Ramsey Theory

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Prerequisites. None (basic concepts of topology)

There are **three** examples sheets.

Books.

1. Bollobás, *Combinatorics*, CUP 1986. (For chapter 3.)

2. Graham, Rothschild, Spencer, *Ramsey Theory*, Wiley 1990. (For chapters 1 & 2.)

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Chapter 1 : Monochromatic Systems

Ramsey's Theorem

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $[n] = \{1, 2, \ldots, n\}$. For a set X, let $X^{(r)} = \{A \subset X : |A| = r\}$.

Suppose we have a 2-**colouring** of $\mathbb{N}^{(2)}$ – i.e. have $c : \mathbb{N}^{(2)} \to \{1, 2\}$. Can we always find an infinite **monochromatic set** – i.e., an infinite $M \subset \mathbb{N}$ such that c is constant on $M^{(2)}$?

Examples. 1. Colour ij (that is, $\{i, j\}$) RED if $i + j$ is even, and BLUE if odd.
Yes: can take $M = \{n : n \text{ even}\}$.

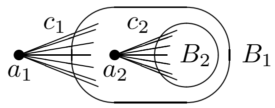
2. Colour ij RED if $\max\{n : 2^n \mid i + j\}$ is even, and BLUE otherwise.
Yes: can take $M = \{4^0, 4^1, 4^2, \ldots\}$.

3. Colour ij RED if $i + j$ has an even number of distinct prime factors, BLUE otherwise.
No explicit M is known! However, the answer is yes, by the following.

Theorem 1 (Ramsey's Theorem). Whenever $\mathbb{N}^{(2)}$ is 2-coloured, there exists an infinite monochromatic set.

Proof. Pick $a_1 \in \mathbb{N}$. There are infinitely many edges from a_1, so we can find an infinite set $B_1 \subset \mathbb{N} - \{a_1\}$ such that all edges from a_1 to B_1 are the same colour c_1.

Now choose $a_2 \in B_1$. There are infinitely many edges from a_2 to points in $B_1 - \{a_2\}$, so we can find an infinite set $B_2 \subset B_1 - \{a_2\}$ such that all edges from a_2 to B_2 are the same colour, c_2.



Continue inductively. We obtain a sequence $a_1, a_2, a_3, \ldots$ of distinct elements of $\mathbb{N}$, and a sequence c_1, c_2, c_3 of colours such that the edge $a_i a_j$ $(i < j)$ has colour c_i. Plainly we must have $c_{i_1} = c_{i_2} = c_{i_3} = \ldots$ for some infinite subsequence. Then $\{a_{i_1}, a_{i_2}, a_{i_3}, \ldots\}$ is an infinite monochromatic set. □

Remarks. 1. Called a '2-pass' proof.

2. The same proof shows that if $\mathbb{N}^{(2)}$ is k-coloured then we get an infinite monochromatic set. Alternatively, we could view '1' and '2 or 3 or … or k' as a 2-colouring of $\mathbb{N}^{(2)}$, and then apply Theorem 1 and use induction on k.

3. An infinite monochromatic set is much more than having arbitrarily large finite monochromatic sets. For example, consider the colouring in which all edges within each of the sets $\{1, 2\}$, $\{3, 4, 5\}$, $\{6, 7, 8, 9\}$, $\{10, 11, 12, 13, 14\}$, … are coloured blue and all other edges are coloured red. Here there is no infinite blue monochromatic set, but there are arbitrarily large finite monochromatic blue sets.

Example. Any sequence $(x_n)_{n \in \mathbb{N}}$ in a totally ordered set has a monotone subsequence: colour $\mathbb{N}^{(2)}$ by giving ij $(i < j)$ the colour UP if $x_i < x_j$ and the colour DOWN otherwise. The result follows by Theorem 1.

What if we colour $\mathbb{N}^{(r)}$ (say for $r = 3, 4, \ldots$)? Given a 2-colouring of $\mathbb{N}^{(r)}$, must there be an infinite monochromatic set?

Example. For $\mathbb{N}^{(3)}$, colour ijk $(i < j < k)$ RED if $i \mid j + k$, BLUE otherwise.
Yes: can take $M = \{2^0, 2^1, 2^2, \ldots\}$.

Theorem 2 (Ramsey for r-sets). Whenever $\mathbb{N}^{(r)}$ is 2-coloured, there exists an infinite monochromatic set.

Proof. The proof is by induction on r, the case $r = 1$ being trivial (i.e., the pigeonhole principle), and $r = 2$ being Theorem 1. So suppose the result holds for $r - 1$.

Given $c : \mathbb{N}^{(r)} \to \{1, 2\}$, pick $a_1 \in \mathbb{N}$. Define a 2-colouring c' of $(\mathbb{N} - \{a_1\})^{(r-1)}$ by $c'(F) = c(F \cup \{a_1\})$ for each $F \in (\mathbb{N} - \{a_1\})^{(r-1)}$. By induction, there exists an infinite monochromatic set $B_1 \subset \mathbb{N} - \{a_1\}$ for c' – that is, there exists a colour c_1 such that $c(F \cup \{a_1\}) = c_1$ for all $F \in B_1^{(r-1)}$.

Now choose $a_2 \in B_1$. In exactly the same way, we get an infinite set $B_2 \subset B_1 - \{a_2\}$ and a colour c_2 such that $c(F \cup \{a_2\}) = c_2$ for all $F \in B_2^{(r-1)}$. Continue inductively.

We get a sequence $a_1, a_2, \ldots$ of distinct elements of $\mathbb{N}$ and colours $c_1, c_2, \ldots$ such that for any $i_1 < \ldots < i_r$ we have $c(\{a_{i_1}, \ldots, a_{i_r}\}) = c_{i_1}$. But we must have $c_{i_1} = c_{i_2} = \ldots$ for some infinite subsequence. So $\{a_{i_1}, a_{i_2}, \ldots\}$ is an infinite monochromatic set. □

Example. We saw that given $(1, x_1)$, $(2, x_2)$, ... in $\mathbb{R}^2$ we could pick a subsequence inducing a monotone (piecewise-linear) function. We can insist that the function is convex or concave: 2-colour $\mathbb{N}^{(3)}$ by colouring ijk $(i < j < k)$ CONVEX or CONCAVE according as the points form a convex or concave triple. The result follows by Theorem 2.

Slightly unexpectedly, we can deduce the finite form of Ramsey's Theorem from Theorem 2.

Theorem 3 (Finite Ramsey). Let $m, r \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that whenever $[n]^{(r)}$ is 2-coloured there is a monochromatic m-set M.

Proof. Suppose not. We construct a 2-colouring of $\mathbb{N}^{(r)}$ without a monochromatic m-set, contradicting Theorem 2. For each $n \geqslant r$, we have a colouring $c_n : [n]^{(r)} \to \{1, 2\}$ with no monochromatic m-set. (We would like to take the 'limit' or union of these, but they may not be nested, i.e. agreeing where each is defined.)

There are only finitely many ways to colour $[r]^{(r)}$ (two in fact), so infinitely many of c_r, c_{r+1}, c_{r+2}, ... agree on $[r]^{(r)}$ – say $c_i|_{[r]^{(r)}} = d_r$ for all i lying in some infinite set A_1, where d_r is some 2-colouring of $[r]^{(r)}$.

Among the c_i for $i \in A_1$, infinitely many must agree on $[r + 1]^{(r)}$, as there are only finitely many ways to colour $[r+1]^{(r)}$ – say $c_i|_{[r+1]^{(r)}} = d_{r+1}$ for all $i \in A_2$, where d_{r+1} is a 2-colouring of $[r + 1]^{(r)}$ and $A_2 \subset A_1$ is infinite. Continue inductively.

We obtain colourings $d_n : [n]^{(r)} \to \{1, 2\}$ for $n = r$, $r + 1$, $r + 2$, ... such that

(i) no d_n has a monochromatic m-set (as there is some k such that $d_n = c_k|_{[n]^{(r)}}$)

(ii) the d_i are nested, i.e. $d_{n'}|_{[n]^{(r)}} = d_n$ for all $n' > n$, by construction.

Define a colouring $c : \mathbb{N}^{(r)} \to \{1, 2\}$ by $c(F) = d_n(F)$ for any $n \geqslant \max F$. This is well-defined by (ii), and has no monochromatic m-set by (i) – a contradiction. □

Remarks. 1. This proof gives no information about the minimal possible $n(m, r)$. There are direct proofs which give upper bounds.

2. This is called a **compactness proof** – what we did was (essentially) show that $\{0, 1\}^{\mathbb{N}}$ with the product topology (i.e. the topology derived from the metric $d(f, g) = 1/\min\{n : f(n) \neq g(n)\}$) is (sequentially) compact.

What if we used infinitely many colours? Suppose we have an arbitrary colouring of $\mathbb{N}^{(r)}$, i.e. we have $c : \mathbb{N}^{(r)} \to X$ for some possibly infinite X. Of course we cannot guarantee an infinite monochromatic set: just colour each edge a different colour. However, such a colouring is injective, so can we guarantee an infinite set on which c is either constant or injective? The answer is no – e.g. give the edge ij $(i < j)$ colour i.

Theorem 4 (The Canonical Ramsey Theorem). Whenever we have a colouring of $\mathbb{N}^{(2)}$ with an arbitrary set of colours, there exists an infinite set M such that

(i) c is constant on $M^{(2)}$, or

(ii) c is injective on $M^{(2)}$, or

(iii) $c(ij) = c(kl)$ iff $i = k$ (for all $i, j, k, l \in M$ with $i < j$ and $k < l$), or

(iv) $c(ij) = c(kl)$ iff $j = l$ (for all $i, j, k, l \in M$ with $i < j$ and $k < l$).

Note that this theorem implies Theorem 1: if we have only a finite set of colours then (ii), (iii) and (iv) are impossible.

Proof. First 2-colour $\mathbb{N}^{(4)}$ by giving $ijkl$ (by which we mean henceforth $i < j < k < l$) colour SAME if $c(ij) = c(kl)$ and colour DIFF if not. By Ramsey for 4-sets, we have an infinite monochromatic set M_1. If M_1 is coloured SAME then M_1 is monochromatic for c (for given any ij and kl in $M_1^{(2)}$, choose any $m < n$ in M_1 with $m > i, j, k, l$, then $c(ij) = c(mn) = c(kl)$). So in this case (i) holds.

Suppose then that M_1 is coloured DIFF. Now 2-colour $M_1^{(4)}$ by giving $ijkl$ colour SAME if $c(il) = c(jk)$ and colour DIFF if not. Again by Ramsey, there exists an infinite $M_2 \subset M_1$ monochromatic for this colouring. We cannot have colour SAME: choose $i < j < k < l < m < n$ in M_2, then $c(jk) = c(in) = c(lm)$, which is a contradiction as $M_2 \subset M_1$ and so $c(jk) \neq c(lm)$.

So M_2 is colour DIFF. Now 2-colour $M_2^{(4)}$ by giving $ijkl$ colour SAME if $c(ik) = c(jl)$ and colour DIFF if not. By Ramsey, we have an infinite monochromatic set $M_3 \subset M_2$. We cannot have colour SAME: choose $i < j < k < l < m < n$ in M_3, then $c(ik) = c(jm) = c(ln)$, which is a contradiction as $M_3 \subset M_1$ and so $c(ik) \neq c(ln)$.

So M_3 is colour DIFF. Now 2-colour $M_3^{(3)}$ by giving ijk colour SAME if $c(ij) = c(jk)$ and colour DIFF if not. We get an infinite monochromatic set $M_4 \subset M_3$. We cannot have colour SAME: choose $i < j < k < l$, then $c(ij) = c(jk) = c(kl)$, which is a contradiction as $M_4 \subset M_1$ and so $c(ij) \neq c(kl)$.

Now 2-colour $M_4^{(3)}$ by giving ijk colour LEFT-SAME if $c(ij) = c(ik)$ and colour LEFT-DIFF if not. We get an infinite monochromatic set $M_5 \subset M_4$. Finally, 2-colour $M_5^{(3)}$ by giving ijk colour RIGHT-SAME if $c(ik) = c(jk)$ and colour RIGHT-DIFF if not. We get an infinite monochromatic set $M_6 \subset M_5$.

If M_6 is LEFT-DIFF and RIGHT-DIFF then (ii) holds.

If M_6 is LEFT-SAME and RIGHT-DIFF then (iii) holds.

If M_6 is LEFT-DIFF and RIGHT-SAME then (iv) holds.

We cannot have M_6 being LEFT-SAME and RIGHT-SAME: choose $i < j < k$, then $c(ij) = c(ik) = c(jk)$, which is a contradiction as $M_6 \subset M_4$. □

Remarks. 1. We could do it all in *one* colouring of $\mathbb{N}^{(4)}$ by colouring $ijkl$ according to the partition of $[4]^{(2)}$ induced by c on $\{i, j, k, l\}$. The number of colours would be the number of partitions of a set of size $\binom{4}{2}$.

2. In the same way, we can show that if we arbitrarily colour $\mathbb{N}^{(r)}$ we get an infinite $M \subset \mathbb{N}$ and a set $I \subset [r]$ such that for any $x_1, x_2, \ldots, x_r$, and $y_1, y_2, \ldots, y_r$ in $M^{(r)}$ we have $c(x_1, x_2, \ldots, x_r) = c(y_1, y_2, \ldots, y_r) \iff x_i = y_i$ for all $i \in I$.

So in Theorem 4, $I = \emptyset$ is (i), $I = \{1, 2\}$ is (ii), $I = \{1\}$ is (iii) and $I = \{2\}$ is (iv). These 2^r colourings are called the **canonical colourings** of $\mathbb{N}^{(r)}$.

Van der Waerden's Theorem

Aim. Whenever $\mathbb{N}$ is 2-coloured, for all $m \in \mathbb{N}$ there exists a monochromatic arithmetic progression of length m (i.e. a, $a + d$, $a + 2d$, $\ldots$, $a + (m - 1)d$ all the same colour).

By the familiar compactness argument, this is the same as:

Aim′. For all $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that whenever $[n]$ is 2-coloured, there exists a monochromatic arithmetic progression of length m.

Indeed, if not, for each n we would have $c_n : [n] \to \{1, 2\}$ with no monochromatic arithmetic progression of length m. Then infinitely many agree on [1], and of those infinitely many agree on [2], etc. Keep going (as before), and hence obtain a 2-colouring of $\mathbb{N}$ with no monochromatic arithmetic progression of length m.

In our proof (of the second form above), we use the following key idea: we show more generally that for all k, $m \in \mathbb{N}$, there exists n such that whenever $[n]$ is k-coloured, there exists a monochromatic arithmetic progression of length m. Note that proving a more general result by induction can actually be *easier*, because the induction hypothesis is correspondingly stronger.

We write $W(m, k)$ for the smallest n if it exists – a 'Van der Waerden number'.

Another idea we use is the following: let $A_1 \ldots, A_r$ be arithmetic progressions of length m – say $A_i = \{a_i, a_i + d_i, \ldots, a_i + (m - 1)d_i\}$. We say that A_1, A_2, $\ldots$, A_r are **focused** at f if $a_i + md_i = f$ for all i. For example, $\{1, 4\}$ and $\{5, 6\}$ are focused at 7.

If in addition each A_i is monochromatic (for a given colouring) and no two are the same colour then we say that they are **colour-focused** at f (for the given colouring).

Thus if $A_1, \ldots, A_k$ are colour-focused arithmetic progressions of length $m-1$ (in a k-colouring) then we have a monochromatic arithmetic progression of length m – by asking 'what colour is the focus?'

Proposition 5. Let $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that whenever $[n]$ is k-coloured, there exists a monochromatic arithmetic progression of length 3.

Note. This result will be contained in Theorem 6. It is included here for motivation and understanding.

Proof. We make the following claim.

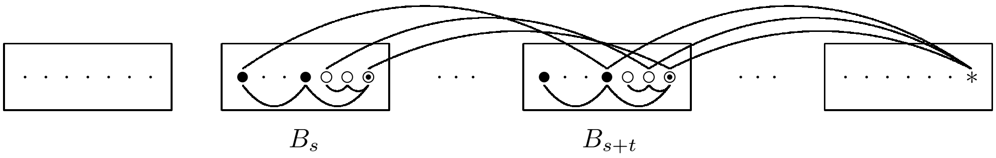
> **Claim.** For all $r \leqslant k$, there exists n such that whenever $[n]$ is k-coloured, we have either
>
> - a monochromatic arithmetic progression of length 3, or
> - r colour-focused arithmetic progressions of length 2.

We will then be done: take $r = k$, then whatever colour the focus is, we get a monochromatic arithmetic progression of length 3.

We prove the claim by induction on r. Note that the case $r = 1$ is trivial – we may simply take $n = k + 1$. So assume that we are given n suitable for $r - 1$. We will show that $(k^{2n} + 1)2n$ is suitable for r.

Given a k-colouring of $[(k^{2n} + 1)2n]$ which does not contain a monochromatic arithmetic progression of length 3, break up $[(k^{2n} + 1)2n]$ into blocks of length $2n$, namely $B_i = [2n(i - 1) + 1, 2ni]$ for $i = 1, 2, \ldots, k^{2n} + 1$, where $[a, b] = \{a, a + 1, \ldots, b\}$. Inside each block, there are $r - 1$ colour-focused arithmetic progressions of length 2 (by our choice of n), together with their focus (as the length of each block is $2n$).

Now there are k^{2n} possible ways to colour a block, so some two blocks, say B_s and B_{s+t}, are coloured identically. Say B_s contains $\{a_i, a_i + d_i\}$ for $i = 1, \ldots, r - 1$, all colour-focused at f. Then B_{s+t} contains $\{a_i + 2nt, a_i + d_i + 2nt\}$ for $i = 1, \ldots, r - 1$, all colour-focused at $f + 2nt$, with corresponding colours the same.



But now $\{a_i, a_i + d_i + 2nt\}$, $i = 1, \ldots, r - 1$, are $r - 1$ arithmetic progressions colour-focused at $f + 4nt$. Also, $\{f, f + 2nt\}$ is monochromatic of a different colour from the $r - 1$ used, so we have r arithmetic progressions of length 2 colour-focused at $f + 4nt$. This completes the induction, the claim, and hence the proof. $\square$

Remarks 1. The idea of looking at the number of ways to colour a block is called a **product argument.**

2. The above proof gives $W(3, k) \leqslant k^{k^{k^{\cdot^{\cdot^{\cdot^{k^{4k}}}}}}}\Big\}^{(k-1)}$, a 'tower-type' bound.

Theorem 6 (Van der Waerden's Theorem). Let m, $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that whenever $[n]$ is k-coloured, there exists a monochromatic arithmetic progression of length m.

Proof. The proof is by induction on m. For all k, the case $m = 1$ is trivial (and $m = 2$ is the pigeonhole principle, and $m = 3$ is Proposition 5).

Now given m, we can assume as our induction hypothesis that $W(m - 1, k)$ exists for all k. We make the following claim.

Claim. For all $r \leqslant k$, there exists n such that whenever $[n]$ is k-coloured, we have either

- a monochromatic arithmetic progression of length m, or
- r colour-focused arithmetic progressions of length $m-1$.

We will then be done: take $r = k$ and look at the focus.

The proof of the claim is by induction on r. For $r = 1$ we may take $n = W(m-1, k)$. So suppose $r > 1$. If n is suitable for $r-1$, we will show that $W(m-1, k^{2n})2n$ is suitable for r.

Given a k-colouring of $[W(m-1, k^{2n})2n]$ with no monochromatic arithmetic progression of length m, we can break up $[W(m-1, k^{2n})2n]$ into $W(m-1, k^{2n})$ blocks of length $2n$, namely B_1, B_2, ..., $B_{W(m-1,k^{2n})}$ where $B_i = [2n(i-1)+1, 2ni]$. Each block can be coloured in k^{2n} ways, so by definition of $W(m-1, k^{2n})$, we can find blocks B_s, B_{s+t}, ..., $B_{s+(m-2)t}$ identically coloured.

Now B_s contains $r-1$ colour-focused arithmetic progressions of length $m-1$ (by choice of n), together with their focus (as the length of each block is $2n$). Say we have A_1, A_2, ..., A_{r-1} colour-focused at f, where $A_i = \{a_i, a_i + d_i, \ldots, a_i + (m-2)d_i\}$. Now look at the arithmetic progression $A_i' = \{a_i, a_i + (d_i + 2nt), \ldots, a_i + (m-2)(d_i + 2nt)\}$ for $i = 1, 2, \ldots, r-1$. Then A_1', A_2', ..., A_{r-1}' are colour-focused at $f + (m-1)2nt$. But $\{f, f + 2nt, \ldots, f + (m-2)2nt\}$ is monochromatic and a different colour, and thus we have r colour-focused arithmetic progressions of length $m-1$.

This completes the induction, the claim, and hence the proof. □

What about bounds on $W(m, k)$? We define the **Ackermann** (or **Grzegorczyk**) **hierarchy** to be the sequence of functions $f_1, f_2, \ldots$, each $\mathbb{N} \to \mathbb{N}$ defined by

$$\begin{aligned} f_1(x) &= 2x \\ \text{and for } n \geqslant 1, \quad f_{n+1}(x) &= f_n^{(x)}(1) \\ &= \underbrace{f_n(f_n(\ldots f_n}_{x}(1)\ldots)) \end{aligned}$$

E.g.,

$$\begin{aligned} f_2(x) &= 2^x \\ f_3(x) &= \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}x \\ f_4(1) &= 2, \ f_4(2) = 2^2 = 4, \ f_4(3) = 2^{2^{2^2}} = 65536, \ f_4(4) = \left.2^{2^{\cdot^{\cdot^{2}}}}\right\}65536, \ \ldots. \end{aligned}$$

Say that a function $f : \mathbb{N} \to \mathbb{N}$ is **of type** n if there exist c and d with $f(cx) \leqslant f_n(x) \leqslant f(dx)$ for all x. Our bound on $W(3, k)$ was of type 3, and on $W(m, k)$ (as a function of k) is of type m, and so our bound on $W(m) = W(m, 2)$ grows faster than f_n for all n. This is often a feature of such double inductions, and it was believed for a long time that perhaps $W(m)$ really does grow this fast. Shelah (1987) found a proof using only induction on m, and gave a bound $W(m, k) \leqslant f_4(m + k)$. Graham offered \$1000 for a proof that $W(m) \leqslant f_3(m)$. Gowers (1998) showed that $W(m) \leqslant 2^{2^{2^{2^{m+9}}}}$, 'almost type 2'. The best lower bound known is $W(m) \geqslant 2^m/8m$.

Corollary 7. Whenever $\mathbb{N}$ is k-coloured, some colour class contains arbitrarily long arithmetic progressions. □

Remark. We cannot guarantee an infinitely long arithmetic progressions. For example:

- 2-colour $\mathbb{N}$ by colouring 1 red, then 2 and 3 blue, then 4, 5 and 6 red then 7, 8, 9 and 10 blue, and so on; or
- Enumerate the infinitely long arithmetic progressions as A_1, A_2, A_3, ... (there are only countably many). Choose $x_1, y_1 \in A_1$ with $x_1 < y_1$, then choose $x_2, y_2 \in A_2$ with $y_1 < x_2 < y_2$, etc. Now colour each x_i red and each y_i blue.

Theorem 8 (Strengthened Van der Waerden). Let $m \in \mathbb{N}$. Whenever $\mathbb{N}$ is finitely-coloured, there is an arithmetic progression of length m that, together with its common difference, is monochromatic (i.e., there exist a, $a+d$, $a+2d$, $\dots$, $a+(m-1)d$ and d all the same colour).

Proof. The proof is by induction on k, the number of colours. The case $k = 1$ is trivial.

We'll show that given n suitable for $k-1$ (i.e., such that whenever $[n]$ is $(k-1)$-coloured there exists a monochromatic arithmetic-progression-with-common-difference of length m), then $W\big(n(m-1)+1, k\big)$ is suitable for k. Indeed, given a k-colouring of $[W\big(n(m-1)+1, k\big)]$, we know there exists a monochromatic arithmetic progression of length $n(m-1)+1$ – say a, $a+d$, $a+2d$, $\dots$, $a+n(m-1)d$ is coloured red.

If d is red, done – look at $a, a+d, \dots, a+(m-1)d$. Indeed, if rd is red for any $1 \leqslant r \leqslant n$, done – look at $a, a+rd, \dots, a+(m-1)rd$. So we are done, unless $\{d, 2d, \dots, nd\}$ is $(k-1)$-coloured, but then we are done by induction (by definition of n). □

Remarks. 1. Henceforth we do not care about bounds.

2. The case $m = 2$ is known as **Schur's Theorem**: whenever $\mathbb{N}$ is k-coloured, we have monochromatic x, y, z such that $x + y = z$. We can also prove Schur's Theorem directly from Ramsey's Theorem. Given a k-colouring c of $\mathbb{N}$, induce a k-colouring c' of $[n]^{(2)}$ (n large) by $c'(ij) = c(j-i)$, for $i < j$. By Ramsey, there exists a monochromatic triangle; i.e. there exist $i < j < k$ with $c'(ij) = c'(ik) = c'(jk)$. So $c(j-i) = c(k-j) = c(k-i)$, and since $(j-i)+(k-j) = (k-i)$ we are done.

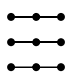
The Hales-Jewett Theorem

Let X be a finite set. A subset L of X^n ('the n-dimensional cube on alphabet X') is called a **line** (or **combinatorial line**) if there exists a non-empty set $I \subset [n]$ and $a_i \in X$ for each $i \notin I$ such that

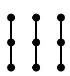
$$L = \{(x_1, \dots, x_n) \in X^n : x_i = a_i \text{ for } i \notin I \text{ and } x_i = x_j \ \forall\, i, j \in I\}.$$

We call I the set of **active coordinates** for L.

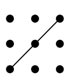
For example, in $[3]^2$ the lines are as follows.



$I = \{1\}$, with lines $\{(1,1),(2,1),(3,1)\}$, $\{(1,2),(2,2),(3,2)\}$, $\{(1,3),(2,3),(3,3)\}$;



$I = \{2\}$, with lines $\{(1,1),(1,2),(1,3)\}$, $\{(2,1),(2,2),(2,3)\}$, $\{(3,1),(3,2),(3,3)\}$;



$I = \{1,2\}$, with line $\{(1,1),(2,2),(3,3)\}$.

For example, in $[3]^3$, we could have lines such as $\{(2,1,1),\ (2,1,2),\ (2,1,3)\}$, for which $I = \{3\}$, or $\{(1,3,1),\ (2,3,2),\ (3,3,3)\}$, for which $I = \{1,3\}$.

Note that the definition of a 'line' does not depend on the ordering of the set X (which is why $\{(1,3),(2,2),(3,1)\}$ is not a 'line' in $[3]^2$).

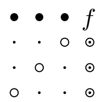
Theorem 9 (The Hales-Jewett Theorem). Let m, $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that whenever $[m]^n$ is k-coloured there exists a monochromatic line.

Remarks. 1. The smallest such n is denoted by $HJ(m,k)$.

2. Thus m-in-a-row Noughts & Crosses played in enough dimensions *cannot* be a draw. (**Exercise.** Show that it is a first-player win.)

3. The Hales-Jewett Theorem implies Van der Waerden's Theorem: we need only embed a cube of sufficiently high dimension linearly into $\mathbb{N}$. For example, given a k-colouring c of $\mathbb{N}$, induce a k-colouring c' of $[m]^n$ (n large) by $c'((x_1, x_2, \dots, x_n)) = c(x_1 + x_2 + \cdots + x_n)$. By Hales-Jewett, there is a monochromatic line, and this corresponds to a monochromatic arithmetic progression of length m in $\mathbb{N}$ (with common difference equal to the number of active coordinates). So we could regard the Hales-Jewett theorem as an 'abstract version' of Van der Waerden's Theorem.

If L is a line in $[m]^n$ write L^- and L^+ for its first and last points (in the ordering on $[m]^n$ given by $x \leqslant y$ if $x_i \leqslant y_i$ for all i). Lines $L_1, L_2, \dots, L_k$ are **focused** at f if $L_i^+ = f$ for all i. They are **colour-focused** (for a given colouring) if in addition each $L_i - \{L_i^+\}$ is monochromatic, no two the same colour.

Example. In $[4]^2$,  has three lines colour-focused at f.

Proof (of Theorem 9). By induction on m; the case $m = 1$ is trivial. Given $m > 1$, we may assume that $HJ(m-1,k)$ exists for all k. We make the following claim:

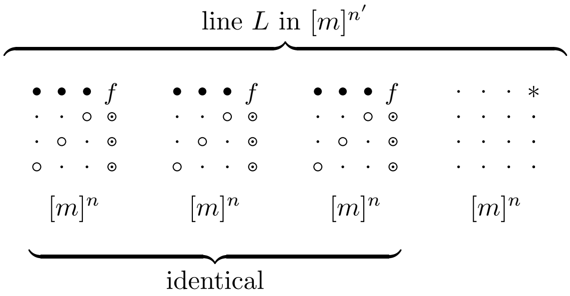
Claim. For all $r \leqslant k$, there exists n such that whenever $[m]^n$ is k-coloured, there exists either

- a monochromatic line, or
- r colour-focused lines.

The result will follow immediately from this claim: put $r = k$ and look at the focus.

The proof of the claim is by induction on r. For $r = 1$ we may take $n = HJ(m-1,k)$. So suppose $r > 1$. Given n suitable for $r-1$, we shall show that $n + HJ(m-1, k^{m^n})$ is suitable for r. Write $n' = HJ(m-1, k^{m^n})$.

Given a k-colouring of $[m]^{n+n'}$ containing no monochromatic line, view $[m]^{n+n'}$ as $[m]^n \times [m]^{n'}$. There are k^{m^n} ways to colour a copy of $[m]^n$. So by our choice of n', we have a line L in $[m]^{n'}$, say with active coordinate set I, such that for all $a \in [m]^n$ and all b, $b' \in L - \{L^+\}$, we have $c((a,b)) = c((a,b'))$ – call this $c'(a)$. By definition of n, there exist $r-1$ colour-focused lines for c' – say $L_1, L_2, \dots, L_{r-1}$, with active coordinate sets $I_1, I_2, \dots, I_{r-1}$ respectively, and focus f.



But now let L'_i be the line through the point (L_i^-, L^-) with active coordinate set $I_i \cup I$ $(i = 1, 2, \ldots, r-1)$. Then $L'_1, L'_2, \ldots, L'_{r-1}$ are colour-focused at (f, L^+). These, together with the line through (f, L^-) with active coordinate set I, give us r colour-focused lines. This completes the induction, the claim, and hence the proof. $\square$

A d**-dimensional subspace** or d**-parameter set** S in X^n is a set of the following form: there exist disjoint non-empty sets $I_1, I_2, \ldots, I_d \subset [n]$ and $a_i \in X$ for each $i \in [n]-(I_1 \cup I_2 \cup \cdots \cup I_d)$ such that

$$S = \left\{ x \in X^n : \begin{array}{l} x_i = a_i \text{ for all } i \notin I_1 \cup I_2 \cup \cdots \cup I_d, \\ x_i = x_j \text{ whenever } i, j \in I_k \text{ for some } k \end{array} \right\}.$$

For example in X^3, $\{(x, y, 2) : x, y \in X\}$ and $\{(x, x, y) : x, y \in X\}$ are 2-parameter sets.

Theorem 10 (The Extended Hales-Jewett Theorem). Let $m, k, d \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that whenever $[m]^n$ is k-coloured, there exists a monochromatic d-parameter set.

'This should be *much* harder than Hales-Jewett, but… '

Proof. Regard X^{dn} as $(X^d)^n$ – a cube on alphabet X^d. Clearly any line in this (on the alphabet X^d) corresponds to a d-parameter set in X^{dn} (on the alphabet X), so we can take $n = dHJ(n^d, k)$. $\square$

Let $S \subset \mathbb{N}^d$ be a finite set. A **homothetic copy** of S is any set of the form $a + \lambda S$ where $a \in \mathbb{N}^d$ and $\lambda \in \mathbb{N}$. For example, in $\mathbb{N}$, a homothetic copy of $\{1, 2, \ldots, m\}$ is precisely an arithmetic progression of length m.

Theorem 11 (Gallai's Theorem). For any finite $S \subset \mathbb{N}^d$ and any k-colouring of $\mathbb{N}^d$, there exists a monochromatic homothetic copy of S.

Proof. Let $S = \{S(1), S(2), \ldots, S(m)\}$. Given a k-colouring c of $\mathbb{N}^d$, define a k-colouring c' of $[m]^n$ (n large) by $c'\big((x_1, \ldots, x_n)\big) = c\big(S(x_1) + \ldots + S(x_n)\big)$. By Hales-Jewett, there is a monochromatic line, giving a monochromatic homothetic copy of S (with λ the number of active coordinates). $\square$

Remarks. 1. Or by a product argument and focusing.

2. For $S = \big\{(x, y) : x, y \in \{0, 1\}\big\}$, Gallai's Theorem tells us that when $\mathbb{N}^2$ is finitely-coloured there exists a monochromatic square. Could we have used Hales-Jewett for 2-parameter sets (on a 2-point alphabet) instead? No, this would only give us a rectangle.

Chapter 2 : Partition Regular Equations

Rado's Theorem

Schur says: whenever $\mathbb{N}$ is finitely-coloured, there are monochromatic x, y, z with $x + y = z$. Strengthened Van der Waerden says: whenever $\mathbb{N}$ is finitely-coloured, there are monochromatic $x_1, x_2, y_1, \ldots, y_m$ such that $x_1 + x_2 = y_1,\ x_1 + 2x_2 = y_2,\ \ldots,\ x_1 + mx_2 = y_m$.

Let A be an $m \times n$ matrix with rational entries. We say that A is **partition regular** (PR) (over $\mathbb{N}$) if whenever $\mathbb{N}$ is finitely-coloured, there is a monochromatic $\mathbf{x} \in \mathbb{N}^n$ with $A\mathbf{x} = \mathbf{0}$.

Examples. 1. Schur states that the matrix $(1\ \ 1\ \ {-1})$ is PR, i.e. that there exist x, y, z of the same colour such that $(1\ \ 1\ \ {-1}) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$.

2. Strengthened Van der Waerden states that $\begin{pmatrix} 1 & 1 & -1 & 0 & \ldots & 0 \\ 1 & 2 & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \ldots & -1 \end{pmatrix}$ is PR.

3. $(2\ \ 3\ \ {-5})$ is PR: take $x = y = z$. But what about $(2\ \ 3\ \ {-6})$ ..?

Notes. 1. Note that A is PR if and only if λA is PR for any $\lambda \in \mathbb{Q} - \{0\}$, so we could restrict our attention to integer matrices if we wished.

2. We also talk about the equation $A\mathbf{x} = \mathbf{0}$ being PR.

3. Not every matrix is PR: for example, $(2\ {-1})$ is not PR. If it were, it would say that we could always find n, $2n$ of the same colour. Clearly false, e.g. by 2-colouring $x \in \mathbb{N}$ with the parity of $\max\{i : 2^i \text{ divides } x\}$. Indeed, $(\lambda\ \ {-1})$ is PR if and only if $\lambda = 1$.

Let A be an $m \times n$ matrix with rational entries, say columns $\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(n)} \in \mathbb{Q}^m$, so

$$A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{c}^{(1)} & \mathbf{c}^{(2)} & \ldots & \mathbf{c}^{(n)} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}.$$

We say that A has the **columns property** (CP) if there is a partition $B_1 \cup \cdots \cup B_r$ of $\{1, 2, \ldots, n\}$ such that

- $\sum_{i \in B_1} \mathbf{c}^{(i)} = \mathbf{0}$,
- $\sum_{i \in B_s} \mathbf{c}^{(i)} \in \left\langle \mathbf{c}^{(j)} : j \in B_1 \cup \cdots \cup B_{s-1} \right\rangle$ for $s = 2, 3, \ldots, r$,

where $\langle\ \ \rangle$ denotes linear span over $\mathbb{R}$. (Or, 'over $\mathbb{Q}$' – if a rational vector is a **real** linear combination of some rational vectors then it is also a **rational** combination of them.)

Examples. 1. The matrix $(1\ \ 1\ \ {-1})$ has CP: take $B_1 = \{1, 3\}$ and $B_2 = \{2\}$.

2. $(2\ \ {-1})$ does not have CP. Indeed, $(\lambda\ \ {-1})$ has CP if and only if $\lambda = 1$.

3. $(2 \;\; 3 \;\; {-5})$ has CP.

4. $\begin{pmatrix} 1 & 1 & -1 & 0 & \dots & 0 \\ 1 & 2 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & m & 0 & 0 & \dots & -1 \end{pmatrix}$ has CP: take $B_1 = \{1, 3, 4, \dots, m+2\}$, $B_2 = \{2\}$.

Our aim is the following:

Rado's Theorem. A rational matrix A is PR if and only if A has CP.

Notes. 1. One strength of this result is that it shows that partition regularity, which does not at first appear to be checkable in finite time, in fact **is** checkable in finite time.

For example, $\begin{pmatrix} 1 & -1 & 3 \\ 2 & -2 & a \\ 4 & -4 & b \end{pmatrix}$ has CP if and only if $(a, b) = (6, 12)$.

2. Neither direction is obvious.

First (for clarity) we show that Rado's Theorem is true for a single equation. Note that $(a_1 \;\; a_2 \; \dots \; a_n)$ has CP if and only if some (non-empty) subset of the non-zero a_i sums to zero (or if $A = 0$). So we may assume that $a_1, a_2, \dots a_n \neq 0$, and must then show:

$$(a_1 \, a_2 \, \dots \, a_n) \text{ is PR} \iff \sum_{i \in I} a_i = 0 \text{ for some non-empty } I \subset [n].$$

Note. Even in this case, neither direction is obvious. However we would expect ($\Leftarrow$) to be harder than ($\Rightarrow$).

Let p be prime. For $x \in \mathbb{N}$, let $d(x)$ be the last non-zero digit in the base p expansion of x. I.e., if $x = d_r p^r + d_{r-1} p^{r-1} + \dots + d_1 p + d_0$, $0 \leqslant d_i < p$ for all i, then $d(x) = d_{L(x)}$ where $L(x) = \min\{i : d_i \neq 0\}$.

For example, if $x = 1002047000$ in base p then $L(x) = 3$ and $d(x) = 7$.

Proposition 1. Let $a_1, a_2, \dots, a_n \in \mathbb{Q} - \{0\}$ such that the matrix $(a_1 \;\; a_2 \;\; \dots \;\; a_n)$ is PR. Then $\sum_{i \in I} a_i = 0$ for some non-empty $I \subset [n]$.

Proof. We may assume, by multiplying up, that $a_1, a_2, \dots, a_n \in \mathbb{Z}$. Fix a large prime p, say with $p > \sum_{i=1}^n |a_i|$, and define a $(p-1)$-colouring of $\mathbb{N}$ by giving x the colour $d(x)$. We know that there are $x_1, x_2, \dots, x_n$ all of the same colour (say colour d) such that $a_1 x_1 + \dots a_n x_n = 0$.

Let $L = \min\{L(x_1), \dots, L(x_n)\}$ and let $I = \{i : L(x_i) = L\}$, so $I \neq \emptyset$. Considering $\sum_{i=1}^n a_i x_i = 0$ performed in base p, we have $\sum_{i \in I} a_i d \equiv 0 \pmod p$ and so, as p prime, $\sum_{i \in I} a_i \equiv 0 \pmod p$. But $p > \sum_{i=1}^n |a_i|$ and so in fact $\sum_{i \in I} a_i = 0$. □

Remarks. 1. Alternatively: for each prime p we get a set I_p with $\sum_{i \in I_p} a_i \equiv 0 \pmod p$, so some fixed set I has $\sum_{i \in I} a_i \equiv 0 \pmod p$ for infinitely many p, and so $\sum_{i \in I} a_i = 0$.

2. We could also colour not by 'end in base p' but by 'start in base p' – but harder.

3. No other way to prove Proposition 1 is known!

For the other direction, we begin with the first non-trivial case, namely $(1 \;\; \lambda \;\; -1)$.

Lemma 2. Let $\lambda \in \mathbb{Q}$. Then whenever $\mathbb{N}$ is finitely-coloured, there exist monochromatic x, y and z with $x + \lambda y = z$.

Proof. If $\lambda = 0$ it is trivial, and if $\lambda < 0$ we may rewrite our equation as $z - \lambda y = x$. So we may assume that $\lambda > 0$. Write $\lambda = r/s$ with $r,\, s \in \mathbb{N}$.

We need to prove that for all k, there exists an n such that, whenever $[n]$ is k-coloured, there exist monochromatic x, y and z with $x + (r/s)y = z$. We use induction on k. For $k = 1$, take $n = \max\{s, r+1\}$ and $(x, y, z) = (1, s, r+1)$.

Suppose $k > 1$. Given n suitable for $k - 1$, we shall show that $W(nr + 1, k)$ is suitable for k. Given a k-colouring of $[W(nr + 1, k)]$ we have a monochromatic arithmetic progression of length $nr + 1$, say a, $a + d$, $\ldots$, $a + nrd$, all of colour c.

Look at sd, $2sd$, $\ldots$, nsd. If any isd has colour c then we are done, as $(a, isd, a + (r/s)isd)$ is monochromatic. Otherwise, the set $\{sd, 2sd, \ldots, nsd\}$ is $(k-1)$-coloured, and we are done by induction. □

Remarks. 1. Very similar to the proof of Strengthened var der Waerden.

2. Lemma 2 seems not to have a proof 'just by Ramsey', unlike the case $\lambda = 1$ (Schur).

Theorem 3 (Rado's Theorem for single equations). Let a_1, a_2, $\ldots$, $a_n \in \mathbb{Q} - \{0\}$. Then $(a_1\, a_2\, \ldots\, a_n)$ is PR if and only if $\sum_{i \in I} a_i = 0$ for some non-empty $I \subset [n]$.

Proof. $(\Rightarrow)$ is Proposition 1.

$(\Leftarrow)$ Given a finite colouring of $\mathbb{N}$, fix $i_0 \in I$. For suitable monochromatic x, y and z, we shall set

$$x_i = \begin{cases} x & \text{if } i = i_0 \\ y & \text{if } i \notin I \\ z & \text{if } i \in I - \{i_0\} \end{cases} .$$

We require that $\sum a_i x_i = 0$, i.e. that

$$a_{i_0} x + \Bigg(\sum_{i \in I - \{i_0\}} a_i \Bigg) z + \Bigg(\sum_{i \notin I} a_i \Bigg) y = 0,$$

i.e.

$$a_{i_0} x - a_{i_0} z + \Bigg(\sum_{i \notin I} a_i \Bigg) y = 0, \quad \text{since } \sum_{i \in I} a_i = 0,$$

i.e.

$$x + \frac{1}{a_{i_0}} \Bigg(\sum_{i \notin I} a_i \Bigg) y - z = 0,$$

and such x, y and z do indeed exist by Lemma 2. □

Rado's Boundedness Conjecture. Let A be an $m \times n$ matrix that is *not* PR. Then there exists a bad colouring – i.e. a k-colouring with no monochromatic solution to $A\mathbf{x} = \mathbf{0}$. Is k bounded (for given m, n)? Equivalently, if there is $k = k(m, n)$ such that an $m \times n$ matrix 'PR for k colours' is PR?

This is known for 1×3 matrices (Fox & Kleitman, 2006) – 24 colours suffice.

Proposition 4. Let A be any matrix with entries in $\mathbb{Q}$. If A is PR then it has CP.

Proof. We may assume that all the entries of A are integers. Let the columns of A be $\mathbf{c}^{(1)}$, $\mathbf{c}^{(2)}$, …, $\mathbf{c}^{(n)}$. For any prime p, colour $\mathbb{N}$ by giving x the colour $d(x)$ (recall = the last non-zero digit in base p). By assumption, there exists a monochromatic $\mathbf{x} \in \mathbb{Z}^n$ with $A\mathbf{x} = \mathbf{0}$, i.e. $x_1\mathbf{c}^{(1)} + x_2\mathbf{c}^{(2)} + \cdots + x_n\mathbf{c}^{(n)} = \mathbf{0}$. Say all the x_i have colour d.

Partition $[n]$ as $B_1 \cup \cdots \cup B_r$ according to the value of $L(x_i)$, as follows:

- $i, j \in B_s$ for some $s \iff L(x_i) = L(x_j)$
- $i \in B_s,\ j \in B_t$ for some $s < t \iff L(x_i) < L(x_j)$

For infinitely many primes p, say all $p \in P$, we get the **same** B_1, …, B_r.

Considering $\sum x_i\mathbf{c}^{(i)} = \mathbf{0}$ performed in base p (for $p \in P$), we have

(i) $\displaystyle\sum_{i \in B_1} d\,\mathbf{c}^{(i)} \equiv \mathbf{0} \pmod{p}$, and

(ii) for all $s \geqslant 2$, $\displaystyle\sum_{i \in B_s} p^t d\,\mathbf{c}^{(i)} + \sum_{i \in B_1 \cup \cdots \cup B_{s-1}} x_i\,\mathbf{c}^{(i)} \equiv \mathbf{0} \pmod{p^{t+1}}$ for some t,

where we have written $\mathbf{u} \equiv \mathbf{v} \pmod{p}$ to mean $u_i \equiv v_i \pmod{p}$ for all i.

From (i), and as d is invertible, we have $\sum_{i \in B_1} \mathbf{c}^{(i)} \equiv \mathbf{0} \pmod{p}$ for infinitely many p, and so $\sum_{i \in B_1} \mathbf{c}^{(i)} = \mathbf{0}$.

From (ii), for all $2 \leqslant s \leqslant r$ we have

$$p^t \sum_{i \in B_s} \mathbf{c}^{(i)} + \sum_{i \in B_1 \cup \cdots \cup B_{s-1}} (d^{-1}x_i)\,\mathbf{c}^{(i)} \equiv \mathbf{0} \pmod{p^{t+1}}.$$

We claim: $\sum_{i \in B_s} \mathbf{c}^{(i)} \in \langle \mathbf{c}^{(i)} : i \in B_1 \cup \cdots \cup B_{s-1} \rangle$.

Suppose not. Then there exists $\mathbf{u} \in \mathbb{Z}^m$ with $\mathbf{u} \cdot \mathbf{c}^{(i)} = 0$ for all $i \in B_1 \cup \cdots \cup B_{s-1}$ but with $\mathbf{u} \cdot \big(\sum_{i \in B_s} \mathbf{c}^{(i)}\big) \neq 0$.

Dot with $\mathbf{u}$: $p^t\mathbf{u} \cdot \big(\sum_{i \in B_s} \mathbf{c}^{(i)}\big) \equiv 0 \pmod{p^{t+1}}$, i.e. $\mathbf{u} \cdot \big(\sum_{i \in B_s} \mathbf{c}^{(i)}\big) \equiv 0 \pmod{p}$. But this holds for infinitely many p, so $\mathbf{u} \cdot \big(\sum_{i \in B_s} \mathbf{c}^{(i)}\big) = 0$, contradiction. □

Let m, p, $c \in \mathbb{N}$. A set $S \subset \mathbb{N}$ is an **(m, p, c)-set** with generators x_1, x_2, …, $x_m \in \mathbb{N}$ if

$$S = \left\{ \sum_{i=1}^{m} \lambda_i x_i : \exists\, j \text{ with } \lambda_i = 0\ \forall\, i < j,\ \lambda_j = c,\ \lambda_i \in [-p, p]\ \forall\, i > j \right\},$$

where $[-p, p] = \{-p, -(p-1), \ldots, p\}$. So S consists of all numbers in the lists:

$$\left.\begin{array}{rl} cx_1 + \lambda_2 x_2 + \lambda_3 x_3 + \cdots + \lambda_m x_m & (\lambda_i \in [-p,p]\ \forall\, i) \\ cx_2 + \lambda_3 x_3 + \cdots + \lambda_m x_m & (\lambda_i \in [-p,p]\ \forall\, i) \\ \vdots & \\ cx_{m-1} + \lambda_m x_m & (\lambda_m \in [-p,p]) \\ cx_m & \end{array}\right\} \text{ the 'rows' of } S$$

'An iterated arithmetic-progression-with-common-difference, or progression of progressions.'

Examples. 1. A $(2,p,1)$-set is the set $\{x_1 - px_2,\, x_1 - (p-1)x_2,\, \ldots, x_1 + px_2,\, x_2\}$ – an arithmetic progression of length $2p+1$ together with its common difference.

2. A $(2,p,3)$-set is an arithmetic progression of length $2p+1$, with middle term divisible by 3, together with three times its common difference.

Theorem 5. Let m, p, $c \in \mathbb{N}$. Whenever $\mathbb{N}$ is finitely-coloured, there exists a monochromatic (m,p,c)-set.

Proof. Given a k-colouring of $\mathbb{N}$, let $M = k(m-1)+1$. 'Go for an (M,p,c)-set with each row monochromatic.'

Take n large (that is, large enough for whatever we need), and look at

$$A_1 = \left\{c,\, 2c,\, \ldots,\, \frac{n}{c}c\right\}.$$

(Officially, $n/c = \lfloor n/c \rfloor$, or: 'choose n large and a multiple of everything we need'.)

By Van der Waerden, A_1 contains a monochromatic arithmetic progression, say

$$R_1 = \big\{cx_1 - n_1 d_1,\, cx_1 - (n_1-1)d_1,\, \ldots,\, cx_1,\, \ldots,\, cx_1 + n_1 d_1\big\},$$

where n_1 is large and R_1 has colour k_1, say. Now we restrict attention to

$$B_1 = \left\{d_1,\, 2d_1,\, \ldots,\, \frac{n_1}{(M-1)p}d_1\right\}.$$

Note that for any integers $\lambda_2,\, \lambda_3,\, \ldots,\, \lambda_M \in [-p,p]$ and $b_2,\, b_3,\, \ldots,\, b_M \in B_1$, we have

$$cx_1 + \lambda_2 b_2 + \lambda_3 b_3 + \cdots + \lambda_M b_M \in R_1,$$

so in particular all sums of this form have colour k_1.

Inside B_1, look at

$$A_2 = \left\{cd_1,\, 2cd_1,\, \ldots,\, \frac{n_1}{(M-1)pc}cd_1\right\}.$$

By Van der Waerden, there is a monochromatic arithmetic progression inside A_2, say

$$R_2 = \{cx_2 - n_2 d_2,\, cx_2 - (n_2-1)d_2,\, \ldots,\, cx_2,\, \ldots,\, cx_2 + n_2 d_2\},$$

where n_2 is large and R_2 has colour k_2, say. Now we restrict attention to

$$B_2 = \left\{d_2,\, 2d_2,\, \ldots,\, \frac{n_2}{(M-2)p}d_2\right\}.$$

Note that for any integers $\lambda_3,\, \lambda_4,\, \ldots,\, \lambda_M \in [-p,p]$, and $b_3,\, b_4,\, \ldots,\, b_M \in B_2$, we have

$$cx_2 + \lambda_3 b_3 + \lambda_4 b_4 + \cdots + \lambda_M b_M \in R_2,$$

so in particular all sums of this form have colour k_2.

Now look at A_3, etc. Keep going M times: we obtain $x_1,\, x_2,\, \ldots,\, x_M$ such that each row of the (M,p,c)-set generated by $x_1,\, x_2,\, \ldots,\, x_M$ is monochromatic. But since $M = k(m-1)+1$, some m of these rows are the same colour, and we are done. □

The case $(m,1,1)$ of Theorem 5 immediately gives

Corollary 6 (Finite Sums/Folkman's/Sanders' Theorem). For $x_1, \ldots, x_m \in \mathbb{N}$, let

$$\mathrm{FS}(x_1, x_2, \ldots, x_m) = \left\{ \sum_{i \in I} x_i : I \subset [m],\, I \neq \emptyset \right\}.$$

Whenever $\mathbb{N}$ is finitely-coloured, there exist $x_1, x_2, \ldots, x_m$ with $\mathrm{FS}(x_1, x_2, \ldots, x_m)$ monochromatic. □

Remarks. 1. This extends Schur ($m = 2$).

2. Similarly, by looking at $\{2^n : n \in \mathbb{N}\}$, we can guarantee a monochromatic

$$\mathrm{FP}(x_1, x_2, \ldots, x_m) = \left\{ \prod_{i \in I} x_i : I \subset [m],\, I \neq \emptyset \right\}.$$

3. How about $\mathrm{FS}(x_1, \ldots, x_m) \cup \mathrm{FP}(x_1, \ldots, x_m)$? Unknown. Not even known for $m = 2$, where we want x, y, $x + y$, xy the same colour. In fact, not even known if there exist xy, $x + y$ the same colour (except for $x = y = 2$)!

Proposition 7. Let A have CP. Then there exist m, p, $c \in \mathbb{N}$ such that every (m, p, c)-set contains a solution to $A\mathbf{x} = \mathbf{0}$, i.e. we can solve $A\mathbf{x} = \mathbf{0}$ with all x_i in the (m, p, c)-set.

Proof. Let A have columns $\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(n)}$. As A has CP, we have a partition $B_1 \cup \cdots \cup B_r$ of $[n]$ such that

$$\sum_{i \in B_s} \mathbf{c}^{(i)} \in \left\langle \mathbf{c}^{(i)} : i \in B_1 \cup \cdots \cup B_{s-1} \right\rangle \text{ for all } s,$$

say

$$\sum_{i \in B_s} \mathbf{c}^{(i)} = \sum_{i \in B_1 \cup \cdots \cup B_{s-1}} q_{is} \mathbf{c}^{(i)}, \text{ some } q_{is} \in \mathbb{Q}.$$

Then for each s we have $\sum_{i \in [n]} d_{is} \mathbf{c}^{(i)} = \mathbf{0}$, where

$$d_{is} = \begin{cases} 0 & \text{if } i \notin B_1 \cup \cdots \cup B_s \\ 1 & \text{if } i \in B_s \\ -q_{is} & \text{if } i \in B_1 \cup \cdots \cup B_{s-1} \end{cases}.$$

(Note: ends with a 1.)

Given $x_1, x_2, \ldots, x_r \in \mathbb{N}$, put $y_i = \sum_{s=1}^{r} d_{is} x_s$ for $i = 1, 2, \ldots, n$. Then

$$\sum_{i=1}^{n} y_i \mathbf{c}^{(i)} = \sum_{i=1}^{n} \sum_{s=1}^{r} d_{is} x_s \mathbf{c}^{(i)} = \sum_{s=1}^{r} x_s \sum_{i=1}^{n} d_{is} \mathbf{c}^{(i)} = \mathbf{0}.$$

So $A\mathbf{y} = \mathbf{0}$. Now we are done: take $m = r$, take c to be the lowest common multiple of the denominators of the q_{is}, and take p to be c times the maximum of the numerators of the q_{is}.

Then $c\mathbf{y}$ is in the (m, p, c)-set generated by $x_1, x_2, \ldots, x_r$ and $A(c\mathbf{y}) = \mathbf{0}$. □

Theorem 8 (Rado's Theorem). Let A be a rational matrix. Then A is partition regular if and only if A has the columns property.

Proof. $(\Rightarrow)$ is Proposition 4.

$(\Leftarrow)$ follows from Theorem 5 and Proposition 7. □

Remarks. 1. Given Rado, things like Schur, Van der Waerden, Finite Sums are trivial CP checks.

2. From the proof, we see if a matrix is PR for all of the 'end in base p' colourings then it is PR for *all* colourings. But no direct proof (i.e. not via Rado) is known.

Theorem 9 (Consistency Theorem). If A, B are PR then the matrix $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$ is PR. In other words, if we can guarantee to solve $A\mathbf{x} = \mathbf{0}$ in some colour class and $B\mathbf{y} = \mathbf{0}$ in some colour class then we can guarantee to solve both in the *same* colour class.

Proof. Trivial by CP. □

Remarks. 1. This is not obvious by considerations of PR alone.

2. It can be proved directly – but harder.

Theorem 10. Whenever $\mathbb{N}$ is finitely coloured, some colour class contains solutions to *all* PR equations.

Proof. Suppose not. Then we have $\mathbb{N} = D_1 \cup D_2 \cup \cdots \cup D_k$, and, for each i, a PR matrix A_i such that D_i does not contain a solution of $A_i\mathbf{x} = \mathbf{0}$. Then the matrix

$$\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{pmatrix}$$

is PR by Theorem 9, but no D_i contains a solution to it, contradiction. □

Rado's Conjecture (1933). Say that $D \subset \mathbb{N}$ is **partition regular** if it contains a solution to every PR equation. So Theorem 10 says that if $\mathbb{N} = D_1 \cup D_2 \cup \cdots \cup D_k$ then some D_i is PR. Rado conjectured that if D is PR and $D = D_1 \cup D_2 \cup \cdots \cup D_k$ then some D_i is PR.

This was proved by Deuber (1973). He introduced (m, p, c)-sets and showed that D is PR if and only if D contains an (m, p, c)-set for all m, p, c (as we know). He showed that for all m, p, c, k there exist n, q, d such that whenever an (n, q, d)-set is k-coloured, there exists a monochromatic (m, p, c)-set (like our proof of Theorem 5 but replacing Van der Waerden with Extended Hales-Jewett) – thus proving Rado's Conjecture.

Ultrafilters

Our next aim is the following:

Hindman's Theorem. For $x_1, x_2, \ldots \in \mathbb{N}$, let $\mathrm{FS}(x_1, x_2, \ldots) = \left\{ \sum_{i \in I} x_i : I \text{ finite}, I \neq \emptyset \right\}$.

Then whenever $\mathbb{N}$ is finitely-coloured, there exists a monochromatic $\mathrm{FS}(x_1, x_2, \ldots)$.

This will be our first *infinite* PR structure of equations.

Idea. A filter is a notion of 'large' for subsets of $\mathbb{N}$; an ultrafilter is more refined such notion.

A **filter** on $\mathbb{N}$ is a non-empty collection $\mathcal{F} \subset \mathbb{P}(N)$ such that

- $\emptyset \notin \mathcal{F}$
- if $A \in \mathcal{F}$ and $B \supset A$ then $B \in \mathcal{F}$ ('$\mathcal{F}$ is an up-set')
- if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$ ('$\mathcal{F}$ is closed under finite intersections').

Examples. The following are (not all) filters:

(i) $\{A \subset \mathbb{N} : 1 \in A\}$ – 'all the weight is at 1: **1** $_2$ $_3$ ...'.
(ii) $\{A \subset \mathbb{N} : 1, 2 \in A\}$.
(iii) $\{A \subset \mathbb{N} : A \text{ infinite}\}$ – this fails, e.g. contains {evens} and {odds}.
(iv) $\{A \subset \mathbb{N} : A^c \text{ finite}\}$ – the **cofinite** filter.
(v) $\{A \subset \mathbb{N} : E - A \text{ finite}\}$, where E is the set of even numbers.

An **ultrafilter** is a maximal filter. For any $x \in \mathbb{N}$, the set $\{A \subset \mathbb{N} : x \in A\}$ is an ultrafilter, the **principal ultrafilter at** x.

Examples. Of the above filters, which are ultrafilters?

(i) is an ultrafilter.
(ii) is not – (i) extends it.
(iv) is not – (v) extends it.
(v) is not – extended by $\{A \subset \mathbb{N} : M - A \text{ finite}\}$ where $M = \{\text{multiples of } 4\}$.

Proposition 11. A filter $\mathcal{F}$ is an ultrafilter $\iff$ for all $A \subset \mathbb{N}$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Proof. ($\Leftarrow$) This is obvious since we cannot add A to $\mathcal{F}$ if we already have A^c.

($\Rightarrow$) Suppose $\mathcal{F}$ is an ultrafilter and $A, A^c \notin \mathcal{F}$ for some $A \subset \mathbb{N}$. By maximality, we must have $B \in \mathcal{F}$ with $B \cap A = \emptyset$ – or else $\mathcal{F}' = \{C \subset \mathbb{N} : C \supset A \cap B \text{ for some } B \in \mathcal{F}\}$ is a filter containing $\mathcal{F}$. Similarly, we must have $C \in \mathcal{F}$ with $C \cap A^c = \emptyset$. But then $B \cap C = \emptyset$, a contradiction. □

Remarks. 1 Or: in the proof, once we have $B \cap A = \emptyset$, then $B \subset A^c$, and so $A^c \in \mathcal{F}$.

2. If $\mathcal{U}$ is an ultrafilter, and $A = B \cup C$ for some $A \in \mathcal{U}$, then $B \in \mathcal{U}$ or $C \in \mathcal{U}$ – for otherwise $B^c, C^c \in \mathcal{U}$ by Proposition 11, whence $A^c = B^c \cap C^c \in \mathcal{U}$, a contradiction.

Theorem 12. Every filter is contained in an ultrafilter.

Proof. For a fixed filter $\mathcal{F}_0$, seek a maximal filter $\mathcal{F}$ with $\mathcal{F} \supset \mathcal{F}_0$. By Zorn's Lemma, it is sufficient to check that every non-empty chain $\{\mathcal{F}_i : i \in I\}$ has an upper bound. Indeed, put $\mathcal{F} = \cup_{i \in I} \mathcal{F}_i$. Then

(i) $\emptyset \notin \mathcal{F}$;
(ii) if $A \in \mathcal{F}$ and $B \supset A$ then $A \in \mathcal{F}_i$ for some i, so $B \in \mathcal{F}_i$ and so $B \in \mathcal{F}$;
(iii) if $A, B \in \mathcal{F}$ then $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$ for some i, j. Say $\mathcal{F}_i \subset \mathcal{F}_j$ (as the $\mathcal{F}_i$ form a chain), so $A \cap B \in \mathcal{F}_j$ and so $A \cap B \in \mathcal{F}$. □

Remarks. 1. Any ultrafilter extending the cofinite filter is non-principal. Also, if $\mathcal{U}$ is non-principal then $\mathcal{U}$ must extend the cofinite filter − else if $A \in \mathcal{U}$ for some finite A then $\{x\} \in \mathcal{U}$ for some $x \in A$ by our remark above about $B \cup C$

2. The Axiom of Choice is needed in some form to get non-principal ultrafilters.

The set of all ultrafilters on $\mathbb{N}$ is denoted $\beta\mathbb{N}$. We define a topology on $\beta\mathbb{N}$ by taking as a base all sets of the form

$$C_A = \{\mathcal{U} \in \beta\mathbb{N} : A \in \mathcal{U}\}, \text{ for each } A \subset \mathbb{N}.$$

This *is* a base: it is sufficient to check that $\bigcup C_A = \beta\mathbb{N}$ and that the intersection of any two of the C_A is another of the C_A. Plainly $\bigcup C_A = \beta\mathbb{N}$, and $C_A \cap C_B = C_{A\cap B}$ as $A,\ B \in \mathcal{U}$ if and only if $A \cap B \in \mathcal{U}$. Thus open sets are of the form

$$\bigcup_{i\in I} C_{A_i} = \{\mathcal{U} \in \beta\mathbb{N} : A_i \in \mathcal{U} \text{ for some } i \in I\}.$$

Note $\beta\mathbb{N} - C_A = C_{A^c}$, since $A \notin \mathcal{U}$ if and only if $A^c \in \mathcal{U}$. So closed sets are of the form

$$\bigcap_{i\in I} C_{A_i} = \{\mathcal{U} \in \beta\mathbb{N} : A_i \in \mathcal{U} \text{ for all } i \in I\}.$$

We can view $\mathbb{N}$ as a subset of $\beta\mathbb{N}$ by identifying $n \in \mathbb{N}$ with the principal ultrafilter $\widetilde{n}$ at n. Each point of $\mathbb{N}$ is isolated: for $\{\widetilde{n}\}$ is open as $\{\widetilde{n}\} = C_{\{n\}}$. Also, $\mathbb{N}$ is dense in $\beta\mathbb{N}$: for every non-empty open set in $\beta\mathbb{N}$ meets $\mathbb{N}$ as $\widetilde{n} \in C_A$ whenever $n \in A$.

Theorem 13. $\beta\mathbb{N}$ is a compact Hausdorff space.

Proof. *Hausdorff.* Given $\mathcal{U} \neq \mathcal{V}$, we have some $A \in \mathcal{U}$ with $A \notin \mathcal{V}$. But then $A^c \in \mathcal{V}$ and so $\mathcal{U} \in C_A$ and $\mathcal{V} \in C_{A^c}$, and $C_A \cap C_{A^c} = \emptyset$.

Compact. Given closed sets F_i $(i \in I)$ with the finite intersections property (i.e. all finite intersections are non-empty), we need to show that $\bigcap_{i\in I} F_i \neq \emptyset$. Assume without loss of generality that each F_i is basic, i.e. that $F_i = C_{A_i}$ for some $A_i \subset \mathbb{N}$.

The sets A_i $(i \in I)$, also have the finite intersections property: for $C_{A_{i_1}} \cap \cdots \cap C_{A_{i_n}} = C_{A_{i_1}\cap\cdots\cap A_{i_n}}$ and so $A_{i_1} \cap \cdots \cap A_{i_n} \neq \emptyset$.

So we can define a filter $\mathcal{F}$ generated by the A_i :

$$\mathcal{F} = \{A \subset \mathbb{N} : A \supset A_{i_1} \cap \cdots \cap A_{i_n} \text{ for some } i_1, \ldots, i_n \in I\}.$$

Let $\mathcal{U}$ be an ultrafilter extending $\mathcal{F}$. Then $A_i \in \mathcal{U}$ for all i, so $\mathcal{U} \in C_{A_i}$ for all i, and so $\cap_{i\in I} C_{A_i} \neq \emptyset$ as desired. □

Remarks. 1. If we view an ultrafilter as a function from $\mathbb{P}(\mathbb{N}) \to \{0,1\}$, i.e. as a point of $\{0,1\}^{\mathbb{P}(\mathbb{N})}$, then we have $\beta\mathbb{N} \subset \{0,1\}^{\mathbb{P}(\mathbb{N})}$. We can check that the topology on $\beta\mathbb{N}$ is the restriction of the product topology, and also that $\beta\mathbb{N}$ is a closed subset of $\{0,1\}^{\mathbb{P}(\mathbb{N})}$, so is compact by Tychonov.

2. Why is $\beta\mathbb{N}$ interesting? It is the largest compact Hausdorff space in which $\mathbb{N}$ is dense. More precisely, for any compact Hausdorff space and $f : \mathbb{N} \to X$, there exists a unique continuous $\widetilde{f} : \beta\mathbb{N} \to X$ extending f. (Not hard to check.)

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{f} & X \\ \downarrow & \nearrow_{\widetilde{f}} & \\ \beta\mathbb{N} & & \end{array}$$

$\beta\mathbb{N}$ is called the **Stone-Čech compactification** of $\mathbb{N}$.

Let $\mathcal{U}$ be an ultrafilter and p a statement. We write $\forall_{\mathcal{U}} x\ p(x)$ to mean $\{x : p(x)\} \in \mathcal{U}$, and say that $p(x)$ holds 'for most x' or 'for $\mathcal{U}$-most x'.

Examples. 1. For $\mathcal{U}$ non-principal, $\forall_{\mathcal{U}} x\ x > 4$.

2. For $\mathcal{U} = \widetilde{n}$ we have $\forall_{\mathcal{U}} x\ p(x) \iff p(n)$.

Ultrafilter quantifiers 'mesh perfectly' with logical connectives, as follows.

Proposition 14. Let $\mathcal{U}$ be an ultrafilter, and p and q statements. Then

(i) $\forall_{\mathcal{U}} x\ \big(p(x) \text{ AND } q(x)\big) \iff \big(\forall_{\mathcal{U}} x\ p(x)\big) \text{ AND } \big(\forall_{\mathcal{U}} x\ q(x)\big)$;

(ii) $\forall_{\mathcal{U}} x\ \big(p(X) \text{ OR } q(x)\big) \iff \big(\forall_{\mathcal{U}} x\ p(x)\big) \text{ OR } \big(\forall_{\mathcal{U}} x\ q(x)\big)$;

(iii) $\forall_{\mathcal{U}} x\ p(x)$ is false $\iff \forall_{\mathcal{U}} x\ \big(\text{NOT } p(x)\big)$.

Proof. Let $A = \{x : p(x)\}$ and let $B = \{x : q(x)\}$. Then

(i) $A \cap B \in \mathcal{U} \iff A \in \mathcal{U}$ and $B \in \mathcal{U}$;

(ii) $A \cup B \in \mathcal{U} \iff A \in \mathcal{U}$ or $B \in \mathcal{U}$;

(iii) $A \notin \mathcal{U} \iff A^c \in \mathcal{U}$. □

Note that $\forall_{\mathcal{U}} x\ \forall_{\mathcal{V}} y\ p(x, y)$ is not in general the same as $\forall_{\mathcal{V}} y\ \forall_{\mathcal{U}} x\ p(x, y)$, even if $\mathcal{U} = \mathcal{V}$. For example, if $\mathcal{U}$ is non-principal then $\forall_{\mathcal{U}} x\ \forall_{\mathcal{U}} y\ x < y$ is true (as *every* x has $\forall_{\mathcal{U}} y\ x < y$), but $\forall_{\mathcal{U}} y\ \forall_{\mathcal{U}} x\ x < y$ is false (as *no* y has $\forall_{\mathcal{U}} x\ x < y$).

For $\mathcal{U}$, $\mathcal{V} \in \beta\mathbb{N}$, define

$$\begin{aligned} \mathcal{U} + \mathcal{V} &= \{A \subset \mathbb{N} : \forall_{\mathcal{U}} x\ \forall_{\mathcal{V}} y\ x + y \in A\} \\ &= \{A \subset \mathbb{N} : \{x \in \mathbb{N} : \{y : x + y \in A\} \in \mathcal{V}\} \in \mathcal{U}\} \qquad \text{('not to be used')} \end{aligned}$$

Example. $\widetilde{m} + \widetilde{n} = \widetilde{m + n}$.

Note that $\mathcal{U} + \mathcal{V}$ *is* an ultrafilter:

- Clearly $\emptyset \notin \mathcal{U} + \mathcal{V}$.
- If $A \in \mathcal{U} + \mathcal{V}$ and $B \supset A$ then clearly $B \in \mathcal{U} + \mathcal{V}$.
- Suppose that A, $B \in \mathcal{U} + \mathcal{V}$, i.e. $(\forall_{\mathcal{U}} x\ \forall_{\mathcal{V}} y\ x + y \in A)$ AND $(\forall_{\mathcal{U}} x\ \forall_{\mathcal{V}} y\ x + y \in B)$.
  Then by Proposition 14(i) twice, we have: $\forall_{\mathcal{U}} x\ \forall_{\mathcal{V}} y\ (x + y \in A \text{ AND } x + y \in B)$.
  And thus: $\forall_{\mathcal{U}} x\ \forall_{\mathcal{V}} y\ (x + y \in A \cap B)$, i.e. $A \cap B \in \mathcal{U} + \mathcal{V}$, as required.
- Suppose that $A \notin \mathcal{U} + \mathcal{V}$, i.e. NOT $\big(\forall_{\mathcal{U}} x\ (\forall_{\mathcal{V}} y\ x + y \in A)\big)$.
  Then by Proposition 14(iii) twice, we have: $\forall_{\mathcal{U}} x\ \forall_{\mathcal{V}} y\ (\text{NOT } x + y \in A)$.
  And thus: $\forall_{\mathcal{U}} x\ \forall_{\mathcal{V}} y\ (x + y \in A^c)$, i.e. $A^c \in \mathcal{U} + \mathcal{V}$, as required.

Next, note that $+ : \beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$ is associative. Indeed, for any $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W} \in \beta\mathbb{N}$,

$$\mathcal{U} + (\mathcal{V} + \mathcal{W}) = \{A \subset \mathbb{N} : \forall_{\mathcal{U}} x\ \forall_{\mathcal{V}} y\ \forall_{\mathcal{W}} z\ \ x + y + z \in A\} = (\mathcal{U} + \mathcal{V}) + \mathcal{W}.$$

Also, + is left-continuous – i.e. for fixed $\mathcal{V}$, the map $\mathcal{U} \mapsto \mathcal{U} + \mathcal{V}$ is continuous. To see this, fix $\mathcal{V}$ and an open set C_A in our base for $\beta\mathbb{N}$. Then

$$\begin{aligned} \mathcal{U} + \mathcal{V} \in C_A &\iff A \in \mathcal{U} + \mathcal{V} \\ &\iff \forall_{\mathcal{U}} x \; \forall_{\mathcal{V}} y \; \{x + y \in A\} \\ &\iff \{x : \forall_{\mathcal{V}} y \; x + y \in A\} \in \mathcal{U} \\ &\iff \mathcal{U} \in C_{\{x : \forall_{\mathcal{V}} y \; x+y \in A\}}, \end{aligned}$$

as required.

Remark. In fact, the operation + is neither commutative nor right-continuous.

We seek an **idempotent** ultrafilter, i.e. some $\mathcal{U} \in \beta\mathbb{N}$ such that $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

(Note that any such $\mathcal{U}$ must be non-principal, as $\tilde{n} + \tilde{n} = \widetilde{2n} \neq \tilde{n}$.)

Lemma 15 (Idempotent Lemma). There exists $\mathcal{U} \in \beta\mathbb{N}$ such that $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

Remark. What we will use about $\beta\mathbb{N}$: it is compact, Hausdorff, non-empty, and + is associative and left-continuous.

Proof. Seek a minimal $M \subset \beta\mathbb{N}$ with $M + M \subset M$, where $M + M = \{x + y : x, y \in M\}$. ('We hope $M = \{x\}$, some x.')

Claim. There exists a minimal compact non-empty $M \subset \beta\mathbb{N}$ with $M + M \subset M$.

Proof of claim. There exist M with $M + M \subset M$, such as $\beta\mathbb{N}$ itself. So by Zorn, it's enough to check that if $\{M_i : i \in I\}$ is a chain of such sets, then so is $M = \bigcap_{i \in I} M_i$.

M is closed: it is an intersection of closed sets (in a compact Hausdorff space, closed $\Leftrightarrow$ compact).

$M \neq \emptyset$: the M_i are closed sets with the finite intersection property, so $\bigcap_{i \in I} M_i \neq \emptyset$ (as $\beta\mathbb{N}$ compact).

$M + M \subset M$: if $x, y \in M$, then $x, y \in M_i \; \forall i$, so $x + y \in M_i \; \forall i$, so $x + y \in M$.

So let M be a minimal such set. Choose $x \in M$. We'll show that $x + x = x$.

Claim. Write $M + x = \{y + x : y \in M\}$. Then $M + x = M$.

Proof of claim. Have $M + x \neq \emptyset$, and $M + x$ is compact (as it is the continuous image of a compact set), and $(M + x) + (M + x) = (M + x + M) + x \subset M + x$. But $M + x \subset M$, so by minimality of M, we have $M + x = M$.

In particular, there exists $y \in M$ with $y + x = x$.

Claim. Let $N = \{y \in M : y + x = x\}$. Then $N = M$.

Proof of claim. We have $N \neq \emptyset$, and N is compact (as it is the preimage of a single point, so is closed). And if $y, z \in N$, then $(y + z) + x = y + (z + x) = y + x = x$, thus $N + N \subset N$. But $N \subset M$, so by minimality of M, we have $N = M$.

In particular, $x + x = x$, as desired. □

Remarks. 1. Thus $M + x = \{x\}$ and so in fact $M = \{x\}$.

2. The **finite subgroup problem.** Does $\beta\mathbb{N}$ have any non-trivial finite subgroups? (E.g., $\mathcal{U}$ with $\mathcal{U}+\mathcal{U} \neq \mathcal{U}$, but $\mathcal{U}+\mathcal{U}+\mathcal{U} = \mathcal{U}$.) Answer: no, by Zelenyum (1996).

3. Can one ultrafilter 'absorb' another? In other words, are there $\mathcal{U} \neq \mathcal{V}$ with $\mathcal{U}+\mathcal{U}$, $\mathcal{U}+\mathcal{V}$, $\mathcal{V}+\mathcal{U}$, $\mathcal{V}+\mathcal{V} = \mathcal{V}$? This is called the **continuous homomorphism problem**. It is unknown.

Theorem 16 (Hindman's Theorem). Whenever $\mathbb{N}$ is finitely-coloured, there exist x_1, x_2, x_3, ... with $\mathrm{FS}(x_1, x_2, x_3, \dots)$ monochromatic.

Note. The ultrafilter is 'making lots of choices and passes for us'.

Proof. Let $\mathcal{U} \in \beta\mathbb{N}$ be an idempotent ultrafilter. Given a finite colouring of $\mathbb{N}$, we have some colour class $A \in \mathcal{U}$. (Think of A as 'the big colour class'.) So $\forall_{\mathcal{U}} y \ y \in A$. So, as $\mathcal{U}$ is idempotent, $\forall_{\mathcal{U}} x \ \forall_{\mathcal{U}} y \ x + y \in A$. So, by Proposition 14, $\forall_{\mathcal{U}} x \ \forall_{\mathcal{U}} y \ \mathrm{FS}(x, y) \subset A$. Pick x_1 with $\forall_{\mathcal{U}} y \ \mathrm{FS}(x_1, y) \subset A$.

Suppose inductively that we have found $x_1, \dots, x_n$ such that $\forall_{\mathcal{U}} y \ \mathrm{FS}(x_1, \dots, x_n, y) \subset A$. For each $z \in \mathrm{FS}(x_1, \dots, x_n, y)$ we have $\forall_{\mathcal{U}} y \ z + y \in A$ and so $\forall_{\mathcal{U}} x \ \forall_{\mathcal{U}} y \ z + x + y \in A$. Thus by Proposition 14, $\forall_{\mathcal{U}} x \ \forall_{\mathcal{U}} y \ \mathrm{FS}(x_1, \dots, x_n, x, y) \subset A$. Let x_{n+1} be such an x.

The result follows by induction. $\square$

Remarks. 1. Very few other infinite PR systems of equations are known. In particular, no Rado-type 'if and only if' theorem is known.

2. The Consistency Theorem *fails* for infinite PR systems. For example, let

$$\mathrm{FS}_{12}(x_1, x_2, \dots) = \left\{ \sum_{i \in I} x_i + 2 \sum_{j \in J} x_j : I, J \text{ finite, non-empty, and } \max I < \min J \right\}.$$

Then whenever $\mathbb{N}$ is finitely-coloured, there exists a monochromatic $\mathrm{FS}_{12}(x_1, x_2, \dots)$ – a special case of the 'Milliken-Taylor' theorem. But it was proved (1995) that this system is inconsistent with Hindman.

3. It follows trivially from Hindman that whenever $\mathbb{N}$ is finitely-coloured there exist $x_1, x_2, \dots$ with $\{x_i\} \cup \{x_i + x_j : i \neq j\}$ monochromatic. Is there a direct proof of this (i.e., not via Hindman)? Unknown!

Chapter 3 : Infinite Ramsey Theory

We know that whenever $\mathbb{N}^{(r)}$ is finitely-coloured there exists a monochromatic set (for any $r = 1, 2, 3, \ldots$). What if we colour the *infinite* subsets?

For infinite $M \subset \mathbb{N}$, write $M^{(\omega)} = \{L \subset M : L \text{ infinite}\}$, the collection of all infinite subsets of M. Given a finite colouring c of $\mathbb{N}^{(\omega)}$, is there an infinite monochromatic set (i.e. such that c is constant on $M^{(\omega)}$)?

Example. Colour $M = \{a_1, a_2, \ldots\}$ RED if $\sum 1/a_i$ converges, and BLUE if not.
Yes: could take $M = \{1, 2, 4, 8, 16, \ldots\}$.

Proposition 1. There is a 2-colouring of $\mathbb{N}^{(\omega)}$ without an infinite monochromatic set.

Proof. We construct a 2-colouring c such that for all $M \in \mathbb{N}^{(\omega)}$ and all $x \in M$ we have $c(M - \{x\}) \neq c(M)$. This is clearly sufficient to prove the proposition.

Define a relation $\sim$ on $\mathbb{N}^{(\omega)}$ by $M \sim N$ if $|M \triangle N|$ is finite (where $\triangle$ means symmetric difference). This is clearly an equivalence relation. Let the equivalence classes be $\{E_i : i \in I\}$, and for each i choose $M_i \in E_i$. Now, for $M \in E_i$, define $c(M)$ to be RED if $|M \triangle M_i|$ is even for some $i \in I$, and to be BLUE if $|M \triangle M_i|$ is odd for some $i \in I$.

It is easy to check that this colouring has the required property. □

Remark. In this proof, we used the Axiom of Choice.

A 2-colouring of $\mathbb{N}^{(\omega)}$ corresponds to a partition $\mathbb{N}^{(\omega)} = Y \cup Y^c$ for some $Y \subset \mathbb{N}^{(\omega)}$. A collection $Y \subset \mathbb{N}^{(\omega)}$ is called **Ramsey** if there exists $M \in \mathbb{N}^{(\omega)}$ with $M^{(\omega)} \subset Y$ or $M^{(\omega)} \subset Y^c$.

So Proposition 1 says that 'not all sets are Ramsey'. Which sets are Ramsey?

We have $\mathbb{N}^{(\omega)} \subset \mathbb{P}(\mathbb{N}) \leftrightarrow \{0,1\}^{\mathbb{N}}$, which has the product topology. So a basic open neighbourhood of $M \in \mathbb{N}^{(\omega)}$ is $\{L \in \mathbb{N}^{(\omega)} : L \cap [n] = M \cap [n]\}$ for some n. Equivalently, we have a metric

$$d(L, M) = \begin{cases} 0 & \text{if } L = M \\ 1/\min(L \triangle M) & \text{if } L \neq M \end{cases} .$$

So the basic open sets are

$$\{M \in \mathbb{N}^{(\omega)} : A \text{ an initial segment of } M\} \quad \text{for } A \text{ finite},$$

where 'initial segment' means 'the first k elements in order, some k'.

We call this the **τ-topology** or **usual topology** or **product topology** on $\mathbb{N}^{(\omega)}$.

First aim. Open sets are Ramsey.

For $M \subset \mathbb{N}$, write $M^{(<\omega)} = \{A \subset M : A \text{ finite}\}$, the collection of finite subsets of M.

For $A \in \mathbb{N}^{(<\omega)}$ and $M \in \mathbb{N}^{(\omega)}$, write

$$(A, M)^{(\omega)} = \{L \in \mathbb{N}^{(\omega)} : A \text{ is an initial segment of } L \text{ and } L - A \subset M\}.$$

(We think of this as the collection of sets which 'start as A and carry on inside M'.)

For fixed $Y \subset \mathbb{N}^{(\omega)}$, we say that M **accepts** A **(into** Y**)** if $(A, M)^{(\omega)} \subset Y$, and that M **rejects** A if no $L \in M^{(\omega)}$ accepts A.

Notes. 1. If M accepts A then every $L \in M^{(\omega)}$ accepts A as well.

2. If M rejects A then every $L \in M^{(\omega)}$ rejects A as well.

3. If M accepts A then M accepts $A \cup B$ for any $B \in M^{(<\omega)}$, providing $\max A < \min B$.

4. M need not accept or reject A.

Lemma 2 (Galvin-Prikry Lemma). Given $Y \subset \mathbb{N}^{(\omega)}$, there exists a set $M \in \mathbb{N}^{(\omega)}$ such that either

(i) M accepts $\emptyset$, or

(ii) M rejects all of its finite subsets.

Proof. Suppose no $M \in \mathbb{N}^{(\omega)}$ accepts $\emptyset$, i.e. that $\mathbb{N}$ rejects $\emptyset$. We shall inductively construct infinite subsets $M_1 \supset M_2 \supset M_3 \supset \cdots$ of $\mathbb{N}$ and a sequence $a_1 < a_2 < a_3 < \cdots \in \mathbb{N}$ with $a_i \in M_i$ for all i, such that M_i rejects all subsets of $\{a_1, \ldots, a_{i-1}\}$. Then we shall be done, for $\{a_1, a_2, a_3, \ldots\}$ rejects all its finite subsets.

Take $M_1 = \mathbb{N}$. Having chosen $M_1 \supset \cdots \supset M_k$ and $a_1, \ldots, a_{k-1}$ as above, we seek $a_k \in M_k$ with $a_k > a_{k-1}$ and $M_{k+1} \subset M_k$ such that M_{k+1} rejects all finite subsets of $\{a_1, \ldots, a_k\}$.

Suppose this is impossible. Pick any $b_1 \in M_k$ with $b_1 > a_{k-1}$. We cannot take $a_k = b_1$ and $M_{k+1} = M_k$, so some $N_1 \subset M_k$ accepts some subset of $\{a_1, \ldots, a_{k-1}, b_1\}$, and this subset must be of the form $E_1 \cup \{b_1\}$, as M_k rejects all subsets of $\{a_1, \ldots, a_{k-1}\}$.

Now choose $b_2 \in N_1$ with $b_2 > b_1$ and try $a_k = b_2$ and $M_{k+1} = N_1$. Some $N_2 \subset N_1$ accepts a subset of $\{a_1, \ldots, a_{k-1}, b_2\}$, say $E_2 \cup \{b_2\}$. Keep going.

We get $M_k \supset N_1 \supset N_2 \supset \cdots$ and $b_1 < b_2 < \cdots$ with $b_i \in N_{i-1}$, together with subsets E_1, E_2,... $\subset \{a_1, \ldots, a_{k-1}\}$, such that $E_i \cup \{b_i\}$ is accepted by N_i for all i. Passing to a subsequence if necessary, we may assume without loss of generality that $E_i = E$ for all i. Then E is accepted by $\{b_1, b_2, \ldots\}$, contradicting the definition of M_k. □

Theorem 3. Let $Y \subset \mathbb{N}^{(\omega)}$. If Y is open then Y is Ramsey.

Proof. Choose $M \in \mathbb{N}$ as given by Galvin-Prikry. If M accepts $\emptyset$ then $M^{(\omega)} \subset Y$, so done.

If M rejects all of its finite subsets, then we'll show that $M^{(\omega)} \subset Y^c$. Indeed, suppose some $L \in M^{(\omega)}$ has $L \in Y$. Since Y is open, we must have $(A, \mathbb{N})^{(\omega)} \subset Y$ for some initial segment A of L. So in particular, we have $(A, M)^{(\omega)} \subset Y$, i.e. M accepts A.

But M rejects A, so we have a contradiction. □

('The 'in particular' is overkill, as is having M reject A when any infinite subset would do.')

Remark. A collection Y is Ramsey if and only if Y^c is Ramsey, so Theorem 3 also says that 'closed sets are Ramsey'.

The **$\star$-topology** or **Ellentuck topology** or **Mathias topology** on $\mathbb{N}^{(\omega)}$ has basic open sets $(A, M)^{(\omega)}$ for $A \in \mathbb{N}^{(<\omega)}$ and $M \in \mathbb{N}^{(\omega)}$. This *is* a base for a topology on $\mathbb{N}^{(\omega)}$:

- $\mathbb{N}^{(\omega)} = (\emptyset, \mathbb{N})^{(\omega)}$ so the union of these is indeed $\mathbb{N}^{(\omega)}$;
- if $(A, M)^{(\omega)}$ and $(A', M')^{(\omega)}$ are basic sets then $(A, M)^{(\omega)} \cap (A', M')^{(\omega)}$ is either $\emptyset$ or $(A \cup A', M \cap M')^{(\omega)}$.

Note. The $\star$-topology is stronger (i.e. has more open sets) than the usual topology.

Theorem 3′. Let $Y \subset \mathbb{N}^{(\omega)}$. If Y is $\star$-open then Y is Ramsey.

Proof. Choose $M \in \mathbb{N}$ as given by Galvin-Prikry. If M accepts $\emptyset$ then $M^{(\omega)} \subset Y$, so done.

If M rejects all of its finite subsets, then we'll show that $M^{(\omega)} \subset Y^c$. Indeed, suppose some $L \in M^{(\omega)}$ has $L \in Y$. Since Y is $\star$-open, we must have $(A, L)^{(\omega)} \subset Y$ for some initial segment A of L. So L accepts A, contradicting 'M rejects A'. □

('No overkill this time.')

We say $Y \subset \mathbb{N}^{(\omega)}$ is **completely Ramsey** if for all $A \in \mathbb{N}^{(<\omega)}$ and all $M \in \mathbb{N}^{(\omega)}$ there is some $L \in M^{(\omega)}$ such that $(A, L)^{(\omega)}$ is contained in either Y or Y^c.

Remark. This is a stronger property than being just Ramsey. For example, let Y be the non-Ramsey set from Proposition 1 and set

$$Z = Y \cup \{M \in \mathbb{N}^{(\omega)} : 1 \notin M\}.$$

Then certainly Z is Ramsey, as $\{2, 3, 4, \dots\}^{(\omega)} \subset Z$. But Z is not completely Ramsey: take $A = \{1\}$ and $M = \mathbb{N}$, then there is no L with $(\{1\}, L)^{(\omega)}$ contained in Z or Z^c.

Theorem 4. Let $Y \subset \mathbb{N}^{(\omega)}$. If Y is $\star$-open then Y is completely Ramsey.

Proof. Given $A \in \mathbb{N}^{(<\omega)}$ and $M \in \mathbb{N}^{(\omega)}$, we seek $L \in M^{(\omega)}$ with $(A, L)^{(\omega)}$ contained in either Y or Y^c. Now view $(A, M)^{(\omega)}$ as a copy of $\mathbb{N}^{(\omega)}$ as follows. We may assume that $\max A < \min M$. Write $M = \{m_1, m_2, m_3, \dots\}$, where $m_1 < m_2 < m_3 < \cdots$, and define a function $f : \mathbb{N}^{(\omega)} \to (A, M)^{(\omega)}$ by $N \mapsto A \cup \{m_i : i \in N\}$. Clearly f is a homeomorphism in the $\star$-topology.

Let $Y' = \{N \in \mathbb{N}^{(\omega)} : f(N) \in Y\}$. Then Y' is $\star$-open since Y is $\star$-open. By Theorem 3′, Y' is Ramsey, and so there exists $L \in \mathbb{N}^{(\omega)}$ with $L^{(\omega)}$ contained in either Y or Y^c. Thus $\{f(N) : N \in L^{(\omega)}\}$ is contained in either Y or Y^c, i.e. $(A, f(L))^{(\omega)}$ is contained in either Y or Y^c. □

Remark. Hence $\star$-closed sets are also completely Ramsey.

So we know that, in the $\star$-topology, all 'locally big' (i.e. open) sets are completely Ramsey. Now we consider 'locally small' (i.e. nowhere-dense) sets.

Recall. Given a space X, we say that $Y \subset X$ is **nowhere-dense** if Y is not dense in any non-empty open subset, i.e. if $\overline{Y}$ has empty interior, i.e. if for any non-empty open O, there is a non-empty open $O' \subset O$ such that $O' \cap A = \emptyset$.

For example, in $\mathbb{R}$, $\mathbb{N}$ is nowhere-dense, as is $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$, as is $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} \cup \{0\}$. But $\mathbb{Q} \cap (0, 1)$ is *not* nowhere dense.

Proposition 5. A set $Y \subset \mathbb{N}^{(\omega)}$ is $\star$-nowhere-dense if and only if for all $A \in \mathbb{N}^{(<\omega)}$ and all $M \in \mathbb{N}^{(\omega)}$, there is some $L \in M^{(\omega)}$ with $(A, L)^{(\omega)} \subset Y^c$.

Remarks. 1. In particular, $\star$-nowhere-dense sets are completely Ramsey.

2. This is good evidence that the topology and the combinatorics are 'meshing nicely'.

Proof. The LHS of the 'iff' says that inside $(A, M)^{(\omega)}$ there is some $(B, L)^{(\omega)}$ missing Y, while the RHS says that inside $(A, M)^{(\omega)}$ there is some $(A, L)^{(\omega)}$ missing Y. So it is immediate that RHS $\Rightarrow$ LHS.

So suppose Y is $\star$-nowhere-dense. Then $\overline{Y}$ has empty interior, so is also $\star$-nowhere-dense (since $\overline{\overline{Y}} = \overline{Y}$). But $\overline{Y}$ is completely Ramsey by Theorem 4 and so inside $(A, M)^{(\omega)}$ there exists some $(A, L)^{(\omega)}$ contained in either $\overline{Y}$ or $(\overline{Y})^c$. But we cannot have $(A, L)^{(\omega)} \subset \overline{Y}$ as $\overline{Y}$ is nowhere-dense, so $(A, L)^{(\omega)} \subset (\overline{Y})^c \subset Y^c$ as required. $\square$

A subset A of a topological space X is called **meagre** or **of first category** if $A = \bigcup_{n=1}^{\infty} A_n$ with each A_n nowhere-dense. For example, any nowhere-dense set is meagre, and $\mathbb{Q}$ is meagre in $\mathbb{R}$, as it's a countable union of singletons. (So meagre sets can be dense.)

We usually think of meagre sets as being 'small': for example, the Baire Category Theorem states that if X is a non-empty complete metric space then X itself is not meagre.

Next aim. $\star$-meagre sets are completely Ramsey.

Theorem 6. Let $Y \subset \mathbb{N}^{(\omega)}$ be $\star$-meagre. Then for all $A \in \mathbb{N}^{(<\omega)}$ and all $M \in \mathbb{N}^{(\omega)}$, there is some $L \in M^{(\omega)}$ such that $(A, L)^{(\omega)} \subset Y^c$. In particular, Y is $\star$-nowhere-dense.

Proof. Suppose we are given $A \in \mathbb{N}^{(<\omega)}$ and $M \in \mathbb{N}^{(\omega)}$. Write $Y = \bigcup_{n=1}^{\infty} Y_n$ with each Y_n $\star$-nowhere-dense.

By Proposition 5, we have $M_1 \subset M$ with $(A, M_1)^{(\omega)} \subset Y_1^c$. Choose $x_1 \in M_1$ with $x_1 > \max A$.

By Proposition 5 twice, we have $M_2' \subset M_1$ with $(A, M_2')^{(\omega)} \subset Y_2^c$ and then $M_2 \subset M_2'$ with $(A \cup \{x_1\}, M_2)^{(\omega)} \subset Y_2^c$. Choose $x_2 \in M_2$ with $x_2 > x_1$.

By Proposition 5 four times (once for each subset of $\{x_1, x_2\}$), we get $M_3 \subset M_2$ such that the sets $(A, M_3)^{(\omega)}$, $(A \cup \{x_1\}, M_3)^{(\omega)}$, $(A \cup \{x_2\}, M_3)^{(\omega)}$ and $(A \cup \{x_1, x_2\}, M_3)^{(\omega)}$ are each contained in Y_3^c. Keep going.

We obtain $M \supset M_1 \supset M_2 \supset \cdots$ and $\max A < x_1 < x_2 < \ldots$ with $x_n \in M_n$ for all n and $(A \cup F, M_n)^{(\omega)} \subset Y_n^c$ for all $F \subset \{x_1, \ldots x_{n-1}\}$. Then $(A, \{x_1, x_2, \ldots\})^{(\omega)} \subset Y_n^c$ for all n and so $(A, \{x_1, x_2, \ldots\})^{(\omega)} \subset Y^c$.

A set Y in a topological space X is a **Baire set**, or has the **property of Baire**, if $Y = O \triangle M$ for some open O and meagre M. We can think of Y as being 'nearly open'.

Examples. 1. Any open set Y is Baire.

2. Any closed set Y is Baire, since $Y = \operatorname{int} Y \triangle (Y - \operatorname{int} Y)$, and $Y - \operatorname{int} Y$ is nowhere-dense as it is closed and contains no (non-empty) open set.

The Baire sets form a σ**-algebra**: they are closed under taking complements and countable unions. Indeed:

- If Y is Baire, write $Y = O \triangle M$ for some open O and meagre M. Then O^c is closed and so by example 2 above is $O' \triangle M'$ for some open O' and meagre M'. So we have $Y^c = O^c \triangle M = (O' \triangle M') \triangle M = O' \triangle (M' \triangle M)$, so Y^c is Baire.
- If the sets $Y_1, Y_2, \ldots$ are Baire, say with $Y_n = O_n \triangle M_n$ for some open O_n and meagre M_n, then their union $\bigcup_{n=1}^{\infty} Y_n = (\bigcup_{n=1}^{\infty} O_n) \triangle M$ for some $M \subset \bigcup_{n=1}^{\infty} M_n$. Clearly $\bigcup O_n$ is open and M is meagre, and so $\bigcup_{n=1}^{\infty} Y_n$ is Baire.

So we can think of Baire as being 'a bit like measurable'.

Theorem 7. Let $Y \subset \mathbb{N}^{(\omega)}$. Then is completely Ramsey if and only if it is $\star$-Baire.

Proof. ($\Leftarrow$) Suppose Y is $\star$-Baire, so $Y = W \triangle Z$ with W open and Z meagre. Given $A \in \mathbb{N}^{(<\omega)}$, $M \in \mathbb{N}^{(\omega)}$, we have $L \subset M$ with $(A, L)^{(\omega)}$ contained in either W or W^c (as W is completely Ramsey, by Theorem 4), and $N \in L^{(\omega)}$ with $(A, N)^{(\omega)} \subset Z^c$ (by Theorem 6).

So either $(A, N)^{(\omega)} \subset W \cap Z^c \subset Y$, or $(A, N)^{(\omega)} \subset W^c \cap Z^c \subset Y^c$, and Y is completely Ramsey as required.

($\Rightarrow$) Suppose conversely that Y is completely Ramsey. Write $Y = \operatorname{int} Y \triangle (Y - \operatorname{int} Y)$. So it will be sufficient for us to show that $Y - \operatorname{int} Y$ is $\star$-nowhere-dense. Given any basic open set $(A, M)^{(\omega)}$, we have $L \in M^{(\omega)}$ with $(A, L)^{(\omega)}$ contained in either Y or Y^c.

If $(A, L)^{(\omega)} \subset Y$ then $(A, L)^{(\omega)}$ is disjoint from $Y - \operatorname{int} Y$ (as $(A, L)^{(\omega)}$ is open).

If $(A, L)^{(\omega)} \subset Y^c$ then clearly $(A, L)^{(\omega)}$ is disjoint from $Y - \operatorname{int} Y$.

So Y is $\star$-Baire, as required. □

Note. Without Theorem 6, the above would show that Y is completely Ramsey if and only if Y is the symmetric difference of a $\star$-open set and a $\star$-nowhere-dense set. But then we would not know that the completely Ramsey sets form a σ-algebra.

Corollary 8. Let $Y \subset \mathbb{N}^{(\omega)}$. If Y is τ-Borel then Y is Ramsey.

The **Borel sets** are the σ-algebra generated by the open sets. We noted above that open sets are Baire, and it follows that any Borel set is Baire.

Proof. $\star$-Baire sets are Ramsey (as they are completely Ramsey), so certainly $\star$-Borel sets are Ramsey, so certainly the τ-Borel sets are Ramsey. □

Example. 2-colour $\mathbb{N}^{(\omega)}$ by giving M colour RED if $\sum_{n \in M} 1/\pi^n \in \mathbb{Q}$, and BLUE otherwise.

This is a τ-Borel colouring (easy check), so there exists an infinite M with $M^{(\omega)}$ red or $M^{(\omega)}$ blue. In fact, can check that red is impossible (as $\mathbb{Q}$ is only countable), so there exists an infinite M such that for every $L \subset M$, $\sum_{n \in L} 1/\pi^n$ is irrational.

1. How many combinatorial lines are there in $[m]^n$?

2. Show that $\mathrm{HJ}(2,k) = k$ for all k.

3. Let A be an infinite subset of the plane, with no three points of A collinear. Prove that A contains an infinite subset B such that no point of B is a convex combination of other points of B.

4. Let c be a 2-colouring of the finite subsets of $\mathbb{N}$. Must there exist an infinite $M \subset \mathbb{N}$ such that, for each r, the colouring c is constant on $M^{(r)}$?

5. Prove that $\{0,1\}^{\mathbb{N}}$ (with the product topology) is compact.

6. By mirroring the proof of van der Waerden's theorem for arithmetic progressions of length 3, show that whenever $\mathbb{N}^2$ is finitely coloured there exist a, b, r such that the set $\{(a,b),(a+r,b),(a,b+r)\}$ is monochromatic. Deduce by a product argument that whenever $\mathbb{N}^2$ is 2-coloured there exist a, b, r such that the square $\{(a,b),(a+r,b),(a,b+r),(a+r,b+r)\}$ is monochromatic. Give an explicit n such that whenever $[n]^2$ is 2-coloured there exists a monochromatic square.

7. Show that for every m there is an n with the following property: whenever $[n]^{(2)}$ is 2-coloured there exists a monochromatic set M of size at least m satisfying $|M| > \min M$.

8. Let A be a subset of $\mathbb{N}$ such that, whenever A is finitely coloured, there is a monochromatic arithmetic progression of length m. Must A contain an arithmetic progression of length $m+1$?

$^+$9. Let A be an uncountable set, and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?

$^+$10. Let c be a colouring of $\mathbb{N}$ using (possibly) infinitely many colours. Prove that, for every m, there is an arithmetic progression of length m on which c is either constant or injective.

Lent 2010 **RAMSEY THEORY – EXAMPLES 2** IBL

1. For which $a, b \in \mathbb{Q}$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & a & 1 & -1 & b \end{pmatrix}$$

partition regular?

2. Deduce from Ramsey's theorem that whenever $\mathbb{N}$ is finitely coloured there exist x, y, z with $\{x, y, z, x+y, y+z, x+y+z\}$ monochromatic.

3. Verify directly that the matrix corresponding to the Finite Sums theorem has the columns property.

4. A rational matrix A is called *partition regular over* $\mathbb{Z}$ (resp. $\mathbb{Q}$) if whenever $\mathbb{Z} - \{0\}$ (resp. $\mathbb{Q} - \{0\}$) is finitely coloured there is a monochromatic vector x with $Ax = 0$. Show that A is partition regular over $\mathbb{Z}$ if and only if it is partition regular over $\mathbb{N}$. If A is partition regular over $\mathbb{Q}$, must it be partition regular over $\mathbb{N}$?

5. For each $k \in \mathbb{N}$, construct a rational matrix A such that A is not partition regular but, whenever $\mathbb{N}$ is k-coloured, there is a monochromatic vector x with $Ax = 0$.

6. For each $m \in \mathbb{N}$, prove that whenever the collection of finite non-empty subsets of $\mathbb{N}$ is finitely coloured there exist disjoint $F_1, \ldots, F_m$ with $\left\{\bigcup_{i \in I} F_i : \emptyset \neq I \subset [m]\right\}$ monochromatic.

7. Do the partition regular subsets of $\mathbb{N}$ form an ultrafilter?

8. Show that the sequence of principal ultrafilters $1, 2, \ldots$ in $\beta\mathbb{N}$ has no convergent subsequence. Is the topology on $\beta\mathbb{N}$ induced by a metric?

9. Prove that $\beta\mathbb{N} - \mathbb{N}$ is not separable.

10. Show that if $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ are distinct ultrafilters on $\mathbb{N}$ then we can find $A \in \mathcal{U}$ such that $A \notin \mathcal{U}_i$ for all i. Show also that if $\mathcal{U}, \mathcal{U}_1, \mathcal{U}_2, \ldots$ are distinct ultrafilters on $\mathbb{N}$ then there need not exist $A \in \mathcal{U}$ such that $A \notin \mathcal{U}_i$ for all i. What happens if we insist that each $\mathcal{U}_i$ is non-principal?

$^{+}$11. How many ultrafilters are there on $\mathbb{N}$?

Lent 2010 **RAMSEY THEORY – EXAMPLES 3** IBL

1. Let $Y = \big\{M \in \mathbb{N}^{(\omega)} :$ no two members of M are coprime$\big\}$. Which $M \in \mathbb{N}^{(\omega)}$ accept $\{3\}$? Which M reject $\{3\}$? Which M reject $\{6\}$?

2. Construct a set $Y \subset \mathbb{N}^{(\omega)}$ such that the finite sets rejected by $\mathbb{N}$ are precisely $\emptyset$ and all sets of the form $\{m, m+1, \ldots, n\}$, $m \leq n$.

3. Let E be the set of even numbers and let P be the set of prime numbers. Show that the set $\big\{M \in \mathbb{N}^{(\omega)} : \ |M \cap E| = \infty, \ |M \cap P| < \infty\big\}$ is not $*$-open, but is a countable intersection of $*$-open sets.

4. Let $f_1, f_2, \ldots : T \to \mathbb{C}$ be bounded complex-valued functions on a set T. For any bounded $f : T \to \mathbb{C}$, write $\|f\|$ for $\sup\{|f(t)| : \ t \in T\}$. Show that there is a subsequence $(f_{n_i})_{i=1}^{\infty}$ of $(f_i)_{i=1}^{\infty}$ such that either for every subsequence $(f_{m_i})_{i=1}^{\infty}$ of $(f_{n_i})_{i=1}^{\infty}$ we have $\limsup_{k\to\infty} \|f_{m_1} + \ldots + f_{m_k}\| \geq 1$ or for every subsequence $(f_{m_i})_{i=1}^{\infty}$ of $(f_{n_i})_{i=1}^{\infty}$ we have $\limsup_{k\to\infty} \|f_{m_1} + \ldots + f_{m_k}\| < 1$.

5. Prove that the operation $+$ on $\beta\mathbb{N}$ is not commutative.

6. Prove that whenever the collection of finite non-empty subsets of $\mathbb{N}$ is finitely coloured there exist disjoint $F_1, F_2, \ldots$ with $\big\{\bigcup_{i\in I} F_i : \ \emptyset \neq I \subset \mathbb{N}, \ I \text{ finite} \ \big\}$ monochromatic.

7. Do the Ramsey subsets of $\mathbb{N}^{(\omega)}$ form a σ-algebra?

8. Is the $*$-topology on $\mathbb{N}^{(\omega)}$ induced by a metric?

$^{+}$9. Show that whenever $\mathbb{N}$ is finitely coloured there exist sets $S_1, S_2, \ldots$, with each S_i an arithmetic progression of length i, such that the set

$$\left\{\sum_{i\in I} x_i : \ \emptyset \neq I \subset \mathbb{N}, \ I \text{ finite}, \ x_i \in S_i \text{ for all } i \in I\right\}$$

is monochromatic.