dense above $q \in \mathbb{P}$ and G is generic in $\mathbb{P}$ over $\mathbb{M}$. Show that if $q \in G$, then $G \cap E \neq \emptyset$.

(iii) Deduce that the following are equivalent for a filter $G \subseteq \mathbb{P} \in \mathbb{M}$ where $\mathbb{M}$ is a transitive model of ZFC.

G is generic in $\mathbb{P}$ over $\mathbb{M}$;
$G \cap D \neq \emptyset$ for every dense open subset D of $\mathbb{P}$;
$G \cap C \neq \emptyset$ for every dense subset C of $\mathbb{P}$;
$G \cap B \neq \emptyset$ for every pre-dense subset B of $\mathbb{P}$;
$G \cap A \neq \emptyset$ for every maximal anti-chain A of $\mathbb{P}$.

(iv) Suppose $\mathbb{M}$ is a CTM, $\mathbb{P} \in \mathbb{M}, E \subseteq \mathbb{P}, E \in \mathbb{M}$, and G is generic in $\mathbb{P}$ over $\mathbb{M}$. Prove that either $G \cap E \neq \emptyset$ or $(\exists q \in G)(\forall r \in E)(r$ and q are incompatible). [HINT. Consider $\{p \in \mathbb{P}: (\exists r \in E)(r \leq p)\} \cup \{q \in \mathbb{P}: (\forall r \in E)(r$ and q are incompatible) $\} \in \mathbb{M}$.]

(v) Suppose $\mathbb{M}$ is a CTM and $\mathbb{P} \in \mathbb{M}$ is a separative forcing. Prove that there are $2^{\aleph_0}$ generic filters in $\mathbb{P}$ over $\mathbb{M}$.

4 The forcing relation $\Vdash_{\mathbb{P}}$

Suppose that $\mathbb{P}$ is a non-trivial forcing, $p, q \in \mathbb{P}$, and φ is a formula in the vocabulary of ZFC which may contain $\mathbb{P}$-names. Show:

(i) if $p \Vdash_{\mathbb{P}} \varphi$ and $p \leq_{\mathbb{P}} q$, then $q \Vdash_{\mathbb{P}} \varphi$;

(ii) if $q \Vdash_{\mathbb{P}} \varphi$ and $p \leq_{\mathbb{P}} q \wedge p \neq q$, then $p \Vdash_{\mathbb{P}} \varphi$;

(iii) if $(\nexists r)(p \leq_{\mathbb{P}} r \wedge r \Vdash_{\mathbb{P}} \varphi)$, then $p \Vdash_{\mathbb{P}} \neg\varphi$;

(iv) $(\exists r)(p \leq_{\mathbb{P}} r)(r$ *decides* $\varphi)$, i.e. either $r \Vdash_{\mathbb{P}} \varphi$ or $r \Vdash_{\mathbb{P}} \neg\varphi$;

(v) if p does not decide φ, then $\bigwedge_{i=1,2}(\exists r_i)(p \leq_{\mathbb{P}} r_i)(r_1 \Vdash_{\mathbb{P}} \varphi) \wedge (r_2 \Vdash_{\mathbb{P}} \neg\varphi)$.

5 Names

Suppose G is generic in $\mathbb{P}$ over $\mathbb{M}$.

(i) Suppose $\sigma, \tau \in \mathbb{M}^{\mathbb{P}}$. Show $\sigma_G \cup \tau_G = (\sigma \cup \tau)_G$.

(ii) Suppose $\tau \in \mathbb{M}^{\mathbb{P}}$ and $range(\tau) \subseteq \{\dot{n} : n \in \omega\}$.
Let $\sigma = \{\langle p, \dot{n}\rangle : (\forall q \in \mathbb{P})(\langle q, \dot{n}\rangle \in \tau \leftrightarrow p$ and q are incompatible) $\}$. Show $\sigma_G = \omega \setminus \cup\tau_G$. [HINT. Show the set $\{r \in \mathbb{P} : (\exists q \leq r)(\langle q, \dot{n}\rangle \in \sigma \vee \langle q, \dot{n}\rangle \in \tau\}$ is dense.]

(iii) Suppose $\sigma, \tau \in \mathbb{M}^{\mathbb{P}}$. Show $\sigma_G \cup \tau_G = (\sigma \cup \tau)_G$.

(iv) Suppose A is an anti-chain in $\mathbb{P}$ and for each $a \in A, \tau_a$ is a $\mathbb{P}$-name. Show there exists a $\mathbb{P}$-name τ such that for every $a \in A$, if $a \in G$, then $\tau[G] = \tau_a[G]$, and $\tau[G] = \emptyset$ if $G \cap A = \emptyset$. [HINT. Suppose $\tau_a = \{(q_{a,j}, \tau_{a,j}) : j < i_a\}$. Consider the $\mathbb{P}$-name $\tau = \{(r, \tau_{a,j}) : a \in A, j < i_a, r \geq q_{a,j}$, and $r \geq a\}$ and refer to Question 3 (i).]

6 Nice Names and Bounds for the Continuum

Suppose G is generic in $\mathbb{P}$ over $\mathbb{M}$. A name $\tau \in \mathbb{M}^{\mathbb{P}}$ is a *nice* name for a subset x of $\sigma \in \mathbb{M}^{\mathbb{P}}$ if $\tau = \bigcup\{A_\pi \times \{\pi\} : \pi \in range(\sigma)\}$, where each A_π is an anti-chain in $\mathbb{P}$.

(i) Prove that if $\mathbb{P} \in \mathbb{M}$, then for all $\sigma, \rho \in \mathbb{M}^{\mathbb{P}}$ there exists a nice name τ such that $\Vdash_{\mathbb{P}} (\rho \subseteq \sigma \to \rho = \tau)$. [HINT. For $\pi \in range(\sigma)$, let A_π be maximal relative to the properties (1) $(\forall p \in A_\pi)(p \Vdash \pi \in \rho)$ and (2) A_π is an anti-chain in $\mathbb{P}$; now use Question 3 (iv) to check τ as defined above works.]

(ii) Suppose ($\mathbb{P}$ is a c.c.c. forcing and λ is a cardinal)$^{\mathbb{M}}$. Let $\kappa^* = (|\ \mathbb{P}\ |^{\lambda})^{\mathbb{M}}$. Then $(2^{\lambda} \leq \kappa^*)^{\mathbb{M}[G]}$. [HINT. For $P(\lambda)^{\mathbb{M}[G]}$, count the number of possible nice names for its members, remembering that $\mathbb{P}$ has the countable chain condition.]

(iii) Deduce that if (λ is a cardinal and $\lambda^{\aleph_0} = \lambda)^{\mathbb{M}}$, then there is a generic extension $\mathbb{M}[H]$ such that $(2^{\aleph_0} = \lambda)^{\mathbb{M}[H]}$.

7 Adding Cohen reals and Suslin trees

(i) A tree $\mathbb{T}$ is *ever-branching* if for every $s \in T$, the set $\{t \in T : s \leq_{\mathbb{T}} t\}$ is not linear ordered. Let $\mathbb{M}$ be a CTM such that ($\mathbb{T}$ is an ever-branching Suslin tree)$^{\mathbb{M}}$. Suppose $(\mathbb{P} = Fn(\lambda \times \omega, 2, \aleph_0) \wedge \lambda \geq \aleph_0)^{\mathbb{M}}$. Prove that for any filter G generic in $\mathbb{P}$ over $\mathbb{M}$, $\mathbb{M}[G] \models$ ($\mathbb{T}$ is a Suslin tree).

(ii) Deduce that there is a model of ZFC in which there is a Suslin tree but CH fails. Remark: So the existence of Suslin trees does not imply CH (nor *a fortiori* $\diamondsuit$). It is a theorem of Shelah that adding a Cohen real adds a Suslin tree.

8 Diamonds and Clubs

(i) † Prove that the theory $ZFC + \diamondsuit$ is relatively consistent. [HINT. It may be slightly easier to verify $\diamondsuit$ in its function form (see Exercise Sheet 3, Question 9): let $I = \{\langle \alpha, \zeta \rangle : \zeta < \alpha < {\omega_1}^{\mathbb{M}}\}$ and consider the forcing $\mathbb{Q} = Fn(I, 2, \omega_1)$. Show that $\mathbb{Q}$ is countably complete, and that if G is generic in $\mathbb{Q}$ over $\mathbb{M}$, then in $\mathbb{M}[G]$, a $\diamondsuit$-sequence is given by $\langle (\bigcup G)_\alpha : \alpha < \omega_1 \rangle$. For this, noticing $\mathbb{Q}$ adds no new ω-sequences and ${\omega_1}^{\mathbb{M}} = {\omega_1}^{\mathbb{M}[G]}$, define a sequence of ordinals and conditions forcing an arbitrary given club to intersect the family of guesses for a function $f : \omega_1 \to \omega_1$. (Refer to K. Kunen, *Set Theory*, chapter VII, or S. Shelah, *Proper and Improper Forcing*, chapter 1, if the abyss looms.)]

(ii) Deduce that $\diamondsuit$ is independent of ZFC.

(iii) Show if (λ is a cardinal and $\lambda^{\aleph_0} = \lambda$ and $\diamondsuit)^{\mathbb{M}}$, then there is a generic extension $\mathbb{M}[H]$ such that $(2^{\aleph_0} = \lambda$ and there is a Suslin tree$)^{\mathbb{M}[H]}$.

(iv) Suppose that ($\mathbb{P}$ is a c.c.c. forcing and $|\mathbb{P}| \leq \aleph_1$ and $\diamondsuit)^{\mathbb{M}}$. Show for every G generic in $\mathbb{P}$ over $\mathbb{M}$, $\mathbb{M}[G] \models \diamondsuit$. [HINT. In $\mathbb{M}$, use $\diamondsuit$ to guess nice names for subsets of ω_1.]

(v) Suppose that ($\mathbb{P}$ is a c.c.c. forcing$)^{\mathbb{M}}$ and $\mathbb{M}[G] \models \diamondsuit$. Show $\mathbb{M} \models \diamondsuit$. [HINT. Remember the equivalent characterization of $\diamondsuit$ from Exercise Sheet 3 and the lemma about approximating functions in c.c.c. generic extensions.]

(vi) Optional Prove that $\clubsuit$ is independent of ZFC.

9 The Generalized Δ-System Lemma

(i) Suppose $\lambda < \kappa = cf(\kappa)$ and $\bigwedge_{\alpha<\kappa} \alpha^\lambda < \kappa$. Prove that if $|A| = \kappa$ and $x \in A$ implies $|x| < \lambda$, then there exists $B \subseteq A$ such that $|B| = \kappa$ and $(\exists r)(\forall x \in B)(\forall y \in B)(x \neq y \rightarrow x \cap y = r)$. [HINT. This is a standard result. WLOG $\bigcup A \subseteq \kappa$ and some $\rho < \lambda$ is the order type of every $x = \langle x(\xi) : \xi < \rho\rangle \in A$. Using $\bigwedge_{\alpha<\kappa} \alpha^\lambda < \kappa$ and $\kappa = cf(\kappa)$, let ξ_0 be the minimal ξ such that $\{x(\xi) : x \in A\}$ is cofinal in κ; let $\sigma = sup\{x(\eta) + 1 : x \in A \wedge \eta < \xi_0\}$, so (*) $x \restriction \xi_0 \subseteq \sigma < \kappa$; now define by induction $\{x_\alpha : \alpha < \kappa\}$ such that $x_\alpha(\xi_0) > max\{\sigma, sup\{x_\beta(\eta) : \beta < \alpha \wedge \eta < \rho\}\}$. Use (*) and $\sigma^\lambda < \kappa$ to refine $\{x_\alpha : \alpha < \kappa\}$ and extract a root $r \subseteq \sigma$.]

(ii) Find a family of $\aleph_\omega$ finite sets such that no subfamily of size $\aleph_\omega$ has a root.

10 The Levy collapse and its Basic Properties †

Suppose $S \subseteq Ord$ and λ is a cardinal. Define $Col(\lambda, S) = \{p : p$ is a function and $|p| < \lambda \wedge dom(p) \subseteq S \times \lambda \wedge (\forall(\alpha,\xi) \in dom(p))(p(\alpha,\xi) = 0 \vee p(\alpha,\xi) \in \alpha)\}$, and $p \leq_{Col(\lambda,S)} q$ if and only if $p \subseteq q$. This forcing is called *Levy forcing*. It adds surjections from λ onto every $\alpha \in S$. For example, if $\kappa > \lambda$, then $Col(\lambda, \{\kappa\})$ collapses $|\kappa|$ to λ. In particular, $Col(\lambda, \kappa)$ makes a cardinal κ into the cardinal successor of λ in a generic extension (so the idiom is slightly misleading, since κ remains a cardinal and only those cardinals strictly between λ and κ are collapsed). Levy forcing is frequently used when κ is an inaccessible cardinal, in order to obtain desirable properties on successor (and other) cardinals in generic extensions. But its applications are fundamental and much more far-reaching in large cardinal theory. In the following, λ is a regular cardinal.

(i) $Col(\lambda, S)$ is λ-closed.

(ii) If $\kappa = cf(\kappa) > \lambda$ and either κ is inaccessible, or $\lambda = \omega$, then $Col(\lambda, \kappa)$ has the κ-chain condition. [HINT. Apply the Generalized Δ-System Lemma, as in the case of the c.c.c. for $Fn(A, B, \aleph_0)$.]

(iii) If $Col(\lambda, \kappa)$ has the κ-chain condition, then forcing with $Col(\lambda, \kappa)$ preserves cardinals $\leq \lambda$ and $\geq \kappa$.

(iv) Suppose $\kappa = cf(\kappa)$, $Col(\lambda,\kappa)$ has the κ-chain condition, and G is $Col(\lambda,\kappa)$-generic. Then for any $f \in \mathbb{M}[G]$ such that $f : \gamma \to Ord$, where $\gamma < \kappa$, there exists $\delta < \kappa$ such that $f \in \mathbb{M}[G \cap Col(\lambda,\delta)]$. [HINT. Let $f = \tau_G$ and for $\alpha < \gamma$, let A_α be a maximal anti-chain such that $(\forall p \in A_\alpha)(\exists \xi)(p \Vdash \tau(\dot{\alpha}) = \xi)$. Deduce $(\exists \delta)(p \in \bigcup_{\alpha<\gamma} A_\alpha \wedge dom(p) \subseteq \delta \times \lambda)$. To see $f \in \mathbb{M}[G \cap Col(\lambda,\delta)]$, observe $f(\alpha) = \xi$ if and only if $p \Vdash \tau(\dot{\alpha}) = \dot{\xi}$ where p is the unique member of $G \cap Col(\lambda,\delta)$ (see Question 3).]

11 Optional: Moderately large cardinals do not decide CH †

Recall the arrow notation from the partition calculus: $\kappa \to (\mu)^\alpha_\beta$ is the statement: for every function $f : [X]^\alpha \to B$, where $\mid X \mid \geq \kappa$ and $\mid B \mid = \beta$, there exists $Y \subseteq X$ such that $\mid Y \mid \geq \mu$ and $f \restriction [Y]^\alpha \equiv b$ for some $b \in B$. In this notation, Ramsey's infinite theorem is written $(\forall n \in \omega)(\forall k \in \omega)(\aleph_0 \to (\aleph_0)^n_k)$. Analogously, $\kappa \to (\mu)^{<\alpha}_\beta$ is the statement: for every function $f : [X]^{<\alpha} \to B$, where $\mid X \mid \geq \kappa$ and $\mid B \mid = \beta$, there exists $Y \subseteq X$ such that $\mid Y \mid \geq \mu$ and $f \restriction [Y]^{<\alpha} \equiv b$ for some $b \in B$, where $[X]^{<\alpha} = \bigcup_{\xi<\alpha}[X]^\xi$. In these cases, the function f is called a *colouring* of the α-element subsets of X with β colours (or a B-colouring of X); Y is called a *homogeneous* set for f. A cardinal κ is *Ramsey* if $\kappa \to (\kappa)^{<\omega}_2$.

(i) Prove that the set ${}^\lambda 2$ with the lexicographic order $\preceq_{lex}$ contains no increasing or decreasing sequences of length λ^+. [HINT. Otherwise, suppose H is e.g. $\preceq_{lex}$-increasing, of size λ^+; WLOG, $H = \{h_\alpha : \alpha < \lambda^+\}$ and for some *least* $\gamma \leq \lambda, \forall g, h \in H, g \restriction \gamma \neq h \restriction \gamma$. Now find $\xi^* < \gamma$ such that $\{h_\alpha \restriction \xi^* : \alpha < \lambda^+\}$ has cardinality λ^+.]

(ii) Prove that $2^\lambda \not\to (\lambda^+)^2_2$. [HINT. Otherwise, consider a homogeneous set Y of size λ^+ for the 2-colouring F of $[{}^\lambda 2]^2$ given by $F(\{f,g\}) = 0$ if and only if $f \preceq_{lex} g$. This is due to Sierpiński and Kurepa independently.]

(iii) Prove that if κ is a Ramsey cardinal, then κ is strongly inaccessible. [HINT. Regularity is easy: use the colouring $c(\{\alpha,\beta\}) = 0 \Leftrightarrow (\exists \xi)(\{\alpha,\beta\} \subseteq X_\xi)$ where $\kappa = \bigcup_{\zeta<\gamma} X_\zeta$, $\mid X_\zeta \mid < \kappa$; use part (ii) for strong inaccessibility.]

(iv) Prove that κ is a Ramsey cardinal if and only if for all $\beta < \kappa$, $\kappa \to (\kappa)^{<\omega}_\beta$. [HINT. For the hard direction, if $f : [\kappa]^{<\omega} \to \beta$, define a 2-colouring g as follows: $g(\{\xi_1,\ldots\xi_m\}) = 0 \Leftrightarrow n = 2m \wedge f(\{\xi_1,\ldots\xi_m\}) = f(\{\xi_{m+1},\ldots\xi_{2m}\})$. Notice if H is homogeneous for g and of cardinality κ, then $g \restriction [H]^n \equiv 0$ and so H is also homogeneous for f.]

(v) Suppose that $\mathbb{M} \models (\kappa$ is a Ramsey cardinal and $\mid \mathbb{P} \mid < \kappa)$. Let G be generic in $\mathbb{P}$ over $\mathbb{M}$. Prove that $\mathbb{M}[G] \models (\kappa$ is a Ramsey cardinal). [HINT. If $\mathbb{M}[G] \models (\tau_G$ is a colouring of $[\kappa)]^{<\omega})$, and this is forced by some condition $p \in G$, consider the colouring $g : [\kappa]^{<\omega} \to P(\mathbb{P} \times 2)$ defined by $g(a) = \{\langle q, x\rangle : p \leq_{\mathbb{P}} q \wedge q \Vdash (\tau(\dot{a}) = \dot{x})\}$. Note $g \in \mathbb{M}$ and $P(\mathbb{P} \times 2)$ has cardinality β for some $\beta < \kappa$ (by strong inaccessibility). Use part (iv) in $\mathbb{M}$ to obtain a homogeneous set $Y \in \mathbb{M}$; show $p \Vdash (\dot{Y}$ is homogeneous for τ).]

(vi) Deduce that CH is independent of $ZFC + (\exists\kappa)(\kappa$ is a Ramsey cardinal).

Remark: This type of result in very general form is due to Levy and Solovay (1967). See A. Kanamori, *The Higher Infinite*, Springer, 2009. The power of large cardinals to decide a statement φ is thus circumscribed by the existence of forcings of relatively small size if such forcings can used to prove the independence of φ. (In the optional 25th lecture, part (v) was proved for measurable cardinals.)

12 Optional: GCH and Diamonds

(i) Let λ be a regular cardinal and E be a stationary subset of λ. Let $\diamondsuit^*_\lambda(E)$ denote the statement: there exists a family $\{S_\alpha \in [P(\alpha)]^{<\lambda} : \alpha \in E\}$ such that $\forall X \subseteq \lambda$ there exists a club $C \subseteq \lambda$ such that $X \cap \alpha \in S_\alpha$ for all $\alpha \in C \cap E$. The sequence $\{S_\alpha : \alpha \in E\}$ is called a $\diamondsuit^*_\lambda(E)$-sequence.

(a) Prove $\diamondsuit^*_\lambda(E)$ implies $\diamondsuit_\lambda(F)$ for every F such that $E \cap F$ is stationary in λ. (Note for a stationary subset S of λ, $\diamondsuit_\lambda(S)$ is $\diamondsuit_S$ in the notation of the lecture notes, where the additional distinguishing detail was not required.)

(ii) Suppose $\lambda = 2^\mu = \mu^+$ and $\kappa = cf(\kappa) < \mu$. Let $E = \{\alpha < \lambda : cf(\alpha) = \kappa\}$.

(a) Show there is an enumeration $\{X_\alpha : \alpha < \lambda\}$ of all the bounded subsets of λ.

(b) Assume $\mu = \mu^\kappa$. For $\alpha \in E$, let $S_\alpha = \{\cup Y : Y \subseteq \{X_\beta \cap \alpha : \alpha < \beta\}$ and $| Y | \leq \kappa\}$. Prove $\{S_\alpha : \alpha \in E\}$ is a $\diamondsuit^*_\lambda(E)$-sequence. [HINT. For all $W \subseteq \lambda$, define a club D as follows: let $0 \in D$; if $\alpha \in D$, let α^s, the successor of α in D, be the least ordinal $\gamma > \alpha$ so that for some $\beta < \gamma, X_\beta = W \cap \alpha$. Now consider the set C of limit points of D and check C is as required.]

(c) Assume μ is singular, $cf(\mu) = \rho \neq \kappa$, and for every $\delta < \mu, \delta^\kappa < \mu$. For $\alpha \in E$, let $\{\alpha_i : i < \kappa\}$ be strictly increasing and cofinal in α. Fix a sequence $\{U^\alpha_j : j < \rho\}$ of sets such that $\alpha = \cup_{j<\rho} U^\alpha_j$ and for all $j, | U^\alpha_j | < \mu$. Let $S_\alpha = \{\cup Y : Y \subseteq \{X_\beta \cap \alpha : \beta \in U^\alpha_j\}$ for some j and $| Y | \leq \kappa\}$. Prove $\{S_\alpha : \alpha \in E\}$ is a $\diamondsuit^*_\lambda(E)$-sequence.[HINT. Consider W and D as previously defined. If $\delta \in E$ is a limit point of D, then there exists $\{\gamma_i < \delta : i < \kappa\}$ such that if $I \in [\kappa]^{<\kappa}$, then $\bigcup_{i \in I} X_{\gamma_i} = W \cap \delta$. Since $\rho \neq \kappa$, there exists $j < \rho$ such that $| U^\delta_j \cap \{\gamma_i < \delta : i < \kappa\} | = \kappa$.]

(iii) Assume GCH. Suppose μ is a cardinal and $cf(\mu) = \rho$.

(a) Suppose $\mu = \rho$. Prove $\diamondsuit^*_{\mu^+}(\{\alpha < \mu^+ : cf(\alpha) \neq \rho\})$ holds. [HINT. Use (ii)(b).]

(b) Suppose $\mu \neq \rho$. Prove $\diamondsuit^*_{\mu^+}(\{\alpha < \mu^+ : cf(\alpha) \neq \rho\})$ holds. [HINT. Use (ii)(c).]

(iv) Deduce GCH implies $\diamondsuit_{\mu^+}$ for all $\mu = cf(\mu) \geq \aleph_1$.

Remark: This exercise sketches an early proof of $\diamondsuit_{\mu^+}$ from GCH due to Gregory (1976) and Shelah (1981); in the course lecture notes, there is a short, much simplified, unpublished proof discovered by Peter Komjath, inspired by a new proof of Shelah: S. Shelah, *Diamonds*, Proc. Amer. Math. Soc. 138 (2010), 2151–2161.

13 Optional

(i) Consider the ordinal ω_1+1 with the order topology and let $X=(\omega_1+1)^\omega$ be the topological space with the product topology. Let A be the set of successor ordinals in ω_1. Show that for $\alpha\in A$, the set $G_\alpha=\{f\in X:\alpha\in range(f)\}$ is dense and open in X. Show that the family $\{H_\alpha:\alpha\in A\}$, where $H_\alpha=\{h\in X:h(0)=\alpha\}$, is a collection of non-empty open pairwise disjoint sets.

(ii) Prove the Continuum Hypothesis is equivalent to the statement: in every compact Hausdorff space, the intersection of less than $2^{\aleph_0}$-many dense open sets is non-empty. [HINT. Forwards, the Baire Category Theorem; backwards, part (i) if CH fails.]

14 Optional

(i) Prove that non-isomorphism (of groups) is not absolute. [HINT. Free groups of different infinite cardinalities; collapse. Now state a generalisation.]

(ii) An infinite abelian group A is called *almost free* if every subgroup B of cardinality less than $|A|$ is free. Show the statement that the group $\mathbb{Z}^\omega$ is almost free is independent of ZFC. [HINT. You may take it as proven that $\mathbb{Z}^\omega$ is $\aleph_1$-free, but is not $\aleph_2$-free.]

15 Optional: Forcing and Partial Isomorphisms

Suppose A and B are τ-structures in a vocabulary τ. Say A and B are *partially isomorphic*, denoted $A\simeq_p B$, if some non-empty family $F\subseteq \textsc{Part}(A,B)$ of the partial isomorphisms from A to B is a *back-and-forth set* for A and B:

$$(\forall f\in F)(\forall a\in A)(\exists g\in F)(f\subseteq g\wedge a\in dom(g)) \text{ and}$$
$$(\forall f\in F)(\forall b\in B)(\exists g\in F)(f\subseteq g\wedge b\in range(g)).$$

(i) Prove if τ, A and B are countable, then $A\simeq_p B$. [Use the idea of your proof of Cantor's theorem from a previous exercise.]

(ii) Show the converse of (i) fails. [HINT. Consider the linear orders $\mathbb{Q}$ and $\mathbb{R}$.]

(iii) Show that if two structures A and B are partially isomorphic, then there is a forcing extension in which A and B are isomorphic.

Remark: Partial isomorphism yields a characterization of elementary equivalence in the infinitary language $L_{\infty\omega}$. For a recent introduction to these ideas, see J. Väänänen, *Models and Games*, Cambridge University Press, 2011.

16 OPTIONAL

(i) Prove Erdős's Theorem: if the Continuum Hypothesis holds then there is an uncountable orbit-countable family of entire holomorphic functions on $\mathbb{C}$. [HINT. This is an ingenious elementary argument; see the original paper: P. Erdős, *An interpolation problem associated with the Continuum Hypothesis*, Mich. Math. J. 11, 9-10 (1964), or pages 335-336 in P. Komjath, V. Totik, *Problems and Theorems in Classical Set Theory*, Springer, 2006.]

(ii) Deduce that the Continuum Hypothesis is equivalent to the assertion that some orbit-countable family is uncountable.

(iii) Deduce that the statement (*) every orbit-countable family is countable is independent of ZFC.

17 Let ZFC^- be the theory ZFC – POWER SET AXIOM.

(i) Suppose $\kappa = cf(\kappa) \geq \aleph_1$. Prove that H_κ is a model of ZFC^-.

(ii) Deduce that no proof of the existence of $\mathbb{R}$ avoids some non-trivial use of the POWER SET AXIOM.

18 OPTIONAL

Let ZFC^- be the theory ZFC – POWER SET AXIOM. Suppose $\kappa = cf(\kappa) \geq \aleph_1$. Let N be a countable elementary submodel of H_κ.

(i) Suppose $\varphi(x)$ is a formula in the vocabulary of ZFC (possibly with parameters from N) such that $ZFC^- \vdash (\exists! x)\varphi(x)$. Show if $a \in H_\kappa$ and $H_\kappa \models \varphi[a]$, then $a \in N$.

(ii) Suppose $f \in H_\kappa, dom(f) = \{a_1, \ldots, a_n\} \subseteq N$, $\bigwedge_{1 \leq i \leq n} f(a_i) \in N$. Prove $f \in N$.

(iii) $\omega \in H_\kappa$; $\omega_1 \in H_\kappa$; $\omega \subseteq H_\kappa$.

(iv) If $\{a, A, B, f\} \subseteq N$, $a \in A$, and $f : A \to B$ is a function (in V), then $f(a) \in N$.

(v) If $X \in N$ and X is countable, then $X \subseteq N$.

(vi) For every ordinal $\alpha \in \omega_1 \cup \{\omega_1\}$, $\alpha \cap N$ is an ordinal.

(vii) If $\overline{X} = \{X_\alpha : \alpha \in \omega_1\} \in N$, then $X_\alpha \in N$ for every $\alpha \in \omega_1 \cap N$.

19 OPTIONAL

For a forcing $\mathbb{P}$, a cardinal κ is *large enough* (for $\mathbb{P}$) if $\kappa = cf(\kappa) > \aleph_1$ and the set of dense subsets of $\mathbb{P}$ is an element of H_κ (so, in particular, $\mathbb{P}$, the conditions in $\mathbb{P}$ and every dense subset of $\mathbb{P}$ all belong to H_κ). For a set N, a condition $p \in \mathbb{P}$ is called N*-generic* if for every $D \in N$ which is a dense subset of $\mathbb{P}$, $D \cap N$ is pre-dense above p.

Suppose κ is large enough for $\mathbb{P}$. Prove the following are equivalent:

(1) $\mathbb{P}$ has the countable chain condition;

(2) for every countable elementary submodel N of H_κ, $0_\mathbb{P}$ is N-generic;

(3) every countable subset X of H_κ is contained in a countable elementary submodel N of H_κ such that $0_\mathbb{P}$ is N-generic.

[HINT. For (1) $\Rightarrow$ (2), consider an $A \in N$ maximal relative to the property of being an anti-chain contained in D. For (3) $\Rightarrow$ (1), show if $A \in N$ is a maximal anti-chain, then $\overline{A} = \{p \in P : (\exists q \in A)(q \leq_\mathbb{P} p)\} \in N$ is dense.]

20 Optional: Martin's Maximum

A forcing $\mathbb{P}$ is called *stationary-preserving* if $\mathbb{P}$ does not destroy stationary subsets of ω_1: if $\mathbb{M} \models$ (S is a stationary subset of ω_1), then $\mathbb{M}[G] \models$ (S is a stationary subset of ω_1), whenever G is generic in $\mathbb{P}$ over $\mathbb{M}$. *Martin's Maximum* is the statement MM: for every stationary-preserving forcing $\mathbb{P}$, if $(\forall \alpha < \omega_1)(D_\alpha$ is dense open in $\mathbb{P})$, then there exists a $\{D_\alpha : \alpha < \omega_1\}$-generic filter G in $\mathbb{P}$.

(i) Show that if $\mathbb{P}$ is c.c.c, then $\mathbb{P}$ is stationary-preserving.

(ii) Give an example of a stationary-preserving forcing that has an uncountable anti-chain.

(iii) MM implies $MA_{\aleph_1}$.

Remark: The relative consistency strength of MM is far stronger than that of MA which is equiconsistent with ZFC; MM requires a large cardinal axiom for its consistency.

21 Optional: Normal Functions and Mahlo Cardinals

A (class) function $G : Ord \to Ord$ is called *normal* if G is increasing ($\alpha < \beta \to G(\alpha) < G(\beta)$) and continuous (for all limit $\delta \in Ord, G(\delta) = \bigcup_{\alpha<\delta} G(\alpha)$).

(i) (a) Prove in ZFC that every normal function G has a fixed point: there exists $\delta \in Ord$ such that $G(\delta) = \delta$.

(b) Call the statement "every normal function has a regular fixed point" the *regular fixed point axiom* $RFPA$. Show that $RFPA$ is not provable in ZFC.

(ii) A strongly inaccessible cardinal κ is called *Mahlo* if $\{\alpha < \kappa : \alpha$ is a regular cardinal$\}$ is stationary in κ.

(a) Suppose κ is Mahlo. Show κ is a regular limit cardinal, and κ is the limit of κ inaccessible cardinals.

(b) Prove that if κ is Mahlo, then $V_\kappa \models RFPA$.

22 Open Question: Banach Spaces and Cofinality ††

The *cofinality* of a Banach space E is the least ordinal ξ such that there exists an increasing chain $\langle E_\alpha : \alpha < \xi \rangle$ of proper closed subspaces of E whose union is dense in E. Does every infinite-dimensional Banach space have cofinality ω?

Remark: This is an equivalent reformulation of the *Separable Quotient Problem*: does every infinite-dimensional Banach space X have a separable infinite-dimensional quotient X/Y? See S. Todorcevic, *Combinatorial dichotomies in set theory*, Bull. Symbolic Logic, 17 (2011), 1–72.

23 Open Question: Abelian groups and the replicating cofinality

Let us call the *replicating cofinality*, $rcf(A)$, of an infinite abelian group A, the least ordinal ξ, if it exists, such that there exists an increasing chain $\langle A_\alpha : \alpha < \xi \rangle$ of proper subgroups of A such that $A_\alpha \cong A$ and $A = \bigcup_{\alpha<\xi} A_\alpha$. [More generally, the *replicating cofinality*, $rcf(B)$, of a structure B, is the least ordinal ξ, if it exists, such that $B = \bigcup_{\alpha<\xi} B_\alpha$ for some increasing chain $\langle B_\alpha : \alpha < \xi \rangle$ of proper submodels of B, where $B_\alpha \cong B$.]

(i) What is the replicating cofinality of the free abelian group $\bigoplus_{\alpha<\kappa} \mathbb{Z}$ of cardinality κ, if $\kappa \geq \aleph_1$?

(ii) Open Question† What is the replicating cofinality of the group $\mathbb{Z}^\omega$? Remark: It is known that $\mathbb{Z}^\omega$ is not the union of a countable chain of proper subgroups each isomorphic to $\mathbb{Z}^\omega$; see J. Irwin, A. Blass, *Baer meets Baire: Applications of category arguments and descriptive set theory to* $\mathbb{Z}^\omega$, in: Arnold, D. M. et al. (eds.), *Abelian Groups and Modules*, New York, NY, Marcel Dekker, Lect. Notes Pure Appl. Math. 182, 1996, pp. 193-202.

24 Open Question: Galvin's Conjecture ††

Let κ_D be the least cardinal κ, if it exists, such that for every partial order $\mathbb{P}$, if every suborder of $\mathbb{P}$ of size less than κ can be decomposed into countably many chains, then $\mathbb{P}$ can also be decomposed into countably many chains. *Galvin's Conjecture* states that $\aleph_2$ is a possible value for κ_D. See S. Todorcevic, *Combinatorial dichotomies in set theory*, Bull. Symbolic Logic, 17 (2011), 1–72.

Remark. After the classic papers of Gődel and Cohen, the following are accessible and list many further suggestions for reading and research:

Woodin, W. H., *The Continuum Hypothesis, part I*, Notices Amer. Math. Soc. 48(2001), 567-576.

Woodin, W. H., *The Continuum Hypothesis, part II*, Notices Amer. Math. Soc. 48(2001), 681-690.

Dehornoy, P., *Recent progress on the Continuum Hypothesis (after Woodin)*;

`http://www.math.unicaen.fr/~dehornoy/Surveys/DgtUS.pdf;`
`http://www.math.unicaen.fr/~dehornoy/Surveys/Dgt.pdf.`

Koellner, P., *The Continuum Hypothesis*, Stanford Encyclopaedia of Philosophy, September 2011;

`http://www.logic.harvard.edu/EFI_CH.pdf.`

Steprans, J., *History of the Continuum in the Twentieth Century*, to appear in: Vol. 6, History of Logic;

`http://www.math.yorku.ca/~steprans/Research/PDFSOfArticles/hoc2INDEXED.pdf`

Remark. For research problems in set theory, there are some treasure houses to visit:

Shelah, S., *On what I do not understand (and have something to say): Part I*, Fund. Math. 166(2000), 1–82;

`http://matwbn.icm.edu.pl/ksiazki/fm/fm166/fm16612.pdf.`

And for the model-theoretic pendant:

Shelah, S., *On what I do not understand (and have something to say), model theory*, Math Japonica 51 (2000), 329–377;

`http://shelah.logic.at/files/702.pdf.`

Fremlin, D.H., *Problems*;

`http://www.essex.ac.uk/maths/people/fremlin/problems.pdf.`

Miller, A.W., *Some interesting problems*;

`http://www.math.wisc.edu/~miller/res/problem.pdf.`

S. Todorcevic, *Combinatorial dichotomies in set theory*, Bull. Symbolic Logic, 17 (2011), 1–72.