Mathematics Tripos Part III
Dr P.H. Haynes, Michaelmas 1996

Perturbation Methods: Sheet 1†

1. Find the rescalings for the roots of

(*a*) $$\epsilon x^4 - x^2 - x + 2 = 0 ,$$

(*b*) $$\epsilon^2 x^3 + x^2 + 2x + \epsilon = 0 .$$

Thence find two terms in the approximation for each root.

2. Find the first two terms of $x(\epsilon)$ the solution near 0 of

(*a*) $$\sqrt{2}\sin\left(\frac{\pi}{4}+x\right) - 1 - x + \tfrac{1}{2}x^2 = -\tfrac{1}{6}\epsilon ,$$

(*b*) $$\tan\left(\frac{\pi}{4}+x\right) = 2x^2 + 2x + 1 + \epsilon .$$

3. Find several terms in an approximation for the solution of

$$e^{-x^2} = \epsilon x .$$

4. Find the first-order perturbations of the eigenvalues of the differential equation

$$y'' + \lambda y + \epsilon y^n = 0$$

in $0 < x < \pi$, with $y(0) = y(\pi) = 0$ for $n = 1$, 2 and 3.

5. Find an asymptotic approximation to the exponential integral as $x \to \infty$,

$$E_n(x) = \int_1^\infty t^{-n} e^{-xt}\, dt ,$$

and estimate the remainder.

6. Evaluate the first two terms as $r \to 0$, and the first 4 terms (counting $\ln r$ and 1 as different orders of magnitude) as $r \to \infty$, of

$$\int_0^\infty \frac{r x\, dx}{(r^2 + x)^{3/2}(1+x)} .$$

7. Evaluate the first two terms as $m \nearrow 1$ of the elliptic integral

$$\int_0^{\pi/2} \frac{d\theta}{(1 - m^2 \sin^2\theta)^{1/2}} .$$

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8. For $0 < \epsilon \ll 1$ deduce the asymptotic behaviour of the integral

$$I(\epsilon) = \int_0^1 \frac{dx}{x(x+\epsilon) + \epsilon^3 \cos x} .$$

Calculate upto and including terms of $O(1)$. The following indefinite integrals may be of help:

$$\int \frac{dx}{x^2(x+1)^2} = \frac{2x^2-1}{x(x+1)} + 2\ln\left(\frac{1+x}{x}\right) , \quad \int \frac{x^2\, dx}{(x+1)^2} = \frac{x(x+2)}{x+1} - 2\ln(1+x) .$$

9. Evaluate the first two terms as $\epsilon \to 0$ of the integral

$$\int_0^1 \frac{\ln x}{\epsilon + x}\, dx .$$

Hint: This question is a little bit of a cheat. Do **not** introduce a δ.

10. Let

$$\alpha_n = \int_0^1 \frac{1-(1-t)^n}{t}\, dt .$$

Show that as $n \to \infty$

$$\alpha_n \sim \ln n + \int_0^1 \frac{1-\exp(-t)}{t}\, dt - \int_1^\infty \frac{\exp(-t)}{t}\, dt .$$

Hence, or otherwise, deduce that

$$\int_0^\infty \ln(t)\exp(-t)\, dt = -\gamma ,$$

where γ is Euler's constant:

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln(n)\right) .$$

Thus find the first two terms of the asymptotic expansion as $\lambda \to \infty$ of

$$\lambda \int_0^1 \frac{\exp(-\lambda t)}{\ln\left(\frac{1}{2}t\right)}\, dt .$$

11.* The function $f(t;\epsilon)$ satisfies the integral equation

$$x = I(x;\epsilon) \equiv \int_{-1}^1 \frac{f(t;\epsilon)}{\epsilon^2 + (t-x)^2}\, dt \quad \text{in} \quad -1 \le x \le 1 .$$

Assuming that f remains $O(\epsilon)$ in the end regions where $(1 - |t|) = \mathrm{ord}(\epsilon)$, obtain the first two terms of an asymptotic approximation for f at fixed $t \neq \pm 1$ as $\epsilon \to 0$. Without any detailed calculation, comment on the contribution to $I(x;\epsilon)$ from the end regions for a point x outside the end regions, and comment on the behaviour of f in the end regions.

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Perturbation Methods: Sheet 2†

1. For real x and $x \to \infty$, find the full asymptotic behaviour of

$$K_0(x) = \int_1^\infty (t^2-1)^{-\frac{1}{2}} e^{-xt}\,dt\ .$$

2. For real x and $x \to \infty$, find the leading-order asymptotic behaviour of

(a)
$$\int_0^1 \sin(t)\, e^{-x\sinh^4 t}\,dt\ ,$$

(b)
$$\int_0^\infty e^{-xt-t^{-1}}\,dt\ .$$

3. For real n and $n \to \infty$, find the leading-order asymptotic behaviour of

$$J_n(n) = \frac{1}{\pi}\int_0^\pi \cos(n\sin t - nt)\,dt\ .$$

4. For real x and $x \to \infty$, find the full asymptotic behaviour of

$$I(x) = \int_0^1 \ln(t) e^{\mathrm{i}xt}\,dt\ .$$

5. Find the asymptotic behaviour of

$$K_\nu(z) = \tfrac{1}{2}\int_{-\infty}^\infty e^{\nu t - z\cosh t}\,dt\ ,$$

for real z and ν, with $z = \mathrm{ord}(1)$ and $\nu \to \infty$.

6. Find the asymptotic behaviour of

$$J_\nu(\nu z) = \frac{1}{2\pi\mathrm{i}}\int_{\infty-\mathrm{i}\pi}^{\infty+\mathrm{i}\pi} e^{\nu z\sinh t - \nu t}\,dt\ ,$$

for real ν and z, as $\nu \to \infty$ with first $0 < z < 1$ and second $z = 1$. (The case of $z = 1$ has a cubic saddle where three ridges meet and $f'' = 0$.)

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7. The function $f(y;\lambda)$ is defined by

$$f(y;\lambda) = \int_C \exp\left(\lambda(1+\mathrm{i}y)z - \tfrac{1}{3}z^3\right) dz,$$

where y and λ are real, and the contour C starts from $z=0$ and extends to $z=\infty$ in the sector $|\arg(z)| < \pi/6$.

(*a*) Show that $f(y;\lambda)$ satisfies the differential equation

$$f_{yy} + \lambda^3(1+\mathrm{i}y)f = -\lambda^2 .$$

(*b*) Deduce the leading-order asymptotic behaviour of $f(0;\lambda)$ as $\lambda \to -\infty$.

(*c*) Deduce the leading-order asymptotic behaviour of $f(0;\lambda)$ as $\lambda \to +\infty$.

(*d*) With reference to the solutions deduced in parts (*b*) and (*c*), and the steepest descent contours for the Airy function included in the notes, find the leading-order asymptotic behaviour of f for $0 \leq y < \infty$. In particular:

(i) state clearly your choice of integration contour;

(ii) for $\lambda \gg 1$ comment on how the asymptotic behaviour of the solution differs according as $0 \leq y < y_c$ and $y_c < y < \infty$, where y_c should be identified.

(*e*) * Why is it that

$$f = -\frac{1}{\lambda(1+\mathrm{i}y)} - \frac{2}{\lambda^4(1+\mathrm{i}y)^4} + \dots ,$$

is not always a uniformly valid asymptotic approximation for $|\lambda| \gg 1$? *

8. (*a*) Let

$$f(\lambda) = \int_C \frac{G(z)}{z-z_0}\exp(-\mathrm{i}\lambda W(z))\, dz ,$$

where G and W are analytic near the contour C and z_0, $W(z)$ has a single saddle point at $z=z_0$, and C passes from one 'valley' of $\Im(W)$ with respect to z_0 to another, avoiding z_0 in a clockwise manner. Show that

$$f(\lambda) \sim -\mathrm{i}\pi G(z_0)\exp(-\mathrm{i}\lambda W(z_0)) .$$

(*b*) The function $g(\theta;\lambda)$ is defined by

$$g(\theta;\lambda) = \int_\Gamma \frac{1}{(z-z_0)(z^2-1)^{\frac{1}{2}}} \exp\left(\mathrm{i}\lambda\left((z^2-1)^{\frac{1}{2}}\theta - z\right)\right) dz ,$$

where θ, z_0 and λ are real and positive, and the contour Γ goes from $-\infty$ to ∞ passing above the three singularities of the integrand. Take the branch cuts for $(z^2-1)^{\frac{1}{2}}$ to be lines drawn towards $\Im(z) = -\infty$ from the points $z = \pm 1$. Obtain the leading-order asymptotic behaviour of $g(\theta;\lambda)$ as $\lambda \to \infty$ on the assumption that $0 < \theta < 1$ and $z_0 > 1$; be careful to discuss all cases.

9. For a contour C enclosing $t = z$, where z is real and positive, find the asymptotic behaviour as $n \to \infty$ of

$$P_n(z) = \frac{1}{2^{n+1}\pi\mathrm{i}} \int_C \frac{(t^2-1)^n}{(t-z)^{n+1}}\, dt .$$

Consider the cases $z < 1$ and $z > 1$ in turn. [*Hint*. Write $z = \cos\alpha$ and $z = \cosh\alpha$ respectively.]

10. Find the leading-order asymptotic behaviour of

$$I(z) = \frac{1}{2}\int_{-1}^{1} e^{-4zt^2+5\mathrm{i}zt-\mathrm{i}zt^3}\, dt$$

as $|z| \to \infty$. [*Hints*. Start with z real and positive, and *carefully* construct steepest descent contours. Then obtain the leading-order asymptotic approximations on each part of the contour. Finally allow z to become complex.]

Perturbation Methods: Sheet 3†

1. *An example of a regular expansion.* The flow down a slightly corrugated channel is described by a function $u(x, y; \epsilon)$ which is periodic in x and which satisfies

$$\nabla^2 u = -1 \quad \text{in} \quad |y| \leq h(x;\epsilon) \equiv 1 + \epsilon \cos kx ,$$
$$\text{subject to} \quad u = 0 \quad \text{on} \quad y = \pm h(x;\epsilon) .$$

Obtain the first three terms for u and hence evaluate correct to ord(ϵ^2) the average flux per unit width

$$\frac{k}{2\pi} \int_{x=0}^{2\pi/k} \int_{y=-h(x;\epsilon)}^{+h(x;\epsilon)} u(x, y; \epsilon)\, dx dy .$$

2. The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + (1+\epsilon)y' + y = 0 \quad \text{in} \quad 0 \leq x \leq 1 ,$$

and is subject to boundary conditions $y = 0$ at $x = 0$ and $y = e^{-1}$ at $x = 1$. Find two terms in the outer approximation, applying only the boundary condition at $x = 1$. Next find two terms in the inner approximation for the ord(ϵ) boundary layer near to $x = 0$; apply only the boundary condition at $x = 0$. Finally determine the constants of integration in the inner approximation by matching.

3. The function $y(x;\epsilon)$ satisfies

$$\epsilon y'' + x^{1/2} y' + y = 0 \quad \text{in} \quad 0 \leq x \leq 1 ,$$

and is subject to the boundary conditions $y = 0$ at $x = 0$ and $y = 1$ at $x = 1$. First find the rescaling for the boundary layer near $x = 0$, and obtain the leading order inner approximation. Then find the leading order outer approximation and match the two approximations.

4. The function $y(x;\epsilon)$ satisfies

$$(x + \epsilon y)y' + y = 1 \quad \text{with} \quad y(1) = 2 .$$

Find $y(0)$ correct to ord(1).

5. Find two terms in ϵ in the outer region, having matched to the inner solutions at both boundaries for

$$\epsilon^2 y'''' - y'' = -1 \quad \text{in} \quad -1 < x < 1 ,$$

with $y = y' = 0$ at $x = -1$ and $x = 1$.

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6. The function $y(x,\epsilon)$ satisfies

$$\epsilon y'' + yy' - y = 0 \quad \text{in} \quad 0 \le x \le 1 \,,$$

and is subject to the boundary condition $y = 0$ at $x = 0$ and $y = 3$ at $x = 1$. Assuming that there is a boundary layer only near $x = 0$, find the leading order terms in the outer and inner approximations and match them.

Suppose now the boundary conditions are replaced by $y = -\frac{3}{4}$ at $x = 0$ and $y = \frac{5}{4}$ at $x = 1$. Show that the boundary layer moves to an intermediate position which is determined by the property of the inner solution that y jumps within the boundary layer from $-M$ to M, for some value M. Find the leading order matched asymptotic expansions.

7. For $0 \le x \le \infty$, the function $G(x;\epsilon)$ satisfies the differential equation

$$(3+2\epsilon)G''' + (1+2\epsilon)GG'' + (1-2\epsilon)G'^2 = 0 \,,$$

and the boundary conditions

$$G(0;\epsilon) = 0 \,, \quad G'(0;\epsilon) = \tfrac{1}{6} \,, \quad G''(0;\epsilon) = 0 \,, \quad G(x;\epsilon) \sim 2x^{\frac{1}{2}}\lambda^{-\frac{1}{2}} \quad \text{as} \quad x \to \infty \,.$$

If $0 < |\epsilon| \ll 1$, determine the eigenvalue λ to leading order. *Hint.* In a regular expansion of the form $G = G_0 + \epsilon G_1 + \ldots$ it is sufficient to identify the asymptotic behaviour of G_1 as $x \to \infty$ (i.e. you do not need to solve for G_1 for all x). The following results *may* prove useful:

$$\int \frac{a\,dx}{a^2 - x^2} = \tanh^{-1}(x/a) \,,$$

$$\int \frac{dx}{\cosh^4(x/a)} = a\tanh(x/a)\left(1 - \tfrac{1}{3}\left(\tanh(x/a)\right)^2\right) \,.$$

8. Consider the following problem which has an outer, an inner and an inner-inner inside the inner

$$x^3 y' \;=\; \epsilon\left((1+\epsilon)x + 2\epsilon^2\right)y^2 \quad \text{in} \quad 0 < x < 1 \,,$$

with $y(1) = 1 - \epsilon$. Calculate two terms of the outer, then two of the inner, and finally one for the inner-inner. At each stage find the rescaling required for the next layer by examining the non-uniformity of the asymptoticness in the current layer.

9. The function $y(x;\epsilon)$ satisfies

$$(\epsilon + x)y' = \epsilon y \quad \text{with} \quad y(1) = 1 \,.$$

Find $y(0)$ correct to $\text{ord}(\epsilon^2)$.

10. For $0 \le x \le 1$, the function $\phi(x;\epsilon)$ satisfies the differential equation

$$\epsilon x^{\frac{1}{2}}(x+\epsilon)\phi_{xx} + (x+\epsilon)(x+1)\phi_x - x\phi = 0 \,,$$

and the boundary conditions

$$\phi(0;\epsilon) = 0 \,, \quad \phi(1;\epsilon) = 2 \,.$$

On the assumption that $0 < \epsilon \ll 1$, find the solution correct to, and including, terms of $O(\epsilon)$ in three asymptotic regions (which are to be identified). The following integral may prove useful

$$2\int_0^x \exp(-2q^{\frac{1}{2}})\,dq = 1 - (1 + 2x^{\frac{1}{2}})\exp(-2x^{\frac{1}{2}}) \,.$$

11. The function $f(r, \epsilon)$ satisfies the equation

$$f_{rr} + \tfrac{2}{r} f_r + \tfrac{1}{2}\epsilon^2(1 - f^2) \;=\; 0 \quad \text{in} \quad r > 1\,,$$

and is subject to the boundary conditions

$$f = 0 \quad \text{at} \quad r = 1 \quad \text{and} \quad f \to 1 \quad \text{as} \quad r \to \infty\,.$$

Using the asymptotic sequence 1, ϵ, $\epsilon^2 \ln \frac{1}{\epsilon}$, ϵ^2, obtain asymptotic expansions for f at fixed r as $\epsilon \searrow 0$ and at fixed $\rho = \epsilon r$ as $\epsilon \searrow 0$. Match the expansions using the intermediate variable $\eta = \epsilon^\alpha r$ with $0 < \alpha < 1$.
Hint. You may quote that the solution to the equation

$$y_{xx} + \tfrac{2}{x} y_x - y \;=\; \frac{e^{-2x}}{x^2}\,,$$

subject to the condition $y \to 0$ as $x \to \infty$, is

$$y \;=\; A\frac{e^{-x}}{x} + \frac{1}{2x}\int_x^\infty \frac{e^{-x-t} - e^{x-3t}}{t}\,dt\,,$$

with A a constant. Further as $x \to 0$

$$y \;\sim\; \frac{2A + \ln 3}{2x} + \ln x - A + \gamma + \tfrac{1}{2}\ln 3 - 1\,.$$

Perturbation Methods: Sheet 4†

1. Obtain equations for the drift in the amplitude and phase in the solution to

$$\ddot{x} + \epsilon\dot{x}(x^2 - 1) + (1 + \epsilon k)x = \epsilon \cos t \ ,$$

with $k = \text{ord}(1)$ as $\epsilon \searrow 0$. [The tough part is then to show that a slave oscillator will lock onto the forcing from a master if the slave is not detuned too much, i.e. if $|k| < k_c$ then R tends to an equilibrium, while if $|k| > k_c$ then R oscillates (the free oscillations beating with the forced response).]

2. Find the leading order approximation to the general solution for $x(t;\epsilon)$ and $y(t;\epsilon)$ satisfying

$$\frac{d^2x}{dt^2} + 2\epsilon y\frac{dx}{dt} + x = 0 \ ,$$
$$\frac{dy}{dt} = \tfrac{1}{2}\epsilon \ln x^2 \ ,$$

which is valid for $t = \text{ord}(1/\epsilon)$ as $\epsilon \to 0$. You may quote the result

$$\frac{1}{2\pi}\int_0^{2\pi} \ln \cos^2\theta \, d\theta = -\ln 4 \ .$$

3. Solve the Mathieu equation

$$\ddot{y} + (\omega^2 + \epsilon \cos t)y = 0 \ ,$$

for the case when $\omega = \frac{1}{2} + \epsilon\omega_1 + \ldots$. Identify the stability boundary correct to $\text{ord}(\epsilon)$.

* Explain why it is necessary to introduce a slow time $\mathcal{T} = \epsilon^{\frac{3}{2}}t$ in order to calculate the stability boundary correct to $\text{ord}(\epsilon^2)$ and perform the calculation.

4. The function $u(t;\epsilon)$ satisfies the governing equation

$$\frac{d^2u}{dt^2} - \lambda\epsilon^2 t\frac{du}{dt} + u = \epsilon\gamma u^2\frac{du}{dt} \ ,$$

and the initial conditions

$$u = 2a \ , \quad \text{and} \quad \frac{du}{dt} = 0 \quad \text{at} \quad t = 0 \ ,$$

where $0 < \epsilon \ll 1$, and λ, γ and a are order one constants. By ascertaining at what order of ϵ a secularity first appears in the regular perturbation expansion for $u(t;\epsilon)$, or otherwise, find a solution for $|u(x,t)|^2$ that is uniformly valid for large times. If

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$\lambda > 0$, sketch typical solutions for $|u|^2$ for both $\gamma > 0$ and $\gamma < 0$. Sketch the squared amplitude as a function of time for different values of a, with special emphasis on the case $|a| \ll 1$.

5. Find the leading order approximation which is valid for times $t = \mathrm{ord}(\epsilon^{-1})$ as $\epsilon \to 0$, to the solution $x(t;\epsilon)$ and $y(t;\epsilon)$ satisfying

$$\begin{aligned} \frac{dx}{dt} + x^2 y \cos t &= \epsilon(x - 2x^2) , \\ \frac{dy}{dt} &= \epsilon\left(1 - \frac{\sin t}{x}\right) , \end{aligned}$$

with $x = 1$ and $y = 0$ at $t = 0$.

6. Use the transformation

$$x(t,\epsilon) = \Re\left[r(\epsilon t, \epsilon)\exp\left(\mathrm{i}\int^t \sigma(\epsilon q, \epsilon)\, dq\right)\right] ,$$

to obtain a higher order approximation correct to $O(\epsilon^2)$ to the equation

$$\ddot{x} + f(\epsilon t)x = 0 .$$

7. Find the large eigenvalue solutions of the equation

$$y'' + \lambda(1 - x^2)^2 y = 0 ,$$

subject to $y = 0$ at $x = \pm 1$. At the ends $x = \pm 1$ you will need to use turning point solutions like

$$(1 - x^2)^{1/2} \mathrm{J}_{1/4}(\lambda^{1/2}(1 - x^2)^2/4),$$

and then use

$$\mathrm{J}_{1/4}(z) \sim (2/\pi z)^{1/2}\cos(z - 3\pi/8) \quad \text{as} \quad z \to \infty.$$

8. Sound waves propagating through a slow-varying mean flow satisfy the equations

$$\rho_0(\tilde{u}_t + (U\tilde{u})_z) = -c_0^2 \tilde{\rho}_z , \quad \tilde{\rho}_t + (U\tilde{\rho})_z = -\rho_0 \tilde{u}_z ,$$

where the wavespeed c_0 and the undisturbed density ρ_0 are constants, $\tilde{u}(z,t)$ and $\tilde{\rho}(z,t)$ are the perturbation velocity and density respectively, and $U(\epsilon z, \epsilon t)$ is the slowly-varying mean flow. By seeking solutions of the form,

$$(\tilde{\rho}, \tilde{u}) = ((A_0, B_0)(\epsilon z, \epsilon t) + \epsilon(A_1, B_1)(\epsilon z, \epsilon t) + \ldots)\exp(\mathrm{i}\theta(\epsilon z, \epsilon t)/\epsilon) + \text{c.c.} ,$$

show that the wave action E_r/ω_r is conserved, where

$$E_r = \frac{c_0^2 |A_0|^2}{2\rho_0} , \quad \text{and} \quad \omega_r = \omega - kU .$$

9. Repeatedly apply the Shanks transform to the 6 terms

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$$

to obtain an estimate for $\pi/4$. What is the error?