Part III Representation Theory

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0 Preliminaries

In this course we will study representations of

(a) (finite) symmetric groups

(b) (infinite) general linear groups

all over $\mathbb{C}$, and apply the theory to "classical" invariant theory.

We require no previous knowledge, except for some commutative algebra, ordinary representation theory and algebraic geometry, as outlined below.

Commutative algebra

References: Part III course, Atiyah–Macdonald.

It is assumed that you will know about rings, modules, homomorphisms, quotients and chain conditions (the ascending chain condition, ACC, and descending chain condition, DCC).

Recall: a **chain**, or **filtration**, of submodules of a module M is a sequence $(M_j : 0 \leq j \leq n)$ such that

$$M = M_0 > M_1 > \cdots > M_n > 0$$

The **length** of the chain is n, the number of links. A **composition series** is a maximal chain; equivalently, each quotient M_{j-1}/M_j is **irreducible**.

Results:

1. Suppose M has a composition series of length n. Then every composition series of M has length n, and every chain in M can be extended to a composition series.

2. M has a composition series if and only if it satisfies ACC and DCC.

Modules satisfying both ACC and DCC are called **modules of finite length**. By (1), all composition series of M have the same length, which we denote by $\ell(M)$ and call the **length** of M.

3. (**Jordan–Hölder**) If (M_i) and (M'_i) are composition series then there exists a one-to-one correspondence between the set of quotients (M_{i-1}/M_i) and the set of quotients (M'_{i-1}/M'_i) such that the corresponding quotients are isomorphic.

Remark. For a vector space V over a field K, the following are equivalent:

(i) finite dimension

(ii) finite length

(iii) ACC

(iv) DCC

and if these conditions are satisfied then $\ell(V) = \dim(V)$.

(Ordinary) representation theory (of finite groups)

References: Part II course, James–Liebeck, Curtis–Riener Vol. 1, §1 below.

Work over $\mathbb{C}$ or any field K where $\text{char}(K) \nmid |G|$. Let V be a finite-dimensional vector space.

- $\mathrm{GL}(V)$ is the **general linear group** of K-linear automorphisms of V.
- If $\dim_K(V) = n$, choose a basis $e_1, \ldots, e_n$ of V over K to identify it with K^n. Then $\theta \in \mathrm{GL}(V) \leftrightarrow A_\theta = (a_{ij}) \in \mathrm{M}_n(K)$, where

$$\theta(e_j) = \sum_{i=1}^{n} a_{ij} e_i$$

  and, in fact, $A_\theta \in \mathrm{GL}_n(K)$. This gives rise to a group isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}_n(K)$.
- Given G and V, a **representation** of G on V is a homomorphism

$$\rho = \rho_V : G \to \mathrm{GL}(V)$$

- G **acts linearly** on V if there is a **linear action** $G \times V \to V$, given by $(g, v) \mapsto g \cdot v$, where
  (a) (**action**) $(g_1 g_2) \cdot v = g_1 \cdot (g_2 \cdot v)$ and $e \cdot v = v$
  (b) (**linearity**) $g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2$, $g \cdot (\lambda v) = \lambda(g \cdot v)$
- If G acts linearly on V then the map $G \to \mathrm{GL}(V)$ given by $g \mapsto \rho_g$, where $\rho_g(v) = g \cdot v$, defines a representation of G on V. Conversely, given a representation $\rho : G \to \mathrm{GL}(V)$ we have a linear action of G on V given by $g \cdot v = \rho(g)v$.

  In this case, we say V is a G-**space** or G-**module**. By linear extension, V is a KG-module, where KG is the **group algebra** – more to come later.

For finite groups you should know

- If S is an irreducible KG-module then S is a composition factor of KG
- *(Maschke's theorem)* If G is finite and $\mathrm{char}(K) \nmid G$, then ever KG-module is completely reducible; or equivalently, every submodule is a summand.
- The number of inequivalent (ordinary) irreducible representations of G is equal to the number of conjugacy classes of G.
- If S is an irreducible $\mathbb{C}G$-module and M is any $\mathbb{C}G$-module, then the number of composition factors of M isomorphic to S is equal to $\dim \mathrm{Hom}_{\mathbb{C}G}(S, M)$.

Algebraic geometry

References: Part II course, Reid, §6 below.

You should know about affine varieties, polynomial functions, the Zariski topology (in particular, Zariski denseness) and the Nullstellensatz.

1 Semisimple algebras

Conventions All rings are associative and have an identity, which we denote by 1 or 1_R. Modules are always left R-modules.

(1.1) Definition A $\mathbb{C}$**-algebra** is a ring R which is also a $\mathbb{C}$-vector space, whose notions of addition coincide, and for which

$$\lambda(rr') = (\lambda r)r' = r(\lambda r')$$

for all $r, r' \in R$ and $\lambda \in \mathbb{C}$. There are obvious notions of **subalgebras** and **algebra homomorphisms**.

Usually, our algebras will be finite-dimensional over $\mathbb{C}$.

(1.2) Examples and remarks

(a) $\mathbb{C}$ is a $\mathbb{C}$-algebra, and $M_n(\mathbb{C})$ is a $\mathbb{C}$-algebra.

If G is a finite group then the **group algebra** $\mathbb{C}G = \{\sum_g \alpha_g g \,:\, \alpha_g \in \mathbb{C}\}$ is a $\mathbb{C}$-algebra with

- pointwise addition: $\sum_g \alpha_g g + \sum_g \beta_g g = \sum_g (\alpha_g + \beta_g) g$
- multiplication: $\sum_g \left(\sum_{hh'=g} \alpha_h \beta_{h'}\right) g$

Recall the correspondence between representations of G over $\mathbb{C}$ and finite-dimensional $\mathbb{C}G$-modules.

(b) If R, S are $\mathbb{C}$-algebras then the **tensor product** $R \otimes S = R \otimes_{\mathbb{C}} S$ is a $\mathbb{C}$-algebra. [More on $\otimes$ can be found in Atiyah–MacDonald p. 30.]

(c) If $1_R = 0_R$ then $R = \{0_R\}$ is the **zero ring**. Otherwise, for $\lambda, \mu \in \mathbb{C}$, we have $\lambda 1_R = \mu 1_R$ if and only if $\lambda = \mu$, so we can identify $\lambda \in \mathbb{C}$ with $\lambda 1_R \in R$.

With the identification $\mathbb{C} \hookrightarrow R$ we can view $\mathbb{C}$ as a subalgebra of R.

(d) An R-module M becomes a $\mathbb{C}$-space via $\lambda m = (\lambda 1_R)m$ for $\lambda \in \mathbb{C}, m \in M$.

Given any R-module N, $\mathrm{Hom}_R(M, N)$ is a $\mathbb{C}$-subspace of $\mathrm{Hom}_{\mathbb{C}}(M, N)$. In particular, it is a $\mathbb{C}$-space, with structure

$$(\lambda\varphi)(m) = \lambda\varphi(m) = \varphi(\lambda m) \quad \forall \lambda \in \mathbb{C},\ m \in M,\ \phi \in \mathrm{Hom}_R(M, N)$$

(e) If M be an R-module. Then $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$ is a $\mathbb{C}$-algebra with multiplication given by composition of maps:

$$(\varphi\psi)(m) = \varphi(\psi(m)) \quad \forall m \in M,\ \phi, \psi \in \mathrm{End}_R(M)$$

In particular, if V is a $\mathbb{C}$-space then $\mathrm{End}_{\mathbb{C}}(V)$ is an algebra. Recall that $\mathrm{End}_{\mathbb{C}}(V) \cong \mathrm{M}_n(\mathbb{C})$, where $n = \dim_{\mathbb{C}}(V)$.

(f) Let R be a $\mathbb{C}$-algebra and $X \subseteq R$ be a subset. Define the **centraliser** of X in R by

$$c_R(X) = \{r \in R \,:\, rx = xr \ \forall x \in X\}$$

It is a subalgebra of R. The **centre** of R is $c_R(R) = Z(R)$.

(g) Let R be $\mathbb{C}$-algebra and M be a R-module. Then the map $\alpha R \to \mathrm{End}_{\mathbb{C}}(M)$ given by $\alpha(r)(m) = rm$ for $r \in R$, $m \in M$ is a map of $\mathbb{C}$-algebras.

Now, $\mathrm{End}_R(M) = c_{\mathrm{End}_{\mathbb{C}}(M)}(\alpha R)$, and so $\alpha R \subseteq c_{\mathrm{End}_{\mathbb{C}}(M)}(\mathrm{End}_R(M))$.

(h) An R-module M is itself naturally an $\mathrm{End}_R(M)$-module, where the action is evaluation, and the action commutes with that of R. The inclusion α as in (g) says that elements of R act as $\mathrm{End}_R(M)$-module endomorphisms of M.

Given an R-module N, $\mathrm{Hom}_R(M, N)$ is also an $\mathrm{End}_R(M)$-module, where the action is composition of maps.

(1.3) Lemma Let R be a finite-dimensional $\mathbb{C}$-algebra. Then there are only finitely many isomorphism classes of irreducible R-modules. Moreover, all the irreducible R-modules are finite-dimensional.

Proof. Let S be an irreducible R-module and pick $0 \neq x \in S$. Define a map $R \to S$ by $r \mapsto rx$. This map has nonzero image, so its image is S by irreducibility. In particular,

$$\dim_{\mathbb{C}}(S) \leq \dim_{\mathbb{C}}(R) < \infty$$

But S must occur in any composition series of R, so by the Jordan–Hölder theorem, there are only finitely many isomorphism classes of irreducible modules. □

(1.4) Lemma *(Schur's lemma)* Let R be a finite-dimensional $\mathbb{C}$-algebra. Then

(a) If $S \not\cong T$ are nonisomorphic irreducible R-modules then $\mathrm{Hom}_R(S,T) = 0$.

(b) If S is an irreducible R-module then $\mathrm{End}_R(S) \cong \mathbb{C}$.

Proof

(a) If $f : R \to S$ is a map of algebras then $\ker(f) \leq R$ and $\mathrm{im}(f) \leq S$. By irreducibility, $\ker(f) = 0$ or R and $\mathrm{im}(f) = 0$ or S. If $\ker(f) = 0$ then $R \cong \mathrm{im}(f) = S$, contradicting nonisomorphism; so we must have $\ker(f) = R$ and $\mathrm{im}(f) = 0$, i.e. $f = 0$.

(b) Clearly $D = \mathrm{End}_R(S)$ is a division ring. Since S is finite-dimensional, so is D. Thus, if $d \in D$, then the elements $1, d, d^2, \dots$ are linearly dependent. Let $p \in \mathbb{C}[X]$ be a nonzero polynomial with $p(d) = 0$.

Since $\mathbb{C}$ is algebraically closed, p factors as

$$p(X) = c(X - a_1) \cdots (X - a_n)$$

for some $0 \neq c \in \mathbb{C}$ and $a_i \in \mathbb{C}$. Hence

$$(d - a_1 1_D) \cdots (d - a_n 1_D) = 0$$

But, being a division ring, D has no zero divisors, so one of the terms must be zero. Thus $d = a_j 1_D \in \mathbb{C}1_D$. But $d \in D$ was arbitrary, so $D = \mathbb{C}1_D \cong \mathbb{C}$. □

(1.5) Definition An R-module is **semisimple** (or **completely reducible**) if it is a direct sum of irreducible submodules. A finite-dimensional $\mathbb{C}$-algebra R is semisimple if it is so as an R-module.

(1.6) Proposition

(a) Every submodule of a semisimple module is semisimple, and every such submodule is a direct summand.

(b) Every quotient of a semisimple module is semisimple.

(c) Direct sums of semisimple modules are semisimple.

In fact, M is semisimple if and only if every submodule of M is a direct summand. The proofs of this and of **(1.6)** are left as an exercise.

Remarks

1. R a semisimple $\mathbb{C}$-algebra $\Rightarrow$ any R-module is semisimple.
2. G a finite group $\Rightarrow$ $\mathbb{C}G$ is semisimple. (This is precisely Maschke's theorem.)

(1.7) Definition If R and S are $\mathbb{C}$-algebras, M is an R-module and N is an S-module, then $M \otimes N$ has the structure of an R-module via

$$r(m \otimes n) = rm \otimes n, \qquad r \in R,\ m \in M,\ n \in N$$

and the structure of an S-module via

$$s(m \otimes n) = m \otimes sn, \qquad s \in S,\ m \in M,\ n \in N$$

and these actions commute:

$$r(s(m \otimes n)) = r(m \otimes sn) = rm \otimes sn = s(rm \otimes n) = s(r(m \otimes n))$$

hence the images of R, S in $\mathrm{End}_{\mathbb{C}}(M \otimes N)$ commute.

If N has basis $\{e_1, \ldots, e_k\}$ then the map $M^{\oplus k} \to M \times N$ given by

$$(m_1, \ldots, m_k) \mapsto m_1 \otimes e_1 + \cdots + m_k \otimes e_k$$

is an isomorphism of R-modules.

Similarly, if M has basis $\{f_1, \ldots, f_\ell\}$ then the map $N^{\oplus \ell} \to M \otimes N$ given by

$$(n_1, \ldots, n_\ell) \mapsto n_1 \otimes f_1 + \cdots + n_\ell \otimes f_\ell$$

is an isomorphism of S-modules.

(1.8) Lemma Let M be a semisimple R-module. Then the **evaluation map**

$$\bigoplus_S S \otimes \mathrm{Hom}_R(S, M) \to M$$

is an isomorphism of R-modules and of $\mathrm{End}_R(M)$-modules. Here S runs over a complete set of nonisomorphic irreducible R-modules and we use the action of R on S and of $\mathrm{End}_R(M)$ on $\mathrm{Hom}_R(S, M)$.

Proof Check that the map is a map of R-modules and of $\mathrm{End}_R(M)$-modules. It's an isomorphism of vector spaces, by observing that we can reduce to the case where M is irreducible and apply Schur's lemma **(1.4)**. □

(1.9) Lemma Let M be a finite-dimensional semisimple R-module. Then

$$\mathrm{End}_R(M) \cong \prod_S \mathrm{End}_{\mathbb{C}} \mathrm{Hom}_R(S, M)$$

where S runs over a complete set of nonisomorphic irreducible R-modules.

Proof Observe that the product $E = \prod_S \mathrm{End}_{\mathbb{C}} \mathrm{Hom}_R(S, M)$ acts naturally on $S \otimes \mathrm{Hom}_R(S, M)$ and, since this action commutes with that of R, there exists a homomorphism $E \to \mathrm{End}_R(M)$, which is injective.

Now count dimensions by **(1.8)**, M is isomorphic to a direct sum of $\dim_{\mathbb{C}} \mathrm{Hom}_R(S, M)$ copies of each irreducible module S, so by **(1.4)**,

$$\dim_{\mathbb{C}} \mathrm{End}_R(S, M) = \textstyle\sum_S (\dim_{\mathbb{C}} \mathrm{Hom}_R(S, M))^2 = \dim_{\mathbb{C}} E \qquad \square$$

(1.10) Theorem *(Artin–Wedderburn Theorem)* Any finite-dimensional semisimple $\mathbb{C}$-algebra is isomorphic to a product

$$R = \prod_{i=1}^{k} \mathrm{End}_{\mathbb{C}}(V_i)$$

where V_i are finite-dimensional vector spaces. Conversely, if R has this form then it is semisimple, the nonzero V_is form a complete set of nonisomorphic irreducible R-modules and, as an R-module, R is isomorphic to a direct sum of $\dim_{\mathbb{C}} V_i$ copies of each V_i.

Proof We'll take this in steps.

Step 1 (Deal with the product) Without loss of generality, assume each V_i is nonzero. The V_i are naturally R-modules, with the factors other than $\mathrm{End}_{\mathbb{C}}(V_i)$ acting as 0.

Now $\mathrm{End}_{\mathbb{C}}$, and hence R, has transitively on $V_i - \{0\}$, thus the V_i are irreducible R-modules.

If V_i has basis $\{e_{i1}, \ldots, e_{im_i}\}$ then the map

$$R \to \underbrace{V_1 \oplus \cdots \oplus V_1}_{m_1 \text{ copies}} \oplus \cdots \oplus \underbrace{V_k \oplus \cdots \oplus V_k}_{m_k \text{ copies}}$$

given by

$$(f_1, \ldots, f_k) \mapsto (f_1(e_{11}), \ldots, f_1(e_{1m_1}), \ldots, f_k(e_{k1}), \ldots, f_k(e_{km_k}))$$

is an injective map of R-modules, and hence is an isomorphism by counting dimensions. Thus R is semisimple.

The V_i form a complete set of irreducible R-modules (by Jordan–Höldeer, as in **(1.3)**), and they are nonisomorphic: if $i \neq j$ then $(0, \ldots, 0, 1, 0, \ldots, 0)$ annihilates V_j but not V_i.

Step 2 Let R be any ring. Now the natural map

$$\alpha : R \to \mathrm{End}_{\mathrm{End}_R(R)}(R)$$

sending r to the map $x \mapsto rx$ is an isomorphism. It's injective, and if $\theta \in \mathrm{End}_{\mathrm{End}_R(R)}(R)$ then it commutes with the endomorphisms $\alpha_r \in \mathrm{End}_R(R)$, where $\alpha_r(x) = xr$. Now if $r \in R$ then

$$\theta(r) = \theta(1r) = \theta(\alpha_r(1)) = \alpha_r(\theta(1)) = \theta(1)r$$

so θ acts by left-multiplication. Hence $\theta = \alpha(\theta(1)) \in \alpha(R)$, so α is surjective.

Assume R is a semisimple $\mathbb{C}$-algebra. Then $\mathrm{End}_R(R)$ is semisimple by **(1.9)**, and so R is a semisimple $\mathrm{End}_R(R)$-module. By **(1.9)** and the isomorphism α, R has the required form. □

(1.11) Lemma If R is semisimple and M is a finite-dimensional R-module then the natural map

$$\alpha : R \to \mathrm{End}_{\mathrm{End}_R(M)}(M)$$

sending r to the map $m \mapsto rm$ is surjective.

Proof Let $I = M^\circ = \{r \in R : rM = 0\}$ be the annihilator of M; then $\ker \alpha = I$. Now, R/I is semisimple and M is an R/I-module, and $\mathrm{End}_R(M) = \mathrm{End}_{R/I}(M)$. So we can replace R by R/I and suppose that M is faithful (and α is injective).

By **(1.9)**, $\mathrm{End}_R(M) = \prod_S \mathrm{End}_{\mathbb{C}} \mathrm{Hom}_R(S, M)$ and, since M is faithful, all the spaces $\mathrm{Hom}_R(S, M)$ are nonzero because they are precisely the simple $\mathrm{End}_R(M)$-modules. By **(1.8)**, M is isomorphic (as an $\mathrm{End}_R(M)$-module) to $\bigoplus_S S \otimes \mathrm{Hom}_R(S, M)$, so it's the direct sum of $\dim_{\mathbb{C}} S$ copies of the simple module $\mathrm{Hom}_R(S, M)$ for each S.

As in **(1.9)**, this implies that

$$\dim_{\mathbb{C}} \mathrm{End}_{\mathrm{End}_R(M)}(M) = \sum_S (\dim_{\mathbb{C}} S)^2$$

But this is equal to $\dim_{\mathbb{C}} R$, so α is an isomorphism. □

Finally, it's worth noting:

(1.12) Lemma Let R be a $\mathbb{C}$-algebra and let $e \in R$ be 'almost idempotent' in the sense that $e^2 = \lambda e$ for some $0 \neq \lambda \in \mathbb{C}$. Then for any R-module M, we have an isomorphism of $\mathrm{End}_R(M)$-modules

$$\mathrm{Hom}_R(Re, M) \cong eM$$

Proof eM is an $\mathrm{End}_R(M)$-submodule of M: if $\varphi \in \mathrm{End}_R(M)$ and $em \in eM$ then $\varphi(em) = e\varphi(m) \in eM$.

Replacing e by $\frac{e}{\lambda}$, we may suppose that e is idempotent. Now ther exists a map of $\mathrm{End}_R(M)$-modules

$$\mathrm{Hom}(Re, M) \to eM$$

given by $\varphi \mapsto \varphi(e)$, whose inverse sends m to the map $r \mapsto rm$. □

Chapter I: Representation theory of the symmetric group

2 Irreducible modules for the symmetric group

Recall the correspondence between representations $\rho : G \to \mathrm{GL}(V)$ and $\mathbb{C}G$-modules given by setting $g \cdot v = \rho(g)v$ for $g \in G$ and $v \in V$. The **trivial representation** $G \to \mathbb{C}^\times = \mathrm{GL}(C)$ is given by $g \mapsto 1$, and the corresponding $\mathbb{C}G$-module is $\mathbb{C}$, sometimes written $\mathbb{C}_G$ to emphasise the context.

If $X \subseteq \{1, \dots, n\}$, write S_X for the subgroup of S_n fixing every number outside of X.

Let $\varepsilon : \mathrm{S}_n \to \{\pm 1\}$ be the **sign representation**, and write

$$\varepsilon_\sigma = \prod_{1 \le i < j \le n} \frac{\sigma(i) - \sigma(j)}{i - j}$$

Throughout this section, write $A = \mathbb{C}\mathrm{S}_n$; this is a finite-dimensional semisimple $\mathbb{C}$-algebra.

(2.1) Definition For $\lambda_i \in \mathbb{N}_0$, we say $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ is a **partition** of n, and write $\lambda \vdash n$, if $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \cdots$ and $\sum_i \lambda_i = n$. Write $(\lambda_1^{a_1}, \lambda_2^{a_2}, \dots, \lambda_k^{a_k})$ to denote

$$(\underbrace{\lambda_1, \dots, \lambda_1}_{a_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{a_2}, \dots, \underbrace{\lambda_k, \dots, \lambda_k}_{a_k}, 0, 0, \dots)$$

with $\lambda_1 > \lambda_2 > \cdots > \lambda_k$.

Partitions of n correspond with conjugacy classes of S_n, e.g.

$$(5, 2^2, 1) \;\leftrightarrow\; (\bullet\ \bullet\ \bullet\ \bullet\ \bullet)(\bullet\ \bullet)(\bullet)$$

If $\lambda, \mu \vdash n$, write $\lambda < \mu$ if and only if there is some $i \in \mathbb{N}$ with $\lambda_j = \mu_j$ for all $j < i$ and $\lambda_i < \mu_i$. This is the **lexicographic** (or **dictionary**) **order** on partitions. It is a total order on the set of partitions of n. E.g.

$$(5) > (4, 1) > (3, 2) > (3, 1^2) > (2^2, 1) > (2, 1^3) > (1^5)$$

(2.2) Definition Partitions $\lambda \vdash n$ may be geometrically represented by their **Young diagram**

$$[\lambda] = \{(i, j) \,:\, i \ge 1,\ 1 \le j \le \lambda_i\} \subseteq \mathbb{N} \times \mathbb{N}$$

If $(i, j) \in [\lambda]$ then it is called a **node** of $[\lambda]$. The k^{th} **row** (resp. **column**) of a diagram consists of those nodes whose first (resp. second) coordinate is k.

Example If $\lambda = (4, 2^2, 1)$ then

$$[\lambda] \quad = \quad \begin{array}{llll} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ \bullet & \bullet & & \\ \bullet & & & \end{array} \qquad \begin{array}{l} \leftarrow \lambda_1 \\ \leftarrow \lambda_2 \\ \\ \leftarrow \lambda_3 \end{array}$$

The **height** (or **length**) of λ is the length of the first column of $[\lambda]$, i.e. $\max\{j \,:\, \lambda_j \ne 0\}$, and is denoted by $\mathrm{ht}(\lambda)$.

(2.3) Definition A λ-**tableau** t_λ is one of the $n!$ arrays of integers obtained by replacing each node in $[\lambda]$ by one of the integers $1, \dots, n$, with no repeats. E.g. the following are $(4, 3, 1)$-tableaux

$$t_1 \quad = \quad \begin{array}{llll} 1 & 2 & 4 & 5 \\ 3 & 6 & 7 & \\ 8 & & & \end{array} \quad , \qquad t_2 \quad = \quad \begin{array}{llll} 4 & 5 & 7 & 3 \\ 2 & 1 & 8 & \\ 6 & & & \end{array}$$

Equivalently, a λ-tableau can be regarded as a bijection $[\lambda] \to \{1, 2, \dots, n\}$.

S_n acts on the set of $[\lambda]$-tableaux, e.g. (1 4 7 8 6)(2 5 3) sends t_1 to t_2. Formally, given a tableau t_λ and $\pi \in S_n$, the composition of the functions π and t_λ gives a new λ-tableau πt_λ, where $(\pi t_\lambda)(x) = \pi(t_\lambda(x))$.

Given a partition $\lambda \vdash n$, we want to pick one representative of all the corresponding λ-tableaux. Let t_λ° be the **standard λ-tableau**, i.e. the tableau numbered in order from left to right and from top to bottom, e.g.

$$t^\circ_{(5,2^2,1)} = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 \\ 8 & 9 \\ 10 \end{matrix}$$

Define the **row stabilizer** R_t and **column stabilizer** C_t of a tableau t by

$$\begin{array}{lcl} \sigma \in R_t & \Leftrightarrow & \text{each } i \in \{1,\ldots,n\} \text{ is in the same row of } t \text{ and } \sigma t \\ \sigma \in C_t & \Leftrightarrow & \text{each } i \in \{1,\ldots,n\} \text{ is in the same column of } t \text{ and } \sigma t \end{array}$$

For instance, if $t = t_1$ as in the examples after **(2.3)** above, then

$$\begin{array}{lcl} R_{t_1} & = & S_{\{1,2,4,5\}} \times S_{\{3,6,7\}} \times S_{\{8\}} \\ C_{t_1} & = & S_{\{1,3,8\}} \times S_{\{2,6\}} \times S_{\{4,7\}} \times S_{\{1\}} \end{array}$$

(2.4) Definition Let t_λ be a λ-tableau. The **Young symmetrizer** is

$$h(t_\lambda) = \sum_{r \in R_{t_\lambda},\ c \in C_{t_\lambda}} \varepsilon_c rc \in A$$

Write $h_\lambda = h(t_\lambda^\circ)$.

Examples

(a) If $\lambda = (n)$, then $h = h(t_{(n)}) = \sum_{\sigma \in S_n} \sigma$, called the **symmetrizer** in A.
Since $\rho \cdot h = h$ for all $\rho \in S_n$, $Ah = \mathbb{C}h$ is the trivial representation.

(b) If $\lambda = (1^n)$, then $h = h(t_{(1^n)}) = \sum_{\sigma \in S_n} \varepsilon_\sigma \sigma$, called the **alternizer** in A.
Since $\rho \cdot h = \varepsilon_\rho h$, $Ah = \mathbb{C}h$ is the sign representation.

Goal The left-ideals in A of the form Ah_λ for $\lambda \vdash n$ (known as **Specht modules**) form a complete set of nonisomorphic irreducible A-modules: see **(2.14)** below.

(2.5) Lemma If $\lambda \vdash n$, $t = t_\lambda$ is a tableau, $\sigma \in S_n$.

(a) $R_t \cap C_t = 1$;

(b) The coefficient of 1 in $h(t_\lambda)$ is 1;

(c) $R_{\sigma(t)} = \sigma R_t \sigma^{-1}$ and $C_{\sigma(t)} = \sigma C_t \sigma^{-1}$;

(d) $h(\sigma t_\lambda) = \sigma h(t_\lambda) \sigma^{-1}$;

(e) $Ah(t_\lambda) \cong Ah_\lambda$ as A-modules.

Proof (a)–(d) are clear.

For (e), $t_\lambda = \sigma t_\lambda^\circ$ for some permutation $\sigma \in S_n$. Postmultiplication by σ then gives an isomorphism $Ah(t_\lambda) \cong Ah_\lambda$. □

The following lemma is sometimes called the *basic combinatorial lemma.*

(2.6) Lemma Let $\lambda, \mu \vdash n$ with $\lambda \geq \mu$, and let t_λ and t_μ be tableaux. Then one of the following is true:

(A) There exist distinct integers i and j occurring in the same row of t_λ and the same column of t_μ;

(B) $\lambda = \mu$ and $t_\mu = rct_\lambda$ for some $r \in R_{t_\lambda}$ and $c \in C_{t_\lambda}$.

Proof Suppose (A) is false. If $\lambda_1 \neq \mu_1$ then $\lambda_1 > \mu_1$, so $[\mu]$ has fewer columns than $[\lambda]$. Hence two of the numbers in the first row of t_λ are in the same column of t_μ, so (A) holds, contradicting our assumption. So $\lambda_1 = \mu_1$.

Since (A) fails, some $c_1 \in C_{t_\mu}$ forces $c_1 t_\mu$ to have the same members in the first row of t_λ. Now ignore the first rows of t_λ and $c_1 t_\mu$. The same argument implies that $\lambda_2 = \mu_2$ and there is $c_2 \in C_{t_\mu}$ such that t_λ and $c_2 c_1 t_\mu$ have the same numbers in each of their first two rows.

Continuing in this way we see that $\lambda = \mu$ and there exists $c' \in C_{t_\lambda}$ such that t_λ and $c' t_\mu$ have the same numbers in each row. Then $rt_\lambda = c' t_\mu$ for some $r \in R_{t_\lambda}$.

Now define

$$c = r^{-1}(c')^{-1}r \in r^{-1}c'C_{t_\mu}(c')^{-1}r = r^{-1}C_{c't_\mu}r = C_{c't_\mu} = C_{t_\lambda}$$

Then $t_\mu = rct_\lambda$. □

(2.7) Lemma If $\sigma \in \mathrm{S}_n$ cannot be written as rc for any $r \in R_{t_\lambda}$ and $c \in C_{t_\lambda}$ then there exist transpositions $u \in R_{t_\lambda}$ and $v \in C_{t_\lambda}$ such that $u\sigma = \sigma v$.

Proof Since **(2.6)**(B) fails for t_λ and σt_λ, there exist integers $i \neq j$ in the same row of t_λ and the same column of σt_λ. Take $u = (i\ j)$ and $v = \sigma^{-1}u\sigma$. □

(2.8) Lemma Given a tableau t_λ and $a \in A$, the following are equivalent:

(1) $rac = \varepsilon_c a$ for all $r \in R_{t_\lambda}$ and $c \in C_{t_\lambda}$;

(2) $a = kh(t_\lambda)$ for some $k \in \mathbb{C}$.

Proof

(2)⇒(1) As r' runs through R_{t_λ} so does rr', and as c runs through C_{t_λ} so does $c'c$ with $\varepsilon_{c'c} = \varepsilon_{c'}\varepsilon_c$. Calculation reveals that $rh(t_\lambda)c = \varepsilon_c h(t_\lambda)$.

(1)⇒(2) Let $a = \sum_{\sigma \in \mathrm{S}_n} a_\sigma \sigma$. If σ is not of the form rc then by **(2.7)** there exist transpositions $u \in R_{t_\lambda}$ and $v \in C_{t_\lambda}$ such that $u\sigma v = \sigma$.

By assumption, $uav = \varepsilon_v a$, and the coefficient of σ gives $a_{u\sigma v} = \varepsilon_v a_\sigma$, so $a_\sigma = -a_\sigma$, and hence $a_\sigma = 0$. The coefficient of rc in (1) gives $a_1 = \varepsilon_c a_{rc} \in \mathbb{C}$. Thus

$$a = \sum_{r,c} a_{rc} rc = \sum_{rc} \varepsilon_c a_1 rc = a_1 h(t_\lambda)$$

as required □

(2.9) Lemma If $a \in A$ then $h(t_\lambda)ah(t_\lambda) = kh(t_\lambda)$ for some $k \in \mathbb{C}$.

Proof $x = h(t_\lambda)ah(t_\lambda)$ satisfies **(2.8)**(1). □

(2.10) Proposition Define $f_\lambda = \dim_{\mathbb{C}} Ah_\lambda$. Then

(a) $h(t_\lambda)^2 = \dfrac{n!}{f_\lambda} h(t_\lambda)$

(b) $f_\lambda \mid n!$

(c) $h(t_\lambda)Ah(t_\lambda) = \mathbb{C}h(t_\lambda)$

In particular, $\dfrac{f_\lambda}{n!} h(t_\lambda)$ is idempotent.

Proof

(a) Let $h = h(t_\lambda)$. We already know that $h^2 = kh$ for some $k \in \mathbb{C}$. Right-multiplication by h induces a linear map $\hat{h} : A \to A$. For $a \in A$, $(ah)h = k(ah)$, so $\hat{k}|_{Ah}$ is multiplication by k.

Extend a basis of Ah to a basis of A. With respect to this basis,

$$\hat{h} \sim \begin{pmatrix} kI_{f_\lambda} & * \\ 0 & 0 \end{pmatrix}$$

and so $\mathrm{tr}(\hat{h}) = kf_\lambda$.

With respect to the basis S_n of A, $\hat{h}$ has matrix H whose entries are

$$H_{\sigma\tau} = \text{coefficient of } \sigma \text{ in } \tau h$$

But $H_{\sigma\tau} = 1$, so $\mathrm{tr}(\hat{h}) = n!$.

Hence $k = \dfrac{n!}{f_\lambda}$.

(b) The coefficient of 1 in $h^2 = kh$ is $\sum \varepsilon_{c_1}\varepsilon_{c_2} \in \mathbb{Z}$ where the sum runs over all $r_1, r_2 \in R_{t_\lambda}$ and $c_1, c_2 \in C_{t_\lambda}$ such that $r_1c_1r_2c_2 = 1$.

(c) By **(2.9)**, the only other possibility is $hAh = 0$; but $h^2 \neq 0$. □

(2.11) Proposition $Ah(t_\lambda)$ is an irreducible A-module.

Proof Put $h = h(t_\lambda)$. Then $Ah \neq 0$ and A is semisimple, so it is enough to show that Ah is indecomposable.

Suppose $Ah = U \oplus V$. Then $\mathbb{C}h = hAh = hU \oplus hV$, so one of hU and hV is nonzero; without loss of generality, $hU \neq 0$. Then $hU = \mathbb{C}h$, so $Ah = AhU \subseteq U$, so $U = Ah$ and $V = 0$. □

(2.12) Lemma If $\lambda > \mu$ are partitions and t_λ, t_μ are tableaux then $h(t_\mu)Ah(t_\lambda) = 0$.

Proof By **(2.6)** there exist integers in the same row of t_λ and the same column of t_μ. Let $\tau \in R_{t_\lambda} \cap C_{t_\mu}$ be the corresponding transposition. Then

$$h(t_\mu)h(t_\lambda) = h(t_\mu)\tau\tau h(t_\lambda) = -h(t_\mu)h(t_\lambda)$$

so $h(t_\mu)h(t_\lambda) = 0$.

Applying this to $(\sigma t_\lambda, t_\mu)$ for $\sigma \in \mathrm{S}_n$ gives

$$0 = h(t_\mu)h(t_\lambda) = h(t_\mu)\sigma h(t_\lambda)\sigma^{-1}$$

and so $h(t_\mu)\sigma h(t_\lambda) = 0$.

Thus $h(t_\mu)Ah(t_\lambda) = 0$.

(2.13) Lemma If $\lambda \neq \mu$ and t_λ, t_μ are tableaux, then $Ah(t_\lambda) \not\cong Ah(t_\mu)$.

Proof Assume without loss of generality that $\lambda > \mu$. If there exists an A-module isomorphism $f : Ah(t_\lambda) \cong Ah(t_\mu)$ then

$$f(\mathbb{C}h(t_\lambda)) \overset{(\mathbf{2.10})}{=} f(h(t_\lambda)Ah(t_\lambda)) = h(t_\lambda)f(Ah(t_\lambda)) = h(t_\lambda)Ah(t_\mu) \overset{(\mathbf{2.12})}{=} 0$$

which is a contradiction. □

(2.14) Theorem The left-ideals $\{Ah_\lambda : \lambda \vdash n\}$ form a complete set of nonisomorphic irreducible A-modules.

Proof They are irreducible by **(2.11)** and nonisomorphic by **(2.13)** and, since

$$|\{\lambda : \lambda \vdash n\}| = |\{\text{ccls of } \mathrm{S}_n\}| = |\{\text{irreducible } \mathbb{C}\mathrm{S}_n\text{-modules}\}|$$

they form a complete set. □

Further reading Chapter 4 of James's lecture notes (Springer) – alternative proof using 'polytabloids' and certain permutation modules.

3 Standard basis of Specht modules

We first redefine what we mean by a 'standard' tableau.

(3.1) Definition t is a **standard** tableau if the numbers increase along the rows (left to right) and columns (top to bottom) of t.

The standard tableaux of shape λ are ordered such that $t_\lambda < t'_\lambda$ if t_λ is smaller than t'_λ in the first place they differ when you read $[\lambda]$ 'like a book'. E.g. the standard $(3,2)$-tableaux are

$$\begin{matrix}1&2&3\\4&5&\end{matrix} \quad < \quad \begin{matrix}1&2&4\\3&5&\end{matrix} \quad < \quad \begin{matrix}1&2&5\\3&4&\end{matrix} \quad < \quad \begin{matrix}1&3&4\\2&5&\end{matrix} \quad < \quad \begin{matrix}1&3&5\\2&4&\end{matrix}$$

Let F_λ be the number of standard λ-tableaux.

Goal $F_\lambda = f_\lambda$; recall from **(2.10)** that $f_\lambda = \dim_{\mathbb{C}} Ah_\lambda$.

First we show $\sum_{\lambda \vdash n} F_\lambda^2 = n!$.

Notation λ/μ means there is some m with $\lambda \vdash m$, $\mu \vdash m-1$ and $[\mu] \subseteq [\lambda] \subseteq \mathbb{N} \times \mathbb{N}$.

This idea will be used in induction arguments.

(3.2) Lemma Let $\lambda \vdash m$. Then $F_\lambda = \sum_{\mu \,:\, \lambda/\mu} F_\mu$.

Proof If t_λ is standard then $t_\lambda^{-1}(\{1,2,\ldots,m-1\})$ is the diagram of a partition $\mu \vdash m-1$; and $t_\lambda|_{[\mu]}$ must be standard. The converse is similar. □

(3.3) Lemma If $\lambda \neq \pi$ are partitions of m then

$$|\{\nu \,:\, \nu/\lambda \text{ and } \nu/\pi\}| = |\{\tau \,:\, \lambda/\tau \text{ and } \pi/\tau\}| \in \{0,1\}$$

Proof If ν/λ and ν/π then $[\nu] \supseteq [\lambda] \cup [\pi]$. Since $[\lambda]$ and $[\pi]$ differ in at least one place, we must in fact have $[\nu] \supseteq [\lambda] \cup [\pi]$.

Likewise, if λ/τ and μ/τ then $[\tau] \supseteq [\lambda] \cap [\pi]$ and we must have $[\tau] = [\lambda] \cap [\pi]$.

Now $[\lambda] \cup [\pi]$ and $[\lambda] \cap [\pi]$ are partitions, so

$$\nu \text{ exists} \;\Leftrightarrow\; |[\lambda] \cup [\pi]| = m+1 \;\Leftrightarrow\; |[\lambda] \cap [\pi]| = m-1 \;\Leftrightarrow\; \tau \text{ exists}$$

This completes the proof. □

(3.4) Lemma If $\lambda \vdash m$ then $(m+1)F_\lambda = \sum_{\nu \,:\, \nu/\lambda} F_\nu$.

Proof by induction on m.

The case $m = 1$ is clear. Suppose the statement holds true for partitions of $m-1$. Then

$$\sum_{\nu \,:\, \nu/\lambda} F_\nu \overset{\textbf{(3.2)}}{=} \sum_{\nu \,:\, \nu/\lambda} \left(\sum_{\pi \,:\, \nu/\pi} F_\pi \right) = |\{\nu \,:\, \nu/\lambda\}| F_\lambda + \sum_{\nu,\pi \,:\, \nu/\lambda, \pi \neq \lambda} F_\pi$$

Using the fact that $|\{\nu \,:\, \nu/\lambda\}| = |\{\tau \,:\, \lambda/\tau\}| + 1$ together with **(3.3)** gives that our expression is equal to

$$(|\{\tau \,:\, \lambda/\tau\}| + 1)F_\lambda + \sum_{\tau,\pi \,:\, \lambda/\tau, \pi/\tau, \pi \neq \lambda} F_\pi$$

$$= F_\lambda + \sum_{\tau,\pi \,:\, \lambda/\tau, \pi/\tau} F_\pi = F_\lambda + \sum_{\tau \,:\, \lambda/\tau} mF_\tau = F_\lambda + mF_\lambda = (m+1)F_\lambda$$

as required. $\square$

(3.5) Lemma $\sum_{\lambda \vdash m} F_\lambda^2 = m!$

Proof by induction on m.

The case $m = 1$ is clear. Suppose the statement holds true for partitions of $m-1$. Then

$$\sum_{\lambda \vdash m} F_\lambda^2 \overset{(3.2)}{=} \sum_{\lambda \vdash m, \lambda / \tau} F_\lambda F_\tau \overset{(3.4)}{=} \sum_{\tau \vdash m-1} m F_\tau^2 \;=\; m \cdot (m-1)! \;=\; m!$$

as required. $\square$

(3.6) Lemma If $t_\lambda > t'_\lambda$ are standard then $h(t_\lambda)h(t'_\lambda) = 0$.

Proof It suffices to show that there are integers $i \neq j$ lying in the same row of t'_λ and the same column of t_λ. For then the corresponding transposition $\tau = (i\,j) \in R_{t'_\lambda} \cap C_{t_\lambda}$ gives the result as in the proof of **(2.12)**.

It remains to find such integers i and j. Let $x \in [\lambda]$ be the first place where t_λ and t'_λ first differ. Let $t'_\lambda = i$ and $y = (t'_\lambda)^{-1}(i) \in [\lambda]$. Then y must be below and to the left of x since t_λ is standard. In particular, x cannot lie in the first column or the last row. Let $z \in [\lambda]$ be in the same row as x and the same column of y, and let $j = t_\lambda(z) = t'_\lambda(z)$ be the common value of t_λ and t'_λ at z.

It is now elementary to verify that i and j satisfy the above assumption. $\square$

(3.7) Theorem $\mathbb{C}S_n = \bigoplus \mathbb{C}S_n h(t_\lambda)$, where the direct sum runs over all standard tableaux t_λ for all partitions $\lambda \vdash n$.

Proof The sum is direct: suppose we have a nontrivial relation $\sum a(t_\lambda)h(t_\lambda)$ with $a(t_\lambda) \in \mathbb{C}S_n$ are not all zero. Choose μ maximal (in the ordering on partitions) such that some $a(t_\mu)h(t_\lambda) \neq 0$, and for this μ pick t'_μ minimal (in the ordering on standard μ-tableaux) such that $a(t'_\mu)h(t'_\mu) \neq 0$. Multiplying the relation on the right by $h(t'_\mu)$ gives $a(t'_\mu)h(t'_\mu)^2 = 0$ by **(3.6)** and **(2.12)**, and hence $a(t'_\mu)h(t'_\mu) = 0$, contradicting our assumption.

It is clear that the $\bigoplus \mathbb{C}S_n h(t_\lambda) \leq \mathbb{C}S_n$. By Artin–Wedderburn, $\mathbb{C}S_n \cong \bigoplus f_\lambda \mathbb{C}S_n h_\lambda$ as $\mathbb{C}S_n$-modules, while in our direct sum there are F_λ copies of $\mathbb{C}h_\lambda$. By Jordan–Hölder, $F_\lambda \leq f_\lambda$. But $f_\lambda \leq F_\lambda$ since, by **(3.5)**, $\sum F_\lambda^2 = n! = \sum f_\lambda^2$. Hence $F_\lambda = f_\lambda$ for each $\lambda \vdash n$, and so $\mathbb{C}S_n \leq \bigoplus \mathbb{C}S_n h(t_\lambda)$. $\square$

(3.8) Corollary The number of standard λ-tableaux is equal to $\dim_{\mathbb{C}}(\mathbb{C}S_n h_\lambda)$.

Proof $H_\lambda = h_\lambda$ as in the proof of **(3.7)**. $\square$

(3.9) Lemma Let M be a finite-dimensional $\mathbb{C}S_n$-module. Then $M \cong \bigoplus h(t_\lambda)M$, where the direct sum runs over all standard tableaux t_λ for all partitions $\lambda \vdash n$.

Proof If $\sum m(t_\lambda) = 0$ is a nontrivial relation with $m(t_\lambda) \in h(t_\lambda)M$ then choose μ minimal and t'_μ maximal with $m(t'_\mu) \neq 0$. Premultiply the relation by $h(t'_\mu)$ to contradict **(3.6)** and **(2.12)**. Finally,

$$\bigoplus h(t_\lambda)M \overset{(1.12)}{\cong} \bigoplus \mathrm{Hom}_{\mathbb{C}S_n}(\mathbb{C}S_n h(t_\lambda), M)$$

$$\cong \mathrm{Hom}_{\mathbb{C}S_n}\left(\bigoplus \mathbb{C}S_n h(t_\lambda), M\right) \cong \mathrm{Hom}_{\mathbb{C}S_n}(\mathbb{C}S_n, M) \cong M$$ $\square$

4 Character formula

Given a finite-dimensional $\mathbb{C}G$-algebra M, its **character** is defined by

$$\chi_M(g) = \operatorname{tr}\begin{pmatrix} M \to M \\ m \mapsto gm \end{pmatrix}$$

and χ_M is a **class function** $G \to \mathbb{C}$, i.e. a function that is constant on the conjugacy classes of G. If α is a conjugacy class in G, write $\chi_M(\alpha)$ for the common value of χ_M on α.

Given $\lambda \vdash n$, denote the character of the irreducible $\mathbb{C}\mathrm{S}_n$-module $\mathbb{C}\mathrm{S}_n h_\lambda$ by χ^λ. We will calculate $\chi^\lambda(\alpha)$ and later use the formula to derive Weyl's character formula for $\mathrm{GL}_n(\mathbb{C})$.

Now S_n-class α comprises all permutations with fixed cycle type $n^{\alpha_n}\cdots 2^{\alpha_2}1^{\alpha_1}$, i.e. having α_n n-cycles, $\ldots$, α_2 2-cycles and α_1 1-cycles. Let $n_\alpha = |\alpha|$.

(4.1) Lemma $\quad n_\alpha = \dfrac{n!}{1^{\alpha_1}2^{\alpha_2}\ldots n^{\alpha_n}\cdot \alpha_1!\alpha_2!\ldots\alpha_n!}$

Proof Any permutation in α is one of the $n!$ of the form

$$\underbrace{(*)(*)\cdots(*)}_{\alpha_1}\underbrace{(**)(**)\cdots(**)}_{\alpha_2}\cdots\underbrace{(\overbrace{*\cdots*}^{n})(\overbrace{*\cdots*}^{n})\cdots(\overbrace{*\cdots*}^{n})}_{\alpha_n}$$

Each such permutation can be represented in $1^{\alpha_1}\ldots n^{\alpha_n}\cdot\alpha_1!\ldots\alpha_n!$ ways: each of the α_k k-cycles can be represented in k ways by cycling their components, and each block of α_k k-cycles can be represented in $\alpha_k!$ ways by permuting the position of the factors. □

(4.2) Theorem *(orthogonality relations)* Let $\lambda, \mu \vdash n$ and let α and β be conjugacy classes in S_n. Then

(a) $\displaystyle\sum_{\text{ccls }\alpha} n_\alpha\chi^\lambda(\alpha)\chi^\mu(\beta) = \begin{cases} n! & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$

(b) $\displaystyle\sum_{\lambda\vdash n}\chi^\lambda(\alpha)\chi^\lambda(\beta) = \begin{cases} n!/n_\alpha & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$

Proof Every element of S_n is real, i.e. conjugate to its inverse, and so

$$\overline{\chi^\lambda(\sigma)} = \chi^\lambda(\sigma^{-1}) = \chi^\lambda(\sigma)$$

so $\chi^\lambda(\sigma) \in \mathbb{R}$ for all $\sigma \in \mathrm{S}_n$. These relations are then just the usual ones for general finite groups. □

Notation Given $x = (x_1, \ldots, x_n) \in \mathbb{C}^m$, $\ell_1, \ldots, \ell_m \in \mathbb{Z}$, define

$$|x^{\ell_1}, \ldots, x^{\ell_m}| = \det(x_i^{\ell_j})$$

Usually $\ell_j \geq 0$, in which case this is a homogeneous polynomial of degree $\displaystyle\sum_{j=1}^m \ell_j$ in $x_1, \ldots, x_m$.

Example The Vandermonde determinant is given by

$$V(x) = |x^{m-1}, \ldots, 1|$$

(4.3) Lemma $\quad V = V(x) = \prod_{i<j}(x_i - x_j)$

Proof Exercise. □

In fact, if $\ell_i \geq 0$ for each i then $|x^{\ell_1}, \ldots, x^{\ell_m}|$ is divisible by V. We define the **Schur polynomial** s_ℓ by

$$s_\ell(x) = \frac{|x^{\ell_1}, \ldots, x^{\ell_m}|}{|x^{m-1}, \ldots, 1|}$$

This is indeed a polynomial in $x_1, \dots, x_m$, provided that $\ell_i \geq 0$.

Example $s_{(1,2,\dots,n)}(x) = x_1 \cdots x_m$.

(4.4) Lemma *(Cauchy's lemma)* If $x_i, y_j \in \mathbb{C}$ for $1 \leq i, j \leq m$ and $x_i y_j \neq 1$ for all i, j, then

$$\det\left(\frac{1}{1 - x_i y_j}\right) = \left|x^{m-1}, \dots, 1\right| \left|y^{m-1}, \dots, 1\right| \prod_{i,j=1}^{m} \frac{1}{1 - x_i y_j}$$

Proof by induction on m. The case $m = 1$ is clear. Let $m > 1$.

Now

$$\frac{1}{1 - x_i y_j} - \frac{1}{1 - x_1 y_j} = \frac{x_i - x_1}{1 - x_1 y_j} \cdot \frac{y_j}{1 - x_i y_j}$$

So subtracting the first row from each other row in the determinant, one can remove the factor $(x_i - x_1)$ from each row $i \neq 1$ and $\dfrac{1}{1 - x_1 y_j}$ from each column. So

$$\det\left(\frac{1}{1 - x_i y_j}\right) = \left(\prod_{i>1}(x_i - x_1)\right)\left(\prod_{j=1}^{m} \frac{1}{1 - x_1 y_j}\right) \underbrace{\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{y_1}{1-x_2 y_1} & \frac{y_2}{1-x_2 y_2} & \cdots & \frac{y_n}{1-x_2 y_n} \\ \frac{y_1}{1-x_3 y_1} & \frac{y_2}{1-x_3 y_2} & \cdots & \frac{y_n}{1-x_3 y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{y_1}{1-x_n y_1} & \frac{y_2}{1-x_n y_2} & \cdots & \frac{y_n}{1-x_n y_n} \end{pmatrix}}_{=\Delta}$$

Now subtract the first column in the matrix above from the other columns and use the fact that

$$\frac{y_j}{1 - x_i y_j} - \frac{y_1}{1 - x_i y_1} = \frac{y_j - y_1}{1 - x_i y_1} \cdot \frac{1}{1 - x_i y_j}$$

This gives

$$\Delta = \left(\prod_{j>1}(y_j - y_1)\right)\left(\prod_{i=2}^{m} \frac{1}{1 - x_i y_1}\right) \det\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ * & \frac{1}{1-x_2 y_2} & \frac{1}{1-x_2 y_3} & \cdots & \frac{1}{1-x_2 y_n} \\ * & \frac{1}{1-x_3 y_2} & \frac{1}{1-x_3 y_3} & \cdots & \frac{1}{1-x_3 y_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \frac{1}{1-x_n y_2} & \frac{1}{1-x_n y_3} & \cdots & \frac{1}{1-x_n y_n} \end{pmatrix}$$

The result now follows by applying the induction hypothesis to this last determinant. □

(4.5) Lemma Choose $x_i, y_j \in \mathbb{C}$ for $1 \leq i, j \leq n$ with modulus < 1. Then

$$\det\left(\frac{1}{1 - x_i y_j}\right) = \sum_{\ell_1 > \ell_2 > \cdots > \ell_m \geq 0} \left|x^{\ell_1}, \dots, x^{\ell_m}\right| \left|y^{\ell_1}, \dots, y^{\ell_m}\right|$$

Proof We use the expansion

$$\frac{1}{1 - x_i y_j} = \sum_{\ell \geq 0} (x_i y_j)^{\ell}$$

The determinant is thus

$$\sum_{\sigma \in \mathrm{S}_m} \varepsilon_\sigma \prod_{i=1}^{m} \left(1 + x_i y_{\sigma(i)} + (x_i y_{\sigma(i)})^2 + \cdots\right)$$

and the monomial $x_1^{\ell_1} \cdots x_m^{\ell_m}$ $(\ell_i \in \mathbb{N})$ occurs with coefficient

$$\sum_{\sigma \in \mathrm{S}_m} \prod_{i=1}^{m} y_{\sigma(i)}^{\ell_i} = \left|y^{\ell_1}, \dots, y^{\ell_m}\right|$$

In particular, it this term is zero unless the ℓ_i are all distinct. So

$$
\begin{aligned}
\det\left(\frac{1}{1-x_iy_j}\right) &= \sum_{\ell_1,\dots,\ell_m \text{ distinct}} x_1^{\ell_1}\dots x_m^{\ell_m}\left|y^{\ell_1},\dots,y^{\ell_m}\right| \\
&= \sum_{\ell_1>\dots>\ell_m\ge 0}\left(\sum_{\pi\in \mathrm{S}_m} x_1^{\ell_{\pi(1)}}\dots x_m^{\ell_{\pi(m)}}\left|y^{\ell_{\pi(1)}},\dots,y^{\ell_{\pi(m)}}\right|\right) \\
&= \sum_{\ell_1>\dots>\ell_m\ge 0}\left(\sum_{\pi\in \mathrm{S}_m} x_1^{\ell_{\pi(1)}}\dots x_m^{\ell_{\pi(m)}}\varepsilon_\pi\left|y^{\ell_1},\dots,y^{\ell_m}\right|\right) \\
&= \sum_{\ell_1>\dots>\ell_m\ge 0}\left|x^{\ell_1},\dots,x^{\ell_m}\right|\left|y^{\ell_1},\dots,y^{\ell_m}\right|
\end{aligned}
$$

which is what we needed. □

Notation If $\lambda = (\lambda_1,\dots,\lambda_m)\in\mathbb{Z}^m$, let $\ell_i = \lambda_i + m - i$, i.e. $\ell_1 = \lambda_1 + m - 1$, …, $\ell_m = \lambda_m$. Sometimes in the literature, the ℓ_i are denoted by β_i and are thus called 'β-numbers'. Note that

(1) $\lambda_1\ge\dots\ge\lambda_m$ if and only if $\ell_1>\dots>\ell_m$

(2) If $\sum_i\lambda_i = n$ and $\lambda_i\ge 0$ then $\dfrac{|x^{\ell_1},\dots,x^{\ell_m}|}{|x^{m-1},\dots,1|}$ is a polynomial of degree m in the x_is.

Write $\Lambda^+(m,n)$ for the set of all partitions λ of n into $\le m$ parts.

(4.6) Definition Given $x_1,\dots,x_m,y_1,\dots,y_m\in\mathbb{C}$, then for any $i\in\mathbb{N}$, set

$$s_i = x_1^i+\dots+x_m^i,\qquad t_i = y_1^i+\dots+t_m^i$$

These are called the **power sums** (sometimes referred to in the literature as **Newton sums**).

(4.7) Lemma $$\sum_{\lambda\in\Lambda^+(m,n)}\frac{|x^{\ell_1},\dots,x^{\ell_m}|}{|x^{m-1},\dots,1|}\frac{|y^{\ell_1},\dots,y^{\ell_m}|}{|y^{m-1},\dots,1|} = \frac{1}{n!}\sum_{\text{ccls }\alpha} n_\alpha s_1^{\alpha_1}\dots s_n^{\alpha_n}t_1^{\alpha_1}\dots t_n^{\alpha_n}$$

Note that both quotients on the left-hand side genuinely are polynomials (in fact, Schur polynomials), so they make sense even if the x_i, y_i are not distinct.

Proof Both sides are polynomials, so without loss of generality we may take all the x_i, y_i to have modulus < 1. Then

$$
\begin{aligned}
\log\prod_{i,j=1}^n\frac{1}{1-x_iy_j} &= \sum_{i,j=1}^n\left(\frac{x_iy_j}{1}+\frac{(x_iy_j)^2}{2}+\frac{(x_iy_j)^3}{3}+\cdots\right) \\
&= \frac{s_1t_1}{1}+\frac{s_2t_2}{2}+\frac{s_3t_3}{3}+\cdots
\end{aligned}
$$

and hence

$$
\begin{aligned}
\prod_{i,j=1}^n\frac{1}{1-x_iy_j} &= \exp\left(\frac{s_1t_1}{1}+\frac{s_2t_2}{2}+\frac{s_3t_3}{3}+\cdots\right) \\
&= \sum_{n=0}^\infty\frac{1}{n!}\left(\frac{s_1t_1}{1}+\frac{s_2t_2}{2}+\frac{s_3t_3}{3}+\cdots\right)^n \\
&= \sum_{(*)}\left(\frac{1}{n!}\cdot\frac{n!}{\alpha_1!\alpha_2!\cdots}\left(\frac{s_1t_1}{1}\right)^{\alpha_1}\left(\frac{s_2t_2}{2}\right)^{\alpha_2}\cdots\right)
\end{aligned}
$$

where $(*)$ denotes the sum over all n and all sequences $\alpha_1,\alpha_2,\dots$ of nonnegative integers with only finitely many nonzero terms and $\alpha_1+\alpha_2+\dots = n$. But then

$$\prod_{i,j=1}^n\frac{1}{1-x_iy_j} = \sum_{(*)}\frac{s_1^{\alpha_1}s_2^{\alpha_2}\cdots t_1^{\alpha_1}t_2^{\alpha_2}\cdots}{1^{\alpha_1}2^{\alpha_2}\cdots\alpha_1!\alpha_2!\cdots}$$

By **(4.4)**(Cauchy) and **(4.5)**, we have

$$\sum_{\ell_1>\ell_2>\cdots>\ell_m\geq 0} \frac{\left|x^{\ell_1},\ldots,x^{\ell_m}\right|}{|x^{m-1},\ldots,1|}\frac{\left|y^{\ell_1},\ldots,y^{\ell_m}\right|}{|y^{m-1},\ldots,1|} = \prod_{i,j=1}^{m}\frac{1}{1-x_iy_j} = \sum_{(*)}\frac{s_1^{\alpha_1}s_2^{\alpha_2}\cdots t_1^{\alpha_1}t_2^{\alpha_2}\cdots}{1^{\alpha_1}2^{\alpha_2}\cdots\alpha_1!\alpha_2!\cdots}$$

so we can equate terms which are of degree n in the x_i to obtain the desired result. □

(4.8) Definition For $\lambda = (\lambda_1,\ldots,\lambda_m) \in \mathbb{Z}^m$ and α a conjugacy class in S_n, let $\psi_\lambda(\alpha)$ be the coefficient of the monomial $x_1^{\lambda_1}\ldots x_m^{\lambda_m}$ in $s_1^{\alpha_1}\ldots s_n^{\alpha_n}$, i.e.

$$s_1^{\alpha_1}\ldots s_n^{\alpha_n} = \sum_{\lambda\in\mathbb{Z}^m}\psi_\lambda(\alpha)x_1^{\lambda_1}\ldots x_m^{\lambda_m}$$

Remarks

(1) ψ_λ is a class function for S_n.

(2) $\psi_\lambda(\alpha)=0$ if any λ_i is negative or if $\sum_{i=1}^{m}\lambda_i \neq n$.

(3) ψ_λ is a symmetric function of the λ_is.

Notation Set

$$\omega_\lambda(\alpha) = \sum_{\pi\in\mathrm{S}_m}\varepsilon_\pi\psi_{(\ell_{\pi(1)}+1-m,\ \ell_{\pi(2)}+2-m,\ \ldots\ ,\ \ell_{\pi(m)})}(\alpha)$$

The idea now is to show that $\omega_\lambda = \chi^\lambda$ whenever $\lambda\vdash n$.

(4.9) Lemma *(Character formula)*

$$s_1^{\alpha_1}\ldots s_n^{\alpha_n}\left|x^{m-1},\ldots,1\right| = \sum_{\lambda\in\Lambda^+(m,n)}\omega_\lambda(\alpha)\left|x^{\ell_1},\ldots,x^{\ell_m}\right|$$

Proof The left-hand side is equal to

$$\sum_{\lambda\in\mathbb{Z}^m}\psi_\lambda(\alpha)x_1^{\lambda_1}\ldots x_m^{\lambda_m}\sum_{\tau\in\mathrm{S}_m}\varepsilon_\tau x_{\tau(1)}^{m-1}\ldots x_{\tau(m)}^{0}$$

which, by combining terms, is equal to

$$\sum_{\lambda\in\mathbb{Z}^m}\sum_{\tau\in\mathrm{S}_m}\varepsilon_\tau\psi_\lambda(\alpha)x_{\tau(1)}^{\lambda_{\tau(1)}+m-1}\ldots x_{\tau(m)}^{\lambda_{\tau(m)}}$$

Let $\ell_i = \lambda_{\tau(i)}+m-i$—note that this is not our usual convention—then since the $\psi_\lambda(\alpha)$ are symmetric, our expression is equal to

$$\sum_{\lambda\in\mathbb{Z}^m}\sum_{\tau\in\mathrm{S}_m}\psi_{(\ell_1+1-m,\ \ldots\ ,\ \ell_m)}(\alpha)x_{\tau(1)}^{\ell_1}\ldots x_{\tau(m)}^{\ell_m}$$

$$= \sum_{\lambda\in\mathbb{Z}^m}\psi_{(\ell_1+1-m,\ \ldots\ ,\ \ell_m)}(\alpha)\left|x^{\ell_1},\ldots,x^{\ell_m}\right|$$

Whenever the ℓ_i are not distinct, the corresponding term gives zero. So setting $\lambda_i = \ell_i+i-m$ and using the fact that $\ell_1>\cdots>\ell_m$ if and only if $\lambda_1\geq\cdots\geq\lambda_m$, our expression is equal to

$$\sum_{\lambda_1\geq\cdots\geq\lambda_m}\sum_{\pi\in\mathrm{S}_m}\psi_{(\ell_{\pi(1)}+1-m,\ \ldots\ ,\ \ell_{\pi(m)})}\varepsilon_\pi\left|x^{\ell_1},\ldots,x^{\ell_m}\right|$$

The terms for which λ is not a partition of n are zero, for if $\lambda_m<0$ then $\ell_m+\pi^{-1}(m)-m<0$. Thus we have the desired equality. □

(4.10) Lemma *(Orthogonality of ω)* Let $\lambda, \lambda' \vdash n$ be partitions of $\leq m$ parts. Then

$$\sum_{\text{ccls } \alpha} n_\alpha \omega_\lambda(\alpha)\omega_{\lambda'}(\alpha) = \begin{cases} n! & \text{if } \lambda = \lambda' \\ 0 & \text{if } \lambda \neq \lambda' \end{cases}$$

Proof By **(4.7)** we have

$$\sum_{\lambda \in \Lambda^+(m,n)} \left|x^{\ell_1}, \ldots, x^{\ell_m}\right| \left|y^{\ell_1}, \ldots, y^{\ell_m}\right| = \frac{1}{n!} \sum_{\text{ccls } \alpha} n_\alpha s_1^{\alpha_1} \ldots s_n^{\alpha_n} t_1^{\alpha_1} \ldots t_n^{\alpha_n} \left|x^{m-1}, \ldots, 1\right| \left|y^{m-1}, \ldots, 1\right|$$

By **(4.9)** this is equal to

$$\frac{1}{n!} \sum_{\text{ccls } \alpha} \left(\sum_{\lambda, \lambda' \in \Lambda^+(m,n)} n_\alpha \omega_\lambda(\alpha)\omega_{\lambda'}(\alpha) \left|x^{\ell_1}, \ldots, x^{\ell_m}\right| \left|y^{\ell'_1}, \ldots, y^{\ell'_m}\right| \right)$$

where ℓ_i come from λ_i and ℓ'_i come from λ_i in the usual way.

As λ, λ' vary, the polynomials $\left|x^{\ell_1}, \ldots, x^{\ell_m}\right|$ and $\left|y^{\ell'_1}, \ldots, y^{\ell'_m}\right|$ are linearly independent in the polynomial ring $\mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_m]$, so the result follows. □

Now that we have trudged through lots of combinatorics we will reintroduce the symmetric group.

(4.11) Lemma If $\lambda \in \Lambda^+(n,m)$ then $\psi_\lambda(\alpha)$ is the character of the $\mathbb{C}\mathrm{S}_n$-module $\mathbb{C}\mathrm{S}_n r_\lambda$, where

$$r_\lambda = \sum_{\sigma \in R_{t_\lambda^\circ}} \sigma$$

Proof (X-rated) Take the character χ and of $\mathbb{C}\mathrm{S}_n r_\lambda$ and suppose $\sigma \in \alpha$, where α is a conjugacy class in S_n. Write $R = R_{t_\lambda^\circ}$, and find a coset decomposition

$$\mathrm{S}_n = \bigcup_{i=1}^{N} g_i R$$

for the transversal $\{1 = g_1, g_2, \ldots, g_N\}$. (Recall: a **transversal** is a minimal set of coset representatives.) Then $\mathbb{C}\mathrm{S}_n r_\lambda$ has basis $\{g_i r_\lambda\}_{1 \leq i \leq N}$, which we now use to calculate the traces. The $\mathbb{C}S_n$-action is given by $\sigma g_i r_\lambda = g_j r_\lambda$ for $\sigma \in \mathrm{S}_n$, where $\sigma g_i \in g_j R$. Thus

$$\chi(\alpha) = \left|\{1 \leq i \leq N \,:\, g_i^{-1}\sigma g_i \in R\}\right|$$

We will now take this and bash it very hard lots of times until enough juice comes out to give us what we want.

Now $g^{-1}\sigma g \in R$ if and only if $g \in g_i R$, where $g_i^{-1}\sigma g_i \in R$, and $|R| = \prod_1^m \lambda_j!$, so

$$\chi(\alpha) = \frac{1}{\prod_1^m \lambda_j!} \left|\{g \in \mathrm{S}_n \,:\, g^{-1}\sigma g \in R\}\right|$$

Now since $g^{-1}\sigma g = h^{-1}\sigma h$ if and only if $gh^{-1} \in C_{\mathrm{S}_n}(\sigma)$, and each value taken by $g^{-1}\sigma g$ for $g \in \mathrm{S}_n$ is taken by $|C_{\mathrm{S}_n}|$ elements. By **(4.1)** we have

$$\chi(\alpha) = \left(\frac{1}{\prod_1^m \lambda_j!} 1^{\alpha_1} 2^{\alpha_2} \ldots n^{\alpha_n} \alpha_1! \alpha_2! \ldots \alpha_n! \right) |\alpha \cap R|$$

since $|C_{\mathrm{S}_n}(\sigma)| = n!/n_\alpha$. Now a permutation $\tau \in \alpha \cap R$ restricts to a permutation of the numbers in the i^{th} row of t_λ°. If this restriction involves, say, α_{ij} j-cycles, then the α_{ij} satisfy

$$\left.\begin{array}{rcll} \alpha_{i1} + 2\alpha_{i2} + \cdots + n\alpha_{in} & = & \lambda_i & (1 \leq i \leq m) \\ \alpha_{1j} + \alpha_{2j} + \cdots + \alpha_{mj} & = & \alpha_j & (1 \leq j \leq n) \end{array}\right\} (*)$$

The number of permutations in R of this type is

$$\left(\frac{\lambda_1!}{1^{\alpha_{11}}\dots n^{\alpha_{1n}}\alpha_{11}!\dots\alpha_{1n}!}\right)\left(\frac{\lambda_2!}{1^{\alpha_{21}}\dots n^{\alpha_{2n}}\alpha_{21}!\dots\alpha_{2n}!}\right)\cdots\left(\frac{\lambda_m!}{1^{\alpha_{m1}}\dots n^{\alpha_{mn}}\alpha_{m1}!\dots\alpha_{mn}!}\right)$$

and so

$$\chi(\alpha)=\sum\left[\left(\frac{\alpha_1!}{1^{\alpha_{11}}\dots n^{\alpha_{1n}}\alpha_{11}!\dots\alpha_{1n}!}\right)\cdots\left(\frac{\alpha_m!}{1^{\alpha_{m1}}\dots n^{\alpha_{mn}}\alpha_{m1}!\dots\alpha_{mn}!}\right)\right] \qquad (\dagger)$$

where the sum runs over all α_{ij} satisfying $(*)$. By the multinomial theorem,

$$s_1^{\alpha_1}\dots s_n^{\alpha_n}=\sum\left[\left(\frac{\alpha_1!x_1^{\alpha_{11}}\dots x_m^{\alpha_{1m}}}{1^{\alpha_{11}}\dots n^{\alpha_{1n}}\alpha_{11}!\dots\alpha_{1n}!}\right)\cdots\left(\frac{\alpha_m!x_1^{\alpha_{n1}}\dots x_m^{\alpha_{nm}}}{1^{\alpha_{m1}}\dots n^{\alpha_{mn}}\alpha_{m1}!\dots\alpha_{mn}!}\right)\right]$$

where the sum runs over all $\alpha_{ij}\in\mathbb{N}$ satisfying

$$\alpha_{1j}+\alpha_{2j}+\cdots+\alpha_{mj}=\alpha_j$$

for all $1\le j\le n$. So $\psi_\lambda(\alpha)$, which is the coefficient of $x_1^{\lambda_1}\dots x_m^{\lambda_m}$, is equal to the right-hand side of $(\dagger)$, and hence it equals $\chi(\alpha)$. □

(4.12) Lemma Let $\lambda,\mu\vdash n$ with $\mu\le\lambda$. The irreducible module $\mathbb{C}\mathrm{S}_n h_\mu$ is isomorphic to a submodule of $\mathbb{C}\mathrm{S}_n r_\lambda$ if and only if $\lambda=\mu$.

Proof With $\mu<\lambda$ and $\sigma\in\mathrm{S}_n$, by **(2.6)** there exist two integers in the same row of t_λ° and the same column of $\sigma^{-1}t_\mu^\circ$, so if τ is their transposition then $\sigma\tau\sigma^{-1}\in C_{t_\mu^\circ}$. And

$$h_\mu\sigma r_\lambda=h_\mu\sigma\tau\sigma^{-1}\sigma\tau r_\lambda=-h_\mu\sigma r_\lambda \quad\Rightarrow\quad h_\mu\sigma r_\lambda=0$$

Thus $0=h_\mu\mathbb{C}\mathrm{S}_n r_\lambda\cong\mathrm{Hom}_{\mathbb{C}\mathrm{S}_n}(\mathbb{C}\mathrm{S}_n h_\mu,\mathbb{C}\mathrm{S}_n r_\lambda)$ by **(1.12)**.

Conversely suppose $\mu=\lambda$. Then

$$h_\lambda r_\lambda\sum_{\sigma\in C_{t_\lambda^\circ}}\varepsilon_\sigma\sigma=h_\lambda^2\neq0$$

so $0\neq h_\lambda\mathbb{C}\mathrm{S}_n r_\lambda\cong\mathrm{Hom}_{\mathbb{C}\mathrm{S}_n}(\mathbb{C}\mathrm{S}_n h_\lambda,\mathbb{C}\mathrm{S}_n r_\lambda)$. □

Note that, in general, $\mathbb{C}\mathrm{S}_n h_\lambda$ is not a submodule of $\mathbb{C}\mathrm{S}_n r_\lambda$.

(4.13) Lemma If $\lambda\in\Lambda^+(n,m)$ then $\omega_\lambda=\chi^\lambda$.

Proof We'll take this in steps.

Step 1 ω_λ is a $\mathbb{Z}$-linear combination of ψ_ν with $\nu\ge\lambda$ and with coefficient of ψ_λ equal to 1.

If $\pi\in\mathrm{S}_n$ and μ_π is the partition with parts

$$\ell_{\pi(1)}+1-m,\dots,\ell_{\pi(m)}$$

i.e. with parts $\lambda_i+\pi^{-1}(i)-1$. Since $\psi_\lambda(\alpha)$ is symmetric in the λ_i, we can write

$$\omega_\lambda=\sum_{\pi\in\mathrm{S}_n}\varepsilon_\pi\psi_{\mu_\pi}$$

If $\pi=1$ then $\mu_\pi=\lambda$, so we're done.

If $\pi\neq1$, then $\lambda_1+\pi^{-1}(1)-1\ge\lambda$ (with equality if and only if $\pi^{-1}(1)=1$; and then $\lambda_2+\pi^{-1}(2)-2\ge\lambda_2$ with equality if and only if $\pi^{-1}(2)=2$; and so on. Thus $\mu_\pi>\lambda$. ◁

Step 2 $\omega_\lambda=\sum_{\nu\vdash n}k_{\lambda\nu}\chi^\nu$ with $k_{\lambda\nu}\in\mathbb{Z}$, $k_{\lambda\lambda}>0$ and $k_{\lambda\nu}=0$ if $\nu<\lambda$.

By **(4.11)** and **(4.12)**, ψ_λ is an $\mathbb{N}$-linear combination χ^μs with $\mu\ge\lambda$ and with nonzero coefficient of χ^λ. Thus ω_λ is a $\mathbb{Z}$-linear combination of χ^νs with $\nu\ge\lambda$ and with positive coefficient of χ^λ. ◁

Step 3 The result follows.

We know by **(4.10)** that

$$\sum_{\text{ccls }\alpha} n_\alpha \omega_\lambda(\alpha)\omega_\mu(\alpha) = \begin{cases} n! & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu \end{cases}$$

In the case $\lambda = \mu$, orthogonality of χ^λ implies $\sum_{\nu \vdash n} k_{\lambda\nu}^2 = 1$, and thus $k_{\lambda\nu} = 0$ if $\lambda \neq \mu$ and $k_{\lambda\lambda} = 1$. ◁

So we're done. □

The $k_{\lambda\nu}$s in the above proof are sometimes referred to as **Kostka numbers** in the literature.

At long last, take $m \in \mathbb{N}$, $x_1, \ldots, x_m \in \mathbb{C}$ arbitrary, $\ell_i = \lambda_i + m - i$ for $\lambda \vdash n$, and $s_i = x_1^i + \cdots + x_m^i$.

(4.14) Theorem $s_1^{\alpha_1} \ldots s_n^{\alpha_n} \left|x^{m-1}, \ldots, 1\right| = \sum_{\lambda \in \Lambda^+(m,n)} \chi^\lambda(\alpha) \left|x^{\ell_1}, \ldots, x^{\ell_n}\right|$

Proof **(4.9)** and **(4.13)**. □

Remarks

(1) With $m \geq n$ we can ensure that the right-hand side of the equation in **(4.14)** involves all partitions of n.

(2) If $\lambda \in \Lambda^+(m,n)$ then $\chi^\lambda(\alpha)$ is the coefficient of the monomial $x_1^{\ell_1} \ldots x_m^{\ell_m}$ in the expansion of $s_1^{\alpha_1} \ldots s_n^{\alpha_n}$.

5 The hook length formula

We already know by **(3.8)** that the dimension of an irreducible $\mathbb{C}S_n$-module is the number of standard tableaux. We give some results which allow for easy calculation of this dimension.

(5.1) Theorem *(Young–Frobenius formula)* If $\lambda \in \Lambda^+(n,m)$ then

$$f_\lambda = \deg \chi^\lambda = n! \cdot \frac{\prod_{1 \le i < j \le n} (\ell_i - \ell_j)}{\prod_{i=1}^n \ell_i!}$$

Proof $\deg \chi^\lambda = \chi^\lambda(1)$ is the coefficient of $x_1^{\ell_1} \dots x_m^{\ell_m}$ in the expansion of

$$(x_1 + \cdots + x_m)^n \left| x^{m-1}, \dots, 1 \right| = \sum_{\tau \in S_n} (x_1 + \cdots + x_m)^n \varepsilon_\tau x_1^{\tau(1)-1} \dots x_m^{\tau(m)-m}$$

By the multinomial theorem, this coefficient equals

$$\begin{aligned} & \sum_{\tau \in S_n} \frac{n!}{(\ell_1 + 1 - \tau(1))! \dots (\ell_m + 1 - \tau(m))!} \\ = \; & n! \left| \frac{1}{(\ell - m + 1)!}, \dots, \frac{1}{(\ell - 1)!}, \frac{1}{\ell!} \right| \\ = \; & n! \cdot \frac{1}{\prod_{j=1}^m \ell_j!} \left| \dots, \ell(\ell-1), \ell, 1 \right| \\ = \; & n! \cdot \frac{1}{\prod_{j=1}^m \ell_j!} \left| \ell^{m-1}, \dots, \ell^2, \ell, 1 \right| \end{aligned}$$

This determinant is a Vandermonde determinant, so applying **(4.3)** gives the result. □

(5.2) Definitions

(a) Let $\lambda \vdash n$. The (i,j)-**hook** of λ is the intersection of an infinite Γ–shape having (i,j) as its corner with $[\lambda]$. For example, the $(2,2)$–hook of $(4^2,3)$ is shown as stars ($\star$) in the following tableau:

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \star & \star & \star \\ \bullet & \star & \bullet & \end{array}$$

(b) The (i,j)-hook of $[\mu]$ comprises the (i,j)-**node** along with the $(\mu_i - j)$ nodes to the right of it (the **arm**) and the $(\mu'_j - i)$ nodes below it (the **leg**). [Here, $[\mu]$ has column lengths $\mu'_1, \mu'_2, \dots$.] The **length** of the (i,j)-hook is $h_{ij} = \mu_i + \mu'_j + 1 - i - j$.

(c) Replacing the (i,j)-node of $[\mu]$ by h_{ij} for each node gives the **hook graph**. For example, the hook graph of $(4^2,3)$ is

$$\begin{array}{cccc} 6 & 5 & 4 & 2 \\ 5 & 4 & 3 & 1 \\ 3 & 2 & 1 & \end{array}$$

(d) A **skew hook** is a connected part of the **rim** of $[\mu]$ which can be removed to leave a proper diagram. For example, the skew 4-hooks of $(4^2,3)$ are

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \star & \star \\ \bullet & \star & \star & \end{array} \quad \text{and} \quad \begin{array}{cccc} \bullet & \bullet & \bullet & \star \\ \bullet & \bullet & \star & \star \\ \bullet & \bullet & \star & \end{array}$$

There is a natural one-to-one correspondence between hooks and skew hooks in a given tableau. For each node (i,j) there is a unique skew hook starting in the i^{th} row and ending in the j^{th} column, and it corresponds with the (i,j)-hook.

Theorem (5.3) *(hook length formula)* [Frame, Robinson & Thrall, 1954]

$$f_\lambda = \frac{n!}{\prod_{(i,j)\in[\lambda]} h_{ij}}$$

Proof Suppose λ has m parts. By **(5.1)** we have to show that

$$\prod_{k=i+1}^{m} (\ell_i - \ell_k) \prod_{j=1}^{\lambda_i} h_{ij} = \ell_i! \quad \text{for each } i$$

The (combined) product on the left-hand side is a product of $\lambda_i + m - 1 = \ell_i$ terms, so it suffices to show that they are precisely $1, 2, \ldots, \ell_i$ in some order. Now

$$\ell_i - \ell_m > \ell_i - \ell_{m-1} > \cdots > \ell_i - \ell_{i+1}$$

$$h_{i1} > h_{i2} > \cdots > h_{i\lambda_i}$$

Since λ has m parts and λ'_1 is the length of the first column, $\lambda'_1 = m$, and $h_{i1} = \lambda_i$. So each term is $\leq \ell_i$. Thus it is enough to show that $h_{ij} \neq \ell_i - \ell_k$ for any j, k.

However if $r = \lambda'_j$ then $\lambda_r \geq j$ and $\lambda_{r+1} < j$, so

$$h_{ij} - \ell_i + \ell_{r+1} = (\lambda_i + r - 1 - j + 1) - (\lambda_i + m - i) + (\lambda_r + m - r) = \lambda_r + 1 - j > 0$$

$$h_{ij} - \ell_i + \ell_{r+1} = (\lambda_i + r - 1 - j + 1) - (\lambda_i + m - i) + (\lambda_{r+1} + m - r - 1) = \lambda_{r+1} - j < 0$$

So $\ell_i - \ell_r < h_{ij} < \ell_i - \ell_{r+1}$. □

This theorem is often referred to as the FRT theorem in the literature.

Example When $n = 11$ and $\lambda = (6, 3, 2)$, the hook graph is given by

8	7	6	5	3	2	1
4	3	1				
2	1					

and so

$$f_\lambda = \frac{11!}{8.7.5.4.3^2.2^2.1^3} = 990$$

Remarks

(1) The determinantal form of the **Young–Frobenius formula** is given by

$$f_\lambda = n! \left| \frac{1}{(\lambda_i - i + j)!} \right|$$

where $\lambda \in \Lambda^+(n, m)$ and the determinant on the right is of an $m \times m$ matrix and we adopt the convention $\dfrac{1}{m!} = 0$ if $m < 0$. For a proof see e.g. Sagan or James §19.

(2) The equation $\sum_{\lambda \vdash n} f_\lambda^2 = n!$ can be proved as a purely combinatorial statement without reference to representations, using the so-called 'RSK algorithm' (for Robinson–Schensted–Knuth).

(3) The **Murnaghan–Nakayama rule** (e.g. example sheet 1, question 9) states that if $\pi\rho \in \mathrm{S}_n$ with ρ an r-cycle and π a permutation of the remaining $n - r$ numbers then

$$\chi^\lambda(\pi\rho) = \sum_\nu (-)^i \chi^\nu(\pi)$$

where ν runs over partitions for which $[\lambda] - [\nu]$ is a skew r-hook of leg length i. (Note: leg length refers to the leg of the hook with which the skew hook corresponds.)

For example,

$$
\begin{aligned}
\chi^{(5,4^2)}(\overbrace{(5,4,3,1)}^{\text{conj. class}}) &= -\chi^{(5,3)} + \chi^{(3^2,2)} && \text{on } (4,3,1)\\
&= \underbrace{\chi^{(2^2)}}_{\text{from } (5,3)} - (\underbrace{\chi^{(3,1)} + \chi^{(2,1^2)}}_{\text{from } (3^2,2)}) && \text{on } (3,1)\\
&= -\chi^{(1)} + 0 + 0 && \text{on } (1)\\
&= -1
\end{aligned}
$$

Chapter II: Representation theory of general linear groups

6 Multilinear algebra and algebraic geometry

Let V and W be finite-dimensional $\mathbb{C}G$-modules, where G is any (possibly infinite) group. Recall the following:

- **Tensor products**: $V \otimes_{\mathbb{C}} W$ is a $\mathbb{C}G$-module with the diagonal action $g(v \otimes w) = gv \otimes gw$.

  Basic properties:
  - $V \otimes \mathbb{C} \cong \mathbb{C}$
  - $V \otimes W \cong W \otimes V$
  - $(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$
  - If $\theta : V \to V'$ and $\varphi : W \to W'$ are $\mathbb{C}G$-maps then there is a $\mathbb{C}G$-map defined by

  $$\begin{array}{rcrcl} \theta \otimes \varphi & : & V \otimes W & \to & V' \otimes W' \\ & & v \otimes w & \mapsto & \theta(v) \otimes \varphi(w) \end{array}$$

- **Hom spaces**: $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is a $\mathbb{C}G$-module via

  $$\left(\sum_i \alpha_i g_i\right)(f)(v) = \sum_i \alpha_i g_i(f(g_i^{-1} v))$$

- **Dual space**: $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$.

  Basic properties:
  - If V is 1-dimensional then $V^* \otimes V \cong \mathbb{C}$.
  - If $\theta : V \to W$ is a $\mathbb{C}G$-map then the dual $\theta^* : W^* \to V^*$ is too.
  - The map

  $$\begin{array}{rcl} V^* \otimes W & \to & \mathrm{Hom}_{\mathbb{C}}(V, W) \\ f \otimes w & \mapsto & (v \mapsto f(v)w) \end{array}$$

    is a $\mathbb{C}G$-isomorphism.

- **Tensor powers**: The n^{th} tensor power of V is defined by

  $$T^n V = V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ copies}}, \quad T^0 V = \mathbb{C}$$

  Basic properties:
  - If V has basis $\{e_i : 1 \leq i \leq m\}$ then $V^{\otimes n}$ has basis

  $$\{e_{i_1} \otimes \cdots \otimes e_{i_n} : 1 \leq i_1, \ldots, i_n \leq m\}$$

    and hence $\dim(V^{\otimes n}) = m^n$.
  - $V^{\otimes n}$ is naturally a left $\mathbb{C}\mathrm{S}_n$-module via

  $$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots v_{\sigma^{-1}(n)}$$

  - The actions of $\mathbb{C}\mathrm{S}_n$ and $\mathbb{C}G$ commute, i.e.

  $$\sigma g x = g \sigma x \quad \text{for all } g \in G,\ \sigma \in \mathrm{S}_n,\ x \in V^{\otimes n}$$

(6.1) Definition The n^{th} **exterior power** of V is $\Lambda^n V = V^{\otimes n}/X$, where

$$X = \mathrm{span}_{\mathbb{C}}\{x - \varepsilon_\sigma \sigma x : x \in V^{\otimes n},\ \sigma \in \mathrm{S}_n\}$$

The image of $v_1 \otimes \cdots \otimes v_n$ in $\Lambda^n V$ under the quotient map will be denoted by $v_1 \wedge \cdots \wedge v_n$.

Define the **antisymmetric tensors** by

$$T^n V_{\text{anti}} = \{x \in V^{\otimes n} : \sigma x = \varepsilon_\sigma x \text{ for all } \sigma \in \mathrm{S}_n\}$$

(6.2) Lemma

(i) $v_1 \wedge \cdots \wedge v_n = \varepsilon_\sigma v_{\sigma^{-1}(1)} \wedge \cdots \wedge v_{\sigma^{-1}(n)}$ for all $\sigma \in \mathrm{S}_n$.

This tells us, by considering transpositions, that $v_1 \wedge \cdots \wedge v_n = 0$ whenever $v_i = v_j$ for some $i \neq j$.

(ii) $T^n V_{\text{anti}}$ and $\Lambda^n V$ are $\mathbb{C}G$-modules.

(iii) $\Lambda^n V$ has basis $\{e_{i_1} \wedge \cdots \wedge e_{i_n} : 1 \leq i_1 < \cdots < i_n \leq n\}$, and so $\dim \Lambda^n V = \binom{m}{n}$.

In particular, $\Lambda^m V$ is one-dimensional and $\Lambda^{m+1} V = \Lambda^{m+2} V = \cdots = 0$.

Proof An easy exercise. □

Remark If V is m-dimensional over a field k then usually we define $\Lambda^n V = V^{\otimes n}/X$, where

$$X = \operatorname{span}_{\mathbb{C}}\{v_1 \otimes \cdots \otimes v_n : v_i = v_j \text{ for some } i \neq j\}$$

If $\operatorname{char} k \neq 2$ then this definition reduces to **(6.2)**.

(6.3) Lemma $T^n V_{\text{anti}} = aV^{\otimes n}$, where $a = \sum_{\sigma \in \mathrm{S}_n} \varepsilon_\sigma \sigma$ is the alternizer [cf. **(2.4)**], and the natural map $T^n V_{\text{anti}} \to \Lambda^n V$ is an isomorphism of $\mathbb{C}G$-modules.

Proof If x is antisymmetric then

$$ax = \sum_{\sigma \in \mathrm{S}_n} \varepsilon_\sigma \sigma x = \sum_{\sigma \in \mathrm{S}_n} \varepsilon_\sigma (\varepsilon_\sigma x) = \sum_{\sigma \in \mathrm{S}_n} x = n!\, x$$

and so $T^n V_{\text{anti}} \subseteq aV^{\otimes n}$.

Conversely, since $\sigma a = \varepsilon_\sigma a$ for all $\sigma \in \mathrm{S}_n$, any element of $aV^{\otimes n}$ must be antisymmetric.

We have a map

$$aV^{\otimes n} \hookrightarrow V^{\otimes n} \to \Lambda^n V$$

which is a $\mathbb{C}G$-homomorphism and whose kernel is $X \cap aV^{\otimes n} \leq aX$, since $x = \frac{1}{n!} ax$ for x antisymmetric. But for $y \in V^{\otimes n}$,

$$a(y - \varepsilon_\sigma \sigma y) = ay - \varepsilon_\sigma(\varepsilon_\sigma a)y = ay - ay = 0$$

so $ax = 0$ (since $x \in X$). Hence the map is injective. And it is surjective: any $x \in \Lambda^n V$ is the image of some $y \in V^{\otimes n}$, but then x is the image of $\frac{1}{n!} ay \in aV^{\otimes n}$. So the map is an isomorphism. □

(6.4) Lemma $\Lambda^n(V^*) \cong (\Lambda^n V)^*$.

Proof The quotient map $V^{\otimes n} \twoheadrightarrow \Lambda^n V$ is surjective, so induces an inclusion

$$(\Lambda^n V)^* \hookrightarrow (T^n V)^* = T^n(V^*)$$

Now by the universal property of $\Lambda^n V$ (i.e. any multilinear map $\underbrace{V \times \cdots \times V}_{n \text{ copies}} \to \mathbb{C}$ factors through $\Lambda^n V$), the image of the inclusion is $T^n(V^*)_{\text{anti}}$, which is isomorphic to $\Lambda^n(V^*)$. □

Now prepare for some déjà vu in **(6.5)**–**(6.8)**...

(6.5) Definition The n^{th} **symmetric power** of V is $\mathrm{S}^n V = V^{\otimes n}/Y$, where

$$Y = \operatorname{span}_{\mathbb{C}}\{x - \sigma x : x \in V^{\otimes n},\ \sigma \in \mathrm{S}_n\}$$

The image of $v_1 \otimes \cdots \otimes v_n$ in $\mathrm{S}^n V$ under the quotient map will be denoted by $v_1 \vee \cdots \vee v_n$.

Define the **symmetric tensors** by

$$T^n V_{\text{sym}} = \{x \in V^{\otimes n} : \sigma x = x \text{ for all } \sigma \in \mathrm{S}_n\}$$

(6.6) Lemma

(i) $v_1 \vee \cdots \vee v_n = v_{\sigma^{-1}(1)} \vee \cdots \vee v_{\sigma^{-1}(n)}$ for all $\sigma \in \mathrm{S}_n$.

(ii) $T^n V_{\mathrm{sym}}$ and $\mathrm{S}^n V$ are $\mathbb{C}G$-modules.

(iii) $\mathrm{S}^n V$ has basis $\{e_{i_1} \vee \cdots \vee e_{i_n} : 1 \leq i_1 \leq \cdots \leq i_n \leq n\}$, and so $\dim \mathrm{S}^n V = \binom{m+n-1}{n}$.

Proof An easy exercise. □

(6.7) Lemma $T^n V_{\mathrm{sym}} = sV^{\otimes n}$, where $s = \sum_{\sigma \in \mathrm{S}_n} \sigma$ is the symmetrizer [cf. **(2.4)**], and the natural map $T^n V_{\mathrm{sym}} \to \mathrm{S}^n V$ is an isomorphism of $\mathbb{C}G$-modules.

Proof If x is symmetric then

$$sx = \sum_{\sigma \in \mathrm{S}_n} \sigma x = \sum_{\sigma \in \mathrm{S}_n} x = n!\, x$$

and so $T^n V_{\mathrm{sym}} \subseteq sV^{\otimes n}$.

Conversely, since $\sigma s = s$ for all $\sigma \in \mathrm{S}_n$, any element of $sV^{\otimes n}$ must be symmetric.

We have a map

$$sV^{\otimes n} \hookrightarrow V^{\otimes n} \to \mathrm{S}^n V$$

which is a $\mathbb{C}G$-homomorphism and whose kernel is $Y \cap aV^{\otimes n} \leq sY$, since $x = \frac{1}{n!} sx$ for x symmetric. But for $y \in V^{\otimes n}$,

$$s(y - \sigma y) = sy - sy = 0$$

so $sx = 0$ (since $x \in Y$). Hence the map is injective. And it is surjective: any $x \in \mathrm{S}^n V$ is the image of some $y \in V^{\otimes n}$, but then x is the image of $\frac{1}{n!} sy \in sV^{\otimes n}$. So the map is an isomorphism. □

(6.8) Lemma $\mathrm{S}^n(V^*) \cong (\mathrm{S}^n V)^*$.

Proof The quotient map $V^{\otimes n} \twoheadrightarrow \mathrm{S}^n V$ is surjective, so induces an inclusion

$$(\mathrm{S}^n V)^* \hookrightarrow (T^n V)^* = T^n(V^*)$$

Now by the universal property of $\mathrm{S}^n V$ (i.e. any multilinear map $\underbrace{V \times \cdots \times V}_{n \text{ copies}} \to \mathbb{C}$ factors through $\mathrm{S}^n V$), the image of the inclusion is $T^n(V^*)_{\mathrm{sym}}$, which is isomorphic to $\mathrm{S}^n(V^*)$. □

Polynomial maps between vector spaces

(6.9) Definition Let V and W be finite-dimensional $\mathbb{C}$-spaces with bases $\{e_i : 1 \leq i \leq m\}$ and $\{f_r : 1 \leq r \leq h\}$, respectively. A function $\varphi : V \to W$ is a **polynomial map** (resp. n-**homogeneous map**) if, for all $X_1, \ldots, X_m \in \mathbb{C}$,

$$\varphi(X_1 e_1 + \cdots + X_m e_m) = \varphi_1(\vec{X}) f_1 + \cdots + \varphi_h(\vec{X}) f_h$$

for some polynomials (resp. homogeneous polynomials of degree n) $\varphi_1, \ldots, \varphi_h$. [Here $\vec{X}$ abbreviates the m-tuple $(X_1, \ldots, X_m)$.]

(6.10) Lemma The definitions in **(6.9)** do not depend on the choice of bases. That is, if there exist such functions φ_i with respect to some bases $\{e_i\}$, $\{f_r\}$, then there exist such functions with respect to any bases $\{e'_i\}$, $\{f'_r\}$.

Proof Write

$$e'_i = \sum_{j=1}^{m} p_{ji} e_j \qquad \text{and} \qquad f_r = \sum_{s=1}^{h} q_{sr} f'_s$$

Then

$$
\begin{aligned}
\varphi\left(\sum_{i=1}^{m} X_i e'_i\right) &= \varphi\left(\sum_{i,j=1}^{m} X_i p_{ji} e_j\right)\\
&= \sum_{r=1}^{h} \varphi_r\left(\sum_{i_1=1}^{m} X_{i_1} p_{1i_1}, \dots, \sum_{i_m=1}^{m} X_{i_m} p_{mi_m}\right) f_r\\
&= \sum_{r,s=1}^{h} \varphi_r\left(\sum_{i_1=1}^{m} X_{i_1} p_{1i_1}, \dots, \sum_{i_m=1}^{m} X_{i_m} p_{mi_m}\right) q_{sr} f'_s
\end{aligned}
$$

and the $\sum_{r=1}^{h} \varphi_r\Big(\cdots\Big) q_{sr}$ are polynomials (resp. n-homogeneous polynomials) like the φ_r. □

Let $\mathrm{P}_{\mathbb{C}}(V, W)$ and $\mathrm{H}_{\mathbb{C},n}(V, W)$ denote the spaces of polynomial maps and n-homogeneous maps, respectively, from V to W. They are $\mathbb{C}$-spaces, as can be easily verified.

(6.11) Lemma The composite of polynomial maps $X \to W \to Z$ is a polynomial map $X \to Z$. Likewise, the composite of an n-homogeneous map $X \to W$ and a n'-homogeneous map $W \to Z$ is an nn'-homogeneous map $X \to Z$.

Proof An easy exercise. □

Basic examples

(1) $\mathrm{H}_{\mathbb{C},0}(V, W) \cong W$

(2) $\mathrm{H}_{\mathbb{C},1}(V, W) = \mathrm{Hom}_{\mathbb{C}}(V, W)$

(3) $\Delta \in \mathrm{H}_{\mathbb{C},n}(V, \mathrm{S}^n V)$, where $\Delta : v \mapsto v \vee \cdots \vee v$. For

$$\Delta\left(\sum_i X_i e_i\right) = \sum_{\vec{i}} X_{i_1} \dots X_{i_n} e_{i_1} \vee \cdots \vee e_{i_n} = \sum_{i_1 \le \cdots \le i_n} \lambda_{\vec{i}} X_{i_1} \dots X_{i_n} e_{i_1} \vee \cdots \vee e_{i_n}$$

for suitable $\lambda_{\vec{i}} \in \mathbb{C}$. [Here $\vec{i}$ abbreviates the n-tuple $(i_1, \dots, i_n)$.]

(6.12) Theorem For $\mathbb{C}$-spaces V and W, the map $\psi \mapsto \psi \circ \Delta$ induces an isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(\mathrm{S}^n V, W) \xrightarrow{\cong} \mathrm{H}_{\mathbb{C},n}(V, W)$$

called **restitution**.

Proof It's injective: suppose $\psi \circ \Delta = 0$. We use descending induction on i to show that $\psi(v_1 \vee \cdots \vee v_n) = 0$ whenever i of the terms are equal. When $i = n$ this is precisely our assumption. Suppose it's true for $i + 1$, then for any $\alpha \in \mathbb{C}$ we have

$$
\begin{aligned}
0 &= \psi(\underbrace{(x + \alpha y) \vee \cdots \vee (x + \alpha y)}_{i+1} \vee v_{i+2} \vee \cdots \vee v_n)\\
&= \sum_{j=0}^{i+1} \alpha^j \binom{i+1}{j} \psi(\underbrace{x \vee \cdots \vee x}_{i+1-j} \vee \underbrace{y \vee \cdots \vee y}_{j} \vee v_{i+2} \vee \cdots \vee v_n)
\end{aligned}
$$

Since this holds true for any $\alpha \in \mathbb{C}$, each term must be zero. In particular

$$\binom{i+1}{1} \psi(\underbrace{x \vee \cdots \vee x}_{i} \vee y \vee v_{i+2} \vee \cdots \vee v_n) = 0$$

If $\dim V = m$ and $\dim W = h$, say, then

$$\dim \mathrm{Hom}_{\mathbb{C}}(\mathrm{S}^n V, W) = h\binom{m+n-1}{n} = \dim \mathrm{H}_{\mathbb{C},n}(V, W)$$

So the map is an isomorphism. $\square$

(6.13) Corollary $\mathrm{S}^n V = \mathrm{span}_{\mathbb{C}}\{v \vee \cdots \vee v \,:\, v \in V\}$

Proof Take $W = \mathbb{C}$ in **(6.12)**. If the elements $v \vee \cdots \vee v$ do not span then there exists a nonzero map $\mathrm{S}^n V \to \mathbb{C}$ whose composition with Δ is zero, contradicting **(6.12)**. $\square$

Remark We can construct the inverse explicitly. Let $\varphi \in \mathrm{H}_{\mathbb{C},n}(V, W)$, so

$$\varphi(X_1 e_1 + \cdots + X_m e_m) = \varphi_1(X_1, \ldots, X_m) f_1 + \cdots + \varphi_k(X_1, \ldots, X_m) f_h$$

with φ_i homogeneous. The **total polarisation** $P\varphi \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{S}^n V, W)$ of φ is

$$P\varphi(e_{i_1} \vee \cdots \vee e_{i_n}) = \sum_{j=1}^{h} \underbrace{\frac{\partial^n \varphi_j}{\partial X_{i_1} \ldots \partial X_{i_n}}}_{\in \mathbb{C}} f_j$$

(The partial derivative is formal, is symmetric in the X_{i_k}, and lies in $\mathbb{C}$ since φ_j has degree N.) Now

$$\begin{aligned}
(P\varphi)(\Delta(X_1 e_1 + \cdots + X_m e_m)) &= \sum_{i_1, \ldots, i_n} X_{i_1} \ldots X_{i_n} (P\varphi)(e_{i_1} \vee \cdots \vee e_{i_n}) \\
&= \sum_j \sum_{i_1, \ldots, i_n} X_{i_1} \ldots X_{i_n} \frac{\partial^n \varphi_j}{\partial X_{i_1} \ldots \partial X_{i_n}} f_j \\
&= \sum_j n!\, \varphi_j(X_1, \ldots, X_m) f_j \\
&= n! \varphi(X_1 e_1 + \cdots + X_m e_m)
\end{aligned}$$

So $P\varphi \circ \Delta = n!\, \varphi$. The penultimate equality follows from **Euler's theorem**, which states that if F is r-homogeneous then $\sum_i \frac{\partial F}{\partial X_i} X_i = rF$.

Example Let $\varphi : V \to \mathbb{C}$ be a quadratic form:

$$\varphi\left(\sum_{i=1}^{m} X_i e_i\right) = \sum_{i,j} a_{ij} X_i X_j, \qquad a_{ij} = a_{ji}$$

Then $(P\varphi)\left[(\sum_i X_i e_i) \vee \left(\sum_j Y_j e_J\right)\right] = 2 \sum_{i,j} a_{ij} X_i Y_j$ is twice the corresponding symmetric bilinear form.

Interlude: some reminders about affine varieties

Let K be an infinite field. Then $R = K[X_1, \ldots, X_n]$ a Noetherian ring by Hilbert's basis theorem.

Define $\mathbb{A}^n K = K^n$, the n-dimensional **affine space** over K.

If $f \in R$ and $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$ then the element $f(p) = f(a_1, \ldots, a_n) \in K$ is the **evaluation** of f at p.

Define a map

$$\begin{array}{ccc} \{\text{ideals } I \text{ of } R\} & \longleftrightarrow & \{\text{subsets of } \mathbb{A}^n\} \\ I & \longmapsto & V(I) \end{array}$$

where $V(I) = \{p \in \mathbb{A}^n \,:\, f(p) = 0 \text{ for all } f \in I\}$.

A subset $X \subseteq \mathbb{A}^n$ is an **algebraic set** if $X = V(I)$ for some ideal I. But I is finitely generated, say $I = (f_1, \ldots, f_m)$, so

$$V(I) = \{p \in \mathbb{A}^n \,:\, f_j(p) = 0 \text{ for each } 1 \le j \le m\}$$

is a locus of points satisfying a finite number of polynomial equations.

Algebraic subsets of $\mathbb{A}^n$ form the closed sets of a topology on $\mathbb{A}^n$, called the **Zariski topology**. The Zariski topology on $\mathbb{A}^n$ induces a topology on any algebraic set $X \subseteq \mathbb{A}^n$: the closed subsets of X are the algebraic subsets of X.

Example Zariski-closed subsets of $\mathbb{A}^1$ are $\mathbb{A}^1$ and finite subsets of $\mathbb{A}^1$; that is, the Zariski topology on $\mathbb{A}^1$ is the **cofinite topology**.

Returning to our correspondence, we have a map in the other direction

$$\begin{array}{ccc} \{\text{ideals } I \text{ of } R\} & \longleftrightarrow & \{\text{subsets of } \mathbb{A}^n\} \\ \mathcal{I}(X) & \longleftarrow & X \end{array}$$

where $\mathcal{I}(X) = \{f \in R \,:\, f(p) = 0 \text{ for all } p \in X\}$.

An algebraic set X is **irreducible** if there does not exist a decomposition $X = X_1 \cup X_2$ with $X_1, X_2 \subsetneq X$ proper algebraic subsets. Note

$$X \text{ is irreducible } \Leftrightarrow \mathcal{I}(X) \text{ is a prime ideal}$$

If $I \trianglelefteq R$ then the **radical** $\sqrt{I}$ of I is given by

$$\sqrt{I} = \{f \in R \,:\, f^m \in I \text{ for some} m\}$$

and I is a **radical ideal** if $I = \sqrt{I}$. Note that prime ideals are automatically radical.

Theorem *(Nullstellensatz)* Let K be algebraically closed. Then

(a) Maximal ideals of R are of the form

$$\mathfrak{m}_j = (X_1 - a_1, \ldots, X_n - a_n)$$

for some $p = (a_1, \ldots, a_n) \in \mathbb{A}^n$.

(b) If $(1) \neq I \trianglelefteq R$ then $V(I) \neq \varnothing$.

(c) $\mathcal{I}(V(I)) = \sqrt{I}$ for all ideals I.

Proof Omitted. □

Let W a finite-dimensional K-space. A map $f : W \to K$ is **regular** (or **polynomial**) if it is defined by a polynomial in the coordinates with respect to a given basis of W. (**Check**: it doesn't matter which basis we pick.)

Define the **coordinate ring** (or **ring of regular functions**) of W, denoted $K[W]$, to be the K-algebra of polynomial functions on W. If $e_1, \ldots, e_m$ is a basis of W and $x_1, \ldots, x_m$ is the corresponding dual basis of W^* then

$$K[W] = K[x_1, \ldots, x_m]$$

We say $f \in K[W]$ is d-**homogeneous** if $f(tw) = t^d f(w)$ for all $d \in \mathbb{C}$ and $w \in W$. Write $K[W]_d$ for the subspace of $K[W]$ of all d-homogeneous polynomials. Then

$$K[W] = \bigoplus_{d \geq 0} K[W]_d$$

is a graded K-algebra, i.e. if f, g are homogeneous of degree d_f, d_g, respectively, then fg is homogeneous of degree $d_f + d_g$.

The monomials $x_1^{d_1} \cdots x_m^{d_m}$ with $d_1 + \cdots + d_m = d$ form a basis of $K[W]_d$. In particular, $K[W]_1 = W^*$. This extends to a canonical identification of $K[W]_d$ with $\mathrm{S}^d(W^*)$, so that

$$K = \bigoplus_{d \geq 0} K[W]_d = \bigoplus_{d \geq 0} \mathrm{S}^d(W^*) = \mathrm{S}(W^*)$$

We call $\mathrm{S}(W^*)$ the **symmetric algebra** of W^*.

Zariski-dense subsets

A subset $X \subseteq W$, with W finite-dimensional, is **Zariski-dense** if every function $f \in K[W]$ vanishing on X is the zero function. So every polynomial function $f \in K[W]$ is completely determined by its restriction $f|_X$ to a Zariski-dense subset $X \subseteq W$.

Now let X be an arbitrary subset of W. As above, the ideal of X

$$\mathcal{I}(X) = \{f \in K[W] \,:\, f(a) = 0 \text{ for all } a \in X\} = \bigcap_{a \in X} \mathfrak{m}_a$$

where $\mathfrak{m}_a = \mathcal{I}(\{a\})$ is the maximal ideal of functions vanishing at a, i.e.

$$\mathfrak{m}_a = \ker \begin{pmatrix} \mathrm{ev}_a & : & K[W] & \to & \mathbb{C} \\ & & f & \mapsto & f(a) \end{pmatrix}$$

We say $X \subseteq Y$ is **Zariski-dense in** Y if $\mathcal{I}(X) = \mathcal{I}(Y)$.

Further reading Chapter 4 of Reid's book.

7 Schur–Weyl duality

Let V be an m-dimensional $\mathbb{C}$-space and let $n \in \mathbb{N}$.

$V^{\otimes n}$ is a right $\mathbb{C}\mathrm{S}_n$-module via the action[1]

$$(v_1 \otimes \cdots \otimes v_n)\sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

So there is a map

$$\begin{array}{ccc} \mathbb{C}\mathrm{S}_n & \to & \mathrm{End}_{\mathbb{C}} V^{\otimes n} \\ \sigma & \mapsto & (v \mapsto v\sigma) \end{array} \qquad \text{—(1)}$$

Regarding V as a representation of $\mathrm{GL}(V)$ in the natural way, $V^{\otimes n}$ is a (left) $\mathbb{C}\mathrm{GL}(V)$-module via the action

$$g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$$

So there is a map

$$\begin{array}{ccc} \mathbb{C}\mathrm{GL}(V) & \to & \mathrm{End}_{\mathbb{C}} V^{\otimes n} \\ g & \mapsto & (v \mapsto gv) \end{array} \qquad \text{—(2)}$$

The actions (1) and (2) commute, so by abusing notation slightly, (1) induces a map

$$\varphi : \mathbb{C}\mathrm{S}_n \to \mathrm{End}_{\mathbb{C}\mathrm{GL}(V)}(V^{\otimes n})$$

and (2) induces a map

$$\psi : \mathbb{C}\mathrm{GL}(V) \to \mathrm{End}_{\mathbb{C}\mathrm{S}_n}(V^{\otimes n})$$

Schur–Weyl duality asserts that the images of $\mathbb{C}\mathrm{S}_n$ and $\mathbb{C}\mathrm{GL}(V)$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ are each other's centralizers.

(7.1) Theorem *(Schur–Weyl duality)* [Schur, 1927]

$$\psi(\mathbb{C}\mathrm{GL}(V)) = \mathrm{End}_{\mathbb{C}\mathrm{S}_n}(V^{\otimes n}) \quad \text{and} \quad \varphi(\mathbb{C}\mathrm{S}_n) = \mathrm{End}_{\mathbb{C}\mathrm{GL}(V)}(V^{\otimes n})$$

Such an assertion looks innocuous, but it is fundamental in explaining why the symmetric group and the general linear group are so closely related.

(7.2) Definition The $\mathbb{C}$-algebra $\mathrm{S}_{\mathbb{C}}(m,n)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ consisting of endomorphisms which commute with the image of $\mathbb{C}\mathrm{S}_n$, i.e.

$$\mathrm{S}_{\mathbb{C}}(m,n) = \mathrm{End}_{\mathbb{C}\mathrm{S}_n}(V^{\otimes n})$$

It is called the **Schur algebra**. [For a definition over arbitrary fields, see J.A. Green's book *Polynomial representations of GL_n*, SLN 830.]

$\mathbb{C}\mathrm{S}_n$ is semisimple and $V^{\otimes n}$ is a finite-dimensional $\mathbb{C}\mathrm{S}_n$-module, so by **(1.9)**, $\mathrm{S}_{\mathbb{C}}(m,n)$ is a semisimple $\mathbb{C}$-algebra.

Put $W = \mathrm{End}_{\mathbb{C}}(V)$. This is a $\mathbb{C}\mathrm{GL}(V)$-module under the conjugation action (check this).

(7.3) Lemma There is a map $\alpha : W^{\otimes n} \to \mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ given by

$$f_1 \otimes \cdots \otimes f_n \mapsto \Big(v_1 \otimes \cdots \otimes v_n \mapsto f_1(v_1) \otimes \cdots \otimes f_n(v_n)\Big)$$

which is an isomorphism both of $\mathbb{C}\mathrm{S}_n$-modules and of $\mathbb{C}\mathrm{GL}(V)$-modules.

[1] Note that, because this is a right-action, $\sigma_1\sigma_2$ denotes the effect of first applying σ_1 and then applying σ_2, which is 'backwards' compared to normal.

Proof

$$\begin{aligned}W^{\otimes n} = W \otimes \cdots \otimes W &\cong (V \otimes V^*) \otimes \cdots \otimes (V \otimes V^*) \\ &\cong (V \otimes \cdots \otimes V) \otimes (V^* \otimes \cdots \otimes V^*) \\ &\cong V^{\otimes n} \otimes (V^{\otimes n})^* \\ &\cong \mathrm{End}_{\mathbb{C}}(V^{\otimes n})\end{aligned}$$

It is left as an exercise to check that this isomorphism is the map we think it is, that $W^{\otimes n}$ has the natural structure of a $\mathbb{C}\mathrm{S}_n$-module and $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ inherits structure from $V^{\otimes n}$ via $\theta \cdot \sigma = \varphi(\sigma)\theta\varphi(\sigma^{-1})$, and finally that α is a $\mathbb{C}\mathrm{S}_n$-map. □

(7.4) Lemma $\mathrm{S}_{\mathbb{C}}(m,n) = \alpha(T^n W_{\mathrm{sym}})$

Proof $\mathrm{S}_{\mathbb{C}}(m,n)$ is the set of all $x \in \mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ fixed under the action of S_n. By **(6.5)**, $T^n W_{\mathrm{sym}}$ is the set of all $y \in W^{\otimes n}$ fixed under the action of S_n (which is induced by α, i.e. $y \cdot \sigma \in W^{\otimes n}$ corresponds with $\varphi(\sigma)\alpha(y)\varphi(\sigma^{-1}) \in \mathrm{End}_{\mathbb{C}}(V^{\otimes n})$. □

Recall affine n-space $\mathbb{A}^n$ from §6.

(7.5) Lemma $\mathbb{A}^n$ is irreducible; i.e. if $\mathbb{A}^n = X \cup Y$ with X, Y Zariski-closed then $X = \mathbb{A}^n$ or $Y = \mathbb{A}^n$.

Proof The ring of regular functions on $\mathbb{A}^n$ is $R = \mathbb{C}[X_1, \ldots, X_n]$. If X and Y are zero-sets of ideals $I, J \trianglelefteq R$ then, by assumption, any maximal ideal contains either I or J. If I, J are both nonzero then pick $0 \neq f \in I$ and $0 \neq g \in J$. Any maximal ideal contains fg, so

$$fg(a_1, \ldots, a_n) = 0 \qquad \text{for all } a_1, \ldots, a_n \in \mathbb{C}$$

By Nullstellensatz, $fg \in \sqrt{\{0\}} = \{0\}$, contradicting the fact that R is an integral domain. □

(7.6) Lemma Let $Y \leq \mathbb{C}^d$ be a subspace. Then Y is Zariski-closed, where $\mathbb{C}^d$ is identified with $\mathbb{A}^d$ in the canonical way.

Proof Chose a basis $\varphi_1, \ldots, \varphi_h$ of $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^d/Y, \mathbb{C})$ and view them as maps $\mathbb{C}^d \to \mathbb{C}$. Then Y is the zero set of the φ_i. □

(7.7) Lemma $T^n W_{\mathrm{Sym}} = \mathrm{span}_{\mathbb{C}}\left(\{\overbrace{\varphi \otimes \cdots \otimes \varphi}^{n} : \varphi \in \mathrm{GL}(V)\}\right)$

Proof Define a subspace X of $T^n W_{\mathrm{sym}}$ by

$$X = \mathrm{span}_{\mathbb{C}}\left(\{\overbrace{\varphi \otimes \cdots \otimes \varphi}^{n} : \varphi \in \mathrm{GL}(V)\}\right)$$

There is a map $\alpha : W \to W^{\otimes n}$ given by $\alpha(\varphi) = \overbrace{\varphi \otimes \cdots \otimes \varphi}^{n}$, which is a regular map of the affine spaces $\mathbb{A}^{m^2} \cong W$ and $\mathbb{A}^{m^{2n}} \cong W^{\otimes n}$.

Now X is a subspace, so is Zariski-closed by **(7.6)**, and hence $\alpha^{-1}(X)$ is Zariski-closed. Thus,

$$W = \alpha^{-1}(X) \cup \{\text{endomorphisms with zero determinant}\}$$

is a union of Zariski-closed subsets.

Since $\mathbb{A}^{m^2}$ is irreducible (by **(7.5)**), $\alpha^{-1}(X) = W$. Thus X contains all maps of the form $\varphi \otimes \cdots \otimes \varphi$ for $\varphi \in W$. But they span $T^n W_{\mathrm{sym}}$ since the images $\varphi \vee \cdots \vee \varphi$ span $\mathrm{S}^n W$ by **(6.13)**. □

In other words...

(7.8) Theorem $\mathrm{S}_{\mathbb{C}}(m,n) = \mathrm{span}_{\mathbb{C}}\left(\{\overbrace{\varphi \otimes \cdots \otimes \varphi}^{n} : \varphi \in \mathrm{GL}(V)\}\right)$ □

Proof of (7.1) The assertion that the image of $\mathbb{C}\mathrm{GL}(V)$ is the centralizer of $\mathbb{C}\mathrm{S}_n$ is a reformulation of the assertion that $\mathrm{S}_{\mathbb{C}}(m,n) = \mathrm{span}_{\mathbb{C}}\left(\{\overbrace{\varphi \otimes \cdots \otimes \varphi}^{n} : \varphi \in \mathrm{GL}(V)\}\right)$, which is **(7.8)**.

Conversely, $\mathrm{S}_{\mathbb{C}}(m,n)$ is a semisimple $\mathbb{C}$-algebra. By **(1.11)**, $\mathbb{C}\mathrm{S}_n$ maps onto $\mathrm{End}_{\mathrm{S}_{\mathbb{C}}(m,n)}(V^{\otimes n})$, and since the image of $\mathrm{GL}(V)$ spans $\mathrm{S}_{\mathbb{C}}(m,n)$, we must have

$$\mathrm{End}_{\mathrm{S}_{\mathbb{C}}(m,n)}(V^{\otimes n}) = \mathrm{End}_{\mathbb{C}\mathrm{GL}(V)}(V^{\otimes n})$$

Thus $\mathbb{C}\mathrm{S}_n$ maps onto $\mathrm{End}_{\mathbb{C}\mathrm{GL}(V)}(V^{\otimes n})$, i.e. the image of $\mathbb{C}\mathrm{S}_n$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes n})$ is the centralizer of the image of $\mathbb{C}\mathrm{GL}(V)$. $\square$

8 Tensor decomposition

Let V be an m-dimensional $\mathbb{C}$-space. We know from **(3.9)** that

$$V^{\otimes n} = \bigoplus h(t_\lambda)V^{\otimes n} \qquad \text{—(1)}$$

where the direct sum runs over standard tableaux t_λ with $\lambda \vdash n$.

Examples If $\lambda = (1^n)$ then $h_\lambda V^{\otimes n} \cong T^n V_{\text{anti}} \cong \Lambda^n V$.

If $\lambda = (n)$ then $h_\lambda V^{\otimes n} \cong T^n V_{\text{Ssym}} \cong \mathrm{S}^n V$.

In particular,when $n = 2$, this tells us that

$$V \otimes V \cong T^2 V_{\text{anti}} \oplus T^2 V_{\text{sym}} \cong \Lambda^2 V \oplus \mathrm{S}^n V$$

Now the actions of S_n and $\mathrm{GL}(V)$ on $V^{\otimes n}$ commute. So if $\lambda \vdash n$ and t_λ is a λ-tableau, then $h(t_\lambda)V^{\otimes n}$ is a $\mathbb{C}\mathrm{GL}(V)$-submodule of $V^{\otimes n}$. Observe that

$$h(t_\lambda)V^{\otimes n} \cong h_\lambda V^{\otimes n} \quad \text{as } \mathbb{C}\mathrm{GL}(V)\text{-modules}$$

since $\sigma h(t_\lambda)\sigma^{-1} = h_\lambda$ for some $\sigma \in \mathrm{S}_n$. So premultiplication by σ induces an isomorphism

$$h(t_\lambda)V^{\otimes n} \xrightarrow{\cong} h_\lambda V^{\otimes n}$$

(8.1) Lemma The nonzero $h_\lambda V^{\otimes n}$ with $\lambda \vdash n$ are nonisomorphic irreducible $\mathbb{C}\mathrm{GL}(V)$-modules. Moreover, if M is an $\mathrm{S}_{\mathbb{C}}(m,n)$-module and M is viewed as a $\mathbb{C}\mathrm{GL}(V)$-module by restriction via the natural map $\mathbb{C}\mathrm{GL}(V) \to \mathrm{S}_{\mathbb{C}}(m,n)$ then M is isomorphic to a direct sum of copies of the $h_\lambda V^{\otimes n}$.

Proof Recall that $\mathrm{S}_{\mathbb{C}}(m,n) = \mathrm{End}_{\mathbb{C}\mathrm{S}_n}(V^{\otimes n})$ and

$$h_\lambda V^{\otimes n} \overset{\textbf{(1.12)}}{\cong} \mathrm{Hom}_{\mathbb{C}\mathrm{S}_n}(\mathbb{C}\mathrm{S}_n h_\lambda, V^{\otimes n})$$

By **(1.9)** and **(1.10)**(Artin–Wedderburn), the nonzero spaces $h_\lambda V^{\otimes n}$ are a complete set of nonisomorphic irreducible $\mathrm{S}_{\mathbb{C}}(m,n)$-modules.

Now $\mathrm{S}_{\mathbb{C}}(m,n)$ is semisimple, so the result follows from (2) and (3) below, both of which derive from the fact (at the end of §7) that the map $\mathbb{C}\mathrm{GL}(V) \to \mathrm{S}_{\mathbb{C}}(m,n)$ is surjective.

(2) If M is an $\mathrm{S}_{\mathbb{C}}(m,n)$-module and N is a $\mathbb{C}\mathrm{GL}(V)$-module of M then N is an $\mathrm{S}_{\mathbb{C}}(m,n)$-submodule of M.

(3) If M and N are $\mathrm{S}_{\mathbb{C}}(m,n)$-modules and $\theta : M \to N$ is a $\mathbb{C}\mathrm{GL}(V)$-module map then it is an $\mathrm{S}_{\mathbb{C}}(m,n)$-map. □

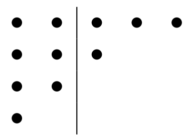
This implies that (1) is a decomposition of $V^{\otimes n}$ into $\mathbb{C}\mathrm{GL}(V)$-submodules which are either zero or irreducible. Which of these submodules are nonzero? We'll prove:

(8.2) Theorem Let $\lambda \vdash n$ and $m = \dim_{\mathbb{C}} V$. Then

$$\dim_{\mathbb{C}} h_\lambda V^{\otimes n} = \begin{cases} 0 & \text{if } \lambda_{m+1} \neq 0 \\ 1 & \text{if } \lambda_{m+1} = 0 \text{ and} \lambda_1 = \cdots = \lambda_m \\ \geq 2 & \text{otherwise} \end{cases}$$

First we have a slightly technical result. Take a partition $\lambda \vdash n$ such that $[\lambda]$ is partitioned into two nonempty parts, say of sizes i and $j = n - i$, by a vertical wall. For example, if $\lambda = (5,3,2,1) \vdash 11$ we could have

$i = 7 \longrightarrow$



$\longleftarrow \; j = 4$

For $\lambda \vdash n$, let t_λ be a tableau whose entries in the left-hand part are $\{1,\ldots,i\}$. Let $\mu \vdash i$ be the corresponding partition, so that $t_\mu = t_\lambda \downarrow_{[\mu]}$. Let $\nu \vdash j$ correspond to the right-hand part, with tableau t_ν. This is a map

$$t_\nu : [\nu] \to \{1',\ldots,j'\} \quad \text{where } k' = k+i \text{ for } 1 \le k \le j$$

(8.3) Lemma There exists a surjective $\mathbb{C}\mathrm{GL}(V)$-module map

$$((h(t_\mu)V^{\otimes i}) \otimes (h(t_\nu)V^{\otimes j}) \to h(t_\lambda)V^{\otimes n} \qquad \text{—}(*)$$

Proof Let $S_i = \mathrm{Aut}(\{1,\ldots,i\})$ and $S_j = \mathrm{Aut}(\{1',\ldots,j'\})$.

S_i and S_j are embedded in S_n and are disjoint, so we regard $\mathbb{C}S_i$ and $\mathbb{C}S_j$ as subsets of $\mathbb{C}\mathrm{S}_n$ which commute. Also

$$C_{t_\lambda} = C_{t_\mu} \times C_{t_\nu} \quad \text{and} \quad H = R_{t_\mu} \times R_{t_\nu} \le R_{t_\lambda}$$

and H permutes each side of the wall. Take a transversal $R_{t_\lambda} = \bigcup_i r_i H$. Then

$$\begin{aligned} h(t_\lambda) &\overset{(2.4)}{=} \sum_{r\in R_{t_\lambda}} \sum_{c \in C_{t_\lambda}} \varepsilon_c r c \\ &= \sum_i \sum_{r'\in R_{t_\mu}} \sum_{r''\in R_{t_\nu}} \sum_{c'\in C_{t_\mu}} \sum_{c''\in C_{t_\nu}} \varepsilon_{c'}\varepsilon_{c''} r_i r' r'' c' c'' \\ &= \sum_i r_i h(t_\mu) h(t_\nu) \end{aligned}$$

So

$$h(t_\lambda)h(t_\mu)h(t_\lambda) = \sum_i r_i h(t_\mu)^2 h(t_\nu)^2 \overset{(2.10)}{=} k h(t_\lambda)$$

where $k = \dfrac{i!j!}{f_\mu f_\nu}$. We thus have a $\mathbb{C}\mathrm{GL}(V)$-map

$$V^{\otimes i} \otimes V^{\otimes j} \to h(t_\lambda)V^{\otimes n}$$

given by premultiplication by $h(t_\lambda)$. Restriction of this map to the left-hand side of $(*)$ is surjective, since

$$h(t_\lambda)(x \otimes y) = h(t_\lambda) \cdot \frac{1}{k}(h(t_\mu)x \otimes h(t_\nu)y) \qquad \square$$

(8.4) Lemma

(1) If $\lambda_{m+1} = 0$ then $h_\lambda V^{\otimes n} \neq 0$

(2) If $n > 0$ and $\lambda_m = 0$ then $\dim_{\mathbb{C}} h_\lambda V^{\otimes n} \ge 2$

Proof

(1) Let i_j be the row in which j occurs in t_λ° and let $x = e_{i_1} \otimes \cdots \otimes e_{i_n}$. For $\sigma \in \mathrm{S}_n$

$$\sigma x = x \quad \Leftrightarrow \quad i_j = i_{\sigma^{-1}(j)}\ \forall j \quad \Leftrightarrow \quad j, \sigma^{-1}(j) \text{ are in the same row} \quad \Leftrightarrow \quad \sigma \in R_{t_\lambda^\circ}$$

So the coefficient of x in the decomposition of $h_\lambda x$ with respect to the standard basis of $V^{\otimes n}$ is $\left|R_{t_\lambda^\circ}\right| \neq 0$. So $h_\lambda x \neq 0$.

(2) Let $y = e_{1+i_1} \otimes \cdots \otimes e_{1+i_n}$ and adapt the argument above, to see that $h_\lambda x$ and $h_\lambda y$ are linearly independent. $\square$

(8.5) Lemma If $\lambda_{m+1} = 0$ and $\lambda_m > 0$ then

$$h_\lambda V^{\otimes n} \cong \Lambda^m V \otimes h_{(\lambda_1 - 1,\ldots,\lambda_m - 1)} V^{\otimes(n-m)}$$

Proof Put a wall in $[\lambda]$ between the first column and the rest. Let t_λ be a tableau whose first column consists of $\{1, \dots, m\}$. By **(8.3)** there exists a surjection

$$\Lambda^m V \otimes h(t_\nu) V^{\otimes(n-m)} \twoheadrightarrow h(t_\lambda) V^{\otimes n}$$

where $\nu = (\lambda_1 - 1, \dots, \lambda_m - 1)$. Hence, with the usual identifications, there exists a map

$$\Lambda^m V \otimes h_\nu V^{\otimes(n-m)} \twoheadrightarrow h_\lambda V^{\otimes n}$$

Both $h_\nu V^{\otimes(n-m)}$ and $h_\lambda V^{\otimes n}$ are nonzero, hence they are irreducible $\mathbb{C}\mathrm{GL}(V)$-modules by **(8.1)**. Since $\Lambda^m V$ is one-dimensional, both sides are irreducible and the map is an isomorphism. □

Proof of (8.2) If $\lambda_{m+1} \neq 0$ then $[\lambda]$ has $i > m$ rows and as in **(8.5)** there exists a surjection

$$\Lambda^i V \otimes h_\nu V^{\otimes(n-i)} \twoheadrightarrow h_\lambda V^{\otimes n}$$

but $\Lambda^i V = 0$, so $h_\lambda V^{\otimes n} = 0$.

If $\lambda_{m+1} = 0$ then iterating **(8.5)** we have

$$\begin{aligned} \dim_{\mathbb{C}} h_\lambda V^{\otimes n} &= \dim_{\mathbb{C}} h_{(\lambda_1 - \lambda_m, \dots, \lambda_{m-1} - \lambda_m)} V^{\otimes(n - m\lambda_m)} \\ &= \begin{cases} 1 & \text{if } \lambda_1 = \dots = \lambda_m \\ \geq 2 & \text{if not, by } \textbf{(8.4)} \end{cases} \end{aligned}$$

□

9 Polynomial and rational representations of $\mathrm{GL}(V)$

Let V be an m-dimensional $\mathbb{C}$-space with basis $e_1, \ldots, e_m$.

(9.1) Definition A finite-dimensional $\mathbb{C}\mathrm{GL}(V)$-module M, with basis $w_1, \ldots, w_n$, is **rational** (resp. **polynomial**, n**-homogeneous**) if there exist rational (resp. polynomial, n-homogeneous) functions $f_{ij}(X_{rs})$ for $1 \le i, j \le h$ in m^2 variables X_{rs} for $1 \le r, s \le m$ such that the map

$$\mathrm{GL}_m(\mathbb{C}) \xrightarrow{\text{basis } e_i} \mathrm{GL}(V) \xrightarrow{\text{representation}} \mathrm{End}_{\mathbb{C}}(M) \xrightarrow{\text{basis } w_j} \mathrm{M}_h(\mathbb{C})$$

sends $(A_{rs})_{1 \le r,s \le m} \mapsto (f_{ij}(A_{rs}))_{1 \le i,j \le h}$.

Exercise Check that the definitions in **(9.1)** are independent of the choice of bases e_i and w_j of V and M.

Examples

(0) The natural representation of $\mathrm{GL}(V)$ on $V^{\otimes n}$ is polynomial. In fact...

(1) $V^{\otimes n}$ is n-homogeneous; but...

(2) $\mathbb{C} \oplus V$ is polynomial but not homogeneous; and...

(3) V^* is rational but not polynomial (or homogeneous).

(9.2) Lemma

(a) Submodules, quotients and direct sums of rational (resp. polynomial, n-homogeneous) modules are rational (resp. polynomial, n-homogeneous).

(b) If M is rational then so is M^*.

(c) If M and N are rational (resp. polynomial, n- and n'-homogeneous) then $M \otimes N$ is rational (resp. polynomial, $(n+n')$-homogeneous).

(d) If $n \ne n'$ and M is both n- and n'-homogeneous then $M = 0$.

(e) if M is rational (resp. polynomial) then $\Lambda^n M$ and $\mathrm{S}^n M$ are rational (resp. polynomial).

Proof Easy exercise. $\square$

(9.3) Definition For $\lambda_1 \ge \cdots \ge \lambda_m \ge 0$ with $\sum_i \lambda_i = n$, set

$$D_{\lambda_1,\ldots,\lambda_m}(V) = h_{(\lambda_1,\ldots,\lambda_m)}(V^{\otimes n})$$

This is sometimes referred to in the literature as the **Weyl module** or **Schur module**.

(9.4) Theorem Every n-homogeneous $\mathbb{C}\mathrm{GL}(V)$-module is a direct sum of irreducible submodules. Moreover, the modules of **(9.3)** form a complete set of nonisomorphic irreducible n-homogeneous $\mathbb{C}\mathrm{GL}(V)$-modules.

Proof By **(8.1)** it suffices to show that any n-homogeneous $\mathbb{C}\mathrm{GL}(V)$-module is obtained from some $\mathrm{S}(m,n)$-module (by restriction).

Let M be an n-homogeneous $\mathbb{C}\mathrm{GL}(V)$-module. M corresponds to a map

$$\rho_0 : \mathrm{GL}(V) \to \mathrm{End}_{\mathbb{C}}(M)$$

Consider the following diagram

$$\begin{array}{ccccccccc}
\mathrm{GL}(V) & \overset{\hookrightarrow}{\longrightarrow} & \mathrm{End}_{\mathbb{C}}(V) & \longrightarrow & \mathrm{S}^n(\mathrm{End}_{\mathbb{C}} V) & \overset{\cong}{\longrightarrow} & T^n(\mathrm{End}_{\mathbb{C}} V)_{\mathrm{sym}} & \overset{\cong}{\longrightarrow} & \mathrm{S}(m,n) \\
\downarrow \rho_0 & & \downarrow \rho_1 & & \downarrow \rho_2 & & \downarrow \rho_3 & & \downarrow \rho_4 \\
\mathrm{End}_{\mathbb{C}}(M) & = & \mathrm{End}_{\mathbb{C}}(M) & = & \mathrm{End}_{\mathbb{C}}(M) & = & \mathrm{End}_{\mathbb{C}}(M) & = & \mathrm{End}_{\mathbb{C}}(M)
\end{array}$$

In the top row all the maps are natural, and the composite $\gamma : \mathrm{GL}(V) \to \mathrm{S}(m,n)$ is the natural map we use for restricting $\mathrm{S}(m,n)$-modules to $\mathbb{C}\mathrm{GL}(V)$-modules.

Claim There exist $\rho_1, \rho_2, \rho_3, \rho_4$ as above making the diagram commute.

Since M is n-homogeneous, we can extend the domain of definition of ρ_0 to obtain an n-homogeneous map ρ_1. For ρ_2, use the universal property of symmetric powers. Then ρ_3, ρ_4 are transported by the isomorphisms.

We check that ρ_4 is a map of $\mathbb{C}$-algebras. Well

$$\rho_4(1) = \rho_4(\gamma(1)) = \rho_0(1) = 1$$

and

$$\rho_4(\gamma(gg')) = \rho_0(gg') = \rho_0(g)\rho_0(g') = \rho_4(\gamma(g))\rho_4(\gamma(g'))$$

for $g, g' \in \mathrm{GL}(V)$. But $\gamma(\mathrm{GL}(V))$ spans $\mathrm{S}(m,n)$ by **(7.8)**.

So M is a $\mathbb{C}\mathrm{S}(m,n)$-module and the restriction to $\mathrm{GL}(V)$ is the module we started with. □

(9.5) Lemma Every polynomial module for $\mathrm{GL}(\mathbb{C}) = \mathbb{C}^\times$ decomposes as a direct sum of submodules on whih $g \in \mathbb{C}^\times$ acts as multiplication by g^r for some r.

Proof If $\rho : \mathbb{C}^\times \to \mathrm{GL}(M) \cong \mathrm{GL}_h(\mathbb{C})$ is a polynomial representation then each $\rho(g)_{ij}$ is polynomial in g. Choose $N \in \mathbb{N}$ such that each $\rho(g)_{ij}$ has degree $< N$. Then M is a $\mathbb{C}G$-module by restriction, where

$$G = \{e^{2\pi \frac{ij}{N}} : 0 \leq j \leq N\} \subseteq \mathbb{C}^\times$$

is a cyclic group of order N. Now

$$M = M_1 \oplus \cdots \oplus M_h$$

as a $\mathbb{C}G$-module with each M_j one-dimensional, and $g \in G$ acts as multiplication by g^{n_j} on M_j for $0 \leq j < N$. (These are the possible irreducible $\mathbb{C}G$-modules.)

Choosing nonzero elements of the M_js gives a basis for M, and if $(\rho(g)_{ij})$ is the matrix of $\rho(g)$ with respect to this basis then

$$\rho(g)_{ij} = \begin{cases} g^{n_i} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

for $g \in e^{2\pi \frac{ij}{N}}$ $(0 \leq j < N)$, and hence for all $g \in \mathbb{C}^\times$ since the $\rho(g)_{ij}$ are polynomials of degree $< N$ in g. □

(9.6) Lemma Every polynomial decomposes as a direct sum of n-homogeneous modules.

Proof Tke a representation $\rho : \mathrm{GL}(V) \to \mathrm{GL}(U)$ be a representation of $\mathrm{GL}(V)$ on a $\mathbb{C}$-space U. We have an inclusion $\mathbb{C}^\times \hookrightarrow \mathrm{GL}(V)$, so that one can regard U as a polynomial representation of $\mathbb{C}^\times$, so by **(9.5)** U will split as

$$U = U_0 \oplus \cdots \oplus U_N$$

with $\alpha 1 \in \mathrm{GL}(V)$ acting as multiplication by α^n on U_n.

Let $u \in U_n$ and let $g \in \mathrm{GL}(V)$. Write

$$gu = u_0 + \cdots + u_N$$

where $u_i \in U_i$ for $0 \leq i \leq N$. Now $(\alpha 1)gu = g(\alpha 1)u$ for $\alpha \in \mathbb{C}^\times$, so

$$u_0 + \alpha u_1 + \cdots + \alpha^N u_N = \alpha^n u_0 + \cdots + \alpha^n u_N$$

and hence $u_i = 0$ for $i \neq n$.

Thus $gu \in U_n$ and the U_n are $\mathbb{C}\mathrm{GL}(V)$-submodules of U. Since $\alpha 1$ acts as multiplication by α^n on U_n, it follows that U_n is an n-homogeneous $\mathbb{C}\mathrm{GL}(V)$-module. $\square$

(9.7) Theorem Every polynomial $\mathbb{C}\mathrm{GL}(V)$-module is a direct sum of irreducible submodules. The modules $D_{\lambda_1,\ldots,\lambda_m}(V)$ with $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$ are a complete set of nonisomorphic irreducible polynomial $\mathbb{C}\mathrm{GL}(V)$-modules. $\square$

(9.8) Definition If $n \in \mathbb{Z}$ then the one-dimensional $\mathbb{C}\mathrm{GL}(V)$-module corresponding to the representation

$$\begin{array}{ccc} \mathrm{GL}(V) & \to & \mathbb{C}^\otimes \\ g & \mapsto & (\det g)^n \end{array}$$

is denoted by $\det^n$. Note that

$$\det{}^1 \cong \Lambda^m V, \quad \det{}^n \cong (\det{}^1)^{\otimes n} \text{ if } n \geq 0, \quad \det{}^n \cong (\det{}^{-n})^* \text{ if } n \leq 0$$

(9.9) Definition If $\lambda_1 \geq \cdots \geq \lambda_m$ but $\lambda_m < 0$ then define

$$D_{\lambda_1,\ldots,\lambda_m} = D_{\lambda_1-\lambda_m,\ldots,\lambda_{m-1}-\lambda_m,0}(V) \otimes \det{}^{\lambda_m}$$

Remark If $\lambda_1 \geq \cdots \geq \lambda_m > 0$ then by iterating **(8.5)** (as in the proof of **(8.2)**) we know that

$$D_{\lambda_1,\ldots,\lambda_m}(V) \cong D_{\lambda_1-\lambda_m,\ldots,\lambda_{m-1}-\lambda_m,0}(V) \otimes \det{}^{\lambda_m}$$

(9.10) Theorem Every rational $\mathbb{C}\mathrm{GL}(V)$-module is a direct sum of irreducible submodules. The modules $D_{\lambda_1,\ldots,\lambda_m}(V)$ for $\lambda_1 \geq \cdots \geq \lambda_m$ form a complete set of nonisomorphic irreducible (rational) $\mathbb{C}\mathrm{GL}(V)$-modules.

Proof The rational functions $f : \mathrm{GL}(V) \to \mathbb{C}$ are of the form $f = \dfrac{p}{\det^i}$ with p a polynomial function. Thus if M is a rational $\mathbb{C}\mathrm{GL}(V)$-module then $M_1 = M \otimes \det^N$ is a polynomial $\mathbb{C}\mathrm{GL}(V)$-module for some N. Since M_1 decomposes as a direct sum of irreducibles, so does M. If M is irreducible then so is M_1, since $\det^N$ is one-dimensional, and thus M_1 is of the form

$$M_1 \cong D_{\mu_1,\ldots,\mu_m}(V) \text{ for some } \mu_1 \geq \cdots \geq \mu_m$$

Finally, $M = D_{\mu_1-N,\ldots,\mu_m-N}(V)$ by the above remark. $\square$

(9.11) Theorem The one-dimensional rational $\mathbb{C}\mathrm{GL}(V)$-modules are precisely the $\det^n$ for $n \in \mathbb{Z}$.

Proof As above, pass to polynomial modules and then apply **(8.2)**. $\square$

Remark The characterisation of irreducible $\mathbb{C}\mathrm{GL}(V)$-modules can be summarised by:

n-homogeneous	$\lambda_1 \geq \cdots \geq \lambda_m > 0$ with $\sum_i \lambda_i = n$	**(9.4)**
polynomial	$\lambda_1 \geq \cdots \geq \lambda_m \geq 0$	**(9.7)**
rational	$\lambda_1 \geq \cdots \geq \lambda_m$	**(9.10)**

Interlude: some reminders about characters of $\mathrm{GL}_n(\mathbb{C})$

Let $T_n \leq \mathrm{GL}_n(\mathbb{C})$ be the n-dimensional torus, i.e. the subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of diagonal matrices.

If $\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}(W)$ is a rational representation then the **character** of ρ is the function

$$\chi_\rho : (x_1, \dots, x_n) \mapsto \operatorname{tr} \rho \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

sometimes denoted by χ_W.

Basic properties

Let ρ, ρ' be rational representations of $\mathrm{GL}_n(\mathbb{C})$ associated with W, W'.

(a) $\chi_\rho \in \mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$, and if ρ is polynomial then $\chi_\rho \in \mathbb{Z}[x_1, \dots, x_n]$.

(b) χ_ρ is a symmetric function under the conjugation action of S_n on T_n, i.e. under permutations of the diagonal entries.

(c) $\rho \cong \rho' \Rightarrow \chi_\rho = \chi_{\rho'}$; $\Leftarrow$ will be **(10.6)**.

(d) $\chi_{W \oplus W'} = \chi_W + \chi_{W'}$ and $\chi_{W \otimes W'} = \chi_W \chi_{W'}$.

(e) If W is an irreducible polynomial representation of degree m then χ_W is a homogeneous polynomial of degree m.

(f) $\chi_{W^*}(x_1, \dots, x_n) = \chi_W(x_1^{-1}, \dots x_n^{-1})$.

(g) When $V = \mathbb{C}^n$ we have

$$\chi_{V^{\otimes n}} = (x_1 + \cdots + x_n)^m,$$

$$\chi_{\mathrm{S}^2 V} = \sum_{i \leq j} x_i x_j, \qquad \chi_{\Lambda^2 V} = \sum_{i<j} x_i x_j,$$

$$\chi_{\det} = x_1 \dots x_n, \qquad \chi_{V^*} = x_1^{-1} + \cdots + x_n^{-1},$$

$$\chi_{\Lambda^{n-1} V} = \sum_{i=1}^{n} x_1 \dots \widehat{x_i} \dots x_n, \qquad \chi_{\mathrm{S}^j V} = h_j(x_1, \dots, x_n) = \sum_{i_1 \leq \cdots \leq i_j} x_{i_1} \dots x_{i_j}$$

The h_j are called **complete symmetric functions**. We could (but won't) study the theory of symmetric functions and apply it here; see R. Stanley's book for more.

10 Weyl character formula

Let V be an m-dimensional $\mathbb{C}$-space. If $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m$ and $\lambda_1 \geq \cdots \geq \lambda_m$ then the character of the $\mathbb{C}\mathrm{GL}(V)$-module $D_{\lambda_1,\ldots,\lambda_m}(V)$ $(= h_\lambda V^{\otimes n}$ if $\lambda_m \geq 0$ and $n = \sum_i \lambda_i)$ will be denoted by ϕ_λ.

(10.1) Lemma If $\xi \in \mathrm{End}_{\mathbb{C}}(V)$ then $\phi_\lambda(\xi)$ is a symmetric rational function of the eigenvalues of ξ.

Proof The function

$$P(x_1, \ldots, x_m) = \phi_\lambda(\mathrm{diag}(x_1, \ldots, x_m))$$

is a rational function of the $x_1, \ldots, x_m$, and it is symmetric since $\mathrm{diag}(x_1, \ldots, x_m)$ is conjugate to $\mathrm{diag}(x_{\tau(1)}, \ldots, x_{\tau(m)})$ for any $\tau \in \mathrm{S}_m$.

Now choose a basis for V such that the matrix A_1 of ξ is in Jordan normal form, and for $t \in \mathbb{C}$ let A_t be the matrix obtained from A_1 by replacing the upperdiagonal 1s by ts. Let ξ_t be the endomorphism corresponding to A_t. For $t \neq 0$, A_t is conjugate to A_1, and so $\phi_\lambda(\xi_t) = \phi_\lambda(\xi)$. Since ϕ_λ is a rational function it is continuous (wherever it is defined), and so

$$\phi_\lambda(\xi) = \lim_{t\to 0} \phi_\lambda(\xi_t) = \phi_\lambda(\lim_{t\to 0} \xi_t) = \phi_\lambda(\xi_0) = P(x_1, \ldots, x_m)$$

which is a symmetric function of the eigenvalues of ξ. □

(10.2) Lemma Let α be an S_n-conjugacy class with cycle type $n^{\alpha_n}, \ldots, 1^{\alpha_1}$, and let ξ be an endomorphism of V with eigenvalues $x_1, \ldots, x_m$. If $s_i = x_1^i + \cdots + x_m^i$ then

$$s_1^{\alpha_1} \ldots s_n^{\alpha_n} = \sum_{\lambda \in \Lambda^+(m,n)} \chi^\lambda(\alpha)\phi_\lambda(\xi)$$

(Compare this with **(4.14)**.)

Proof Pick $g \in \alpha$. Suppose ξ has matrix $\mathrm{diag}(x_1, \ldots, x_m)$ with respect to the standard basis $e_1, \ldots, e_m$ of V. We compute the trace of the endomorphism of $V^{\otimes n}$ sending $x \mapsto (g\xi)x = \xi g x$ in two ways.

Way 1 $V^{\otimes n}$ is a $\mathbb{C}\mathrm{S}_n$-module, so by **(1.8)**,

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} \mathbb{C}\mathrm{S}_n h_\lambda \otimes \mathrm{Hom}_{\mathbb{C}\mathrm{S}_n}(\mathbb{C}\mathrm{S}_n h_\lambda, V^{\otimes n})$$

which, by **(1.12)**, becomes

$$V^{\otimes n} \cong \bigoplus_{\lambda \vdash n} (\mathbb{C}\mathrm{S}_n h_\lambda \otimes h_\lambda V^{\otimes n})$$

This is an isomorphism both of $\mathbb{C}\mathrm{S}_n$- and $\mathrm{End}_{\mathbb{C}\mathrm{S}_n}(V^{\otimes n})$-modules. Since the action of $\mathrm{GL}(V)$ on $V^{\otimes n}$ commutes with that of S_n, the corresponding action of $g\xi$ on the right-hand side is given by the action of g on $\mathbb{C}\mathrm{S}_n h_\lambda$ and of ξ on $h_\lambda V^{\otimes n}$. So the trace of the action is

$$\textstyle\sum_{\lambda \in \Lambda^+(n,m)} \xi^\lambda(\alpha)\phi_\lambda(\xi) \qquad \text{—(1)}$$

Way 2 (direct computation)

$$g\xi(e_{i_1} \otimes \cdots \otimes e_{i_n}) = x_{i_1} \ldots x_{i_n} e_{i_{g^{-1}(1)}} \otimes \cdots \otimes e_{i_{g^{-1}(n)}}$$

and so

$$\mathrm{tr}(g\xi) = \sum_D x_{i_1} \ldots x_{i_n}$$

where $D = \{(i_1, \ldots, i_n) : 1 \leq i_1, \ldots, i_n \leq m \text{ and } i_{g^{-1}(j)} = i_j \ \forall j\}$.

The condition $i_{g^{-1}(j)} = i_j$ for all j is equivalent to requiring that the function $j \mapsto i_j$ be constant on the cycles involved in g. Hence

$$\mathrm{tr}(g\xi) = s_1^{\alpha_1} \ldots s_n^{\alpha_n} \qquad \text{—(2)}$$

Equating (1) and (2) gives the result. □

(10.3) Theorem Let $\lambda \in \Lambda^+(m,n)$ and let $\xi \in \mathrm{End}_{\mathbb{C}}(V)$ with eigenvalues $x_1, \ldots, x_m$. If $s_i = x_1^i + \cdots + x_m^i$, then

$$\phi_\lambda(\xi) = \sum_{\text{ccls } \alpha} \frac{\chi^\lambda(\alpha)}{\alpha_1! \ldots \alpha_n!} \left(\frac{s_1}{1}\right)^{\alpha_1} \cdots \left(\frac{s_n}{n}\right)^{\alpha_n}$$

Proof Take the formula of **(10.2)**, multiply it by $n_\alpha \chi^\mu(\alpha)$ and sum over α; then use orthogonality of the χs. □

(10.4) Theorem *(Weyl's character formula for* GL_n*)* Let $\xi \in \mathrm{GL}(V)$ with eigenvalues $x_1, \ldots, x_m$. Then

$$\phi_\lambda(\xi) = \frac{|x^{\ell_1}, \ldots, x^{\ell_m}|}{|x^{m-1}, \ldots, 1|}$$

where $\ell_i = \lambda_i + m - i$.

Proof

(a) Suppose $\lambda_i \geq 0$ for all i, so that $\lambda \in \Lambda^+(n,m)$. By **(10.2)** and the character formula for the symmetric group **(4.14)** we know that

$$s_1^{\alpha_1} \ldots s_n^{\alpha_n} = \sum_{\lambda \in \Lambda^+(n,m)} \chi^\lambda(\alpha) \frac{|x^{\ell_1}, \ldots, x^{\ell_m}|}{|x^{m-1}, \ldots, 1|} = \sum_{\lambda \in \Lambda^+(n,m)} \chi^\lambda(\alpha) \phi_\lambda(\alpha)$$

Then the orthogonality of χ^λs allows us to equate coefficients.

(b) For general λ, since

$$D_\lambda \overset{\textbf{(9.9)}}{\cong} D_{\lambda_1 - \lambda_m, \ldots, \lambda_m - \lambda_m}(V) \otimes \det{}^{\lambda_m}$$

it follows that

$$\phi_\lambda(\xi) = \frac{|x^{\ell_1 - \lambda_m}, \ldots, x^{\ell_m - \lambda_m}|}{|x^{m-1}, \ldots, 1|} (x_1 \ldots x_m)^{\lambda_m} = \frac{|x^{\ell_1}, \ldots, x^{\ell_m}|}{|x^{m-1}, \ldots, 1|}$$

□

Remark This short proof uses the character formula of S_n from §4. One can do this in reverse, using integration to compute the Weyl formula for the compact subgroup U_n of unitary matrices in $\mathrm{GL}_m(\mathbb{C})$, and then translate that to $\mathrm{GL}_m(\mathbb{C})$. [See also Telemann's notes for the 2005 Part II course.] You then use **(10.2)** to pass to S_n. See also Weyl's book.

(10.5) Theorem *(Weyl's dimension formula)*

$$\deg \phi_\lambda = \dim D_{\lambda_1, \ldots, \lambda_m}(V) = \frac{\prod_{1 \leq i < j \leq m} (\ell_i - \ell_j)}{\prod_{1 \leq i < j \leq m} (j - i)}$$

Proof For $t \in \mathbb{C}$ set $x_m = 1, x_{m-1} = e^t, \ldots, x_1 = e^{(m-1)t}$. Then

$$|x^{\ell_1}, \ldots, x^{\ell_m}| = \prod_{1 \leq i < j \leq m} (e^{\ell_i t} - e^{\ell_j t})$$

since it's the transpose of a Vandermonde matrix. The term of lowest degree is $\prod_{i<j}((\ell_i - \ell_j)t)$.

Also $|x^{m-1}, \ldots, 1| = \prod_{i<j}(e^{(m-i)t} - e^{(m-j)t})$ and the term of lowest degree in t is $\prod_{i<j}((j-i)t)$.

If $\xi_t = \mathrm{diag}(x_1, \ldots, x_m)$ then by **(10.4)**,

$$\deg \phi_\lambda = \phi_\lambda(1) = \lim_{t \to 0} \phi_\lambda(\xi_t) = \frac{\prod_{i<j}(\ell_i - \ell_j)}{\prod_{i<j}(j-i)}$$ □

(10.6) Theorem If two rational $\mathbb{C}\mathrm{GL}(V)$-modules have the same characters then they are isomorphic.

Proof As λ varies, the rational functions in Weyl's character formula are linearly independent elements of $\mathbb{C}(x_1, \ldots, x_m)$. $\square$

(10.7) Lemma $D_{\lambda_1,\ldots,\lambda_m}(V)^* \cong D_{-\lambda_m,\ldots,-\lambda_1}(V)$

Proof Let ψ be the character of $D_{\lambda_1,\ldots,\lambda_m}(V)$. Then

$$\psi(\xi) = \phi_\lambda(\xi^{-1}) = \frac{\left|x^{-\ell_1}, \ldots, x^{-\ell_m}\right|}{\left|x^{-(m-1)}, \ldots, 1\right|} = \frac{\left|x^{-\ell_m}, \ldots, x^{-\ell_1}\right|}{\left|1, \ldots, x^{-(m-1)}\right|}$$
$$= \frac{\left|x^{m-1-\ell_m}, \ldots, x^{m-1-\ell_1}\right|}{\left|x^{m-1}, \ldots, 1\right|} = \phi_\mu(\xi)$$

where $\mu = (-\lambda_m, \ldots, -\lambda_1)$. Hence we have the required isomorphism by **(10.6)**. $\square$

Finally we have the famous Clebsch–Gordan formula.

(10.8) Theorem *(Clebsch–Gordan formula)* Let $V = \mathbb{C}^2$ and $p, q \in \mathbb{N}$. Then

$$D_{p,0}(V) \otimes D_{q,0}(V) \cong \bigoplus_{r=0}^{\min\{p,q\}} D_{p+q-r,r}(V)$$

Proof Exercise; write down the characters and use **(10.6)**. $\square$

(10.9) Examples of rational $\mathbb{C}\mathrm{GL}(V)$-modules

- $\mathbb{C} \cong D_{0,0,\ldots,0,0}(V)$
- $\det^i \cong D_{i,i,\ldots,i,i}(V)$
- $V \cong D_{1,0,\ldots,0,0}(V)$
- $\mathrm{S}^n V \cong D_{n,0,\ldots,0,0}(V)$
- $\Lambda^n V \cong D_{1,\ldots,1,0,\ldots,0}(V)$ where n '1's and $m - n$ '0's appear, and $n \leq m$
- $V^* \cong D_{0,0,\ldots,0,-1}(V)$
- $(\mathrm{S}^n V)^* \cong D_{0,0,\ldots,0,-n}(V)$
- $(\Lambda^n V)^* \cong D_{0,\ldots,0,-1,\ldots,-1}(V)$ where $m - n$ '0's and n '-1's appear, and $n \leq m$

Chapter III: Invariant theory

11 Introduction to invariant theory and first examples

Let W be a finite-dimensional $\mathbb{C}$-space and $\mathbb{C}[W]$ be its coordinate ring (cf. §6).

Facts about $\mathbb{C}[W]$

- $\mathbb{C}[W]$ is a commutative $\mathbb{C}$-algebra, which is infinite-dimensional if $W \neq 0$, with
$$(\lambda f)(w) = \lambda f(w), \qquad (f+g)(w) = f(w) + g(w),$$
$$(fg)(w) = f(w)g(w), \qquad 1_{\mathbb{C}[W]}(w) = 1$$
for all $\lambda \in \mathbb{C}$, $f, g \in \mathbb{C}[W]$ and $w \in W$.
- $\mathbb{C}[W]$ is the ring of regular functions of the affine variety $\mathbb{A}^{\dim W} \cong W$.
- $$\mathbb{C}[W] \cong \bigoplus_{n\geq 0} \mathrm{H}_{\mathbb{C},n}(W,C) \overset{\textbf{(6.12)}}{\cong} \bigoplus_{n \geq 0} \mathrm{S}^n(W^*) = \mathrm{S}(W^*)$$
This latter isomorphism is known as **polarisation**.
- If W^* has basis $w_1^*, \dots, w_h^*$ then the map
$$\begin{array}{ccc} \mathbb{C}[X_1, \dots, X_h] & \to & \mathbb{C}[W] \\ X_i & \mapsto & w_i^* \end{array}$$
is a $\mathbb{C}$-algebra homomorphism.
- If W is a $\mathbb{C}G$-module then $\mathbb{C}[W]$ is a $\mathbb{C}G$-module via the action
$$gf(w) = f(g^{-1}w)$$
for $g \in G$, $f \in \mathbb{C}[W]$ and $w \in W$.
- For $g \in G$, the map $\begin{array}{ccc} \mathbb{C}[W] & \to & \mathbb{C}[W] \\ f & \mapsto & gf \end{array}$ is a $\mathbb{C}$-algebra automorphism, since
$$g(ff')(w) = (ff')(g^{-1}w) = f(g^{-1}w)f'(g^{-1}w) = (gf)(w)(gf')(w) = ((gf)(gf'))(w)$$
and
$$g1_{\mathbb{C}[W]}(w) = 1_{\mathbb{C}[W]}(g^{-1}w) = 1 = 1_{\mathbb{C}[W]}(w)$$

(11.1) Definition If V, W are $\mathbb{C}G$-modules then $f : W \to V$ is **concomitant** if
$$f(gw) = gf(w) \quad \text{for all } w \in W,\ g \in G$$
A G-**invariant** for the $\mathbb{C}G$-module W is a concomitant map $f : W \to \mathbb{C}$.

Remarks

(1) f is invariant if and only if it is a fixed point under the G-action, i.e. $gf = f$ for all $g \in G$.

(2) linear concomitant $\equiv$ $\mathbb{C}G$-homomorphism

(3) The map $\Delta : \begin{array}{ccc} V & \to & \mathrm{S}^n W \\ w & \mapsto & w \vee \cdots \vee w \end{array}$ is an n-homogeneous concomitant.

(4) $\mathbb{C}[W]^G = \{f \in \mathbb{C}[W] : gf = f \text{ for all } g \in G\}$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}[W]$, called the **polynomial invariant ring**. It is a graded $\mathbb{C}$-algebra:
$$\mathbb{C}[W]^G = \bigoplus_{d \geq 0} \mathbb{C}[W]_d^G$$
with grading inherited by that of $\mathbb{C}[W]$.

(11.2) Problem Compute $\mathbb{C}[W]^G$, describing its generators and relations.

What follows are some classical results which we will state but not prove.

- [Noether] If G is finite then $\mathbb{C}[W]^G$ is finitely generated as a $\mathbb{C}$-algebra.
- [Noether, 1916] If $\operatorname{char} K = 0$ then for any representation W of the finite group G, the invariant ring $K[W]^G$ is generated by the invariants of degree $\leq |G|$.
- [Hilbert] If W is a rational $\mathbb{C}\mathrm{GL}(V)$-module then $\mathbb{C}[W]^{\mathrm{GL}(V)}$ and $\mathbb{C}[W]^{\mathrm{SL}(V)}$ are finitely-generated $\mathbb{C}$-algebras. In fact [Hochster–Roberts, 1974] they are also Cohen–Macaulay rings. Also, $\mathbb{C}[W]^{\mathrm{SL}(V)}$ is a unique factorisation domain, and hence it is a Gorenstein ring (i.e. it has finite injective dimension).
- [Popov, *Astérisque*, 1988] There is an explicit bound on the number of generators for $\mathbb{C}[W]^{\mathrm{SL}(V)}$.

Examples

(11.3) Symmetric polynomials $W = \mathbb{C}^n$ with basis $e_1, \ldots, e_n$, is naturally a $\mathbb{C}\mathrm{S}_n$-module. If $\xi_1, \ldots, \xi_n$ is the corresponding dual basis for W^*, then the isomorphism

$$\begin{array}{ccc} \mathbb{C}[X_1, \ldots, X_n] & \xrightarrow{\cong} & \mathbb{C}[W] \\ X_i & \mapsto & \xi_i \end{array}$$

allows an identification $\mathbb{C}[W] \leftrightarrow \mathbb{C}[X_1, \ldots, X_n]$. The polynomial invariant ring of the $\mathbb{C}\mathrm{S}_n$-module W is

$$\mathbb{C}[X_1, \ldots, X_n]^{\mathrm{S}_n} = \{p \in \mathbb{C}[X_1, \ldots, X_n] \,:\, p \text{ symmetric in the } X_i\text{s}\}$$

We have **elementary symmetric functions** $e_j(X_1, \ldots, X_n) \in \mathbb{C}[X_1, \ldots, X_n]^{\mathrm{S}_n}$ such tha

$$(t + X_1) \cdots (t + X_n) = t^n + e_1(\vec{X})t^{n-1} + \cdots + e_n(\vec{X})$$

where $\vec{X} = (X_1, \ldots, X_n)$. That is,

$$e_1(\vec{X}) = X_1 + \cdots + X_n, \qquad e_i(\vec{X}) = \sum_{j_1 < j_2 < \cdots < j_i} X_{j_1} X_{j_2} \ldots X_{j_i}, \qquad e_n(\vec{X}) = X_1 \ldots X_n$$

The **fundamental theorem of symmetric functions** computes the polynomial invariants for W; namely, it shows that the map

$$\begin{array}{ccc} \mathbb{C}[Y_1, \ldots, Y_n] & \to & \mathbb{C}[X_1, \ldots, X_n]^{\mathrm{S}_n} \\ Y_i & \mapsto & e_i(\vec{X}) \end{array}$$

is a $\mathbb{C}$-algebra isomorphism. The standard way to prove this is by showing that $\mathbb{C}[X_1, \ldots, X_n]^{\mathrm{S}_n} = \mathbb{C}[e_1, \ldots, e_n]$ by induction on n.

Recall for $p(t) \in \mathbb{C}[t]$ monic of degree n, say

$$p(t) = t^n + a_1 t^{n-1} + \cdots + a_n = \prod_{i=1}^{n} (t + \lambda_i)$$

its **discriminant is**

$$\operatorname{disc}(p) = \prod_{i<j} (\lambda_i - \lambda_j)^2$$

Now, since the polynomial $\left|X^{n-1}, \ldots, 1\right|^2 = \prod_{i<j}(X_i - X_j)^2$ is symmetric in the X_is, it may be represented as a polynomial in the e_is, say

$$\prod_{i<j} (X_i - X_j)^2 = D(e_1(\vec{X}), \ldots, e_n(\vec{X}))$$

Also, disc(p) is a polynomial in the coefficients of p, for example

$$\mathrm{disc}(t^2 + bt + c) = b^2 - 4c, \qquad \mathrm{disc}(t^3 + bt + c) = -4b^3 - 27b^2$$

(11.4) Alternating groups

Restriction in **(11.3)** enables consideration of W as a representation of A_n ($n \geq 2$). The ring of invariants is $\mathbb{C}[X_1, \ldots, X_n]^{\mathrm{A}_n}$. Consider the Vandermonde determinant

$$\left|X^{n-1}, \ldots, 1\right| = \prod_{i<j}(X_i - X_j)$$

$\left|X^{n-1}, \ldots, 1\right|^2$ is S_n-invariant since it is equal to $D(e_1, \ldots, e_n)$ as above.

Claim The $\mathbb{C}$-algebra map

$$\begin{array}{rccc} \theta: & \mathbb{C}[Y_1, \ldots, Y_n, Z] & \to & \mathbb{C}[X_1, \ldots, X_n]^{\mathrm{A}_n} \\ & Y_i & \mapsto & e_i \\ & Z & \mapsto & \left|X^{n-1}, \ldots, 1\right| \end{array}$$

is surjective, and $\ker(\theta) = (Z^2 - D(Y_1, \ldots, Y_n))$.

Proof We prove the claim in three steps.

Step 1 θ is surjective. Take a transposition $\tau \in \mathrm{S}_n$. Let $f \in \mathbb{C}[X_1, \ldots, X_n]^{\mathrm{A}_n}$. Then $f_{\mathrm{sym}} = f + \tau f$ is a symmetric polynomial, since S_n is generated by τ and A_n, and

$$\tau f_{\mathrm{sym}} = \tau f + \tau^2 f = \tau f + f = f_{\mathrm{sym}}$$

and if $\sigma \in \mathrm{A}_n$ then

$$\sigma f_{\mathrm{sym}} = \sigma f + \tau(\tau^{-1}\sigma\tau) f = f + \tau f = f_{\mathrm{sym}}$$

Similarly, $f_{\mathrm{alt}} = f - \tau f$ is an alternating polynomial.

Observe that $f_{\mathrm{alt}}(X_1, \ldots X_n) = 0$ whenever $X_i = X_j$ for $i \neq j$. Hence by Nullstellensatz

$$f_{\mathrm{alt}} \in \sqrt{(X_i - X_j)} = (X_i - X_j)$$

and so $(X_i - X_j) \mid f_{\mathrm{alt}}$ for all $i \neq j$. In particular,

$$f_{\mathrm{alt}} = \left|X^{n-1}, \ldots, 1\right| g$$

for some polynomial g. Now g is symmetric, and hence

$$f = \frac{1}{2}(f_{\mathrm{sym}} + f_{\mathrm{alt}}) = \frac{1}{2}(f_{\mathrm{sym}} + \left|X^{n-1}, \ldots, 1\right| g)$$

Since f_{sym} and g lie in $\mathrm{im}(\theta)$, so does f.

Step 2 $(Z^2 - D(\vec{Y})) \subseteq \ker(\theta)$. This is clear since

$$\theta(Z^2 - D(\vec{Y})) = \left|X^{n-1}, \ldots, 1\right|^2 - D(e_1, \ldots, e_n) = D(e_1, \ldots, e_n) - D(e_1, \ldots, e_n) = 0$$

Step 3 $\ker(\theta) \subseteq (Z^2 - D(\vec{Y}))$. Any polynomial $P(\vec{Y}, Z)$ is of the form

$$P(\vec{Y}, Z) = \underbrace{Q(\vec{Y}, Z)}_{\text{quotient}}(Z^2 - D(\vec{Y})) + \underbrace{A(\vec{Y}) + B(\vec{Y})Z}_{\text{remainder}}$$

We show that if $P \in \ker(\theta)$ has the form $A(\vec{Y}) + B(\vec{Y})Z$ then $P = 0$.

Suppose $a_1, \ldots, a_n \in \mathbb{C}$, and let

$$f(t) = t^n + a_1 t^{n-1} + \cdots + a_n = (t + \lambda_1) \cdots (t + \lambda_n)$$

Then

$$P(a_1,\dots,a_n,\left|\lambda^{n-1},\dots,1\right|) = \theta(P)(\lambda_1,\dots,\lambda_n) = 0$$

since $P \in \ker(\theta)$. Exchanging λ_1 and λ_2 changes the sign of the Vandermonde determinant, so $P(a_1,\dots,a_n,\pm\delta) = 0$, where $\delta = \mathrm{disc}(t^n + a_1t^{n-1} + \dots + a_n)^{\frac{1}{2}}$. Using the given form for P,

$$A(a_1,\dots,a_n) \pm B(a_1,\dots,a_n)\delta = 0$$

and hence $A = B = 0$ on the Zariski-dense open subset

$$\{(a_1,\dots,a_n) \in \mathbb{C}^n \,:\, \mathrm{disc}(t^n + a_1t^{n-1} + \dots + a_n) \neq 0\}$$

Then $A = B = 0$ everywhere on $\mathbb{C}^n$, so $P = 0$. □

(11.5) Characteristic polynomial

Let V be an m-dimensional $\mathbb{C}$-space. $\mathrm{GL}(V)$ acts naturally on $U = \mathrm{End}_{\mathbb{C}}(V)$ by conjugation, i.e.

$$(g\cdot\theta)(v) = g\cdot\theta(g^{-1}\cdot v)$$

(or, in shorthand, $g\cdot\theta = g\theta g^{-1}$).

Suppose $\chi_\theta(t) = \det(t1_V + \theta)$ is the characteristic polynomial of θ and $c_n(\theta)$ is the coefficient of t^{m-n} in $\chi_\theta(t)$. Then

$$\chi_{g\cdot\theta}(t) = \det(t1_V + g\theta g^{-1}) = \det(g(t1_V+\theta)g^{-1}) = \chi_\theta(t)$$

so $c_n : U \to \mathbb{C}$ is an (n-homogeneous) invariant.

If θ has eigenvalues $lambda_1,\dots,\lambda_m$, then putting θ in Jordan normal form gives

$$\chi_\theta(t) = \prod_{i=1}^{m}(t+\lambda_i)$$

so that $c_n(\theta) = e_n(\lambda_1,\dots,\lambda_m)$; in particular

$$c_1(\theta) = \lambda_1 + \dots + \lambda_m = \mathrm{tr}(\theta) \quad\text{and}\quad c_m(\theta) = \lambda_1\dots\lambda_m = \det(\theta)$$

Claim The $\mathbb{C}$-algebra map $\begin{matrix}\mathbb{C}[Y_1,\dots,Y_m] & \to & \mathbb{C}[U]^{\mathrm{GL}(V)} \\ Y_i & \mapsto & c_i\end{matrix}$ is an isomorphism.

Sketch proof If f is a polynomial invariant for U then $f(\theta)$ is a symmetric polynomial function of the eigenvalues of θ. Mimic the proof that the characters of rational $\mathbb{C}\mathrm{GL}(V)$-modules are symmetric rational functions of the eigenvalues of $g \in \mathrm{GL}(V)$ as done in **(10.1)**. □

(11.6) Discriminant of quadratic forms

Let V be an m-dimensional $\mathbb{C}$-space and let

$$U = \mathrm{H}_{\mathbb{C},2}(V,\mathbb{C}) = \{\text{quadratic forms on } V\}$$

By polarisation (see end of §6) identify U with the symmetric $m\times m$ matrices via $f \mapsto A$, where

$$f(x) = x^tAx$$

for $x \in \mathbb{C}^m$ a column vector.

Define $\mathrm{disc}(f) = \det(A)$.

$\mathrm{GL}_m(\mathbb{C})$ acts on U by

$$(gf)(x) = f(g^{-1}x) = x^t({g^{-1}}^tAg^{-1})x, \qquad f\in U, g\in \mathrm{GL}_m(\mathbb{C}), x\in\mathbb{C}^m$$

so

$$\mathrm{disc}(gf) = \det({g^{-1}}^tAg^{-1}) = (\det g)^{-2}\,\mathrm{disc}(f)$$

and therefore $\mathrm{disc} : U \to \mathbb{C}$ is an $\mathrm{SL}_m(\mathbb{C})$-invariant (but *not* a $\mathrm{GL}_m(\mathbb{C})$-invariant).

Claim The $\mathbb{C}$-algebra map $\begin{array}{ccc} \mathbb{C}[X] & \to & \mathbb{C}[U]^{\mathrm{SL}_m(\mathbb{C})} \\ X & \mapsto & \mathrm{disc} \end{array}$ is an isomorphism.

Proof It is injective: if there is a polynomial P such that $P(\mathrm{disc}\, f) = 0$ for all $f \in U$ then $P(\lambda) = 0$ for all $\lambda \in \mathbb{C}$, since the quadratic form

$$f_\lambda(x_1, \dots, x_m) = x_1^2 + \cdots + x_{m-1}^2 + \lambda x_m^2$$

has discriminant λ, so $P = 0$.

Let $\theta : U \to \mathbb{C}$ be a polynomial $\mathrm{SL}_n(\mathbb{C})$-invariant and define

$$F : \begin{array}{ccc} \mathbb{C} & \to & \mathbb{C} \\ \lambda & \mapsto & \theta(f_\lambda) \end{array}$$

This is a polynomial function since θ is a polynomial map. We prove that $\theta(f) = F(\mathrm{disc}\, f)$. It is enough to show this for f with $\mathrm{disc}\, f \neq 0$. For, if this is case is known, then

$$U = \{f \,:\, \mathrm{disc}\, f = 0\} \cup \{f \,:\, \theta(f) = F(\mathrm{disc}\, f)\}$$

is a union of Zariski-closed subsets, but U is irreducible, so that $U = \{f \,:\, \theta(f) = F(\mathrm{disc}\, f)\}$.

Recall that any symmetric matrix is congruent (over $\mathbb{C}$) to a matrix of the form

$$\mathrm{diag}(1, \dots, 1, 0, \dots, 0) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

Thus if $f \in U$ corresponds to the matrix A and $\lambda = \mathrm{disc}\, f = \det A \neq 0$ then there exists $B \in \mathrm{GL}_m(\mathbb{C})$ such that $B^t A B = I$. Now if $C = \mathrm{diag}(1, \dots, 1, (\det B)^{-1})$, then $BC \in \mathrm{SL}_m(\mathbb{C})$, and

$$(BC)^t A (BC) = C^t B^t A B C = \mathrm{diag}(1, \dots, 1, (\det B)^{-2}) = \mathrm{diag}(1, \dots, 1, \lambda)$$

and so $\theta(f) = \theta(f_\lambda) = F(\lambda)$, as required. □

(11.7) Discriminant of binary forms

Define

$$U = \mathrm{H}_{\mathbb{C},n}(\mathbb{C}^2, \mathbb{C}) = \left\{ \begin{array}{c} \text{homogeneous polynomials of degree } n \\ \text{in two variables } X_1, X_2 \end{array} \right\}$$

Take $f \in U$. Then f has the form

$$\begin{aligned} f(X_1, X_2) &= a_0 X_1^n + a_1 X_1^{n-1} X_2 + \cdots + a_n X_2^n \\ &= b(\lambda_1 X_1 + \mu_1 X_2)(\lambda_2 X_1 + \mu_2 X_2) \cdots (\lambda_n X_1 + \mu_n X_2) \end{aligned}$$

and define

$$\mathrm{disc}(f) = b^{2n-2} \prod_{i<j} (\lambda_i \mu_j - \lambda_j \mu_i)^2$$

This is well-defined since it is unchanged if two terms are exchanged or if one term is enlarged and another is reduced by the same factor.

Example ($n = 3$) $\mathrm{disc}(f) = -27a_0^2 a_3^2 + 18 a_0 a_1 a_2 a_3 - 4 a_0 a_2^3 - 4 a_1^3 a_3 + a_1^2 a_2^2$

Claim $\mathrm{disc} : U \to \mathbb{C}$ is a $(2n-2)$-homogeneous $\mathrm{SL}_2(\mathbb{C})$-invariant.

Proof Exercise for the daring. □

(11.8) Definition) A **covariant** for a $\mathbb{C}\mathrm{SL}(V)$-module U is a polynomial invariant of the form $U \oplus V \to \mathbb{C}$.

Examples

- Every invariant θ for U gives rise to a covariant $\begin{array}{ccc} U \oplus V & \to & \mathbb{C} \\ (u,v) & \mapsto & \theta(u) \end{array}$

- If $U = \mathrm{H}_{\mathbb{C},n}(V,\mathbb{C})$ then there is an 'evaluation covariant' ev : $\begin{array}{ccc} U \oplus V & \to & \mathbb{C} \\ (f,v) & \mapsto & f(v) \end{array}$.

(11.9) The Hessian

If f is a function of $X_1, \ldots, X_m$ then the **Hessian** of f is

$$H(f) = \det \left(\frac{\partial^2 f}{\partial X_i \partial X_j} \right)$$

It is itself a function of $X_1, \ldots, X_m$.

Let $U = \mathrm{H}_{\mathbb{C},n}(\mathbb{C}^m, \mathbb{C})$, the n-homogeneous polynomials in $X_1, \ldots, X_m$. The Hessian defines a polynomial map

$$\begin{array}{ccc} U \oplus \mathbb{C}^m & \to & \mathbb{C} \\ (f,v) & \mapsto & H(f)(v) \end{array}$$

Now U is naturally a $\mathbb{C}\mathrm{GL}_m(\mathbb{C})$-module via the action

$$(gf)(v) = f(g^{-1}v), \qquad f \in U,\ g \in \mathrm{GL}_m(\mathbb{C}),\ v \in \mathbb{C}^m$$

By the chain rule,

$$H(gf)(gv) = (\det g)^{-2} H(f)(v)$$

and so H is an $\mathrm{SL}_2(\mathbb{C})$-covariant.

12 First fundamental theorem of invariant theory

The first fundamental theorem of invariant theory gives generators for the set of polynomial invariants when the module is the direct sum of copies of V and V^*. The hope was then *in principle* to compute invariants of an arbtirary rational module, but *in practice* this proved to be impossible. We will state and prove the first fundamental theorem for $\mathrm{GL}(V)$ and state it for $\mathrm{SL}(V)$.

As usual, V is an m-dimensional $\mathbb{C}$-space with basis $e_1, \ldots, e_m$.

(12.1) Theorem *(multilinear first fundamental theorem for the general linear group)*

For $n, r \in \mathbb{N}$,

(i) $\mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(T^n(V^*) \otimes T^r(V), \mathbb{C}) = 0$ if $n \neq r$;

(ii) $\mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(T^n(V^*) \otimes T^n(V), \mathbb{C}) = \mathrm{span}_{\mathbb{C}}\{\mu_\sigma : \sigma \in \mathrm{S}_n\}$, where

$$\mu_\sigma(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes v_1 \otimes \cdots \otimes v_n) = \varphi_{\sigma(1)}(v_1) \cdots \varphi_{\sigma(n)}(v_n)$$

Remark A proof in arbitrary characteristic can be found in de Cocini and Procesi (1976).

Proof Let X, Y be $\mathbb{C}G$-modules, where G is some group. We know that

$$\mathrm{Hom}_{\mathbb{C}}(X^* \otimes Y, \mathbb{C}) = (X^* \otimes Y)^* \cong X^{**} \otimes Y^* \cong X \otimes Y^* \cong \mathrm{Hom}_{\mathbb{C}}(Y, X)$$

Taking G-fixed points, we obtain

$$\mathrm{Hom}_{\mathbb{C}G}(X^* \otimes Y, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}G}(Y, X)$$

With this and the isomorphism $T^n(V^*) \cong (T^n V)^*$ we have an isomorphism

$$\pi \;:\; \begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(T^r V, T^n V) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(T^n(V^*) \otimes T^r(V), \mathbb{C}) \\ f & \longmapsto & \Big(\varphi_1 \otimes \cdots \otimes \varphi_n \otimes \vec{v} \mapsto (\varphi_1 \otimes \cdots \otimes \varphi_n)(f(v))\Big) \end{array}$$

Now

(i) If $n \neq r$ then $\mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(T^r V, T^n V) = 0$ because $\mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(S', S) = 0$ if S' (resp. S) is an irreducible r-homogeneous (resp. n-homogeneous) $\mathbb{C}\mathrm{GL}(V)$-module.

(ii) is just a restatement of Schur–Weyl duality. To see this, recall that $\mathrm{End}_{\mathbb{C}\mathrm{GL}(V)}(T^n V)$ is spanned by λ_σ for $\sigma \in \mathrm{S}_n$, where

$$\lambda_\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

and now $\pi(\lambda_\sigma) = \mu_\sigma$. $\square$

Notation For $\mathbb{C}G$-modules U and W, let $\mathrm{Hom}_{\mathbb{C}G,n}(U, W)$ denote the vector space of all n-homogeneous concomitants $U \to W$ (cf. **(11.1)**).

Note that $\mathrm{Hom}_{\mathbb{C}G,n}(U, W)$ is a $\mathbb{C}G$-module with conjugation action

$$(gf)(u) = gf(g^{-1}u), \qquad g \in G,\ u \in U,\ f \in \mathrm{Hom}_{\mathbb{C}G,n}(U, W)$$

Note also that $\mathrm{Hom}_{\mathbb{C}G,n}(U, W) = \mathrm{H}_{\mathbb{C},n}(U, W)^G$.

(12.2) Lemma If U is a $\mathbb{C}G$-module then

$$\mathbb{C}[U]^G \cong \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathbb{C}G}(U, \mathbb{C})$$

Proof We know that

$$\mathbb{C}[U] \cong \bigoplus_{n \geq 0} \mathrm{H}_{\mathbb{C},n}(U, \mathbb{C})$$

is an isomorphism of $\mathbb{C}G$-modules; now take G-fixed points. □

(12.3) Lemma If U and W are CCG-modules then there exist invrse isomorphisms of vector spaces

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}G}(\mathrm{S}^n U, W) & \to & \mathrm{Hom}_{\mathbb{C}G,n}(U, W) \\ \psi & \mapsto & \psi \circ \Delta \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}G,n}(U, W) & \to & \mathrm{Hom}_{\mathbb{C}G}(\mathrm{S}^n U, W) \\ f & \mapsto & \frac{1}{n!} Pf \end{array}$$

where $\Delta : \begin{array}{ccc} U & \mapsto & \mathrm{S}^n U \\ u & \mapsto & u \vee \cdots \vee u \end{array}$ and Pf is the total polarisation of f (see after **(6.13)**).

Proof Clearly the maps induce isomorphisms between $\mathrm{Hom}_{\mathbb{C}}(\mathrm{S}^n U, W)$ and $\mathrm{H}_{\mathbb{C},n}(U, W)$, so we'll prove:

(i) ψ concomitant $\Rightarrow \psi \circ \Delta$ concomitant;

(ii) f concomitant $\Rightarrow \frac{1}{n!} Pf$ concomitant.

So. . .

(i) is clear since Δ is a concomitant.

(ii) To prove this, there are (at least) three possible approaches.

Approach 1 Use the formula for Pf (not recommended).

Approach 2 If $f \in \mathrm{Hom}_{\mathbb{C}G,n}(U, W)$ then $f = \left(\frac{1}{n!} Pf\right) \circ \Delta$, so

$$\begin{aligned} \left(\frac{1}{n!} Pf\right)(g(u \vee \cdots \vee u)) &= \left(\frac{1}{n!} Pf\right)(g\Delta(u)) \\ &= \left(\frac{1}{n!} Pf\right)\Delta(g(u)) \\ &= f(g(u)) \\ &= gf(u) \\ &= g\left(\frac{1}{n!} Pf\right)(u \vee \cdots \vee u) \end{aligned}$$

Approach 3 $\mathrm{Hom}_{\mathbb{C}}(\mathrm{S}^n U, W)$ and $\mathrm{H}_{\mathbb{C},n}(U, W)$ are $\mathbb{C}G$-modules, with G acting by conjugation, and $\psi \mapsto \psi \circ \Delta$ is a $\mathbb{C}G$-map and an isomorphism. So its inverse $f \mapsto \frac{1}{n!} Pf$ is also a $\mathbb{C}G$-map. These maps restrict to isomorphisms between sets of G-fixed points. □

Remark Δ factors as $U \xrightarrow{\delta} T^n U \xrightarrow{\text{natural}} \mathrm{S}^n U$, where

$$\delta(u) = \underbrace{u \otimes \cdots \otimes u}_{n}$$

so composition with δ induces a surjection

$$\mathrm{Hom}_{\mathbb{C}G}(T^n U, W) \longrightarrow \mathrm{Hom}_{\mathbb{C}G}(U, W)$$

We'll use this reformulation of **(12.3)** to prove

(12.4) Theorem *(first fundamental theorem for the general linear group)*

Let $U = \underbrace{V \oplus \cdots \oplus V}_{p} \oplus \underbrace{V^* \oplus \cdots \oplus V^*}_{q}$. Then $\mathbb{C}[U]^{\mathrm{GL}(V)}$ is generated as a $\mathbb{C}$-algebra by elements ρ_{ij} for $1 \le i \le p$, $1 \le j \le q$, where

$$\rho_{ij}(v_1, \dots, v_p, \varphi_1, \dots, \varphi_q) = \varphi_j(v_i)$$

Proof Clearly the ρ_{ij} are polynomial invariants, since

$$g\varphi_j(gv_i) = \varphi_j(g^{-1}gv_i) = \varphi_j(v_i)$$

By **(12.2)** we must show that any n-homogeneous invariant f is a linear combination of products of the ρ_{ij}. By **(12.3)** there exists a surjection

$$\mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(T^n U, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V),n}(U, \mathbb{C})$$

Put $V_1 = \cdots = V_p = V$ and $V_{p+1} = \cdots = V_{p+q} = V^*$, so that $U = \bigoplus_{i=1}^{p+q} V_i$.

This induces a decomposition of $T^n U$ as

$$T^n U = \bigoplus_{i_1=1}^{p+q} \bigoplus_{i_2=1}^{p+q} \cdots \bigoplus_{i_n=1}^{p+q} V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}$$

Thus

$$\mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(T^n U, \mathbb{C}) = \bigoplus_{i_1,\ldots,i_n} \mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(V_{i_1} \otimes \cdots \otimes V_{i_n}, \mathbb{C})$$

Focus on one of the summands. By **(12.1)**, either $\mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(V_{i_1} \otimes \cdots \otimes V_{i_n}, \mathbb{C}) = 0$, or $n = 2k$, k of the i_j are $\leq p$ and k of the i_j are $> p$.

In the latter case, say $i_j \leq p$ for $j = \alpha_1, \ldots, \alpha_k$ and $i_j > p$ for $j = \beta_1, \ldots, \beta_k$. In this case, $\mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(V_{i_1} \otimes \cdots \otimes V_{i_n}, \mathbb{C})$ is spanned by the $k!$ maps τ_σ, for $\sigma \in \mathrm{S}_k$, given by

$$\tau_\sigma(v_1 \otimes \cdots \otimes v_n) = \varphi_{\beta_{\sigma(1)}}(v_{\alpha_1}) \cdots \varphi_{\beta_{\sigma(k)}}(v_{\alpha_k})$$

The n-homogeneous invariant of U corresponding to τ_σ is just

$$\rho_{i_{\alpha_1}, i_{\beta_{\sigma(1)}} - p} \cdots \rho_{i_{\alpha_k}, i_{\beta_{\sigma(k)}} - p}$$

and we're done. □

(12.5) Theorem If $r \in \mathbb{N}$ and $U = \underbrace{\mathrm{End}_{\mathbb{C}} V \oplus \cdots \oplus \mathrm{End}_{\mathbb{C}} V}_{r}$ then $\mathbb{C}[U]^{\mathrm{GL}(V)}$ is generated as a $\mathbb{C}$-algebra by invariants

$$t_{i_1,\ldots,i_k} : \begin{array}{ccc} U & \to & \mathbb{C} \\ (\theta_1, \ldots, \theta_r) & \mapsto & \mathrm{tr}(\theta_{i_1} \cdots \theta_{i_k} \end{array}$$

where $k \geq 1$ and $1 \leq i_1 < \cdots < i_k \leq r$.

Remarks

(1) [Procesi, 1976] In fact, $\mathbb{C}[U]^{\mathrm{GL}(V)}$ is generated by the $t_{i_1,\ldots,i_k}$ with $1 \leq k \leq 2^m - 1$.

(2) Procesi also shows that **(12.5)** is equivalent to the first fundamental theorem. This links invariant theory to the representation theory of *bouquets*, a certain type of *quiver*.

Proof By **(12.2)** and **(12.3)** we must compute $\mathrm{Hom}_{\mathbb{C}\mathrm{GL}(V)}(T^n(\mathrm{End}_{\mathbb{C}} V, \mathbb{C})$ which, since $\mathrm{End}_{\mathbb{C}} V \cong V^* \otimes V$, is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(T^n V^* \otimes T^n V, \mathbb{C})$. By the multilinear first fundamental theorem **(12.1)**, this is spanned by the μ_σ for $\sigma \in \mathrm{S}_n$, where

$$\mu_\sigma(\varphi_1 \otimes \varphi_n \otimes v_1 \otimes \cdots \otimes v_n) = \varphi_{\sigma(1)}(v_1) \cdots \varphi_{\sigma(n)}(v_n)$$

We now compute the corresponding map

$$\nu_\sigma : T^n(\mathrm{End}_{\mathbb{C}} V) \longrightarrow \mathbb{C}$$

Let V have basis $e_1, \ldots, e_m$ and V^* have dual basis $\eta_1, \ldots, \eta_m$. Let (A_{ij}) be the matrix of $\theta \in \mathrm{End}_{\mathbb{C}} V$ with respect to these bases, i.e.

$$\theta(v) = \sum_{i,j} A_{ij} \eta_j(v) e_i$$

so that the corresponding element of $V^* \otimes V$ is $\sum_{i,j} A_{ij} \eta_j \otimes e_i$.

If θ_k has matrix A^k_{ij} then $\theta_1 \otimes \cdots \otimes \theta_n$ corresponds with

$$\sum_{a_1,\ldots,a_n} \sum_{b_1,\ldots,b_n} A^1_{a_1 b_1} \cdots A^n_{a_n b_n} \eta_{b_1} \otimes \cdots \otimes \eta_{b_n} \otimes e_{a_1} \otimes \cdots \otimes e_{a_n} \in T^n V^* \otimes T^n V$$

so we have

$$\begin{aligned}\nu_\sigma(\theta_1 \otimes \cdots \otimes \theta_n) &= \sum_{\vec{a},\vec{b}} A^1_{a_1 b_1} \cdots A^n_{a_n b_n} \eta_{b_{\sigma(1)}}(e_{a_1}) \cdots \eta_{b_{\sigma(n)}}(e_{a_n}) \\ &= \sum_{\vec{b}} A^1_{b_{\sigma(1)} b_1} \cdots A^n_{b_{\sigma(n)} b_n}\end{aligned}$$

[The latter equality comes from the definition of dual basis.] Writing $\sigma = (i_1 \cdots i_k)(j_1 \cdots j_\ell) \cdots$ as a product of disjoint cycles, we reorder this to get

$$\begin{aligned}\nu_\sigma(\theta_1 \otimes \cdots \otimes \theta_n) &= \sum A^{i_1}_{b_{i_2} b_{i_1}} A^{i_2}_{b_{i_3} b_{i_2}} \cdots A^{i_k}_{b_{i_1} b_{i_k}} A^{j_1}_{b_{j_2} b_{j_1}} \cdots A^{j_\ell}_{b_{j_1} b_{j_\ell}} \cdots \\ &= \operatorname{tr}(\theta_{i_k} \cdots \theta_{i_1}) \operatorname{tr}(\theta_{j_\ell} \cdots \theta_{j_1}) \cdots\end{aligned}$$

which is what we required. □

Theorem *(first fundamental theorem for the special linear group)*

Let V be an m-dimensional $\mathbb{C}$-space with basis $e_1, \ldots, e_m$ and let V^* have dual basis $\eta_1, \ldots, \eta_m$. If

$$U = \underbrace{V \oplus \cdots \oplus V}_{p} \oplus \underbrace{V^* \oplus \cdots \oplus V^*}_{q}$$

then $\mathbb{C}[U]^{\mathrm{SL}(V)}$ is generated as a $\mathbb{C}$-algebra by the polynomial invariants which send

- $(v_1, \cdots, v_p, \varphi_1, \cdots, \varphi_q) \mapsto \varphi_j(v_i)$ for $1 \le i \le p$, $1 \le j \le q$;
- $(v_1, \cdots, v_p, \varphi_1, \cdots, \varphi_q) \mapsto |v_{i_1}, \ldots, v_{i_m}|$ for $1 \le i_1 < \cdots < i_m \le p$
- $(v_1, \cdots, v_p, \varphi_1, \cdots, \varphi_q) \mapsto |\varphi_{j_1}, \ldots, \varphi_{j_m}|$ for $1 \le j_1 < \cdots < j_m \le q$

where $|v|$ is the determinant of the matrix whose n^{th} column is the coordinate of the v_{i_n} with respect to $e_1, \ldots, e_m$ and $|\varphi|$ likewise with respect to $\eta_{j_1}, \ldots, \eta_{j_m}$.

End of course.

MATHEMATICAL TRIPOS PART III, 2013
REPRESENTATION THEORY

This sheet deals with the representation theory and combinatorics of S_n and of GL_n. Work over $\mathbb{C}$ unless told otherwise.

1 Let H be a subgroup of S_n; define the element $H^- = \sum_{h\in H} \varepsilon_h h$ which lies in the group algebra $\mathbb{C}S_n$. Prove that

$$\pi H^- = H^-\pi = \varepsilon_\pi H^- \quad \text{for any } \pi \in H,$$

and

$$\pi H^- \pi^{-1} = (\pi H \pi^{-1})^- \quad \text{for any } \pi \in S_n.$$

2 Let r be a natural number. Let $\lambda = (\lambda_1, \ldots \lambda_\ell)$ be a sequence of positive integers. We call λ a *composition* of r into ℓ parts if $\sum_{i=1}^{\ell} \lambda_i = r$. The elements λ_i are called the *parts* of λ.

(a) Find a formula for the number of compositions of r.

(b) Find a formula for the number of compositions of r with at most k parts.

3 Let r, s be non-negative integers. A partition of the form $\lambda = (r, 1^s)$ is called a *hook* partition; we will write $H(n)$ for the set of all hook partitions of n. Let f^λ be the number of standard tableaux of shape λ.

(a) Find a formula for the number of standard tableaux of shape $\lambda = (r, 1^s)$ where λ is a partition of n. Justify your answer.

(b) Prove that $\sum_{\lambda\in H(n)} f^\lambda = 2^{n-1}$.

(c) Prove that $\sum_{\lambda\in H(n)} (f^\lambda)^2 = \binom{2(n-1)}{n-1}$.

4 Let $n \in \mathbb{N}$, and let $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ and $\mu = (\mu_j)_{j\in\mathbb{N}}$ be partitions of n. The *conjugate partition* λ' is defined to be the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ where $\lambda'_i = |\{j|\lambda_j \geqslant i\}|$. The *dominance order* on the set of partitions of n is defined as follows:

$\lambda \trianglelefteq \mu$ if and only if for all $i \in \mathbb{N}$:

$$\sum_{j\leqslant i} \lambda_j \leqslant \sum_{j\leqslant i} \mu_j.$$

(a) Prove that $\lambda \trianglerighteq \mu$ if and only if $\mu' \trianglerighteq \lambda'$.

(b) We say that a partition λ is *self-conjugate* if $\lambda = \lambda'$. Show that there are as many self-conjugate partitions of n as there are partitions of n with distinct odd parts (i.e. all λ_i are odd and distinct).

5 (a) Assuming $k \geqslant 3$, prove that then $f^{(3^k)} \geqslant 3k$.

(b) Let $\lambda = (l, \ldots, l) = (l^k)$ be a partition of $n = kl$ and $\tilde{\lambda} = (l+1, \ldots, l+1)$ be a partition of $n+k$. Prove (by using the Young-Frobenius formula (5.1)) that

$$\frac{f^{\tilde{\lambda}}}{f^\lambda} = \frac{(n+k)!}{n!} \cdot \frac{l!}{(l+k)!}.$$

(c) Let $k, l \geqslant 3$ with $kl = n$. Prove that then $f^{(l^k)} \geqslant n$.

6 Let C_k be the set of all standard tableaux of shape (k,k) whose entry in box (2,1) is some odd number.

(a) Find $|C_k|$.

(b) For a partition $\lambda = (r,s)$ with two parts, write down $f^{(r,s)}$ (e.g. from the Young-Frobenius formula). Can you find $\lim_{k\to\infty} |C_k|/f^{(k,k)}$?

7 In this question $\lambda = (\lambda_1\lambda_2,\ldots,\lambda_k)$ is a partition of n, and M^λ is the permutation module with basis the λ-tabloids (a λ-tabloid is an equivalence class of numberings of a Young diagram (with distinct numbers 1,...,n), two being equivalent if the corresponding rows containing the same entries.)

(a) Show that M^λ is isomorphic to the permutation module $\mathbb{C}\Delta$ where $\Delta = \cos(S_n : S_\lambda)$, (the coset space with S_λ the Young subgroup).

(b) Show that

$$\dim M^\lambda = \frac{n!}{\prod_{i=1}^k \lambda_i!} .$$

(c) In the case $\lambda = (n-2,2)$ has two parts, show that $M^\lambda \cong \mathbb{C}\Omega^{(2)}$ where $\Omega^{(2)}$ is the set of 2-element subsets of $\Omega = \{1,\ldots,n\}$ (with the natural action).

8 For a λ-tableau T with n boxes, define the signed column sum $\kappa_T = \sum_{\sigma\in C_T} \varepsilon_\sigma \sigma$. Let $e_T = \kappa_T\{T\}$ be the λ-polytabloid. Let 1 denote the identity element of S_n.

(a) List the columns of T as $C_1, C_2, \ldots, C_s$. Prove that $\kappa_T = \kappa_{C_1}\cdots\kappa_{C_s}$.

(b) Let $H \leqslant S_n$ and let $(a\ b) \in H$. Define $H^- = \sum_{h\in H} \varepsilon_h h$, as in Q.1. Prove that $H^- = s\cdot(1-(a\ b))$ for some $s \in \mathbb{C}S_n$.

(c) Let $\pi \in S_n$ and $\sigma \in C_T$. Show that

$$\kappa_{\pi T} = \pi\kappa_T\pi^{-1}, \quad \kappa_T\sigma = \sigma\kappa_T = \varepsilon_\sigma\kappa_T$$

and

$$e_{\pi T} = \pi e_T; \ ;\sigma e_T = \varepsilon_\sigma e_T.$$

9 *[The Murnaghan-Nakayama Rule.]

Part (a) below is an optional task for enthusiasts only. You will need to read about James' method (lecture notes pp 79–83) using the Littlewood-Richardson Rule. A good account appears in Sagan's Springer GTM, p.182. This beautiful result has generalisations to the irreducible characters of exotic structures called Iwahori-Hecke algebras of type A_{n-1} due to Ram (who appealed to Schur-Weyl duality), and it appears to have been known to Young in his work on hyperoctahedral groups (the so-called Weyl groups of type B_n).

(a) Consider the character $\chi^\lambda(w) =: \chi^\lambda_\mu$ where $w \in S_n$ has cycle type $(\mu_1,\mu_2,\ldots,\mu_\ell)$. Suppose there are precisely t partitions $\lambda(1),\ldots,\lambda(t)$ which are obtained by removing a rim hook of length μ_1 from $[\lambda]$. Find an expression for χ^λ_μ in terms of $\chi^{\lambda(i)}_{(\mu_2,\mu_3\ldots,\mu_\ell)}$.

(b) Let $n \geqslant 6$. Prove that

$$\chi^{(n-3,3)}_{(2,1^{n-2})} = (n-3)(n-4)(n-5)/6.$$

10 Given some $\mathbb{C}S_n$-module M, with action written as $\pi \cdot m$ ($\pi \in S_n, m \in M$). Define a new $\mathbb{C}S_n$-module M^ε as the module whose underlying vector space is M and with new action $\pi * m := \varepsilon_\pi \pi \cdot m$ ($\pi \in S_n, m \in M$).

(a) Verify that M^ε is a $\mathbb{C}S_n$-module.

(b) Prove that M is irreducible if and only if M^ε is irreducible.

(c) Let χ be the character of M, and let $\chi^{(1^n)}$ be the character of the Specht module $S^{(1^n)}$. Prove that the character χ^ε of M^ε is given by $\chi^\varepsilon(\pi) = \chi^{(1^n)}(\pi)$ for $\pi \in S_n$.
[Why do you now know that the trivial module $S^{(n)}$ and the sign module $S^{(1^n)}$ are the only 1-dimensional $\mathbb{C}S_n$-modules.]

11 Let ρ :GL(V) $\to$GL$_n(\mathbb{C})$ be an irreducible polynomial representation. Show that the matrix entries $\rho_{ij} \in \mathbb{C}[\mathrm{GL}(V)]$ are homogeneous polynomials of the same degree. More generall, show that every polynomial representation ρ is a direct sum of representations $\rho^{(i)}$ whose matrix entries are homogeneous polynomials of degree i.

12 Give a proof of the result proved in lectures that affine n-space $\mathbb{A}^n$ is irreducible <u>without</u> appealing to the Nullstellensatz.

13 Prove that the 1-dimensional rational representations of $\mathbb{C}^* =$GL$_1(\mathbb{C})$ are power maps of the form $u \mapsto u^r$ where $r \in \mathbb{Z}$.

14 (a) Show that the set of diagonalisable matrices is Zariski dense in M$_n(\mathbb{C})$ (hence every invariant function on M$_n(\mathbb{C})$ is completely determined by its restriction to T_n, the diagonal matrices. [Actually this is nothing more than Jordan decomposition; if you work a bit harder you can show this is true over any field, not necessarily algebraically closed].

(b) Let ρ :GL$_n$ $\to$GL(W) be a rational representation. Show that ρ is polynomial if and only if $\rho|_{T_n}$ is polynomial. Show also that ρ is polynomial if and only if its character χ_ρ is a polynomial.

(c) Deduce that every 1-dimensional rational representation of GL(V) is of the form $\det^r$:GL(V) $\to$GL$_1$, $r \in \mathbb{Z}$.

15 Consider the linear action of GL(V) on End(V) by conjugation.

(a) What are the orbits of this action? Any element $\alpha \in$End(V) has a characteristic polynomial of the form $p_\alpha(t) = t^n + \sum_{i=1}^n (-1)^i e_i(\alpha) t^{n-i}$, where $n = \dim V$, showing that the e_i are invariant polynomial functions on End(V).

(b) [HARD] Find a proof of the fact that the ring of invariants for the conjugation action is generated by the s_i without using properties of symmetric functions.

(c) Show that the functions Tr$_1 \ldots$,Tr$_n$ generate the invariant ring $\mathbb{C}[\mathrm{End}(V)]^{GL(V)}$. Here Tr$_k$:End$(V) \to \mathbb{C}$ is the map $A \mapsto$TrA^k, $(k = 1, 2, \ldots)$.

SM, Lent Term 2013
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MATHEMATICAL TRIPOS PART III, 2013
REPRESENTATION THEORY

This sheet deals mostly with polynomial invariant theory. Work over $\mathbb{C}$.

1 Let W be finite-dimensional over $\mathbb{C}$. Check that the coordinate functions $x_1, \dots, x_m \in \mathbb{C}[W]$ are algebraically independent (equivalently, if $f(a_1, \dots, a_m) = 0$ for a polynomial f and all $a = (a_1, \dots, a_m) \in \mathbb{C}^m$ then f is the zero polynomial). Hint: use induction on the number of variables and the fact that a non-zero polynomial in one variable has only finitely many zeroes.

2 (a) Show that the natural representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ is self-dual (i.e. equivalent to its dual representation). Hint: find a non-singular matrix P such that $(A^{-1})^t = PAP^{-1}$.

(b) Show that this representation has two orbits.

3 Consider GL_2 acting on $\mathbb{C}[X, Y]$ (induced by the natural representation of GL_2 on $\mathbb{C}^2$).

(a) What is the image of X and Y under the action of any $g \in \mathrm{GL}_2$?

(b) Calculate $\mathbb{C}[X, Y]^{\mathrm{GL}_2}$ and $\mathbb{C}[X, Y]^{\mathrm{SL}_2}$.

(c) Calculate $\mathbb{C}[X, Y]^U$ where U is the subgroup of upper triangular unipotent matrices.

4 Consider the representation of $\mathrm{SL}_n(\mathbb{C})$ in its action on $\mathrm{M}_n(\mathbb{C})$, as discussed in lectures. Prove that the invariant ring is generated by the determinant. You may need the fact that GL_n is Zariski-dense in M_n.

5 Determine the invariant rings of $\mathbb{C}[\mathrm{M}_2(\mathbb{C})]^U$ and $\mathbb{C}[\mathrm{M}_2(\mathbb{C})]^T$ under left multiplication by the subgroup U of upper triangular unipotent matrices and the 2-torus $T = \mathrm{diag}(t, t^{-1})$ of diagonal matrices of SL_2.

6 Let T_n be the subgroup of GL_n of non-singular diagonal matrices. Choose the standard basis in $W = \mathbb{C}^n$ and the dual basis in W^* and thus identify the coordinate ring $\mathbb{C}[W \oplus W^*]$ with $\mathbb{C}[x_1, \dots x_n, \zeta_1, \dots, \zeta_n]$. Show that

$$\mathbb{C}[W \oplus W^*]^{T_n} = \mathbb{C}[x_1\zeta_1, \dots, x_n\zeta_n].$$

What happens if you replace T_n by the subgroup T'_n of diagonal matrices with determinant 1?

7 Recall that for each $j \geqslant 1$ the symmetric functions $n_j(\mathbf{x}) = x_1^j + x_2^j + \cdots x_n^j$ are known as *power sums* (or *Newton functions*). As before the e_j are the elementary symmetric functions.

(a) Show that

$$(-1)^{j+1} j e_j = n_j - e_1 n_{j-1} + e_2 n_{j-2} - \cdots + (-1)^{j-1} e_{j-1} n_1$$

for all $j = 1 \ldots, n$. This is known as the Newton-Girard identity. [Hint: let $j = n$, consider $f(t) = \prod_i (t - x_i)$ and calculate $\sum_i f(x_i)$. For $j < n$ what can you say about the right-hand side?]

(b) Here is Weyl's original proof[1] of the identity in (a). Define the polynomial

$$\psi(t) = \prod_{i=1}^{n} (1 - t x_i) = 1 - e_1 t + e_2 t^2 - \cdots + (-1)^n e_n t^n,$$

where the e_i are the elementary symmetric functions. Determine its logarithmic derivative $-\frac{\psi'(t)}{\psi(t)}$ as a formal power series. Deduce (a).

(c) Show that (in characteristic 0) the power sums generate the symmetric functions.

8 (a) Let A be a commutative algebra and let G be a group algebra of automorphisms of A. Assume the representation of G on A is completely reducible. Show that the subalgebra A^G of invariants has a canonical G-stable complement and the corresponding G-equivariant projection $\pi : A \to A^G$ satisfies $\pi(hf) = h\pi(f)$ for $h \in A^G, f \in A$. This projection is sometimes called the *Reynolds operator*.

(b) Let $A = \oplus_{j \geqslant 0} A_j$ be a graded K-algbbra (meaning $A_i A_j \subset A_{i+j}$). Assume the idea $A^+ = \oplus_{j>0} A_j$ is finitely-generated. Show that A is finitely-generated as an algebra over A_0. (In fact if the ideal A^+ is generated by the homogeneous elements $a_1, \ldots, a_n$, then $A = A_0[a_1, \ldots, a_n]$.)

(c) Deduce Hilbert's theorem: if W is a G-module and the representation of G on $K[W]$ is completely reducible then the invariant ring $K[W]^G$ is finitely-generated. [Hint: you will need to use Hilbert's Basis Theorem.] This shows also that $K[W]^G$ is noetherian.

(d) Hilbert's Theorem is applicable to finite groups provided Maschke's Theorem holds (i.e we require that $|G|$ is invertible in K). In the special case of characteristic 0, show that for any representation W of G, the ring of invariants $K[W]^G$ is generated by invariants of degree less than or equal to $|G|$ (Noether's Theorem).

Schmidt introduced a numerical invariant $\beta(G)$ for every finite group G. It is defined to be the minimal number m such that for every representation W of G, the invariant ring $K[W]^G$ is generated by the invariants of degree $\leqslant m$. With this definition, Noether's Theorem asserts that $\beta(G) \leqslant |G|$. In fact Schmidt showed that $\beta(G) = |G|$ if and only if G is cyclic[2]. It is also a fact that if G is actually abelain then $\beta(G)$ coincides with the so-called Davenport constant.

In the next few questions we'll write $V = \mathbb{C}^2$, $G = \mathrm{SL}(V) = \mathrm{SL}_2(\mathbb{C})$, and

$$C_n = H_{\mathbb{C},n}(V, \mathbb{C}) \cong (S^n V)^* \cong S^n(V^*),$$

which can be identified with the set of homogeneous polynomials of degree n in variables X_1, X_2. Note that $\{C_n : n \in \mathbb{N}\}$ are the non-isomorphic simple $\mathbb{C}G$-modules. For a fixed n, recall that a *covariant* for C_n is a polynomial $\mathbb{C}\mathrm{SL}(V)$-invariant $C_n \oplus V \to \mathbb{C}$. The following questions sketch a classification of the generators for $\mathbb{C}[C_n \otimes V)]^G$.

[1]Weyl, The Classical Groups, II.A.3

[2]see *Finite groups and invariant theory* in Séminaire d'Algèbre Paul Dubreil et M.-P. Malliavin (Springer Lecture Notes 1478, Springer 1991)

9 Suppose f, g are functions of X_1, X_2 and $r \in \mathbb{N}$. Define the rth *transvectant* of f, g as

$$\tau_r(f,g) = \sum_{j=0}^{r} \frac{(-1)^j}{j!(r-j)!} \frac{\partial^r f}{\partial X_1^{r-j} \partial X_2^j} \frac{\partial^r g}{\partial X_1^j \partial X_2^{r-j}}.$$

$$= \frac{1}{r!} \left(\frac{\partial}{\partial X_1} \frac{\partial}{\partial Y_2} - \frac{\partial}{\partial X_2} \frac{\partial}{\partial Y_1} \right)^r (f(X_1, X_2) g(Y_1, Y_2) |_{Y_1 = X_1, Y_2 = X_2}.$$

You should observe $\tau_0(f,g) = fg$, $\tau_1(f,g)$ is the Jacobian of f, g and $\tau_2(f,f)$ is the Hessian of f. Show that

(a) if f, g are homogeneous polynomials of degrees p, q then $\tau_r(f,g) = 0$ unless $r \leqslant \min\{p,q\}$, in which case it is a homogeneous polynomial of degree $p+q-2r$;

(b) if $r \leqslant \min\{p,q\}$ then the map $\tau_r : C_p \otimes C_q \to C_{p+q-2r}$ sending $f \otimes g \mapsto \tau_r(f,g)$ is a non-zero map of $\mathbb{C}G$-modules;

(c) any $\mathbb{C}G$-module map $C_p \otimes C_q \to C_{p+q-2r}$ is a multiple of τ_r. [Hint: use Clebsch-Gordan];

(d) the τ_r give an isomorphism of $\mathbb{C}G$-modules

$$C_p \otimes C_q \to \bigoplus_{r=0}^{\min\{p,q\}} C_{p+q-2r}.$$

10 Let $p, q, n, r \in \mathbb{N}$ and $r \leqslant \min\{q, n\}$. Set $N = \max\{0, r-p\}$. Show that there are scalars $\lambda_N, \ldots, \lambda_r \in \mathbb{C}$, with $\lambda_r \neq 0$, such that

$$f\tau_r(g,h) = \sum_{j=N}^{r} \lambda_j \tau_j(\tau_{r-j}(f,g), h),$$

for all $f \in C_p, g \in C_q, h \in C_n$. [HINT: Schur's Lemma.] Thus the τ_j are trying hard to be associative.

11 Fix $n \in \mathbb{N}$. Let $R = \mathbb{C}[C_n \oplus V]$, so that R^G is the set of covariants of C_n. If $\varphi \in R$ and $f \in C_n$, define $\varphi(f) \in \mathbb{C}[V]$ by $\varphi(f)(x) = \varphi(f,x)$. If $r \in \mathbb{N}$ and $\phi, \psi \in R$ define $\tau_r(\phi, \psi)$ as the map $C_n \oplus V \to \mathbb{C}$ by $(f,x) \mapsto \tau_r(\phi(f), \psi(f))(x)$. Finally if $d, i \in \mathbb{N}$, let R_{di} be the set of all $\phi \in R$ which are homogeneous, of degree d in C_n and degree i in V. Show that

(a) R_{di} are $\mathbb{C}G$-submodules of R, and R is graded, $R = \oplus_{d,i=0}^{\infty} R_{di}$, hence so too is R^G;

(b) $R_{di} R_{ej} \subseteq R_{d+e,i+j}$ and $\tau_r(R_{di}, R_{ej}) \subseteq R_{d+e,i+j-2r}$ for $r \leqslant \min\{i,j\}$;

(c) the assignment $\phi \mapsto (f \mapsto \phi(f))$ induces an isomorphism

$$R_{di}^G \cong \hom_{\mathbb{C}G,d}(C_n, C_i),$$

and deduce that

(d) any covariant $\theta \in R_{di}^G$ $(d \geqslant 1)$ can be expressed as a linear combination

$$\theta = \sum_{r=0}^{\min\{n,i\}} \tau_{n-r}(\phi_r, E)$$

with $\phi_r \in R_{d-1,n+i-2r}^G$ and E is the evaluation map $E(f,v) = f(v)$ [you should note that (c) says that $R_{1,i}^G = \mathbb{C}.E$ when $i = n$ and 0 if $i \neq n$.]

12 [Gordan's Theorem[3] - a weak form] If S is a $\mathbb{C}$-subalgebra of R^G with the property that $\tau_r(\phi, E) \in S$ whenever $r \in \mathbb{N}$ and $\phi \in S$, then $S = R^G$. [Remark: after the last question the proof should reduce to a two- line argument.]

13 [REALLY HARD] This uses Gordan's Theorem to find a set of generators of R^G in the case $n = 3$. Given $n = 3$, show that R^G is generated by the covariants

$E \in R^G_{13}$;
$H = \tau_2(E, E)$, the Hessian, $\in R^G_{22}$;
$t = \tau_1(H, E) \in R^G_{33}$;
$D = \tau_3(t, E)$, the discriminant ($\times 48$), $\in R^G_{40}$

14 [REALLY REALLY HARD[4]] Take $n = 4$ and repeat the last question to generate R^G using quartic forms.

Classically, the (still unsolved) problem of computing all covariants of binary forms of degree n was tackled by something called the symbolic method (using polarisation to reduce it to the FFT). Associated to the symbolic method is the symbolic notation, designed to make the calculations easier, but still non-trivial. You can consult any old text on Invariant Theory, such as Grace and Young if interested.

SM, Lent Term 2013
Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`

[3]Paul Gordan, dubbed the 'King of Invariant Theory' is perhaps better known for being Emmy Noether's thesis advisor. The result appeared in 1868 in Crelle's Journal as *Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Funktion mit numerischen Coeffizienten einer endlichen Anzahl solcher Formen ist.*

[4]J.J. Sylvester's collected works (four volumes, edited by H.F. Baker) include tables with details about higher degree forms. However his published tables for forms of degree larger than 6 appear to be totally wrong. Corrections appear in von Gall (1888) and Dixmier/Lazard (1986), Shioda (1967) and Brouwer/Popoviciu (2010).