Perturbation Methods 1: Algebraic Equations†

1.1 Regular expansions and iteration

Consider

$$x^2 + \varepsilon x - 1 = 0 \ . \tag{1.1}$$

Exact solution:

$$x = -\frac{1}{2}\varepsilon \pm \left(1 + \frac{1}{4}\varepsilon^2\right)^{\frac{1}{2}} \ .$$

If $|\varepsilon| < 2$, then can expand in a convergent series:

$$x = \begin{cases} 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 - \frac{1}{128}\varepsilon^4 + \cdots \\ -1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \frac{1}{128}\varepsilon^4 + \cdots \end{cases}$$

Since the series is convergent, for $|\varepsilon| < 2$ we can increase the accuracy by taking more terms. We have

SOLVED THE EQUATION and then APPROXIMATED THE SOLUTION

However, we cannot always solve the equation exactly, so can we

APPROXIMATE then SOLVE THE EQUATION?

Iterative Method (liked by Pure Mathematicians)

$$x_{n+1} = g(x_n)$$

Suppose $x_n = x^* + \delta_n$ where $x^* = g(x^*)$. Then by Taylor Series

$$\delta_{n+1} = g'(x^*)\,\delta_n + \mathcal{O}\left(\delta_n^2\right) \ .$$

If we have a good guess, so that $|\delta_n|$ is small, this is convergent if

$$|g'(x^*)| < 1 \ .$$

Rearrange (1.1):

$$x^2 = 1 - \varepsilon x \ .$$

† Corrections and suggestions can be emailed to me at P.H.Haynes@damtp.cam.ac.uk.

For the root near $x = 1$ try

$$\begin{aligned}
x_{n+1} &= (1 - \varepsilon x_n)^{\frac{1}{2}} \\
x_0 &= 1 \\
x_1 &= (1-\varepsilon)^{\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + \cdots \\
x_2 &= \left(1 - \varepsilon (1-\varepsilon)^{\frac{1}{2}}\right)^{\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + \frac{1}{8}\varepsilon^3 + \cdots \\
x_3 &= \left(1 - \varepsilon \left(1 - \varepsilon (1-\varepsilon)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&= 1 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2 + 0 + \mathcal{O}\left(\varepsilon^4\right) + \cdots
\end{aligned}$$

Hard work for the higher terms — also, how many terms are correct?

Expansion Method

For $\varepsilon = 0$, the roots are $x = \pm 1$. For the root near $x = 1$ try

$$x(\varepsilon) = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots$$

Substitute into equation (1.1):

$$\begin{array}{llllll}
1 + 2\varepsilon x_1 & + 2\varepsilon^2 x_2 + \varepsilon^2 x_1^2 & + 2\varepsilon^3 x_3 + 2\varepsilon^3 x_1 x_2 & + \cdots & \\
\quad + \varepsilon & + \varepsilon^2 x_1 & + \varepsilon^3 x_2 & + \cdots & \\
-1 & & & & = 0
\end{array}$$

Equate powers of ε:

$$\begin{array}{lrll}
\varepsilon^0 : & 1 - 1 & = 0 & \\
\varepsilon^1 : & 2x_1 + 1 & = 0 & , \quad x_1 = -\frac{1}{2} \\
\varepsilon^2 : & 2x_2 + x_1^2 + x_1 & = 0 & , \quad x_2 = \frac{1}{8} \\
\varepsilon^3 : & 2x_3 + 2x_1 x_2 + x_2 & = 0 & , \quad x_3 = 0
\end{array}$$

Easier than the iterative method for higher terms, but you need to guess the expansion correctly.

1.2 Singular Perturbations and Rescaling

Consider

$$\varepsilon x^2 + x - 1 = 0 \,. \tag{1.2}$$

$$\begin{array}{lcl}
\varepsilon = 0 & : & \text{one solution} \\
\varepsilon \neq 0 & : & \text{two solutions}
\end{array}$$

The limit process $\varepsilon \to 0$ is said to be *singular*.

Exact solution: $\dfrac{-1 \pm (1+4\varepsilon)^{\frac{1}{2}}}{2\varepsilon}$.

Expansion for $|\varepsilon| < \frac{1}{4}$:

$$x = \begin{cases} 1 - \varepsilon + 2\varepsilon^2 - 5\varepsilon^3 + \cdots & (1.3a) \\ -\dfrac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + \cdots & (1.3b) \end{cases}$$

The singular (i.e. extra) root $\to \infty$ as $\varepsilon \to 0$.

Iterative method

(a) For the non-singular root try

$$x_{n+1} = 1 - \varepsilon x_n^2 \ .$$

(b) For the singular root, we need to keep the 'εx^2' term as a major player. The leading order approximation is

$$\varepsilon x^2 + x \approx 0 \ ;$$

so try rearranging (1.2) to

$$x_{n+1} = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x_n} \ .$$

Exercise: Confirm (1.3) by iteration.

Note that in (b)

$$x_{n+1} = g(x_n) \ , \qquad \text{where} \qquad g(x) = -\frac{1}{\varepsilon} + \frac{1}{\varepsilon x} \ .$$

Hence

$$g'(x) = -\frac{1}{\varepsilon x^2} \ , \qquad \left| g'\left(-\frac{1}{\varepsilon}\right) \right| = \varepsilon < 1 \quad \text{if} \quad 0 < \varepsilon < 1 \ .$$

Expansion method

For one root try

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \ , \qquad (1.4a)$$

and for the other try

$$x = \frac{x_{-1}}{\varepsilon} + x_0 + \varepsilon x_1 + \cdots \ . \qquad (1.4b)$$

Substitute (1.4b) into (1.2):

$$\begin{array}{rcccl} & \frac{x_{-1}^2}{\varepsilon} + 2x_{-1}x_0 + \varepsilon\left(x_0^2 + 2x_{-1}x_1\right) + \cdots \\ + & \frac{x_{-1}}{\varepsilon} + \quad x_0 + \quad \varepsilon x_1 + \cdots \\ - & 1 \qquad\qquad = 0 \end{array}$$

$$\begin{array}{llcl} \varepsilon^{-1}: & x_{-1}^2 + x_{-1} = 0 & ; & x_{-1} = 0\ ,\ -1 \\ \varepsilon^{0}: & (2x_{-1}+1)\,x_0 - 1 = 0 & ; & x_0 = 1\ ,\ -1 \\ \varepsilon: & x_0^2 + 2x_{-1}x_1 + x_1 = 0 & ; & x_{-1} = -1\ ,\ 1 \\ & & & \qquad\ \uparrow \qquad\ \uparrow \\ & & & \quad (1.3a) \quad (1.3b) \end{array}$$

Rescaling before expansion

How do you decide on the expansion if you do not know the solution?

Seek rescaling[s] to convert the singular equation into a regular equation. Try

$$x = \delta(\varepsilon)\,X$$

need to choose suitable δ ↗ ↖ strictly order 'unity'; say $X = \text{ord}(1)$.

(1.2) becomes

$$\varepsilon\delta^2X^2 + \delta X - 1 = 0\ .$$

Consider the possibilities for different choices of δ ($|\varepsilon| \ll 1$):

$\delta \ll 1$:			small	+	small	−	1	=	0	⁕
$\delta = 1$:			small	+	X	−	1	=	0	regular root
$1 \ll \delta \ll \frac{1}{\varepsilon}$:	$\frac{\text{LHS}}{\delta}$	=	small	+	X	+	small	=	0	⁕
										(since $X = \text{ord}(1)$)
$\delta = \frac{1}{\varepsilon}$:	$\frac{\text{LHS}}{\delta}$	=	X^2	+	X	+	small	=	0	singular root
$\delta \gg \frac{1}{\varepsilon}$:	$\frac{\text{LHS}}{\varepsilon\delta^2}$	=	X^2	+	small	+	small	=	0	⁕

The distinguished choices are therefore:

$\delta = 1$: $\quad \varepsilon X^2 + X - 1 = 0 \quad ; \quad X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots$

$\delta = \frac{1}{\varepsilon}$: $\quad X^2 + X - \varepsilon = 0 \quad ; \quad X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \ldots$

1.3 Non Integral Powers

Inter alia, double roots can cause problems. Consider

$$(1-\varepsilon)\,x^2 - 2x + 1 = 0\;. \tag{1.5}$$

When $\varepsilon = 0$, there is a double root at $x = 1$. Try an expansion:

$$x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$$

then

$$\begin{array}{llll} & 1 + 2\varepsilon x_1 + \varepsilon^2\left(2x_2 + x_1^2\right) & + \cdots & \\ & \quad - \varepsilon \qquad - \varepsilon^2\,(2x_1) & & \\ - & 2 - 2\varepsilon x_1 - 2\varepsilon^2 x_2 & + \cdots & \\ + & 1 & & = 0 \end{array}$$

$$\begin{array}{lrll} \varepsilon^0: & 1-2+1 & =0 & \\ \varepsilon^1: & 2x_1 - 1 - 2x_1 & =0 & \quad * \end{array}$$

We need 'εx_1' to be larger.

Exact solution: $\quad x = \dfrac{1 \pm \varepsilon^{\frac{1}{2}}}{1-\varepsilon}$.

We should have expanded in powers of $\varepsilon^{\frac{1}{2}}$:

$$x = 1 + \varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} + \varepsilon x_1 + \varepsilon^{\frac{3}{2}} x_{\frac{3}{2}} + \cdots$$

$$\begin{array}{llllll} & 1 & + \; 2\varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} & + \; 2\varepsilon x_1 & + \; \varepsilon x_{\frac{1}{2}}^2 & \\ & & & - \; \varepsilon & & \\ - & 2 & - \; 2\varepsilon^{\frac{1}{2}} x_{\frac{1}{2}} & - \; 2\varepsilon x_1 & & \\ + & 1 & & & & = 0 \end{array}$$

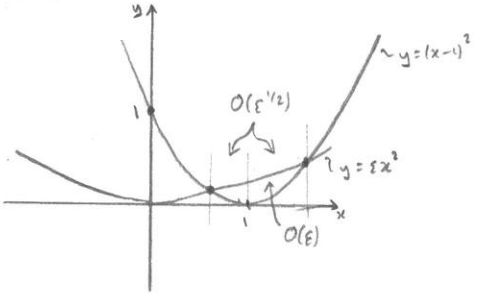
$$\begin{array}{lrll} \varepsilon^0: & 1-2+1 & =0 & \\ \varepsilon^{\frac{1}{2}}: & 2x_{\frac{1}{2}} - 2x_{\frac{1}{2}} & =0 & \quad \textbf{no information} \\ \varepsilon^1: & 2x_1 + x_{\frac{1}{2}}^2 - 1 - 2x_1 & =0 & \quad x_{\frac{1}{2}} = \pm 1 \end{array}$$

We must work to $\mathcal{O}(\varepsilon)$ to obtain the solution to $\mathcal{O}(\varepsilon^{\frac{1}{2}})$.

From the original equation

$$(x-1)^2 = \varepsilon x^2 \ ,$$

we see that a change in the ordinate by $\mathrm{ord}(\varepsilon)$ changes the position of the root by $\mathrm{ord}(\varepsilon^{\frac{1}{2}})$.



In general we must derive (guess) the expansion required, e.g. try

$$\begin{aligned} x(\varepsilon) &= 1 + \delta_1(\varepsilon)\, x_1 + \delta_2(\varepsilon)\, x_2 + \cdots \\ & 1 \gg \delta_1 \gg \delta_2 \gg \cdots \\ & x_j = \mathrm{ord}(1). \end{aligned}$$

Substitute into (1.5):

$$\begin{array}{l} \quad 1 + 2\delta_1 x_1 + 2\delta_2 x_2 + \cdots + \delta_1^2 x_1^2 \ + \cdots + 2\delta_1\delta_2 x_1 x_2 + \cdots \\ \qquad\qquad -\varepsilon \qquad\qquad\quad - 2\varepsilon\delta_1 x_1 + \cdots \\ -2 - 2\delta_1 x_1 - 2\delta_2 x_2 + \cdots \\ +1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad = 0 \end{array}$$

The leading order terms are $\delta_1^2 x_1^2$ and $-\varepsilon$.
Hence take $\delta_1 = \varepsilon^{\frac{1}{2}}$, allow x_1 to absorb any multiple roots.

Exercise: Show that the choices $\delta_1^2 \gg \varepsilon$, or $\delta_1^2 \ll \varepsilon$, lead to a $\divideontimes$.

Cancelling off these two terms, the leading-order terms become

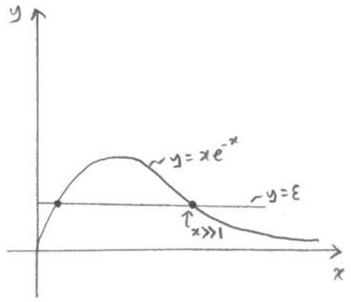
$$2\delta_1\delta_2 x_1 x_2 \qquad \text{and} \qquad -2\varepsilon\delta_1 x_1 \ .$$

Repeating the argument $\Rightarrow \delta_2 = \varepsilon$ (and $x_2 = 1$).

1.4 Logarithms

Solve

$$xe^{-x} = \varepsilon \ . \tag{1.6}$$



One root is close to $x = \varepsilon$, the other root is between

$$x = \ln\frac{1}{\varepsilon} \qquad (xe^{-x} = \varepsilon\ln\tfrac{1}{\varepsilon} > \varepsilon)$$

and

$$x = 2\ln\frac{1}{\varepsilon} \qquad (xe^{-x} = 2\varepsilon^2\ln\tfrac{1}{\varepsilon} < \varepsilon, \text{ for } \varepsilon \text{ small}).$$

Note: doubling x reduces the e^{-x} factor by an order of magnitude.

The expansion method is unclear, so try the iteration scheme.

Consider a rearrangement that emphasises the e^{-x} factor:

$$e^x = \frac{x}{\varepsilon}$$

so try

$$x_{n+1} = \log\frac{1}{\varepsilon} + \log x_n \ .$$

Then

$$x_0 = \log\frac{1}{\varepsilon}$$

$$x_1 = \underbrace{\log\frac{1}{\varepsilon}}_{L_1} + \underbrace{\log\log\frac{1}{\varepsilon}}_{L_2}$$

$$\begin{aligned} x_2 &= L_1 + \log(L_1 + L_2) \\ &= L_1 + L_2 + \frac{L_2}{L_1} - \frac{L_2^2}{2L_1^2} + \frac{L_2^3}{3L_1^3} + \cdots \\ x_3 &= L_1 + \log\left(L_1 + L_2 + \frac{L_2}{L_1} - \frac{L_2^2}{2L_1^2} + \frac{L_2^3}{3L_1^3} + \cdots\right) \\ &= L_1 + L_2 + \frac{L_2}{L_1} + \frac{-\frac{1}{2}L_2^2 + L_2}{L_1^2} + \frac{\frac{1}{3}L_2^3 - \frac{3}{2}L_2^2}{L_1^3} + \cdots \end{aligned}$$

The iterative method can give more than one term per iteration.

Numerical Disaster Percentage errors for the truncated series:

ε	L_1	L_2	L_2/L_1	$-L_2^2/2L_1^2$	L_2/L_1^2
10^{-1}	36%	12%	2%	4%	0.03%
10^{-3}	24%	3%	0.02%	0.04%	0.04%
10^{-5}	19%	1%	0.04%	0.1%	0.001%

Do not separate terms like $-L_2^2/2L_1^2$ & L_2/L_1^2.

A very small ε is needed before this is tolerable.

Check convergence

$$\begin{aligned} x_{n+1} &= g(x_n) \\ g(x) &= \log\frac{1}{\varepsilon} + \log x \\ g'(x) &= \frac{1}{x} \\ g'(x_*) &\approx \frac{1}{\log\frac{1}{\varepsilon}} \end{aligned}$$

$\uparrow$ need ε very small for $|g'| \ll 1$.

Perturbation Methods 2: Asymptotic Approximations †

2.1 Convergence and asymptoticness

An expansion $\sum_{n=0}^{\infty} f_n(z)$ converges for a fixed z if, given $\varepsilon > 0$, $\exists\, N(z,\varepsilon)$ s.t.

$$\left|\sum_{m}^{n} f_n(z)\right| < \varepsilon \qquad \forall\, m,n > N \; .$$

Convergent series can be useful analytically, but hopeless in practice. For instance, consider

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt \; .$$

We know that

$$e^{-t^2} = \sum_0^\infty \frac{\left(-t^2\right)^n}{n!}$$

is analytic in the entire complex plane. Hence we have uniform convergence on any bounded part of the plane $\Rightarrow$ we can integrate term by term:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_0^\infty \frac{(-)^n z^{2n+1}}{(2n+1)n!} \; .$$

$\downarrow$ also has ∞ radius of convergence

To obtain an accuracy of 10^{-5} we need

8 terms up to $z = 1$
16 terms up to $z = 2$
31 terms up to $z = 3$
75 terms up to $z = 5$

However, intermediate terms can be large $\Rightarrow$ problems due to round-off error on computers.

An alternative for large z is to proceed as follows. First rewrite the integral:

$$\operatorname{erf}(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt \; .$$

Then integrate by parts:

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$$\int_z^\infty e^{-t^2}\,dt = \frac{e^{-z^2}}{2z} - \int_z^\infty \frac{1}{2t^2}e^{-t^2}\,dt$$

$$\vdots$$

$$= \left(1 - \frac{1}{2z^2} + \frac{1.3}{(2z^2)^2} - \frac{1.3.5}{(2z^2)^3}\right)\frac{e^{-z^2}}{2z} + R$$

where

$$\begin{aligned} R &= \int_z^\infty \frac{105}{16}\frac{e^{-t^2}}{t^8}\,dt \\ &\leq \frac{105}{32z^9}\int_z^\infty d\left(-e^{-t^2}\right) = \frac{105}{32}\frac{e^{-z^2}}{z^9} \end{aligned}$$

The series in z^{-1} is divergent (due to the odd factorial in the numerator), but the truncated series is useful, e.g. 10^{-5} accuracy with 3 terms for $z = 2.5$
2 terms for $z = 3$.

"First term is essentially the answer, while subsequent terms are minor corrections."

Problem: What if the leading term is not sufficiently accurate (e.g. in reality ε is not sufficiently small)? Adding a few extra terms *may* help, but there is a limit to the number of useful extra terms if the series diverges as $N \to \infty$ at fixed ε. It is not sensible to include extra terms once they stop decreasing in magnitude. By suitable truncation, one can obtain exponential accuracy (see §3.1 and Q5 on example sheet 1).

2.2 Definitions

The expansion $\sum_0^N f_n(\varepsilon)$ is an *asymptotic approximation* of $f(\varepsilon)$ as $\varepsilon \to 0$, if $\forall\, m \leq N$,

$$\frac{\sum_0^m f_n(\varepsilon) - f(\varepsilon)}{f_m(\varepsilon)} \to 0 \qquad \text{as } \varepsilon \to 0$$

i.e. the remainder is less than the last included term.

If we can let $N \to \infty$ (in principle) then we have an *asymptotic expansion*.

If $f_n = a_n\varepsilon^n$, then we have an *asymptotic power series*; however we frequently need more general expansions involving terms like ε^α, $\left(\ln\frac{1}{\varepsilon}\right)^{-1}$, etc. We write these as

$$\sum_{n=0}^{N} a_n \delta_n(\varepsilon) \tag{2.1}$$

where the δ_n form an asymptotic sequence:

$$\frac{\delta_{n+1}}{\delta_n} \to 0 \qquad \text{as } \varepsilon \to 0 \ .$$

Note that sometimes we need to restrict to one sector of the complex ε plane to keep the δ_n single valued.

Often ε is real and positive. A useful set of asymptotic functions are then Hardy's logarithm–exponential functions obtained by a finite number of $+, -, *, /, \exp$ & $\log$ operations, with all intermediate quantities real.

This class has the property that it can be ordered, i.e. either $f(\varepsilon) = \mathrm{o}\,(g(\varepsilon))$, or $g(\varepsilon) = \mathrm{o}\,(f(\varepsilon))$ or $f(\varepsilon) = \mathrm{ord}\,(g(\varepsilon))$.

2.3 Uniqueness and manipulation

If f can be expanded asymptotically for a given asymptotic sequence, then the expansion is unique. For if the expansion exists it has the form

$$f(\varepsilon) \sim \sum_n a_n \delta_n(\varepsilon) \ ,$$

and

$$\begin{aligned} a_0 &= \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{\delta_0(\varepsilon)} \\ a_n &= \lim_{\varepsilon \to 0} \left\{ \frac{f(\varepsilon) - \sum_0^{n-1} a_m \delta_m}{\delta_n} \right\} \ . \end{aligned}$$

However, a single function can have different asymptotic expansions for different sequences:

$$\begin{aligned} \tan(\varepsilon) &\sim \varepsilon + \frac{1}{3}\varepsilon^3 + \frac{2}{15}\varepsilon^5 + \cdots \\ &\sim \sin\varepsilon + \frac{1}{2}(\sin\varepsilon)^3 + \frac{3}{8}(\sin\varepsilon)^5 + \cdots \\ &\sim \varepsilon \cosh\sqrt{\frac{2}{3}}\varepsilon + \frac{31}{270}\left(\varepsilon\cosh\sqrt{\frac{2}{3}}\varepsilon\right)^5 + \cdots \ . \end{aligned}$$

Part of the 'art' of obtaining an effective asymptotic solution is choosing the most appropriate asymptotic sequence.

Worse: two functions can have the same asymptotic expansion:

$$\begin{aligned} \exp\varepsilon &\sim \sum_0^\infty \frac{\varepsilon^n}{n!} \qquad &\text{as } \varepsilon \to 0 \\ \exp\varepsilon + \exp\left(-\frac{1}{\varepsilon}\right) &\sim \sum_0^\infty \frac{\varepsilon^n}{n!} \qquad &\text{as } \varepsilon \searrow 0 \ . \end{aligned}$$

Exercise: Does $f = x^2 + e^{-x^2(1-\sin x)}$ have an asymptotic expansion as $x \to \infty$?

- Asymptotic expansions can be added, multiplied and divided to produce asymptotic expansions for the sum, product and quotient (if necessary one may need to enlarge the asymptotic sequence).
- If appropriate, one can try to substitute an asymptotic expansion into another – but care is needed, e.g. if

$$f(z) = e^{z^2}, \qquad z(\varepsilon) = \frac{1}{\varepsilon} + \varepsilon$$

then

$$\begin{aligned} f(z(\varepsilon)) &= \exp\left[\frac{1}{\varepsilon^2} + 2 + \varepsilon^2\right] \\ &\sim e^{1/\varepsilon^2} e^2 \left\{1 + \varepsilon^2 + \frac{\varepsilon^4}{2} + \cdots\right\}, \end{aligned}$$

but if we just work to leading order

$$z \sim \frac{1}{\varepsilon}$$

$$f(z) \not\sim e^{1/\varepsilon^2}$$

↑missing e^2; the leading-order approximation in z is inadequate for the leading-order approximation in $f(z)$.

- Integration w.r.t. ε of asymptotic expansions is allowed term by term producing the correct result.
- Differentiation is not allowed in principle because $\mathcal{O}$ and o estimates do not survive differentiation. For instance:

(a)

$$f = e^{ix^2} = \mathcal{O}(1) \qquad \text{as } x \to \infty$$

$$\frac{df}{dx} = 2ixe^{ix^2} = \mathcal{O}(x) \qquad \text{as } x \to \infty$$

(b)

$$f = 1 + e^{-1/x^2} \sin\left(e^{1/x^2}\right) \sim 1 + \cdots \qquad \text{as } x \to 0$$

$$\frac{df}{dx} = \underbrace{-\frac{2}{x^3}\cos\left(e^{1/x^2}\right)} + \frac{2}{x^3}e^{-1/x^2}\sin\left(e^{1/x^2}\right)$$

No asymptotic expansion as $x \to 0$.

However: (i) If $f'(x)$ exists and is integrable, and $f(x) \sim \sum_{n=0}^{N} a_n x^n$ as $x \to 0$, then

$$f' \sim \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \text{as } x \to 0.$$

(ii) If $f(z)$ is analytic in $\theta_1 \leq \arg z \leq \theta_2$, $0 < |z| < R$ and

$$f \sim \sum_{n=0}^{\infty} a_n z^n \qquad \text{as } z \to 0 \ (\theta_1 \leq \arg z \leq \theta_2)$$

then

$$f' \sim \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad \text{as } z \to 0 \ (\theta_1 \leq \arg z \leq \theta_2).$$

(iii) There are lots more special cases. For instance

- Asymptotic expansions of solutions to differential equations. Suppose that y is the solution to

$$y'' + qy = 0 \tag{$*$}$$

where q has an asymptotic expansion as $x \to 0$.

Assume y has an asymptotic expansion as $x \to 0$;
then from $(*)$ y'' has an asymptotic expansion (multiplication OK)
thus y' has an asymptotic expansion (integration OK)
thus y has an asymptotic expansion (integration OK) .

Hence if y has an asymptotic expansion, the equation ensures that its differentials have asymptotic expansions (the *proof* that y has an asymptotic expansion is often tricky).

2.4 Parametric Expansions

For functions of 2 (or more) variables, e.g. $f(x, \varepsilon)$ (as might arise in solutions to pdes, etc.), we make the obvious generalisation of (2.1) to allow the a_n to be functions of x:

$$f(x, \varepsilon) \sim \sum_{n=0}^{N} a_n(x)\delta_n(\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{2.2}$$

If the approximation is asymptotic as $\varepsilon \to 0$ for each x, then it is called a Poincaré/classical asymptotic expansion.

The above pointwise asymptoticness may not be uniform in x, e.g. it may require $\varepsilon < x$ (restrictive as $x \to 0$). Such problems sometimes need a further extension:

$$f(x, \varepsilon) \sim \sum_{n} a_n(x, \varepsilon)\delta_n(\varepsilon) \tag{2.3}$$

$$\text{e.g. } a_n(x, \varepsilon) = b_n\left(\tfrac{x}{\varepsilon}\right).$$

Uniqueness extends to (2.2), but not to (2.3), etc.

2.5 Stokes Phenomena in the Complex Plane

If a power series is asymptotic to a single valued function as $z \to z_0$ in a 2π disk about $z = z_0$, and if z_0 is at worst an isolated singularity, then the series is a Taylor series. Thus if an asymptotic power series is divergent, it can only be valid in a sector of angle $< 2\pi$. Hence a single function may possess several asymptotic expansions, each restricted to a different sector: the *Stokes phenomenon.*

E.g.

$$\operatorname{erf} z \sim 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \qquad \text{as } z \to \infty,\ z \text{ real}\ .$$

One can extend this approximation into the complex plane as long as the contour for

$$\int_z^\infty e^{-t^2}\, dt$$

is kept in the sector where $e^{-z^2} \to 0$ as $z \to \infty$. Hence

$$\operatorname{erf} z \sim 1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \qquad \text{as } z \to \infty,\ |\arg z| < \pi/4.$$

But erf is an odd function, so

$$\operatorname{erf} z \sim -1 - \frac{e^{-z^2}}{\sqrt{\pi} z} \qquad \text{as } z \to \infty,\ 3\pi/4 < |\arg z| < 5\pi/4.$$

For $\pi/4 < \arg z < 3\pi/4$ use $\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\, dt$ to find that

$$\operatorname{erf} z \sim -\frac{e^{-z^2}}{\sqrt{\pi} z} \qquad \text{as } z \to \infty,\ \pi/4 < |\arg z| < 3\pi/4.$$

erf is analytic everywhere, but there is a non-analytic essential singularity at ∞.

Terminology.

- The line where a term that is subdominant in one sector becomes comparable with a term that is dominant in that sector, is called an *anti-Stokes line* by some (e.g. Stokes and modern trendies), and a *Stokes lines* by others (e.g. Bender & Orszag). In the case above the anti-Stokes lines are at

$$\arg z = (2n + 1)\pi/4\ .$$

- The lines where the leading behaviours of the two terms are most unequal are called *Stokes line* by some (e.g. Stokes and modern trendies), and a *anti-Stokes lines* by others (e.g. Bender & Orszag). In the case above the Stokes lines are at

$$\arg z = n\pi/2\ .$$

  Stokes lines are important since the coefficient of the subdominant term can jump at them.

3. Asymptotic Expansions of Integrals

3.1 Elementary Examples

Example 1. Rewrite an integral so that we can use a Taylor series. For instance:

$$I = \int_x^\infty e^{-t^4}\, dt \qquad \text{as} \qquad x \to 0\ .$$

Then

$$\begin{aligned} I &= \int_0^\infty e^{-t^4}\, dt - \int_0^x e^{-t^4}\, dt \\ &= \Gamma(5/4) - \int_0^x \sum_{n=0}^\infty \frac{(-t^4)^n}{n!}\, dt \\ &= \Gamma(5/4) - \sum_{n=0}^\infty \frac{(-)^n x^{4n+1}}{(4n+1)n!}\ . \end{aligned}$$

Example 2. Use a Taylor series even when we cannot! For instance:

$$I = \int_0^\infty \frac{e^{-t}}{x+t}\, dt \qquad \text{as} \qquad x \to \infty\ .$$

Then

$$\begin{aligned} I &= \frac{1}{x}\int_0^\infty e^{-t}\left(1+\frac{t}{x}\right)^{-1} dt \\ &= \frac{1}{x}\int_0^\infty e^{-t}\left(1-\frac{t}{x}+\frac{t^2}{x^2}-\frac{t^3}{x^3}+\dots\right) dt \\ &= \frac{1}{x}\left(1-\frac{1!}{x}+\frac{2!}{x^2}-\frac{3!}{x^3}+\dots\right)\ . \end{aligned}$$

↑naughty, since invalid for $t > x$.

↑Divergent

Estimate the remainder using

$$1-\frac{t}{x}+\frac{t^2}{x^2}+\dots+\left(-\frac{t}{x}\right)^{m-1} = \frac{1-\left(-\frac{t}{x}\right)^m}{1+\frac{t}{x}}\ .$$

Then

$$I = \frac{1}{x}\sum_{n=0}^{m-1}\int_0^\infty \left(-\frac{t}{x}\right)^{m-1} e^{-t}\, dt + R_m(x)\ ,$$

where

$$R_m(x) = \frac{1}{x^{m+1}}\int_0^\infty \frac{(-t)^m e^{-t}}{\left(1+\frac{t}{x}\right)}\, dt\ ,$$

and

$$|R_m(x)| \leq \frac{1}{|x^{m+1}|} \int_0^\infty t^m e^{-t}\, dt = \frac{m!}{x^{m+1}} \,.$$

Hence

$$I = \frac{1}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} + \ldots + \frac{m!}{(-x)^m} + \mathcal{O}\left(\frac{(m+1)!}{x^{m+1}}\right)\right)$$

Truncate the series when the remainder has the smallest bound, i.e. stop one before smallest term when $x \sim m$. The error when we truncate is then

$$|R_m| \sim \frac{x!}{x^{x+1}} \sim \frac{(2\pi)^{1/2} e^{-x}}{x^{3/2}} \,,$$

i.e. the error is exponentially small.

3.2 Integration by Parts

Integrals of the form $\int f(t)g(t)\, dt$ can be integrated by parts and **may** so yield asymptotic expansions; one automatically obtains the remainder.

Example 1. See §2.1 for $\mathrm{erf}(z)$.

Example 2. Consider the *exponential integral*

$$E_1(x) \equiv \int_x^\infty \frac{e^{-t}}{t}\, dt = e^{-x} \int_0^\infty \frac{e^{-t}\, dt}{x+t} \,.$$

Then integrating by parts

$$\begin{aligned} E_1(x) &= \left[-\frac{e^{-t}}{t}\right]_x^\infty - \int_x^\infty \frac{e^{-t}}{t^2}\, dt \\ &= \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} + \ldots + \frac{m!}{(-x)^m}\right) + r_m(x) \,, \end{aligned}$$

where

$$r_m(x) = (-)^{m+1}(m+1)! \int_x^\infty \frac{e^{-t}}{t^{m+2}}\, dt \,.$$

As in §~~2~~3.1, the size of this remainder ~~can be shown to be~~ is asymptotically smaller than the retained terms.

Example 3. The sine and cosine integrals.

$$\begin{aligned} -\mathrm{Ci}(x) - i\,\mathrm{si}(x) = -\mathrm{Ci}(x) + i\left(\frac{\pi}{2} - \mathrm{Si}(x)\right) &\equiv \int_x^\infty \frac{e^{it}\, dt}{t} \\ &= \frac{e^{ix}}{ix}\left(1 + \frac{1}{ix} + \frac{2!}{(ix)^2} + \ldots + \frac{m!}{(ix)^m}\right) + r_m(x) \,, \end{aligned}$$

where

$$r_m(x) = i(m+1)! \int_x^\infty \frac{e^{it}\,dt}{(it)^{m+2}} \; .$$

If we proceed as before

$$|r_m| \le (m+1)! \int_x^\infty \frac{dt}{t^{m+2}} = \frac{m!}{x^{m+1}} = \mathcal{O}(\text{last term}) \; ,$$

but this does not demonstrate asymptoticness. We seek an improved error estimate by integrating by parts:

$$r_m = \left[\frac{(m+1)!}{(it)^{m+2}} \right]_x^\infty + i(m+2)! \int_x^\infty \frac{e^{it}\,dt}{(it)^{m+3}} \; ,$$

hence

$$|r_m| \le \frac{(m+1)!}{x^{m+2}} + \frac{(m+1)!}{x^{m+2}} = \mathcal{O}\left(\frac{1}{x^{m+2}} \right) \; .$$

3.3 Integrals with Algebraic Parameter Dependence

Example 1. Consider the integral

$$I(\varepsilon) = \int_0^1 \frac{1}{(x+\varepsilon)^{\frac{1}{2}}}\,dx = 2\left(\sqrt{1+\varepsilon} - \sqrt{\varepsilon}\right).$$

The leading-order ($\varepsilon \to 0$) estimate is just

$$I(0) = \underbrace{\int_0^1 \frac{1}{x^{\frac{1}{2}}}\,dx}_{\substack{\text{global contribution from} \\ \text{all of integration range}}} = 2 \; .$$

In order to obtain an improved estimate, one cannot expand $(1+\varepsilon/x)^{-1/2}$ as

$$(1+\varepsilon/x)^{-1/2} = 1 - \varepsilon/2x + \dots \; ,$$

throughout the range; for instance when $0 \le x \ll \varepsilon$ the expansion is not convergent. Further, we note that when $x = \text{ord}(\varepsilon)$, the integrand is ord$\left(\varepsilon^{-1/2}\right) \Rightarrow$ contribution to the integral for this range of x with be ord$\left(\varepsilon^{1/2}\right)$.

To account for this correction, one could subtract the leading-order estimate exactly; then

$$I = 2 + \underbrace{\int_0^1 \left[\frac{1}{(x+\varepsilon)^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}} \right] dx}$$

$x = \text{ord}(\varepsilon)$, integrand $= \text{ord}\left(\varepsilon^{-1/2}\right)$, contribution to $\int = \text{ord}\left(\varepsilon^{1/2}\right)$
$x = \text{ord}(1)$, integrand $= \text{ord}\left(\varepsilon\right)$, contribution to $\int = \text{ord}\left(\varepsilon\right)$

The major contribution is from near $x = 0$, so put $x = \varepsilon\xi$ ($\xi = \text{ord}(1)$), then

$$
\begin{aligned}
I &= 2 + \varepsilon^{\frac{1}{2}} \int_0^{\frac{1}{\varepsilon}\approx\infty} \left[\frac{1}{(1+\xi)^{\frac{1}{2}}} - \frac{1}{\xi^{\frac{1}{2}}}\right] d\xi \\
&\approx 2 - 2\varepsilon^{\frac{1}{2}}
\end{aligned}
$$

Further corrections can be obtained by now subtracting out this contribution, but this method is tedious and difficult! There must be a better way.

Alternative 1: Solve a differential equation. Let

$$J(x) = \int_0^x \frac{1}{(q+\varepsilon)^{\frac{1}{2}}} dq \ .$$

Then we need to find $J(1)$. This can be done by solving the differential equation

$$\frac{dJ}{dx} = \frac{1}{(x+\varepsilon)^{\frac{1}{2}}}$$

subject to the initial condition $J(0) = 0$. We will discover how to do this in §4.

Alternative 2: Divide & Conquer. In this method we split the range of integration.

Split $[0, 1]$ at $x = \delta$ where $\varepsilon \ll \delta \ll 1$.

$$
\begin{aligned}
I &= \int_0^\delta \frac{dx}{(x+\varepsilon)^{\frac{1}{2}}} + \int_\delta^1 \frac{dx}{(x+\varepsilon)^{\frac{1}{2}}} \\
&= \varepsilon^{\frac{1}{2}} \int_0^{\delta/\varepsilon} \frac{d\xi}{(1+\xi)^{\frac{1}{2}}} + \int_\delta^1 \frac{1}{x^{\frac{1}{2}}}\left(1 - \frac{\varepsilon}{2x} + \frac{3\varepsilon^2}{8x^2} + \dots\right) dx \\
&= 2\varepsilon^{\frac{1}{2}}\left(\left(\frac{\delta}{\varepsilon}+1\right)^{\frac{1}{2}} - 1\right) + 2 - 2\delta^{\frac{1}{2}} + \varepsilon - \frac{\varepsilon}{\delta^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon^2, \frac{\varepsilon^2}{\delta^{\frac{3}{2}}}\right) \\
&= 2\delta^{\frac{1}{2}} + \frac{\varepsilon}{\delta^{\frac{1}{2}}} - 2\varepsilon^{\frac{1}{2}} + 2 - 2\delta^{\frac{1}{2}} + \varepsilon - \frac{\varepsilon}{\delta^{\frac{1}{2}}} + \mathcal{O}\left(\varepsilon^2, \frac{\varepsilon^2}{\delta^{\frac{3}{2}}}\right) \\
&= 2 - 2\varepsilon^{\frac{1}{2}} + \varepsilon + \mathcal{O}\left(\varepsilon^2, \frac{\varepsilon^2}{\delta^{\frac{3}{2}}}\right) \ .
\end{aligned}
$$

The error term is definitely small if $\varepsilon^{\frac{2}{3}} \ll \delta \ll 1$, but in fact since δ is arbitrary, all terms containing a δ must cancel.

To organise the algebra it is sometimes helpful to tie δ to ε, e.g.

$$\delta = K\varepsilon^{\frac{3}{4}} \ ,$$

and then the answer must be independent of K.

Example 2. Suppose that we wish to estimate the integral

$$I(m,\varepsilon) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1-m^2\cos^2\theta)^2\sin^2\theta+\varepsilon^2}\,d\theta \qquad 0<m<\infty\ ,$$

for $0<\varepsilon\ll 1$. It turns out that there are three cases to consider: $0<m<1$; $|m-1|\ll 1$; $m>1$.

(a) $\quad 0<m<1$

θ	integrand	contribution to $\int$
ord(1)	ord(1)	ord(1)
ord(ε)	ord(1)	ord(ε)

$\uparrow (1-m^2\cos^2\theta)^2\sin^2\theta\sim\varepsilon^2$

We will find the solution correct to $\mathcal{O}\left(\varepsilon^2\right)$; to this end let $0<\varepsilon\ll\delta\ll 1$. Then

$$I = \varepsilon\int_0^{\frac{\delta}{\varepsilon}} \frac{\sin^2(\varepsilon u)}{(1-m^2\cos^2(\varepsilon u))^2\sin^2(\varepsilon u)+\varepsilon^2}\,du + \int_\delta^{\frac{\pi}{2}} \frac{\sin^2\theta}{(1-m^2\cos^2\theta)^2\sin^2\theta+\varepsilon^2}\,d\theta$$

$$= \varepsilon\int_0^{\frac{\delta}{\varepsilon}} \frac{u^2\,du}{(1-m^2)^2u^2+1} + \int_\delta^{\frac{\pi}{2}} \frac{1}{(1-m^2\cos^2\theta)^2}\,d\theta + \mathcal{O}\left(\varepsilon^2\right)$$

$$= \varepsilon\left[\frac{(1-m^2)u-\tan^{-1}\left((1-m^2)u\right)}{(1-m^2)^3}\right]_0^{\frac{\delta}{\varepsilon}} + \frac{(2-m^2)\pi}{4(1-m^2)^{\frac{3}{2}}} - \int_0^\delta \frac{d\theta}{(1-m^2\cos^2\theta)^2} + \mathcal{O}\left(\varepsilon^2\right)$$

via a $\quad \tan\theta = t = (1-m^2)^{\frac{1}{2}}\tan\psi \quad$ substitution

$$= \frac{\delta}{(1-m^2)^2} - \frac{\varepsilon\pi}{2(1-m^2)^3} + \frac{(2-m^2)\pi}{4(1-m^2)^{\frac{3}{2}}} - \frac{\delta}{(1-m^2)^2} + \mathcal{O}\left(\varepsilon^2,\delta^2,\frac{\varepsilon^2}{\delta}\right)$$

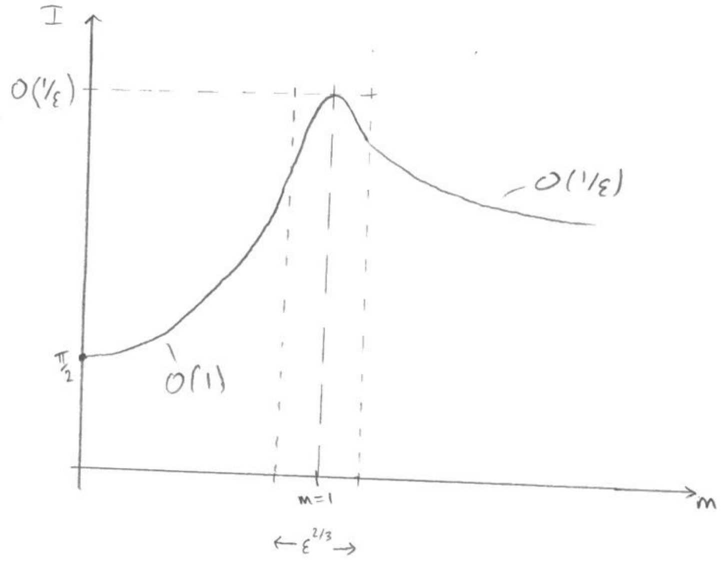
$$I = \underbrace{\frac{(2-m^2)\pi}{4(1-m^2)^{\frac{3}{2}}}}_{\text{global}} - \underbrace{\frac{\varepsilon\pi}{2(1-m^2)^3}}_{\text{local}} + \dots \tag{3.1}$$

Note that this is a non-uniform approximation as $m\to 1$. There is a loss of ordering of the series solution when

$$\frac{1}{(1-m^2)^{\frac{3}{2}}} \sim \frac{\varepsilon}{(1-m^2)^3}$$

i.e. when

$$(1-m^2)\sim\varepsilon^{\frac{2}{3}} \qquad \text{and} \qquad I\sim\frac{1}{\varepsilon}\,.$$



(b) This suggests that when $|m-1| \ll 1$, we should introduce a scaled parameter: viz.

$$m = 1 - \varepsilon^{\frac{2}{3}}\lambda \,. \tag{3.2a}$$

First let us examine the local contribution from near $\theta = 0$ (since on the basis of the estimates above it will be leading order). Put $\theta = \varepsilon^{\beta} u$, then

$$\underbrace{\left(1 - m^2\cos^2\theta\right)^2}_{=1-m^2+m^2\sin^2\theta} \sin^2\theta + \varepsilon^2 = \left(\varepsilon^{2\beta}u^2 + 2\varepsilon^{\frac{2}{3}}\lambda\right)^2 \varepsilon^{2\beta}u^2 + \varepsilon^2 + \ldots \tag{3.2b}$$

All leading order terms balance if $\beta = \frac{1}{3}$; this is referred to as a *distinguished scaling*. As a first guess, let us assume that this is the scaling in θ to consider. Then

$\theta = \text{ord}(\varepsilon^{\frac{1}{3}})$; integrand $= \text{ord}\left(\varepsilon^{\frac{2}{3}}/\varepsilon^2\right)$; contribution to $\int = \text{ord}\,(1/\varepsilon)$

$\theta = \text{ord}(1)$; integrand $= \text{ord}\,(1)$; contribution to $\int = \text{ord}\,(1)$

The 'local' contribution dominates. Hence introduce $\varepsilon^{\frac{1}{3}} \ll \delta \ll 1$, and split the integral:

$$I = \int_0^{\delta} \ldots \, d\theta + \int_{\delta}^{\frac{\pi}{2}} \ldots \, d\theta \quad = \ldots = \int_0^{\delta\varepsilon^{-1/3}} \frac{\varepsilon^{2/3} u^2 \varepsilon^{1/3}\, du}{\varepsilon^2\left((u^2+2\lambda)^2 u^2+1\right)}$$

$$\sim \frac{1}{\varepsilon}\int_0^{\delta\varepsilon^{-\frac{1}{3}}} \frac{u^2\, du}{\left(u^2+2\lambda\right)^2 u^2 + 1} \sim \frac{1}{\varepsilon} f(\lambda)$$

where

$$= \frac{1}{\varepsilon}\left[\int_0^{\infty} \frac{u^2 du}{(u^2+2\lambda)^2 u^2+1} + O\left((\delta\varepsilon^{-1/3})^3\right)\right] \sim \frac{1}{\varepsilon} f(\lambda)$$

$$f(\lambda) = \int_0^\infty \frac{u^2\,du}{(u^2+2\lambda)^2\,u^2+1}\ .$$

Hence for a given λ (and m), we have a leading order asymptotic estimate. However, we should check that as $\lambda \to \infty$, we obtain the same estimate as in (a). In particular, when $\lambda \gg 1$

$$u = \text{ord}(1)\ \ ,\ \text{integrand} = \text{ord}\left(1/\lambda^2\right),\ \text{contribution to} \int = \text{ord}\left(1/\lambda^2\right)$$
$$u = \text{ord}(\lambda^{\frac{1}{2}}),\ \text{integrand} = \text{ord}\left(1/\lambda^2\right),\ \text{contribution to} \int = \text{ord}\left(1/\lambda^{\frac{3}{2}}\right)$$

This suggests that the largest contribution will come from where $v = \lambda^{-\frac{1}{2}}u = \text{ord}(1)$. Hence estimate f in this range:

$$f(\lambda) \sim \frac{1}{\lambda^{\frac{3}{2}}}\int_0^\infty \frac{dv}{(2+v^2)^2} = \frac{\pi}{4\,(2\lambda)^{\frac{3}{2}}}\ ,$$

and

$$I \sim \frac{\pi}{4\varepsilon\,(2\lambda)^{\frac{3}{2}}} \sim \frac{\pi}{4\,(1-m^2)^{\frac{3}{2}}} \tag{3.3}$$

$\downarrow$agrees with (3.1) for $m \approx 1$

We might also be interested in the other limit, i.e. $\lambda \to -\infty$. This estimate is a little more tricky, since $\left(u^2+2\lambda\right)$ can now have a zero (when $|\lambda| \gg 1$, this term normally dominates the denominator). First we test for a significant contribution from near this zero by introducing a scaled coordinate:

$$u = (-2\lambda)^{\frac{1}{2}} + (-\lambda)^{\gamma} w\ .$$

Then

$$1 + u^2\left(u^2+2\lambda\right)^2 \sim 1 + (-2\lambda)\left(2\,(-2\lambda)^{\frac{1}{2}}\,(-\lambda)^{\gamma} w + \ldots\right)^2$$
$$\sim 1 + 16\lambda^2(-\lambda)^{2\gamma} w^2 + \ldots\ .$$

There is a distinguished scaling for the choice $\gamma = -1$, in which case the contribution to the integral from near the zero can be estimated as follows:

$$u = (-2\lambda)^{\frac{1}{2}} + \text{ord}\left(1/\left|\lambda\right|\right);\ \text{integrand} = \text{ord}\left(\left|\lambda\right|/1\right);\ \text{contribution to} \int = \text{ord}(1)\ .$$

This is a much larger contribution than we found for $\lambda \gg 1$. In order to estimate the contribution set

$$u = (-2\lambda)^{\frac{1}{2}} + \frac{w}{(-\lambda)} \tag{3.4}$$

then

$$f(\lambda) = \int_{-2^{1/2}(-\lambda)^{3/2}\approx-\infty}^{\infty} \frac{(-2\lambda + \dots)\,dw}{(-\lambda)\,[1+16w^2\dots]} \sim \frac{\pi}{2} \,.$$

Hence as $\lambda \to -\infty$, the value of the integral tends to a large constant, viz.

$$I \sim \frac{\pi}{2\varepsilon} \,. \qquad (3.5)$$

(c) Finally consider the case when $m > 1$.

The limit $\lambda \to -\infty$ (i.e. $0 < (m-1) \ll 1$) suggests that the main contribution will be local, and will come from the region close to the point where

$$m^2 \cos^2\theta = 1 \,.$$

Define

$$\theta_m = \cos^{-1}\left(\frac{1}{m}\right) \qquad \left(0 < \theta_m < \frac{\pi}{2}\right) \,.$$

In order to deduce the coordinate scaling that is appropriate close to θ_m, we note from (3.2) & (3.4) that the 'inner' scaling for $0 < m - 1 \ll 1$ can be written in the form

$$\theta = \varepsilon^{\frac{1}{3}}\left((-2\lambda)^{\frac{1}{2}} + \frac{w}{(-\lambda)}\right) = (2(m-1))^{\frac{1}{2}} + \frac{\varepsilon w}{(m-1)} \sim \theta_m + \frac{2\varepsilon w}{\theta_m^2} \,.$$

This suggests that for $(m-1) = \mathcal{O}(1)$ we should try the scaling

$$\theta = \theta_m + \varepsilon t \,,$$

in which case

$$\left(1 - m^2\cos^2\theta\right)^2 \sin^2\theta + \varepsilon^2 \sim 4\varepsilon^2 m^2 \sin^4\theta_m t^2 + \dots + \varepsilon^2$$

and

$$\begin{aligned} I &\sim \int_{-\frac{1}{\varepsilon}\theta_m\approx-\infty}^{\frac{1}{\varepsilon}\left(\frac{\pi}{2}-\theta_m\right)\approx+\infty} \frac{\varepsilon\sin^2(\theta_m + \varepsilon t)\,dt}{\varepsilon^2\left(4m^2t^2\sin^4\theta_m + 1\right) + \dots} \\ &\sim \frac{1}{\varepsilon}\cdot\frac{\pi}{2m} \end{aligned} \qquad (3.6)$$

We note that (3.6) agrees with (3.5) in the limit $m \to 1$.

3.4 Logarithms

Consider

$$\int_0^a f(x,\varepsilon)\,dx \quad \text{with} \quad f(x,\varepsilon) = \begin{cases} \text{ord}\,(\varepsilon^{-\alpha}) & x = \text{ord}(\varepsilon) \\ x^{-\alpha} & \varepsilon \ll x \ll 1 \\ \text{ord}(1) & x = \text{ord}(1). \end{cases}$$

e.g.

$$f = \frac{1}{(x+\varepsilon)^\alpha}\frac{1}{1+x}\,.$$

There are 3 possibilities for the leading-order contribution depending on the value of α:

(i) $\alpha < 1$. Dominant contribution from $x = \text{ord}(1)$, e.g.

$$\int_0^\infty \frac{dx}{(x+\varepsilon)^{\frac{1}{2}}(1+x)} \sim \int_0^\infty \frac{dx}{x^{\frac{1}{2}}(1+x)}\,.$$

(ii) $\alpha > 1$. Dominant contribution from $x = \text{ord}(\varepsilon)$, e.g.

$$\int_0^\infty \frac{dx}{(x+\varepsilon)^{\frac{3}{2}}(1+x)} \sim \int_0^\infty \frac{d\xi}{\varepsilon^{\frac{1}{2}}(1+\xi)^{\frac{3}{2}}} \qquad (x = \varepsilon\xi)\,.$$

(iii) $\alpha = 1$. Dominant contribution *not* from $x = \text{ord}(\varepsilon)$ or $x = \text{ord}(1)$ but from the *intermediate* region between. Easiest to see by splitting the integration region:

$$\begin{aligned}
\int_0^\infty \frac{dx}{(x+\varepsilon)(1+x)} &= \int_0^{\frac{\delta}{\varepsilon}} \frac{d\xi}{(1+\xi)(1+\varepsilon\xi)} + \int_\delta^\infty \frac{dx}{(x+\varepsilon)(1+x)} \\
&= \Big[\log(1+\xi) - \varepsilon\,[\xi - \log(1+\xi)] + \ldots\Big]_0^{\frac{\delta}{\varepsilon}} \\
&\quad + \left[\log\left(\frac{x}{x+1}\right) + \frac{\varepsilon}{x} - \varepsilon\log\left(\frac{x+1}{x}\right) + \ldots\right]_\delta^\infty \\
&\sim (1+\varepsilon)\,(\log\delta - \log\varepsilon) + \frac{\varepsilon}{\delta} + \ldots \\
&\quad - \log\delta - \frac{\varepsilon}{\delta} - \varepsilon\log\delta + \ldots \\
&\sim (1+\varepsilon)\,\log\left(\frac{1}{\varepsilon}\right) + \ldots\,.
\end{aligned}$$

$\uparrow$

'fortunate' ord(1) cancellation

3.5 Integrals with Exponential Power Dependence

General case: limit as $\lambda \to \infty$ of integrals of type

$$I(\lambda) = \underbrace{\int_a^b}_{\text{paths in } \mathbb{C}} e^{\lambda\phi(z)} \underbrace{f(z,\lambda)}_{\substack{\uparrow \\ \text{weak algebraic} \\ \text{dependence on } \lambda}} \, dz \quad .$$

3.5.1 Watson's Lemma

Assume a, b, λ, ϕ, f, and the path of the integral, are real.

There are then different cases to consider depending on whether the maximum of ϕ is at an end point (Watson's Lemma), or in the interior of the integration range (steepest descents).

In this section we assume the maximum is at an end point; wlog $z = a$. Write

$$x = \phi(a) - \phi(z) \quad , \quad F(x) = -\frac{f(z)}{\phi'(z)} e^{\lambda\phi(a)} \quad , \quad c = \phi(a) - \phi(b) > 0 \ ,$$

then

$$I(\lambda) = \int_0^c e^{-\lambda x} F(x) \, dx \ .$$

Assume that F is analytic in some sector S of the complex plane, and that as $x \to 0$,

$$F(x) \sim \sum_{k=0}^{N} a_k x^{\alpha_k} \qquad -1 < \alpha_0 < \alpha_1 < \dots \ .$$

Also assume that c is in S, and that F is bounded in S; let $F_{max} = \max_{z \in S} |F(z)|$. Then

$$\int_0^c e^{-\lambda x} F(x) \, dx \sim \sum_0^N a_n \frac{\Gamma(\alpha_n + 1)}{\lambda^{\alpha_n + 1}} \ .$$

Proof. For a given $\varepsilon > 0$, $\exists\ \delta(\varepsilon)$ s.t.

$$\left| F(x) - \sum_{k=0}^{N} a_k x^{\alpha_k} \right| < \varepsilon \, |x^{\alpha_N}| \qquad \forall \ x \text{ in } S \text{ with } |x| < \delta.$$

Split range of integral at $\delta(\varepsilon)$. Then

$$\int_\delta^c e^{-\lambda x} F(x) \, dx < F_{max} \, e^{-\lambda\delta} \ ,$$

and

$$
\begin{aligned}
&\left|\int_0^\delta e^{-\lambda x} F(x)\,dx - \sum_{k=0}^{N} a_k \lambda^{-\alpha_k - 1}\Gamma(\alpha_k+1)\right| \\
&\quad = \left|\int_0^\delta e^{-\lambda x} F(x)\,dx - \sum_{k=0}^{N} a_k \int_0^\infty x^{\alpha_k} e^{-\lambda x}\,dx\right| \\
&\quad \le \left|\varepsilon \int_0^\delta e^{-\lambda x}\,|x|^{\alpha_N}\,dx\right| + \left|\sum_{k=0}^{N} a_k \int_\delta^\infty x^{\alpha_k} e^{-\lambda x}\,dx\right| \\
&\quad \le \varepsilon \left|\frac{\Gamma(\alpha_N+1)}{\lambda^{\alpha_N+1}}\right| + \left|e^{-(\lambda-1)\delta}\right| \int_\delta^\infty \sum_{k=0}^{N} |a_k x^{\alpha_k}|\,|e^{-x}|\,dx\ .
\end{aligned}
$$

Hence as $\lambda \to \infty$,

$$
\text{error} = \mathcal{O}\left(\frac{\varepsilon}{|\lambda^{\alpha_N+1}|}, \exp\right)\ .
$$

This proves the result since ε can be arbitrarily small (and λ arbitrarily large). This proof can be extended to the case when λ is complex, by deforming the integration contour so that $x\lambda$ is real.

How to obtain a practical answer!

The introduction of the coordinate x is not always simple. If all that is required is a few leading-order terms, then it is possible to proceed as follows (for $\phi'(a) < 0$):

$$
\begin{aligned}
I &= \int_a^b e^{\lambda\phi(x)} f(x)\,dx \\
&= \int_0^{\lambda(b-a)} f\left(a+\frac{t}{\lambda}\right) \exp\left(\lambda\phi\left(a+\frac{t}{\lambda}\right)\right)\frac{dt}{\lambda} \qquad \left(x = a+\frac{t}{\lambda}\right) \\
&= \int_0^{\lambda(b-a)} \left[f(a) + \frac{t}{\lambda} f'(a) + \dots\right] \exp\left(\lambda\phi(a) + t\phi'(a) + \frac{1}{2}\frac{t^2}{\lambda}\phi''(a) + \dots\right)\frac{dt}{\lambda} \\
&= \int_0^{\lambda(b-a)\approx\infty} \left[f(a) + \frac{t}{\lambda} f'(a) + \dots\right] e^{\lambda\phi(a)} e^{t\phi'(a)} \left[1 + \frac{t^2}{2\lambda}\phi''(a) + \dots\right]\frac{dt}{\lambda} \\
&\approx \frac{e^{\lambda\phi(a)}}{\lambda}\left[-\frac{f(a)}{\phi'(a)} + \frac{1}{\lambda}\left(\frac{f'(a)}{[\phi'(a)]^2} - \frac{\phi''(a) f(a)}{[\phi'(a)]^3}\right) + \mathcal{O}\left(\frac{1}{\lambda^2}\right)\right] + \exp\ .
\end{aligned}
$$

3.5.2 Intermediate Maximum (Laplace's Method)

$$I = \int_a^b e^{\lambda\phi(x)} f(x)\, dx \; .$$

Suppose that (a) $\max\limits_{x\in[a,b]} \phi = \phi(c)$;

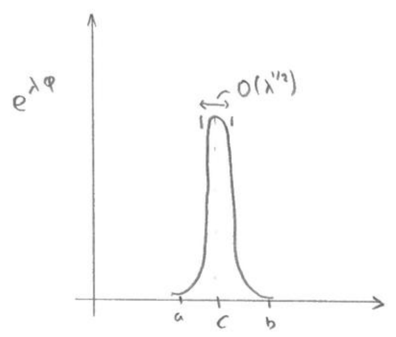
(b) $\phi'(c) = 0, \quad \phi''(c) < 0$;

(c) $x = c + \dfrac{t}{\lambda^{\frac{1}{2}}}$.

The scaling (c) can be justified by expanding ϕ close to the maximum at $x = c$:

$$\phi(x) \sim \phi(c) + (x-c)\phi'(c) + \frac{1}{2}(x-c)^2\phi''(c) + \frac{1}{6}(x-c)^3\phi'''(c) + \dots \; .$$

We introduce a rescaling such that $\lambda(x-c)^2 = \mathrm{ord}(1)$, so that the decay of the exponential occurs over an ord(1) scaled distance.



Then

$$\begin{aligned}
I &= \frac{1}{\lambda^{\frac{1}{2}}} \int_{(a-c)\lambda^{\frac{1}{2}}}^{(b-c)\lambda^{\frac{1}{2}}} f\left(c + \frac{t}{\lambda^{\frac{1}{2}}}\right) \exp\left(\lambda\phi\left(c + \frac{t}{\lambda^{\frac{1}{2}}}\right)\right) dt \\
&= \frac{1}{\lambda^{\frac{1}{2}}} \int_{\lambda^{\frac{1}{2}}(a-c)}^{\lambda^{\frac{1}{2}}(b-c)} \left(f(c) + \frac{t}{\lambda^{\frac{1}{2}}} f'(c) + \dots\right) \exp\left(\lambda\phi(c) + \frac{t^2}{2}\phi''(c) + \frac{t^3}{6\lambda^{\frac{1}{2}}}\phi'''(c) + \dots\right) dt \\
&\approx \frac{1}{\lambda^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(c) e^{\lambda\phi(c)} e^{\frac{1}{2}t^2\phi''(c)} \left(1 + \mathcal{O}\left(\lambda^{-\frac{1}{2}}\right)\right) dt + \exp \\
&\approx \left(\frac{2\pi}{-\lambda\phi''(c)}\right)^{\frac{1}{2}} f(c) e^{\lambda\phi(c)} + \dots \; .
\end{aligned}$$

Example: Stirling's Formula

$$\Gamma(\lambda) = \int_0^\infty e^{-x} x^{\lambda-1}\, dx \qquad \lambda \to \infty$$

$$f(x) = \frac{e^{-x}}{x}, \quad \phi(x) = \log x$$
$$\max \phi(x) = \infty \quad \text{for } 0 < x < \infty.$$

Method seems invalid! Write

$$\Gamma(\lambda) = \int_0^\infty \frac{1}{x} \exp\big(\underbrace{-x + \lambda \log x}\big)\, dx$$

$$\phi(x) = \lambda \log x - x$$
$$\phi'(x) = \tfrac{\lambda}{x} - 1\,, \quad \phi' = 0 \text{ at } x = \lambda\,.$$

Let $x = \lambda s$.

$$\Gamma(\lambda) = \int_0^\infty \frac{ds}{s} \exp\big(-\lambda s + \lambda \log \lambda + \lambda \log s\big)$$
$$= \lambda^\lambda \int_0^\infty \frac{ds}{s} \exp\big(-\lambda\,(s - \log s)\big)$$

$$f(s) = \tfrac{1}{s} \quad , \quad \phi(s) = \log s - s$$
$$\phi' = \tfrac{1}{s} - 1 \quad , \quad c = 1$$
$$\phi'' = -\tfrac{1}{s^2} \quad , \quad \phi''(c) = -1$$

$$\Gamma(\lambda) \sim \left(\frac{2\pi}{\lambda}\right)^{\frac{1}{2}} \lambda^\lambda e^{-\lambda} + \dots\,.$$

3.5.3 Stationary Phase

Let $\phi(x) = i\psi(x)$, with $\psi(x)$ real. Consider

$$I(x) = \int_a^b f(x) e^{i\lambda\psi(x)}\, dx$$

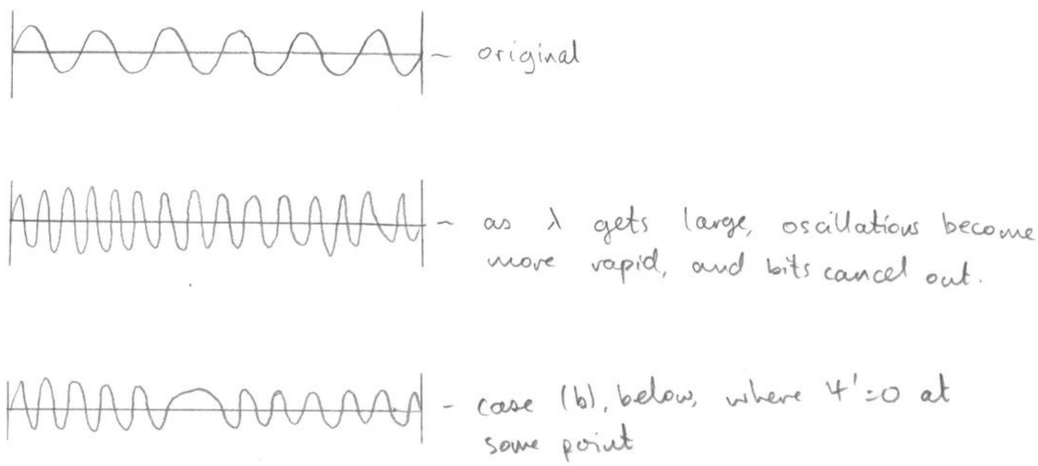
Generalised Fourier Integral.

(a) $\psi' \neq 0$ on $[a,b]$ In this case integrate by parts:

$$I(x) = \left[\frac{f}{i\lambda\psi'} e^{i\lambda\psi}\right]_a^b - \int_a^b \left(\frac{f}{i\lambda\psi'}\right)' e^{i\lambda\psi}\, dx$$
$$= \frac{i}{\lambda}\left[\frac{f(a)}{\psi'(a)} e^{i\lambda\psi(a)} - \frac{f(b)}{\psi'(b)} e^{i\lambda\psi(b)}\right] + \underbrace{\frac{i}{\lambda}\int_a^b \left(\frac{f}{\psi'}\right)' e^{i\lambda\psi}\, dx}_{J}\,.$$

Riemann-Lebesgue Lemma

$$\int_a^b f(x)e^{i\lambda x}\,dx \to 0 \qquad \text{as} \qquad \lambda \to \infty \quad \text{if} \qquad \int_a^b |f(x)|\,dx \qquad \text{exists} .$$



Generalised Riemann-Lebesgue Lemma

$$I(x) \to 0 \qquad \text{as} \qquad \lambda \to \infty ,$$

if (a) $|f(x)|$ is integrable,
(b) $\psi(x)$ is continuously differentiable, and
(c) $\psi(x)$ is not constant on any subinterval.

J satisfies the conditions for the generalised Riemann-Lebesgue Lemma if $f(x)/\psi'(x)$ is smooth; hence

$$I(x) \sim \frac{i}{\lambda}\left[\frac{f(a)}{\psi'(a)}e^{i\lambda\psi(a)} - \frac{f(b)}{\psi'(b)}e^{i\lambda\psi(b)}\right] .$$

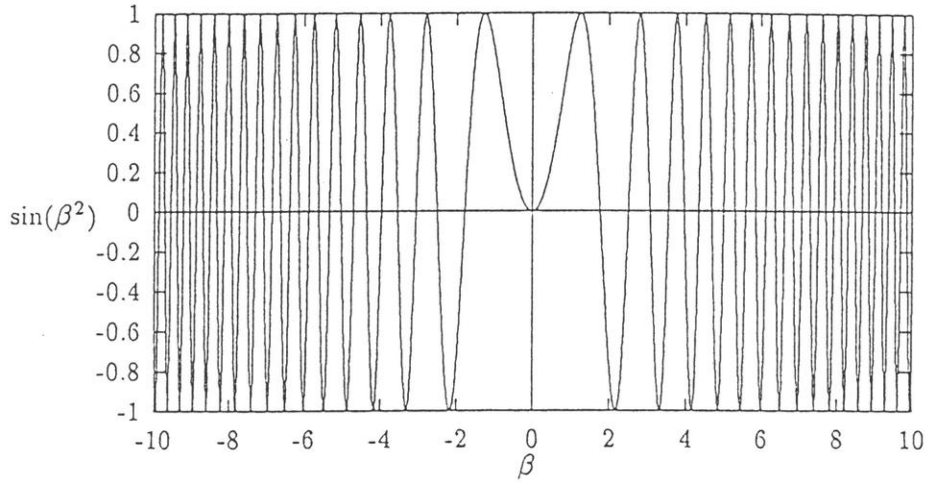
If we can continue to integrate by parts, we can obtain higher order terms.

(b) $\psi' = 0$ on $[a, b]$ Assume a unique zero at $x = c$:

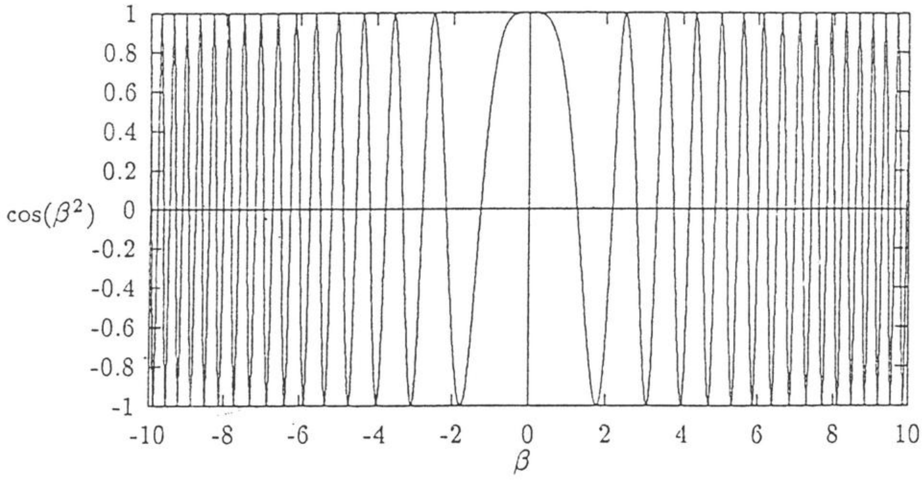
$$\psi'(c) = 0, \qquad \psi''(c) \neq 0 .$$

Cancellation is much reduced near $x = c$. Put

$$x = c + \frac{y}{\lambda^{\frac{1}{2}}} .$$



"Stationary phases".



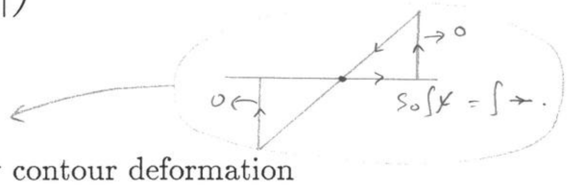
$$I(x) = \int_{(a-c)\lambda^{\frac{1}{2}}}^{(b-c)\lambda^{\frac{1}{2}}} f\left(c + \frac{y}{\lambda^{\frac{1}{2}}}\right) \exp\left(i\lambda\psi\left(c + \frac{y}{\lambda^{\frac{1}{2}}}\right)\right) \frac{dy}{\lambda^{\frac{1}{2}}}$$

$$= \int_{(a-c)\lambda^{\frac{1}{2}} \approx -\infty}^{(b-c)\lambda^{\frac{1}{2}} \approx \infty} \left[f(c) + \frac{y}{\lambda^{\frac{1}{2}}} f'(c) + \dots\right] \exp\left[i\lambda\psi(c) + \frac{iy^2}{2}\psi''(c) + \frac{iy^3}{6\lambda^{\frac{1}{2}}}\psi'''(c) + \dots\right] \frac{dy}{\lambda^{\frac{1}{2}}}$$

$$= \frac{f(c)}{\lambda^{\frac{1}{2}}} e^{i\lambda\psi(c)} \int_{-\infty}^{\infty} \exp\left(\frac{i\psi''(c)y^2}{2}\right) dy \left(1 + \mathcal{O}\left(\lambda^{-\frac{1}{2}}\right)\right)$$

substitute $y = \left(\frac{2}{|\psi''(c)|}\right)^{\frac{1}{2}} t$, $s = \operatorname{sgn}[\psi''(c)]$

$$\sim \left(\frac{2}{\lambda|\psi''(c)|}\right)^{\frac{1}{2}} f(c) e^{i\lambda\psi(c)} \underbrace{\int_{-\infty}^{\infty} e^{ist^2}\, dt}_{\pi^{\frac{1}{2}} e^{is\pi/4} \text{ by contour deformation}}$$



$$\sim \left(\frac{2\pi}{\lambda|\psi''(c)|}\right)^{\frac{1}{2}} f(c) \exp\left(i\lambda\psi(c) + i\operatorname{sgn}[\psi''(c)]\frac{\pi}{4}\right) .$$

↑ leading order; next order approximation can come from end points, etc.

Note that one can tighten up the 'proof' by changing variables at the start:

$$\psi(x) = \psi(c) + \frac{1}{2}\psi''(c)Y^2 .$$

3.5.4 Steepest Descents

This is a method for estimating integrals of the form

$$I = \int_C f(z) e^{\lambda\phi(z)}\, dz ,$$

where C is a contour in the complex z-plane, f and ϕ are analytic functions of z, and λ is complex.

(a) The idea is to deform the contour and then use Laplace's method **or** Watson's Lemma.

Let $\phi = u + iv$.

Then (i) $u_x = v_y$, $u_y = -v_x$ Cauchy Riemann
(ii) $\nabla^2 u = 0 = \nabla^2 v$.

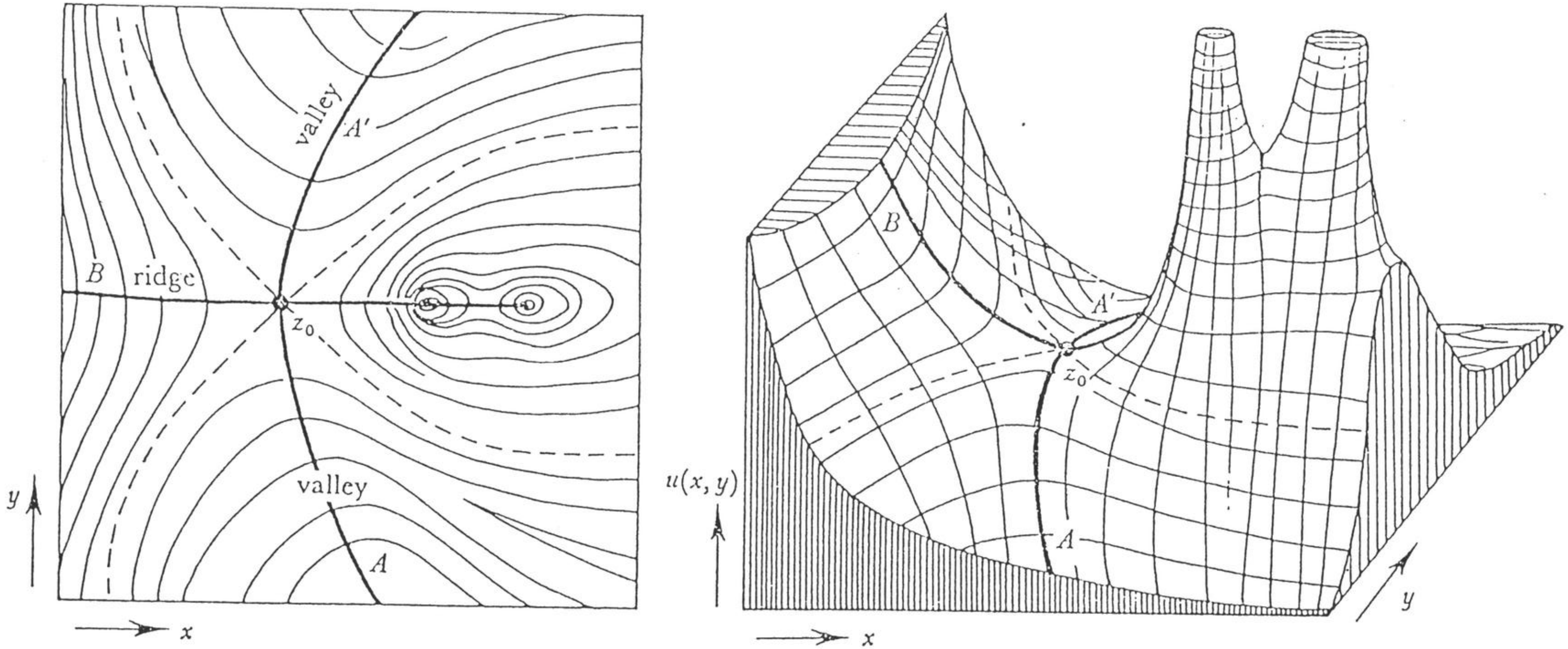


Figure 3–13 Topography of the surface $u = \mathrm{Re}\, f(z)$ near the saddle point z_0, for a typical function $f(z)$. The heavy solid curves follow the centers of the ridges and valleys from the saddle point, and the dashed curves follow level contours, $u = u(x_0, y_0) =$ constant. The curve AA' is the path of steepest descent

(b) From stationary phase we have seen that rapid oscillations can cause cancellation. This makes estimation of the integral difficult and in particular means that the dominant contribution to I may not come from the part of C where $\Re(\lambda\phi(z))$ is largest. We eliminate such oscillations by choosing a contour with

$$\Im(\phi) = v = \text{constant} \ .$$

The Cauchy-Riemann equations imply that

$$\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = 0 \ .$$

Thus the $v =$ constant contours are $\|$ to $\boldsymbol{\nabla} u$. It follows that the $v =$ constant contours are paths of steepest ascent/descent of u. [Note that we need the steepest descent path to obtain 'all' terms of the series].

(c) The major contribution to the integral I then comes from close to the 'highest' point (w.r.t. u) on the contour. *Those of you who already know about the method of steepest descents need to remember this — do not just go for the nearest turning point!*

(d) For the case of an interior turning point (when it is appropriate to apply Laplace's method) we are interested in points where

$$\Re(\phi'(z)) = u'(z) = 0 \ .$$

Since $\nabla^2 u = 0$, these points can only be saddles:

$$\Delta = u_{xx}u_{yy} - (u_{xy})^2 = -(u_{xx})^2 - (u_{xy})^2 \leq 0 \ .$$

Example. Find an asymptotic expansion for

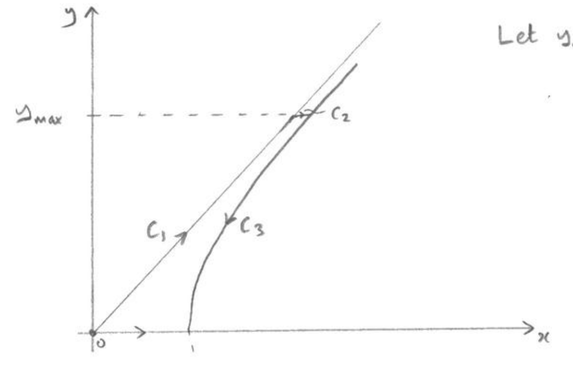
$$I = \int_0^1 e^{i\lambda z^2}\, dz \qquad \text{as} \qquad \lambda \to \infty \ .$$

The leading-order approximation can be obtained by a stationary phase calculation near $z = 0$. To obtain a full expansion try to use steepest descent contours. From above

$$\phi = iz^2 = i(x^2 - y^2) - 2xy \ ,$$
$$u = -2xy, \quad v = x^2 - y^2 \ .$$

Hence

steepest contours through $z = 0$:	$v = 0$,	$x = \pm y$,	$u = \mp 2y^2$
S.D. contour through $z = 0$:		$x = +y$,	$u = -2y^2$
steepest contours through $z = 1$:	$v = 1$,	$x = \pm\sqrt{1+y^2}$,	$u = \mp 2y\sqrt{1+y^2}$
S.D. contour through $z = 1$:		$x = \sqrt{1+y^2}$,	$u = -2y\sqrt{1+y^2}$



Let $y_{max} \to 0$. Then $\int_{C_2} \to 0$

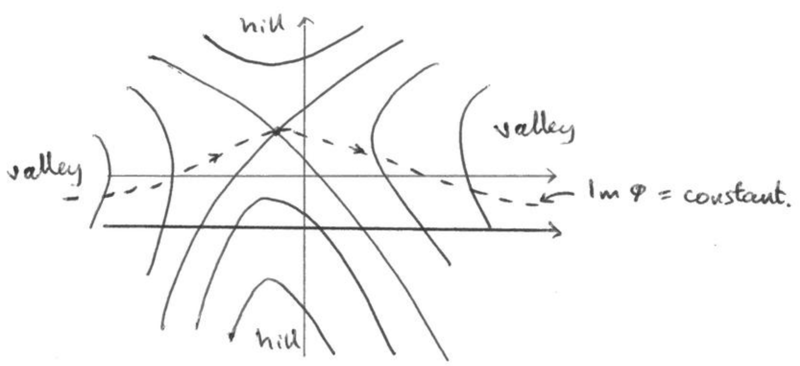
The contribution from C_2 vanishes as $y_{\max} \to \infty$; thus

$$
\begin{aligned}
I &= \int_{C_1} e^{i\lambda z^2}\, dz + \int_{C_3} e^{i\lambda z^2}\, dz \\
&= (1+i)\int_0^\infty e^{-2\lambda y^2}\, dy - \frac{i}{2}\int_0^\infty \frac{e^{i\lambda} e^{-\lambda s}\, ds}{(1+is)^{\frac{1}{2}}}
\end{aligned}
$$

substitute $iz^2 = i - s$

$$
\begin{aligned}
&= \left(\frac{\pi}{4\lambda}\right)^{\frac{1}{2}} e^{\frac{i\pi}{4}} - \frac{ie^{i\lambda}}{2}\int_0^\infty ds\, e^{-\lambda s} \sum_{n=0}^{\infty} \frac{(-is)^n \Gamma\left(n+\frac{1}{2}\right)}{n!\,\Gamma\left(\frac{1}{2}\right)} \\
&\sim \left(\frac{\pi}{4\lambda}\right)^{\frac{1}{2}} e^{\frac{i\pi}{4}} + \frac{e^{i\lambda}}{2}\sum_{n=0}^{\infty}\left(\frac{-i}{\lambda}\right)^{n+1} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\,.
\end{aligned}
$$

The Local Contribution from a Saddle



Close to the saddle at $z = z_s$

$$
\phi(z) \sim \phi(z_s) + (z - z_s)\phi'(z_s) + \frac{1}{2}(z-z_s)^2\phi''(z_s) + \frac{1}{6}(z-z_s)^3\phi'''(z_s) + \dots\,.
$$

As in Laplace's method introduce a rescaling such that $\lambda(z-z_s)^2 = \text{ord}(1)$;

$$z = z_s + \frac{w}{\lambda^{\frac{1}{2}}}, \qquad \lambda\phi(z) \sim \lambda\phi(z_s) + \frac{1}{2}\phi''(z_s)w^2 + \mathcal{O}\left(\lambda^{-\frac{1}{2}}\right)$$

$$\int_C f(z)e^{\lambda\phi(z)}\,dz = \int_C f(z_s)e^{\lambda\phi(z_s)+\frac{1}{2}\phi''(z_s)w^2}\left(1+\mathcal{O}\left(\lambda^{-\frac{1}{2}}\right)\right)\frac{dw}{\lambda^{\frac{1}{2}}}$$

$$\sim f(z_s)e^{\lambda\phi(z_s)}\left(\frac{-2\pi}{\lambda\phi''(z_s)}\right)^{\frac{1}{2}} + \dots.$$

by evaluating the integral using Laplace's method on S.D. path.

The Airy Function and Stokes Phenomenon

$$\text{Ai}(\lambda) = \frac{1}{2\pi i}\int_C e^{\lambda z - \frac{1}{3}z^3}\,dz$$

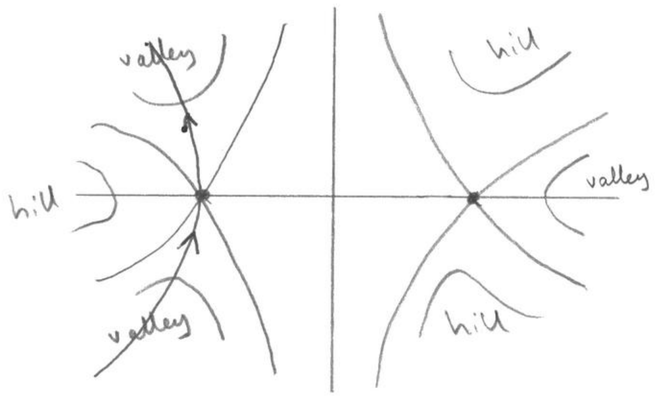
C starts from ∞ with $\arg(z) = -2\pi/3$

ends at ∞ with $\arg(z) = +2\pi/3$.

$$\lambda\phi = \Phi = \lambda z - \frac{1}{3}z^3$$

$$\Phi' = \lambda - z^2 \qquad \text{— saddles at } z_s = \pm\lambda^{\frac{1}{2}}$$

$$\Phi(z_s) = \pm\frac{2}{3}\lambda^{\frac{3}{2}}\ .$$

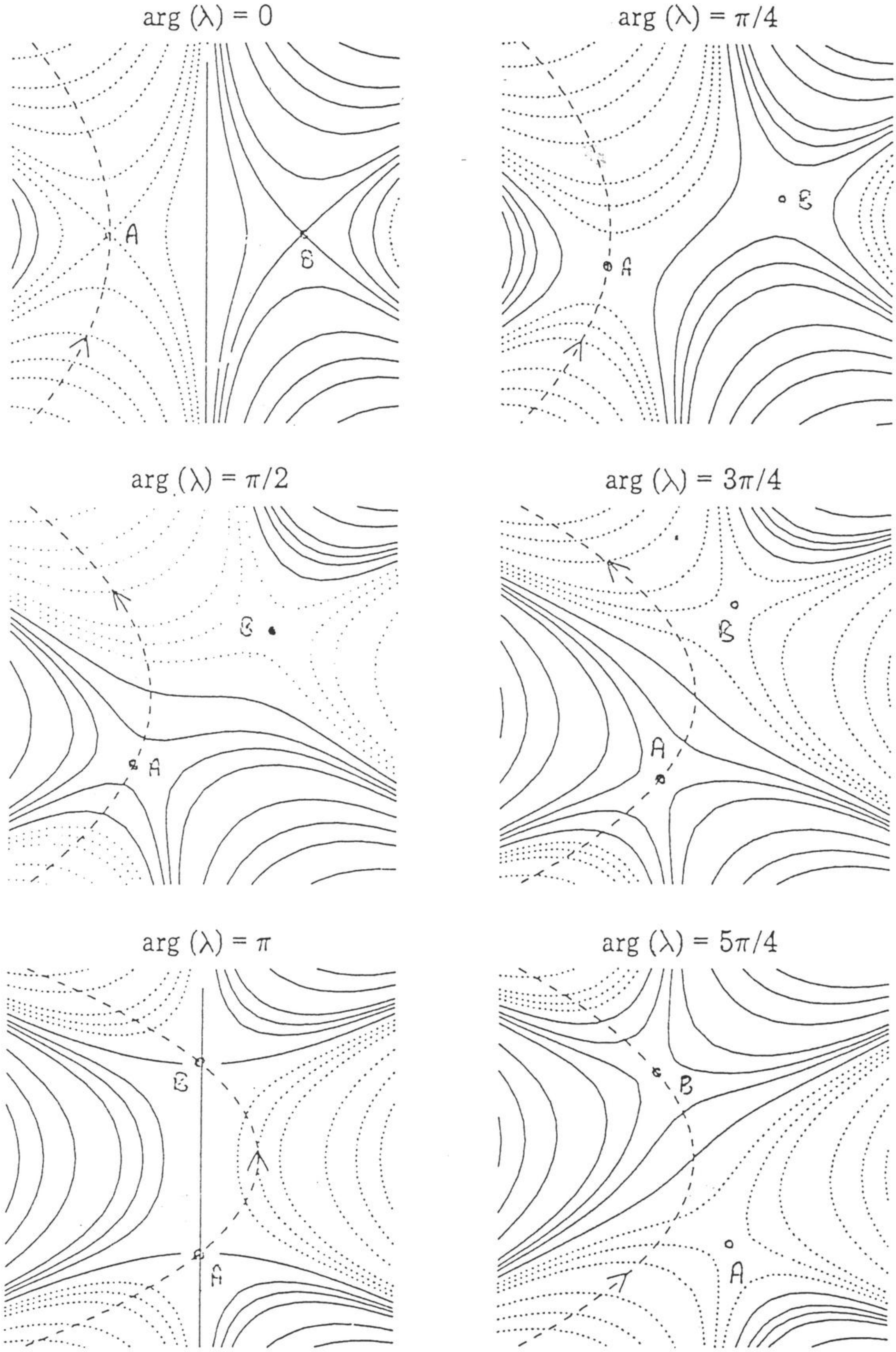


It is only necessary to go over the lower left hand saddle to go over the ridge separating the fixed end points of integration.

Seek a local contribution from near the saddle. Write:

$$z = -\lambda^{\frac{1}{2}} + i\lambda^{\beta}w$$

$$\lambda z - \frac{1}{3}z^3 = -\frac{2}{3}\lambda^{\frac{3}{2}} - \lambda^{\frac{1}{2}+2\beta}w^2 + i\frac{1}{3}\lambda^{3\beta}w^3\ .$$



The changing contour C for Ai when λ is complex.

Continuous curves : positive values of $\mathrm{Re}(\lambda z - \frac{1}{3}z^3)$

Dotted curves : negative values of $\mathrm{Re}(\lambda z - \frac{1}{3}z^3)$

To apply Laplace's method put $\beta = -\frac{1}{4}$;

$$\begin{aligned}
\mathrm{Ai}(\lambda) &= \frac{1}{2\pi\lambda^{\frac{1}{4}}} \int_C e^{-\frac{2}{3}\lambda^{\frac{3}{2}}} e^{-w^2} \left(1 + \frac{iw^3}{\lambda^{\frac{3}{4}}} - \frac{w^6}{18\lambda^{\frac{3}{2}}} + \dots\right) dw \\
&\sim \frac{e^{-\frac{2}{3}\lambda^{\frac{3}{2}}}}{2\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}} \left(1 - \frac{5}{48\lambda^{\frac{3}{2}}} + \dots\right) \qquad |\arg\lambda| < \pi .
\end{aligned}$$

- Consider complex values of λ.
- Positions of saddles rotate anticlockwise. Saddles swap dominance at

$$\arg(\lambda) = \frac{\pi}{3} + \frac{2}{3}n\pi .$$

- To go from valley at $\infty e^{-2\pi i/3}$ to valley at $\infty e^{2\pi i/3}$ only need to go over left-hand saddle up to $\arg\lambda = \pi$.
- For $\arg\lambda = \pi$ we need to go over both saddles:

$$\begin{aligned}
\mathrm{Ai} &\sim \frac{e^{-\frac{2}{3}\lambda^{\frac{3}{2}}}}{2\pi^{\frac{1}{2}}\lambda^{\frac{1}{4}}} + \text{c.c.} \\
&\sim \frac{1}{(-\lambda)^{\frac{1}{4}}\pi^{\frac{1}{2}}} \sin\left(\frac{2}{3}(-\lambda)^{\frac{3}{2}} + \frac{\pi}{4}\right) . \qquad (*)
\end{aligned}$$

- For $\pi < \arg\lambda < 5\pi/3$ we need to go through the other saddle, but $(*)$ is still an asymptotic approximation. (When $\arg\lambda = 5\pi/3$ the second saddle becomes subdominant.)
- An example of the *Stokes phenomenon.*

Perturbation Methods 4. Singular Perturbation Problems: Matched Asymptotic Expansions (MAEs).†

These are mainly used for solving differential equations. They are often needed when the highest-order derivative is multiplied by a small parameter. We will apply them to ODEs, but they are equally applicable to PDEs.

4.1 Regular Perturbation Problems: An Example.

$$y'' + 2\varepsilon y' + (1+\varepsilon^2)y = 1\ , \qquad y(0)=0, \quad y\left(\frac{\pi}{2}\right) = 0\ .$$

4.1.1 Exact Solution

$$\begin{aligned} y &= \frac{1}{1+\varepsilon^2}\left[1 - e^{-\varepsilon(x-\pi/2)}\sin x - e^{-\varepsilon x}\cos x\right] \\ &= (1-\sin x - \cos x) + \varepsilon\left[\left(x-\frac{\pi}{2}\right)\sin x + x\cos x\right] \\ &\quad - \varepsilon^2\left[1 - \cos x - \sin x + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 \sin x + \frac{1}{2}x^2\cos x\right] + \dots\ . \end{aligned}$$

4.1.2 Perturbation Solution

Try

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots\ .$$

Then

$$\begin{aligned} y_0'' + y_0 &= 1\ , & y_0(0) &= 0, \quad y_0\left(\frac{\pi}{2}\right) = 0\ , \\ y_1'' + y_1 &= -2y_0'\ , & y_1(0) &= 0, \quad y_1\left(\frac{\pi}{2}\right) = 0\ , \\ y_2'' + y_2 &= -2y_1' - y_0\ , & y_2(0) &= 0, \quad y_2\left(\frac{\pi}{2}\right) = 0\ . \end{aligned}$$

Hence

$$\begin{aligned} y_0 &= 1 - \sin x - \cos x\ , \\ y_1 &= \left(x - \frac{\pi}{2}\right)\sin x + x\cos x\ , \\ y_2 &= -1 + \cos x + \sin x - \frac{1}{2}\left(x-\frac{\pi}{2}\right)^2\sin x - \frac{1}{2}x^2\cos x\ . \end{aligned}$$

□

4.2 Singular Perturbation: Example.

$$\varepsilon y'' + y' = -e^{-x}\ , \qquad y(0)=0, \quad y \to 0 \quad \text{as } x\to\infty.$$

† Corrections and suggestions can be emailed to me at P.H.Haynes@damtp.cam.ac.uk.

4.2.1 Exact Solution

$$y = \frac{e^{-x} - e^{-x/\varepsilon}}{1-\varepsilon} .$$

Limit $\varepsilon \to 0$, x fixed:

$$y \sim e^{-x}\left(1 + \varepsilon + \varepsilon^2 + \dots\right) . \tag{4.1}$$

This expansion satisfies the boundary condition as $x \to \infty$, but does not satisfy the boundary condition $y(0) = 0$.

The limit $\varepsilon \to 0$, with x fixed, is a non-uniform limit since

$$e^{-x/\varepsilon} \ll \varepsilon^m \quad \text{only if} \quad |x| \gg m\varepsilon \log \varepsilon ;$$

hence we cannot put $x = 0$ in (4.1).

For x small we obtain an asymptotic expansion by first setting $x = \varepsilon\xi$, and then expanding:

$$y \sim \left(1 - e^{-\xi}\right) + \varepsilon\left(1 - e^{-\xi} - \xi\right) + \varepsilon^2\left(1 - e^{-\xi} - \xi + \frac{1}{2}\xi^2\right) + \dots . \tag{4.2}$$

Now

$$y(0) = 0 + \varepsilon 0 + \varepsilon^2 0 + \dots ,$$

while

$$y \to 1 + \varepsilon(1-\xi) + \varepsilon^2\left(1 - \xi + \frac{1}{2}\xi^2\right) + \dots \qquad \text{as } \xi \to \infty.$$

'Outer' ($\varepsilon \to 0$, x fixed) expansion satisfies the $x \to \infty$ boundary condition,
'Inner' ($\varepsilon \to 0$, ξ fixed) expansion satisfies the $\xi = 0$ boundary condition.

Exercise: Put $x = \varepsilon^{\frac{1}{2}}\eta$ in (4.1) and expand to $\mathcal{O}(\varepsilon)$;
$\xi = \varepsilon^{-\frac{1}{2}}\eta$ in (4.2) and expand to $\mathcal{O}(\varepsilon)$.
Compare the results.

4.2.2 Expansion Solution

Outer Approximation. Pose a Poincaré expansion for x fixed ($\neq 0$) and $\varepsilon \to 0$:

$$y = \sum_{n=0}^{\infty} \varepsilon^n y_n(x) .$$

Then

$$\begin{aligned} y_0' &= -e^{-x} , & y_0 &= A_0 + e^{-x} , \\ y_0'' + y_1' &= 0 , & y_1 &= A_1 + e^{-x} , \\ y_{n-1}'' + y_n' &= 0 , & y_n &= A_n + e^{-x} . \end{aligned}$$

We wish to apply two boundary conditions at each order, but have only one unknown constant. From comparison with the exact solution we choose not to satisfy the boundary condition at $x = 0$. From applying the boundary condition as $x \to \infty$, it follows that $A_n = 0$, and

$$y = e^{-x}\left(1 + \varepsilon + \varepsilon^2 + \dots\right) . \tag{4.3}$$

This is in agreement with (4.1).

Inner Approximation. Since we wish to apply two boundary conditions, we need the $\varepsilon y''$ term to be important somewhere at leading order. Note that

$$\left.\begin{aligned} \varepsilon y'' &\sim \frac{\varepsilon y}{(x-x_0)^2} \\ y' &\sim \frac{y}{(x-x_0)} \end{aligned}\right\} \quad \text{this suggests rescaling for } (x-x_0)\sim\varepsilon.$$

Hence try

$$x = x_0 + \varepsilon\xi \ ,$$

$$y = \sum_{n=0}^{\infty} \varepsilon^n Y_n(\xi) \ .$$

From substituting into the governing equation it follows that

$$Y_0'' + Y_0' = 0 \qquad , \qquad Y_0 = B_0 + C_0 e^{-\xi} \ ,$$

$$Y_1'' + Y_1' = -e^{-x_0} \qquad , \qquad Y_1 = B_1 + C_1 e^{-\xi} - \xi e^{-x_0} \ .$$

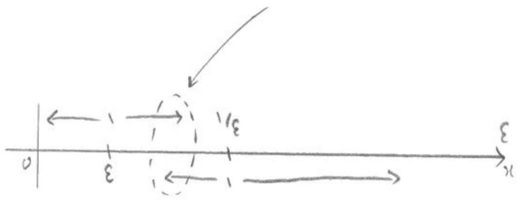
Since we need to satisfy the boundary condition at $x = 0$, take $x_0 = 0$. Then

$$\left.\begin{aligned} Y_0 &= B_0\left(1-e^{-\xi}\right) \ , \\ Y_1 &= B_1\left(1-e^{-\xi}\right) - \xi \ , \\ Y_2 &= B_2\left(1-e^{-\xi}\right) - \xi + \frac{1}{2}\xi^2 \ , \\ &\dots \ . \end{aligned}\right\} \tag{4.4}$$

Match.

We have two asymptotic expansions valid in x fixed, i.e. (4.3), and ξ fixed, i.e. (4.4). They must represent the same function in the intermediate region

$$\varepsilon \ll x \ll 1 \ , \text{ i.e. } 1 \ll \xi \ll \varepsilon^{-1} \ .$$



Forcing the two expansions to be identical determines the B_j. To this end introduce an 'intermediate variable'

$$\eta = \frac{x}{\varepsilon^\alpha} = \varepsilon^{1-\alpha}\xi \ , \quad (0<\alpha<1, \text{ e.g. } \alpha = \tfrac{1}{2}).$$

When $\eta = \operatorname{ord}(1)$, then as required $\varepsilon \ll x \ll 1$. Expand both outer and inner asymptotic expansions in powers of η:

$$
\begin{array}{llllll}
\text{Outer:} & y \sim 1 & -\,\varepsilon^{\alpha}\eta & +\,\frac{1}{2}\varepsilon^{2\alpha}\eta^{2} & +\,\frac{1}{6}\varepsilon^{3\alpha}\eta^{3} + \ldots \\
& \quad\ \boxed{1} & \quad\ \boxed{2} & \quad\ \boxed{3} & \\
& +\,\varepsilon & -\,\varepsilon^{1+\alpha}\eta & +\,\varepsilon^{1+2\alpha}\frac{1}{2}\eta^{2} + \ldots & \\
& \quad\ \boxed{4} & \quad\ \boxed{5} & & \\
& +\,\varepsilon^{2} & -\,\varepsilon^{2+\alpha}\eta + \ldots & & \\
& \quad\ \boxed{6} & & & \\
& +\,\varepsilon^{3} + \ldots & & & \\
& + \ldots\ , & & &
\end{array}
$$

$$
\begin{array}{llllll}
\text{Inner:} & y \sim B_0 & +\,\exp & & \\
& \quad\ \boxed{1} & & & \\
& +\,\varepsilon B_1 & -\,\varepsilon^{\alpha}\eta & +\,\exp & \\
& \quad\ \boxed{4} & \quad\ \boxed{2} & & \\
& +\,\varepsilon^{2} B_2 & -\,\varepsilon^{\alpha+1}\eta & +\,\frac{1}{2}\varepsilon^{2\alpha}\eta^{2} & +\,\exp \\
& \quad\ \boxed{6} & \quad\ \boxed{5} & \quad\ \boxed{3} & \\
& + \ldots\ . & & &
\end{array}
$$

After reordering the expansions should be the same; hence

$$B_0 = 1\,, \qquad B_1 = 1\,, \qquad B_2 = 1\,.$$

Terms jump order when matching. This indicates that there are terms in the governing equation that, although small in one region, are to be treated as dominant in the next region.

$$x = \mathcal{O}(1) \qquad\qquad \xi = \mathcal{O}(1)$$

$$\underbrace{-\varepsilon y''}_{\text{small}} = e^{-x} + \underbrace{y' \qquad y'}_{\text{common term}} + \varepsilon y'' = \underbrace{-e^{-x}}_{\text{small}}$$

Note that if the largest ignored term in the inner expansion, i.e. the $\mathcal{O}\left(\varepsilon^{(Q+1)\alpha}\right)$ term, is to be formally smaller than the last retained known term, i.e. the $\mathcal{O}\left(\varepsilon^{Q}\right)$ term, we require

$$\frac{Q}{Q+1} < \alpha < 1\,;$$

i.e. for $Q = 2$ we require $\frac{2}{3} < \alpha < 1$.

4.3 Van Dyke's Matching Rule.

This can be simpler than using an intermediate variable, but sometimes fails (beware of logs).

Notation

$$E_n y = \text{Outer limit } (x \text{ fixed, } \varepsilon \downarrow 0) \text{ of } y \text{ retaining } n \text{ terms} = \sum_{r=0}^{n} \varepsilon^r y_r(x)$$

$$H_m y = \text{Inner limit } (\xi \text{ fixed, } \varepsilon \downarrow 0) \text{ of } y \text{ retaining } m \text{ terms} = \sum_{r=0}^{m} \varepsilon^r Y_r(x)$$

Van Dyke's rule is

$$E_n H_m y = H_m E_n y \ .$$

↑ Take $(m+1)$ terms of the inner expansion, re-express ξ in terms of x, and then take $(n+1)$ terms of the resulting expansion.

Forcing equality determines the unknown constants. We illustrate this using our model problem:

$$\begin{aligned}
E_1 H_1 y &= E_1 \left(B_0 \left(1 - e^{-\xi}\right) + \varepsilon B_1 \left(1 - e^{-\xi}\right) - \varepsilon\xi\right) \\
&= E_1 \left(B_0 \left(1 - e^{-x/\varepsilon}\right) + \varepsilon B_1 \left(1 - e^{-x/\varepsilon}\right) - x\right) \\
&= B_0 - x + \varepsilon B_1 \ , \\
H_1 E_1 y &= H_1 \left(e^{-x} + \varepsilon e^{-x}\right) \\
&= H_1 \left(e^{-\varepsilon\xi} + \varepsilon e^{-\varepsilon\xi}\right) \\
&= 1 - \varepsilon\xi + \varepsilon \ .
\end{aligned}$$

Hence

$$B_0 - x + \varepsilon B_1 = 1 - \underbrace{\varepsilon\xi}_{x} + \varepsilon \ ,$$

and

$$B_0 = 1 = B_1 \ .$$

Exercise: Do for general m and n.

4.4 The Choice of Scaling.

There is no magic law that enables one to make the correct choice of scaling. However, there are tips:†

† In a forest, a fox bumps into a little rabbit, and enquires, 'Hi, what are you up

(a) First find 'the' regular solution:

$$y = y_0 = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots \ .$$

If for some x it happens that, $\varepsilon y_1 \sim y_0$ or $\varepsilon^2 y_2 \sim \varepsilon y_1$ or ..., then the solution is no longer asymptotic. This often suggests a rescaling for x. For instance suppose that the regular-perturbation solution yields

$$y = 1 + \frac{2\varepsilon}{(x - x_0)^2} + \frac{7\varepsilon^2}{(x - x_0)^4} + \dots \ .$$

This breaks down when $(x - x_0) \sim \varepsilon^{\frac{1}{2}}$, which suggests that an appropriate rescaling would be $x = x_0 + \varepsilon^{\frac{1}{2}}\xi$.

(b) Look at the equation and see if one can predict the scaling from there, i.e. seek distinguished limits. For instance consider the problem

$$(x + \varepsilon y)\frac{dy}{dx} + y = 1 \ , \qquad y(1) = 2 \ .$$

This has the leading-order (i.e. $\varepsilon = 0$) solution

$$x\frac{dy_0}{dx} + y_0 = 1 \ , \quad y_0 = 1 + \frac{1}{x} \ . \tag{4.5}$$

Now, using(4.5), compare the size of the terms in the equation:

$$\underbrace{x\left(\frac{y}{x}\right) \ ; \ \frac{\varepsilon y^2}{x}}_{\text{comparable when } y \sim \frac{x}{\varepsilon}.} \ ; \ y \ ; \ 1$$

Hence the neglected term is comparable with the largest retained term when $\frac{1}{x} \sim \frac{x}{\varepsilon}$, i.e. when $x \sim \varepsilon^{\frac{1}{2}}$.

---

to?'. 'I'm writing a dissertation on how rabbits eat foxes', says the rabbit. 'Come now rabbit, you know that's impossible', replies the fox. 'Well, follow me and I'll show you', says the rabbit. They both go into the rabbit's dwelling and after a while the rabbit emerges with a satisfied expression on his face.

Along comes a wolf who asks, 'Hello, what are you doing these days?'. 'I'm writing the second chapter of my thesis, on how rabbits devour wolves', says the rabbit. 'Are you crazy! Where is your academic honesty?' explodes the wolf. 'Come with me and I'll show you', says the rabbit. As before the rabbit comes out of his dwelling with a satisfied expression on his face, and with a diploma in his paw.

Switch to the rabbit's dwelling to find a huge lion sitting next to some bloody and furry remnants of the fox and the wolf. The moral: it's your supervisor that really counts.

4.5 Where is the 'Boundary Layer'?

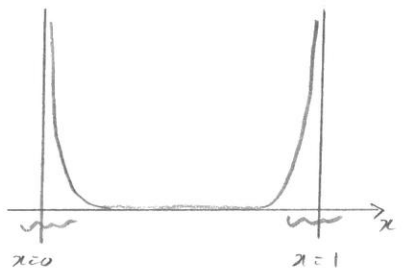
The 'boundary layer' could be anywhere! *One* way to try and track it down is to look at regular solution and see where it breaks down. However, this method does not always work, as illustrated by the following examples.

Example 1. Consider the problem

$$\varepsilon y'' - y = 0 \, , \quad y(0) = y(1) = 1 \, .$$

For $\varepsilon > 0$ this has solution

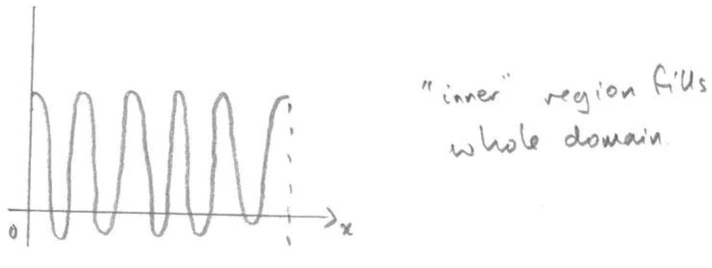
$$y = \left(\frac{1 - e^{-1/\varepsilon^{\frac{1}{2}}}}{1 - e^{-2/\varepsilon^{\frac{1}{2}}}} \right) \left[e^{-x/\varepsilon^{\frac{1}{2}}} + e^{(x-1)/\varepsilon^{\frac{1}{2}}} \right] .$$



There are boundary layers at both $x = 0$ and $x = 1$.

Now consider the case $\varepsilon < 0$. This has solution

$$y = \frac{\sin\left(x/\left|\varepsilon\right|^{\frac{1}{2}}\right) - \sin\left((x-1)/\left|\varepsilon\right|^{\frac{1}{2}}\right)}{\sin\left(1/\left|\varepsilon\right|^{\frac{1}{2}}\right)} .$$



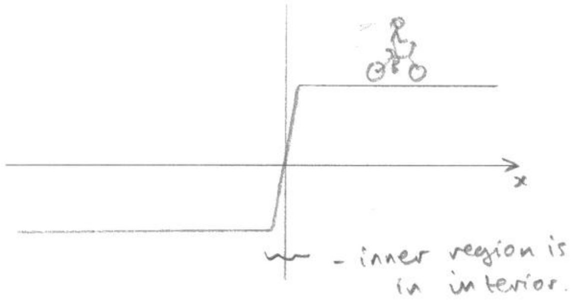
In this case there are boundary layers everywhere. What happens if $\sin\left(1/\left|\varepsilon\right|^{\frac{1}{2}}\right) = 0$?

Example 2.

$$\tfrac{1}{2}\varepsilon^2 f'' - f(f^2 - 1) = 0 \, , \qquad f(\infty) = 1 \, , \; f(-\infty) = -1 \, .$$

$\varepsilon = 0$: $\quad f(f^2-1)=0$, hence $f=-1$ or 0 or $+1$.

$\varepsilon \neq 0$: $\quad$ the exact solution is $f = \tanh\left(\dfrac{x}{\varepsilon}\right)$. $\qquad$ (4.6)



There is a boundary layer in the interior of width $\mathcal{O}(\varepsilon)$. Within the 'boundary layer'

$$\varepsilon^2 f'' \sim f(f^2-1) ,$$

i.e. the boundary layer is confined to a region where $(x - x_0) \sim \varepsilon$.

Exercise: Is (4.6) unique?

4.6 Composite Expansions.

The outer solution in (4.3) fails as $x \to 0$ due to the missing $e^{-x/\varepsilon}$ term.

The inner solution in (4.4) fails as $\xi \to \infty$ due to the missing $\frac{\varepsilon^n \xi^n}{n!}$ terms.

By correcting either one we can obtain a uniformly valid asymptotic expansion called a *composite expansion* — this is useful for real answers/comparison with experiment.

It takes little effort to obtain the composite when using Van Dyke's matching rule — just use the composite operator:

$$C_{nm}y = E_n y + H_m y - E_n H_m y .$$

Note:

$$E_n C_{nm} y = E_n y ,$$
$$H_m C_{nm} y = H_m y .$$

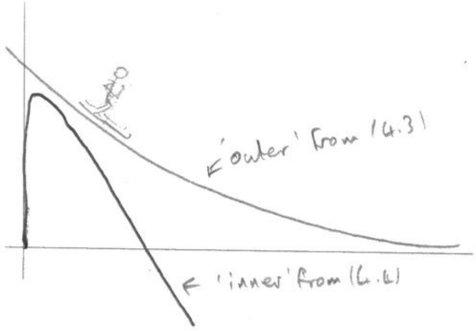
For the example we have been considering

$C_{11}y = \quad Ey \quad + \quad H_1 y \quad - E_1 H_1 y.$

$$C_{11}y = \left(e^{-x} + \varepsilon e^{-x}\right) + \left(\left(1 - e^{-x/\varepsilon}\right) + \varepsilon\left(1 - e^{-x/\varepsilon}\right) - x\right) - 1 + x - \varepsilon$$
$$= (1+\varepsilon)\left(e^{-x} - e^{-x/\varepsilon}\right) .$$

(i) This is correct to $\mathcal{O}(\varepsilon)$. Such expansions tend to be accurate to $\mathcal{O}\left(\varepsilon^{\min(m,n)}\right)$.

(ii) The expansion is not of Poincaré form — so it is not unique.



Other rules exist, for instance *multiplicative composition*:

$$C_{nm}y = \frac{E_n y H_m y}{E_n H_m y} .$$

Alternatively, suppose that F is a sufficiently smooth functional with an inverse, then a composite expansion can be defined by

$$C_{nm}y = F^{-1}\Big\{F(E_n y) + F(H_m y) - F(E_n H_m y)\Big\} .$$

Additive composition corresponds to $F = 1$, while multiplicative composition corresponds to $F = \log$.

4.7 Matching Involving Logarithms.

4.7.1 Model equation

We consider a model equation which can be thought of representing heat conduction outside a spherical cavity with a weak nonlinear heat source. The equation can be written in two forms. In the first form the small parameter ε occurs in the equation

$$f_{rr} + \left(\frac{n-1}{r}\right) f_r + \varepsilon f f_r = 0 \quad , \quad f(1) = 0, \quad f \to 1 \quad \text{as} \quad r \to \infty , \tag{4.7a}$$

while in the second form, with $\rho = \varepsilon r$, ε occurs in one of the boundary conditions

$$f_{\rho\rho} + \left(\frac{n-1}{\rho}\right) f_\rho + f f_\rho = 0 \quad ; \quad f(\varepsilon) = 0, \quad f \to 1 \quad \text{as} \quad \rho \to \infty . \tag{4.7b}$$

4.7.2 The case $n = 3$

First seek a regular expression (r fixed, $\varepsilon \downarrow 0$):

$$f(r, \varepsilon) \sim f_0(r) + \varepsilon f_2(r) + \dots .$$

Then from substituting into (4.7a) we find that

ε^0: $$f_0'' + \frac{2}{r} f_0' = 0 ,$$

$$f_0 = 1 - \frac{1}{r}\,, \qquad f_0(1) = 0, \quad f_0 \to 1 \quad \text{as} \quad r \to \infty;$$

ε^1: $$\frac{1}{r^2}\left(r^2 f_2'\right)' = -f_0 f_0' \,.$$

On integrating and applying $f_2(1) = 0$, we obtain

$$f_2 = A_2\left(1 - \frac{1}{r}\right) - \ln r\left(1 + \frac{1}{r}\right). \tag{4.8}$$

The boundary condition at ∞, i.e. $f_2 \to 0$ as $r \to \infty$, cannot be satisfied for any choice of A_2. As a result the expansion cannot be uniformly asymptotic at large r. In fact for $r \gg 1$

$$f_0'' \sim -\frac{2}{r^3} \quad , \quad \varepsilon f_0 f_0' \sim \frac{\varepsilon}{r^2}.$$

Hence the $\mathcal{O}(\varepsilon)$ term is no longer a small correction to the equation when

$$r = \mathcal{O}\left(\frac{1}{\varepsilon}\right).$$

Since $\varepsilon f_2 \sim \ln(1/\varepsilon)$ when $r = \mathcal{O}(\varepsilon^{-1})$, we try the asymptotic sequence

$$1, \quad \varepsilon \ln(1/\varepsilon), \quad \varepsilon, \; \ldots \; .$$

Note that we can view the $\ln(1/\varepsilon)$ term as coming from the particular integral:

$$f_2 = -\int_0^r \frac{ds}{s^2} \underbrace{\int_0^s t^2 f_0(t) f_0'(t)\, dt}_{\sim s \quad \text{as} \quad s\to\infty} .$$

Asymptotic expansion for r fixed and $\varepsilon \downarrow 0$. Try the Poincaré expansion:

$$f \sim f_0 + \varepsilon \ln(1/\varepsilon)\, f_1 + \varepsilon f_2 + \ldots\,, \quad f_j(1) = 0\,. \tag{4.9a}$$

Substitute into (4.7a) and solve:

ε^0: $$f_0 = \left(1 - \frac{1}{r}\right);$$

$\varepsilon \ln(1/\varepsilon)$: The same linear equation is obatined as for f_0, hence

$$f_1 = A_1\left(1 - \frac{1}{r}\right);$$

ε: This is the same as (4.8), viz.

$$f_2 = A_2\left(1 - \frac{1}{r}\right) - \ln r\left(1 + \frac{1}{r}\right).$$

The constants A_1 & A_2 are to be determined by matching.

Asymptotic expansion for ρ fixed, and $\varepsilon \downarrow 0$. Try the Poincaré expansion:

$$f \sim 1 + \varepsilon \ln(1/\varepsilon)\, g_1(\rho) + \varepsilon\, g_2(\rho) + \dots\ , \qquad g_j(\infty) = 0\ . \tag{4.9b}$$

We obtain the same equation for g_1 and g_2:

$$g_j'' + \frac{2}{\rho} g_j' + g_j' = 0\ ,$$

$$\left(\rho^2 e^{\rho} g_j'\right)' = 0\ ,$$

$$g_j = B_j \int_{\rho}^{\infty} \frac{e^{-\tau}}{\tau^2}\, d\tau\ , \qquad g_j(\infty) = 0\ .$$

Match by intermediate variable to fix A_1, A_2, B_1 & B_2. First observe that

$$\int_{\rho}^{\infty} \frac{e^{-\tau}}{\tau^2}\, d\tau \sim \frac{1}{\rho} + (\ln\rho + \gamma - 1) - \frac{1}{2}\rho + \mathrm{o}(\rho) \quad \text{as} \quad \rho \to 0.$$

Introduce $\eta = \varepsilon^{\alpha} r = \varepsilon^{\alpha-1}\rho$, with $0 < \alpha < 1$. Take the limit of η fixed, $\varepsilon \downarrow 0$:

(4.9a)

$$f \sim \underset{\boxed{1}}{1} \quad \underset{\boxed{3}}{-\frac{\varepsilon^{\alpha}}{\eta}} \quad + \dots \qquad \underset{\boxed{5}}{+\,\varepsilon \ln(1/\varepsilon)\, A_1 \left(1 - \frac{\varepsilon^{\alpha}}{\eta}\right)} + \dots$$

$$+\,\varepsilon \left[\underset{\boxed{6}}{A_2 \left(1 - \frac{\varepsilon^{\alpha}}{\eta}\right)} - \underset{\boxed{5}\qquad\quad\ \ \boxed{6}}{\left(\alpha \ln(1/\varepsilon) + \ln\eta\right)} \left(1 + \frac{\varepsilon^{\alpha}}{\eta}\right) + \dots \right] + \dots\ ;$$

(4.9b)

$$f \sim \underset{\boxed{1}}{1} \ + \varepsilon \ln(1/\varepsilon)\, B_1 \left[\underset{\boxed{2}}{\frac{\varepsilon^{\alpha-1}}{\eta}} + \underset{\boxed{4}}{(\alpha - 1)\ln(1/\varepsilon)} + \ln\eta + (\gamma - 1) + \dots \right]$$

$$+\,\varepsilon B_2 \left[\underset{\boxed{3}}{\frac{\varepsilon^{\alpha-1}}{\eta}} \qquad + \underset{\boxed{5}}{(\alpha - 1)\ln(1/\varepsilon)} + \underset{\boxed{6}}{\ln\eta} + \underset{\boxed{6}}{(\gamma - 1)} + \dots \right] + \dots\ .$$

Make the expansions agree:

ε^0	:	$1 = 1\ ;$	
$\varepsilon^{\alpha} \ln(1/\varepsilon)$	:	$0 = B_1\ ,$ — as ≠ a 2nd (2)	$B_1 = 0\ ;$
ε^{α}	:	$-1 = B_2\ ,$	$B_2 = -1\ ;$
$\varepsilon(\ln(1/\varepsilon))^2$	:	$0 = B_1\ ,$	consistent ;
$\varepsilon \ln(1/\varepsilon)$	:	$A_1 - \alpha = (\alpha - 1)B_2\ ,$	$A_1 = 1\ ;$
ε	:	$A_2 - \ln\eta = B_2 \ln\eta + (\gamma - 1)B_2\ ,$	$A_2 = 1 - \gamma\ .$

Hence

r fixed: $\quad f \sim (1-\frac{1}{r}) + \varepsilon \ln(1/\varepsilon)\,(1-\frac{1}{r}) + \varepsilon\Big((1-\gamma)\,(1-\frac{1}{r}) - \ln r\,(1+\frac{1}{r})\Big) + \dots\ ,$

ρ fixed: $\quad f \sim 1 - \varepsilon \displaystyle\int_\rho^\infty \frac{e^{-\tau}}{\tau^2}\,d\tau \dots\ .$

Match using Van Dyke's rule.

Identify E and H with the coordinates r and ρ respectively. Then

$$
\begin{aligned}
H_1E_1f &= H_1\left[\left(1-\frac{1}{r}\right) + \varepsilon\ln(1/\varepsilon)\,A_1\left(1-\frac{1}{r}\right)\right] \\
&= 1 + \varepsilon\ln(1/\varepsilon)\,A_1\ , \\
E_1H_1f &= E_1\left(1 + \varepsilon\ln(1/\varepsilon)\,B_1\int_\rho^\infty e^{-\tau}\,\frac{d\tau}{\tau}\right) \\
&= 1 + \frac{B_1}{r}\ln(1/\varepsilon) - \varepsilon\ln^2(1/\varepsilon)\,B_1 + B_1\varepsilon\ln(1/\varepsilon)\,(\ln r + \gamma - 1)\ .
\end{aligned}
$$

If these two expansions are to agree then $A_1 = 0$ and $B_1 = 0$, which is **WRONG**. The trouble is a $\ln\rho$ in the ε term when $\rho = \mathcal{O}(1)$ — this changes to a $\varepsilon\ln(1/\varepsilon)$ term in the intermediate scaling.

In general, terms like $(\ln r)^p$ lead to failures near to the diagonal where $|n-m| < p$. However, in general there is success sufficiently far from the diagonal, e.g.

$$
\begin{aligned}
H_2E_1f &= 1 + \varepsilon\ln(1/\varepsilon)\,A_1 - \frac{\varepsilon}{\rho}\ , \\
E_1H_2f &= E_1\left(1 + (\varepsilon\ln(1/\varepsilon)\,B_1 + \varepsilon B_2)\int_\rho^\infty \frac{e^{-\tau}}{\tau^2}d\tau\right) \\
&= 1 + \frac{1}{r}\,(B_1\ln(1/\varepsilon) + B_2) - \varepsilon\ln(1/\varepsilon)\,\big(\ln(1/\varepsilon)\,B_1 + B_2\big) \\
&\quad + \varepsilon\ln(1/\varepsilon)\,B_1\,(\ln r + \gamma - 1)\ ;
\end{aligned}
$$

so $B_1 = 0,\ \ B_2 = -1$, and $A_1 = 1$ as before.

It is best to apply Van Dyke's rule (and composite expansions) only at changes in the power of ε:

1	$\varepsilon\ln(1/\varepsilon)$	ε	$\varepsilon^2\ln^2(1/\varepsilon)$	$\varepsilon^2\ln(1/\varepsilon)$	ε^2	$\dots$
$\uparrow$		$\uparrow$			$\uparrow$	

Apply Van Dyke's rule only at the arrowed orders — DO NOT SPLIT LOGS!

<u>*4.7.3 The case $n=2$*</u>

In this case the governing equation is

$$
f_{rr} + \frac{1}{r}f_r + \varepsilon f f_r = 0\ , \quad f(1) = 0\ , \quad f \to 1 \quad \text{as } r \to \infty.
$$

Try a regular expansion, $f \sim f_0 + \varepsilon f_1 + \ldots$; then

ε^0: $$f_0'' + \frac{1}{r} f_0' = 0 \quad , \quad f_0 = A_0 \ln r + C_0 \ .$$

No choice of A_0 or C_0 will satisfy the boundary conditions both at $r = 1$ and as $r \to \infty$. Choose to satisfy the boundary condition at $r = 1$, i.e. set $C_0 = 0$.

ε^1: $$f_1'' + \frac{1}{r} f_1' = -f_0 f_0' \quad , \quad f_1 = A_1 \ln r + C_1 - A_0^2 (r \ln r - 2r + 2) \ .$$

Again satisfy the boundary condition at $r = 1$ (i.e. $f_1(1) = 0$) – this time by setting $C_1 = 0$. Note that if $A_0 \neq 0$, then f_1 has even worse behaviour as $r \to \infty$ than f_0. By comparing where the expansion for f becomes non-asymptotic, it follows that we should introduce $\rho = \varepsilon r$ as the stretched variable.

Note that when $r = \mathcal{O}\left(\frac{1}{\varepsilon}\right)$,
$$f_0 \sim A_0 \ln(1/\varepsilon) \ .$$

Since $f_0 \sim 1$ as $\rho \to \infty$, this suggests trying $A_0 = \frac{1}{\ln(1/\varepsilon)}$, and the asymptotic sequence

$$1 \quad , \quad \frac{1}{\ln(1/\varepsilon)} \quad , \quad \frac{1}{(\ln(1/\varepsilon))^2} \quad , \quad \ldots \ .$$

Asymptotic expansion for r fixed and $\varepsilon \downarrow 0$.

We propose the asymptotic expansion

$$f(r,\varepsilon) \sim 0 + \frac{f_1(r)}{\ln(1/\varepsilon)} + \frac{f_2}{(\ln(1/\varepsilon))^2} + \ldots \ . \tag{4.10a}$$

Then
$$f_n'' + \frac{1}{r} f_n' = 0 \ , \quad \text{and} \quad f_n = A_n \ln r \ .$$

Note that the $\varepsilon f f'$ term never enters into the expansion.

Asymptotic expansion for ρ fixed and $\varepsilon \downarrow 0$.

In this case we propose

$$f \sim 1 + \frac{g_1(\rho)}{\ln(1/\varepsilon)} + \frac{g_2(\rho)}{(\ln(1/\varepsilon))^2} + \ldots \ . \tag{4.10b}$$

Then

$$g_1'' + \left(\frac{1}{\rho} + 1\right) g_1' = 0 \ ,$$
$$g_1 = B_1 \int_\rho^\infty \frac{e^{-\tau}}{\tau} \, d\tau = B_1 E_1(\rho) \ ;$$
$$g_2'' + \left(\frac{1}{\rho} + 1\right) g_2' = -g_1 g_1' \ ,$$
$$g_2 = B_2 E_1(\rho) - B_1^2 \left(e^{-\rho} E_1(\rho) - 2E_1(2\rho)\right) \ .$$

Match using the intermediate variable

$$\eta \sim \varepsilon^{\alpha} r = \varepsilon^{\alpha-1}\rho \qquad (0<\alpha<1),$$

and the asymptotic expansion

$$E_1(\rho) \to -\gamma - \ln\rho + \rho + \mathcal{O}\left(\rho^2\right) \quad \text{as} \quad \rho \to 0\,.$$

Then

$$\text{(4.10}a\text{):}\qquad f \sim \frac{1}{\ln(1/\varepsilon)}A_1\left(\alpha\ln(1/\varepsilon)+\ln\eta\right) + \frac{1}{\left(\ln(1/\varepsilon)\right)^2}A_2\left(\alpha\ln(1/\varepsilon)+\ln\eta\right)+\ldots \tag{4.11a}$$

$$\begin{aligned}\text{(4.10}b\text{):}\qquad f \sim 1 &+ \frac{B_1}{\ln(1/\varepsilon)}\left(-(\alpha-1)\ln(1/\varepsilon)-\ln\eta-\gamma+\varepsilon^{1-\alpha}\eta+\ldots\right)\\ &+\frac{1}{\left(\ln(1/\varepsilon)\right)^2}\Big(B_2\left[-(\alpha-1)\ln(1/\varepsilon)-\ln\eta-\gamma+\ldots\right]\\ &\qquad + B_1^2\left[-(\alpha-1)\ln(1/\varepsilon)-\gamma-\ln\eta-\ln4+\ldots\right]\Big)\,.\end{aligned} \tag{4.11b}$$

On equating equal orders of ε we find that

$\ln^0(1/\varepsilon)$: $\qquad \alpha A_1 = 1-(\alpha-1)B_1\,,$

— if this is true $\forall\alpha$ then $B_1=-1,\ A_1=1$;

$\ln^{-1}(1/\varepsilon)$: $\qquad A_1\ln\eta+\alpha A_2 = -B_1\left(\ln\eta+\gamma\right)-B_2\left(\alpha-1\right)-B_1^2\left(\alpha-1\right)\,,$

— if this is true $\forall\alpha,\eta$ then $B_2=-(1+\gamma),\ A_2=\gamma$.

Match by Van Dyke's Rule (if you must).

Put $\alpha=1$ and $\eta=\rho$ in (4.11a), and $\alpha=0$ and $\eta=r$ in (4.11b). Then Van Dyke's rule gives

$E_0H_0=1\,,$	$H_0E_0=0\,,$	WRONG ;
$E_1H_0=1\,,$	$H_0E_1=A_1\,,$	$A_1=1$;
$E_0H_1=1+B_1\,,$	$H_1E_0=0\,,$	$B_1=-1$.

Similarly

$$\left.\begin{aligned}E_1H_1 &= 1+B_1-\frac{B_1}{\ln(1/\varepsilon)}\left(\ln r+\gamma\right)\,,\\ H_1E_1 &= A_1\left(1+\frac{\ln\rho}{\ln(1/\varepsilon)}\right)=\frac{A_1\ln r}{\ln(1/\varepsilon)}\,,\end{aligned}\right\}\quad\text{WRONG;}$$

$$E_2H_1 = 1+B_1-\frac{B_1}{\ln(1/\varepsilon)}\left(\ln r+\gamma\right)\,,$$

$$H_1E_2 = A_1+\frac{1}{\ln(1/\varepsilon)}\left(A_1\ln\rho+A_2\right)=\frac{A_1\ln r}{\ln(1/\varepsilon)}+\frac{A_2}{\ln(1/\varepsilon)}\,,$$

hence

$$A_1=1\,,\quad B_1=-1\,,\quad A_2=\gamma\,.$$

As before, Van Dyke's rule works if $n\neq m$.

4.7.4 A 'terrible' problem

Consider the equation with $n = 2$ plus a new term:

$$f_{rr} + \frac{1}{r} f_r + f_r^2 + \varepsilon f f_r = 0 , \quad f(1) = 0, \quad f \to 1 \quad \text{as} \quad r \to \infty .$$

First compare the size of terms using the solution calculated in §4.7.3:

$$r = \text{ord}(1), \qquad f \sim \frac{1}{\ln(1/\varepsilon)} , \quad f_r^2 \sim \left(\frac{1}{\ln(1/\varepsilon)}\right)^2, \quad f_{rr} \sim \frac{1}{\ln(1/\varepsilon)} ,$$

$$\rho = \text{ord}(1), \qquad f \sim 1 , \quad f_\rho \sim \frac{1}{\ln(1/\varepsilon)} , \quad f_\rho^2 \sim \left(\frac{1}{\ln(1/\varepsilon)}\right)^2, \quad f_{\rho\rho} \sim \frac{1}{\ln(1/\varepsilon)} .$$

From this comparison of terms we might expect a small perturbation to the previous answer.

Asymptotic expansion for r fixed.

As in \$ 4.7.3 propose the asymptotic expansion

$$f \sim \frac{1}{\ln(1/\varepsilon)} f_1 + \frac{1}{(\ln(1/\varepsilon))^2} f_2 + \frac{1}{(\ln(1/\varepsilon))^3} f_3 + \dots . \qquad (4.12a)$$

The from substituting into the equation we find:

$$\ln^{-1}(1/\varepsilon): \quad f_1'' + \frac{1}{r} f_1' = 0 , \quad f_1 = A_1 \ln r ,$$

$$\ln^{-2}(1/\varepsilon): \quad f_2'' + \frac{1}{r} f_2' = -f_1'^2 , \quad f_2 = A_2 \ln r - \tfrac{1}{2} A_1^2 \ln^2 r ,$$

$$\ln^{-3}(1/\varepsilon): \quad f_3'' + \frac{1}{r} f_3' = -2 f_1' f_2' , \quad f_3 = A_3 \ln r + \tfrac{1}{3} A_1^3 \ln^3 r - A_1 A_2 \ln^2 r .$$

By induction, one can show that as $r \to \infty$,

$$f_n \sim (-)^n \left(-\frac{1}{n} A_1^n \ln^n r + A_1^{n-2} A_2 \ln^{n-1} r \right) ,$$

and hence by summation that

$$f \sim \ln \left[1 + \left(\frac{A_1}{\ln(1/\varepsilon)} + \frac{A_1}{(\ln(1/\varepsilon))^2} + \dots \right) \ln r \right] \qquad \text{as} \quad r \to \infty .$$

Lemma (for future reference).

Instead of adopting the above approach, ignore §4.7.3 and assume

$$f = f_0 + \dots .$$

Then

$$f_0'' + \frac{1}{r} f_0' + f_0'^2 = 0 \quad \Rightarrow \quad f_0 = \ln(1 + A \ln r) \quad \text{if} \quad f_0(1) = 0 .$$

If $A = \mathcal{O}\left(\ln^{-1}(1/\varepsilon)\right)$, this suggests that the natural variable is

$$\frac{\ln r}{\ln(1/\varepsilon)} . \tag{4.13}$$

Asymptotic expansion for ρ fixed.

In this variable ε does not appear in the equation:

$$f_{\rho\rho} + \frac{1}{\rho} f_\rho + f_\rho^2 + f f_\rho = 0 .$$

We pose the Poincaré expansion:

$$f \sim 1 + \frac{g_1}{\ln(1/\varepsilon)} + \frac{g_2}{\left(\ln(1/\varepsilon)\right)^2} + \dots . \tag{4.12b}$$

Substitute, equate, etc:

$\ln^{-1}(1/\varepsilon)$:
$$g_1'' + \frac{1}{\rho} g_1' + g_1' = 0 ,$$
$$g_1 = B_1 \int_\rho^\infty \frac{e^{-\tau}}{\tau}\, d\tau \quad \text{(setting } g_1(\infty) = 0\text{)};$$

$\ln^{-2}(1/\varepsilon)$:
$$g_2 = B_2 E_1(\rho) + B_1^2 \left(2E_1(2\rho) - \frac{1}{2} E_1^2(\rho) - e^{-\rho} E_1(\rho)\right) .$$

As $\rho \to 0$ we have

$$g_1 \sim B_1 \left(-\ln\rho - \gamma\right) ,$$
$$g_2 \sim B_2 \left(-\ln\rho - \gamma\right) + B_1^2 \left(-\frac{1}{2}\ln^2\rho - (\gamma + 1)\ln\rho - \frac{1}{2}\gamma^2 - \gamma - \ln 4\right) .$$

The leading-order behaviour as $\rho \to 0$ comes from the balance

$$g_2'' + \frac{1}{\rho} g_2' \sim -{g_1'}^2 \sim \frac{B_1^2}{\rho^2} .$$

Similarly we can show that for small ρ

$$g_3'' + \frac{1}{\rho} g_3' \sim -2g_1' g_2' \sim -2B_1^3 \frac{\ln\rho}{\rho^2} - \left(2B_1^3(\gamma+1) + 2B_1 B_2\right) \frac{1}{\rho^2} ,$$
$$g_3 \sim -\frac{1}{3} B_1^3 \ln^3\rho - \left(B_1^3(\gamma + 1) + B_1 B_2\right) \ln^2\rho .$$

By induction it is possible to conclude that as $\rho \to 0$

$$g_n \sim -\frac{1}{n} B_1^n \ln^n\rho - \left(B_1^n(\gamma+1) + B_1^{n-2} B_2\right) \ln^{n-1}\rho .$$

Match using the intermediate variable

$$\eta = \varepsilon^{\alpha} r = \rho \varepsilon^{\alpha - 1} \ .$$

Then

$$\begin{aligned}
(4.12a): \quad f \sim & \frac{1}{\ln(1/\varepsilon)} A_1 \left(\alpha \ln(1/\varepsilon) + \ln \eta\right) \\
& + \frac{1}{\ln^2(1/\varepsilon)} \left[-\frac{1}{2} A_1^2 \left(\alpha \ln(1/\varepsilon) + \ln \eta\right)^2 + \ldots \right] + \ldots \\
& + \frac{1}{\ln^n(1/\varepsilon)} \left[\frac{(-)}{n}^{n+1} A_1^n \left(\alpha \ln(1/\varepsilon) + \ln \eta\right)^n + \ldots \right] + \ldots \ ;
\end{aligned}$$

$$\begin{aligned}
(4.12b): \quad f \sim 1 + & \frac{B_1}{\ln(1/\varepsilon)} \left[-(\alpha - 1) \ln(1/\varepsilon) - \ln \eta - \gamma + \ldots \right] \\
& + \frac{1}{\ln^2(1/\varepsilon)} \left[-\frac{B_1^2}{2} \left(((\alpha - 1) \ln(1/\varepsilon) + \ln \eta)^2 + \ldots \right) + \ldots \right] + \ldots \\
& + \frac{1}{\ln^n(1/\varepsilon)} \left[-\frac{B_1^2}{n} \left((\alpha - 1) \ln(1/\varepsilon) + \ln \eta\right)^n + \ldots \right] + \ldots \ .
\end{aligned}$$

Equate these two expansions. At leading order

$$\ln^0(1/\varepsilon): \quad \alpha A_1 - \frac{1}{2} \alpha^2 A_1^2 + \frac{1}{3} \alpha^3 A_1^3 + \ldots = 1 - B_1(\alpha - 1) - \frac{1}{2} B_1^2 (\alpha - 1)^2 + \ldots \ .$$

or from summing the series

$$\ln\left(1 + \alpha A_1\right) = 1 + \ln\left[1 - (\alpha - 1) B_1\right] \ .$$

This must be true $\forall \alpha$, hence

$$e(1 + B_1) - 1 = 0 \quad , \quad A_1 + e B_1 = 0 \ ,$$

i.e.

$$B_1 = -\left(\frac{e - 1}{e}\right) \quad , \quad A_1 = (e - 1) \ .$$

Note that in matching an infinite number of terms jumped order — hence the need for general expressions for f_n & g_n.

Is there an easier way?

Recall from earlier that a natural variable is

$$t = \frac{\ln r}{\ln(1/\varepsilon)} \ . \tag{4.14}$$

Note that

$$\{r = 1\} \equiv \{t = 0\} \ ,$$

and that

$$\rho = \text{ord}(1) \qquad \text{when} \qquad r = \frac{k}{\varepsilon} \quad \text{for} \quad k = \text{ord}(1) \ ,$$

i.e. when

$$t = \underset{\substack{\uparrow \\ \text{finite value}}}{1} + \frac{\ln k}{\ln(\varepsilon^{-1})} \quad \text{for} \quad k = \text{ord}(1).$$

Let $\tau = 1 - t$, so that $\rho = \text{ord}(1)$ when $\tau = \text{ord}(\ln^{-1}(\varepsilon^{-1}))$, and substitute into the equation:

$$f_{\tau\tau} + f_\tau^2 = -\ln(1/\varepsilon)\, e^{-\tau \ln(1/\varepsilon)}\, f f_\tau \ .$$

Seek a Poincaré expansion for $\tau > 0$ (so that the r.h.s. is exponentially small):

$$f = f_0 + \frac{1}{\ln(1/\varepsilon)} f_1 + \dots \ , \tag{4.15a}$$

then

$$f_{0\tau\tau} + f_{0\tau}^2 = 0 \ .$$

If we require $f_0(1) = 0$, then

$$f_0 = \log\left(1 + \alpha_0\left(1 - \tau\right)\right) \ .$$

We need to match with the outer solution that is valid for $\rho = \text{ord}(1)$, i.e. for $\tau = \text{ord}(\ln^{-1}(\varepsilon^{-1}))$. Since

$$\tau = 1 + \frac{\ln 1/r}{\ln(1/\varepsilon)} = \frac{\ln 1/\rho}{\ln(1/\varepsilon)} \ ,$$

introduce

$$s = \ln \rho = -(\ln(1/\varepsilon))\tau$$

and seek an expansion

$$f = 1 + \frac{G_1(s)}{\ln(1/\varepsilon)} + \frac{G_2(s)}{\left(\ln(1/\varepsilon)\right)^2} + \dots \ . \tag{4.15b}$$

As before

$$G_1 = B_1 \int_{e^s}^{\infty} \frac{e^{-\tau}}{\tau}\, d\tau \ ,$$

$$G_1 \to B_1\left(-s - \gamma + \dots\right) \qquad \text{as} \quad s \to -\infty.$$

Now try matching by Van Dyke's rule using $s = -(\ln(1/\varepsilon))\tau$:

$$H_2 E_1 f = H_2\left[\log\left(1 + \alpha_0 + \frac{\alpha_0 s}{\ln(1/\varepsilon)}\right)\right] = \ln(1+\alpha_0) + \frac{\alpha_0 s}{(1+\alpha_0)\ln(1/\varepsilon)} \ ,$$

$$E_1 H_2 f = E_1\left[1 + \frac{B_1}{\ln(1/\varepsilon)}\left((\ln(1/\varepsilon))\tau - \gamma + \dots\right)\right] = 1 + B_1\tau = 1 - \frac{B_1 s}{\ln(1/\varepsilon)} \ .$$

Hence, as before,

$$\alpha_0 = e - 1 \quad , \quad B_1 = \frac{1-e}{e} \ .$$

4.7.5 Strained coordinates

The method of strained co-ordinates is a better, but less general way, of solving certain singular perturbation problems. However, usually such problems can also be solved either by using MAEs, or by means of the method of 'Multiple Scales'.

Perturbation Methods 5. Method of Multiple Scales.[†]

MAE: Two or more 'processes' with different scales; processes act separately in different regions.

MS : Two or more processes each with own scale; processes act simultaneously.

Multiple scales is a useful technique for a number of problems. For instance, it underlies much of the theory of '*ray-tracing*'.

One of the simpler, if important, uses of multiple scales is to describe the evolution of linear waves through slowly varying media (e.g. sound waves through the atmosphere). For such examples, the different scales are often immediately apparent (e.g. the wavelength of sound, and the depth of the troposphere). For further discussion of such problems see the *Waves and Stability* course.

We will concentrate on nonlinear problems where the need for two (or more) scales is necessary, but not immediately apparent.

5.1 Van der Pol oscillator.

The Van der Pol oscillator is described by the equation

$$\ddot{x} + \underbrace{\varepsilon\dot{x}(x^2-1)}_{\substack{\text{nonlinear friction}\\ -\text{ve}:\ |x|<1\\ +\text{ve}:\ |x|>1}} + x = 0\ , \qquad t \geq 0\ . \tag{5.1}$$

Typical initial conditions are $x = 1$, $\dot{x} = 0$ at $t = 0$. Solutions are found to tend to a finite amplitude oscillation, during which energy losses when $|x| > 1$ are balanced by energy gains when $|x| < 1$.

5.1.1 Regular perturbation.

Try

$$x = x_0 + \varepsilon x_1 + \dots\ .$$

Then at leading order

$$\ddot{x}_0 + x_0 = 0 \quad \Rightarrow \quad x_0 = \cos t\ .$$

At the next order

$$\ddot{x}_1 + x_1 = \dot{x}_0(1 - x_0^2) = -\sin^3 t$$
$$= -\tfrac{3}{4}\sin t + \tfrac{1}{4}\sin 3t\ ,$$

[†] Corrections and suggestions can be emailed to me at P.H.Haynes@damtp.cam.ac.uk.

and

$$x_1 = \tfrac{3}{8}\left(t\cos t - \sin t\right) - \tfrac{1}{32}\left(\sin 3t - 3\sin t\right) \ .$$

Note that the expansion loses its asymptoticness when

$$\varepsilon x_1 = \mathrm{ord}(x_0) \quad \text{i.e. when} \quad t = \mathrm{ord}\left(\tfrac{1}{\varepsilon}\right) \ .$$

The 'problem' is that the ε-damping term slowly changes the oscillation amplitude on a time scale of $\mathrm{ord}(\varepsilon^{-1})$ by the slow accumulation of small effects.

5.1.2 Multiple scales expansion.

The oscillator has two processes:

Harmonic oscillation on time scale of ord(1).	Slow drift in amplitude (and possible phase) on time scale of $\mathrm{ord}(\varepsilon^{-1})$.
$\tau = t$	$T = \varepsilon t$
The 'fast' time scale.	The 'slow' time scale.

We treat τ and T as independent variables:

- the rapidly changing features are modelled by τ,
- the slowly changing features are modelled by T.

Hence we seek a solution with the form

$$x(t;\varepsilon) = x(\tau, T; \varepsilon) \ ,$$

where the two variables are introduced as an artifice in order to remove secular effects. We use the chain rule to compute derivatives:

$$\frac{d}{dt}x(t;\varepsilon) = \frac{\partial x}{\partial \tau}(\tau, T;\varepsilon) + \varepsilon\frac{\partial x}{\partial T}(\tau, T;\varepsilon) \ ,$$
$$\ddot{x} = x_{\tau\tau} + 2\varepsilon x_{\tau T} + \varepsilon^2 x_{TT} \ .$$

We now seek an asymptotic expansion of the form

$$x = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \dots \ ,$$

and require the expansion to be valid for $T = \mathrm{ord}(1)$, i.e. $t = \mathrm{ord}(\varepsilon^{-1})$. Then at leading order

ε^0:

$$x_{0\tau\tau} + x_0 = 0 \ , \quad t \geq 0 \ ,$$
$$x_0 = 1 \ , \quad x_{0\tau} = 0 \ , \quad \text{at} \quad t = 0 \ .$$

This has solution

$$x_0 = R_0(T)\,\cos\left(\tau + \theta_0(T)\right)\ ,$$

where, in order to satisfy the initial conditions,

$$R_0(0) = 1 \quad , \quad \theta_0(0) = 0\ .$$

The functions R_0 and θ_0 are not fixed at this stage — we need equations for them. At next order we have that

$$\begin{aligned} \varepsilon^1: \qquad x_{1\tau\tau} + x_1 &= -x_{0\tau}\left(x_0^2 - 1\right) - 2x_{0\tau T} \\ &= 2R_0\theta_{0T}\cos\left(\tau + \theta_0\right) + \left(2R_{0T} + \tfrac{1}{4}R_0^3 - R_0\right)\sin(\tau + \theta_0) \\ &\qquad + \tfrac{1}{4}R_0^3\sin 3(\tau + \theta_0)\ , \end{aligned}$$

together with the initial conditions

$$x_1 = 0 \quad , \quad x_{1\tau} = -x_{0T} = -R_{0T} \quad \text{at} \quad t = 0\ .$$

The solution is

$$\begin{aligned} x_1 = R_0\theta_{0T}\,\tau\sin(\tau + \theta_0) - \tfrac{1}{2}\left(2R_{0T} + \tfrac{1}{4}R_0^3 - R_0\right)\tau\cos(\tau + \theta_0) \\ - \tfrac{1}{32}R_0^3\sin 3(\tau + \theta_0) + R_1\sin\left(\tau + \theta_1(T)\right)\ . \end{aligned}$$

However, the asymptotic expansion will not be valid for $\tau = \mathrm{ord}(\varepsilon^{-1})$ unless

$$R_0\theta_{0T} = 0 \quad , \quad 2R_{0T} + \tfrac{1}{4}R_0^3 - R_0 = 0\ . \tag{5.2}$$

This is the 'secularity' or 'integrability' condition of Poincaré. Using the initial conditions we deduce that

$$\theta_0 = 0 \quad , \quad R_0 = \frac{2}{(1 + 3e^{-T})^{\frac{1}{2}}}\ .$$

In particular note that $R_0 \to 2$ as $T \to \infty$.

It follows that the solution for x_1 becomes

$$x_1 = R_1\sin\left(\tau + \theta_1(T)\right) - \tfrac{1}{32}R_0^3\sin 3\tau\ ,$$

while the initial conditions for R_1 and θ_1 become

$$\begin{gathered} R_1(0)\,\sin\left(\theta_1(0)\right) = 0\ , \\ R_1(0)\,\cos\left(\theta_1(0)\right) - \tfrac{3}{32}R_0^3(0)\,\cos\left(3\theta_1(0)\right) = -R_{0T}(0)\ , \end{gathered}$$

i.e.

$$\theta_1(0) = 0 \quad , \quad R_1(0) = -\tfrac{9}{32}\ .$$

The equations governing R_1 & θ_1 are determined by the secularity condition for the x_2 problem. However, we then find that there is insufficient freedom in R_1 and θ_1 to avoid breaking the asymptoticness when $T = \text{ord}(1)$. This problem can be avoided by introducing a super slow time scale, $T_2 = \varepsilon^2 t$.

- *Alternative approach to deriving (5.2).* Instead of solving explicitly for x_1, we could use a condition that requires x_1 to be periodic over the time scale τ. For instance, we could require that (cf. inner products and Sturm-Liouville operators)

$$\int_0^{2\pi} (x_{1\tau\tau} + x_1) \, {}^{\sin}_{\cos} (\tau + \theta_0) \, d\tau = 0 \ ,$$

  i.e. that

$$\int_0^{2\pi} \left(x_{0\tau}(x_0^2 - 1) + 2x_{0\tau T}\right) {}^{\sin}_{\cos} (\tau + \theta_0) \, d\tau \ .$$

  On performing the integrals, (5.2) is again recovered.

5.1.3 A simple example of super slow time scale.

Consider the exact solution to the equation

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0 \ ,$$

i.e.

$$x = e^{-\varepsilon t} \cos\left((1 - \varepsilon^2)^{\frac{1}{2}} t\right) \ .$$

This has:

(*a*) an oscillation on the time scale $t = \text{ord}(1)$,

(*b*) an amplitude drift on the time scale $t = \text{ord}(\varepsilon^{-1})$, and

(*c*) a phase drift on the time scale $t = \text{ord}(\varepsilon^{-2})$.

In general, when working to $\text{ord}\left(\varepsilon^k\right)$ on a time scale $\text{ord}\left(\varepsilon^{k-n}\right)$, one must expect to have a hierarchy of n slow time scales.

5.2 Mathieu Equation.

As a further example of multiple-scales consider solutions to the Mathieu equation:

$$\ddot{y} + \left(\omega^2 + \varepsilon \cos t\right) y = 0 \ .$$

The coefficients are 2π-periodic. This equation describes the small amplitude oscillations of a pendulum whose length changes slightly in time. If the natural oscillation frequency is near a multiple of *half* the forcing frequency, then the amplitude of the pendulum will increase in time. This is an example of **parametric excitation.**

5.2.1 *Floquet Theory (for second order ODEs).*

First note that, since the coefficients of the Mathieu equation are 2π periodic, if $y(t)$ is a solution, then $y(t + 2\pi)$ is also a solution. Further since the equation is second order, we can write the general solution as

$$y = Ay_1(t) + By_2(t) \ .$$

Combining these results we see that

$$y_j(t + 2\pi) = \alpha_j y_1(t) + \beta_j y_2(t) \ ,$$

and hence

$$\begin{aligned} y(t + 2\pi) &= (A\alpha_1 + B\alpha_2)\, y_1(t) + (A\beta_1 + B\beta_2)\, y_2(t) \\ &= A' y_1 + B' y_2 \ . \end{aligned}$$

In matrix notation

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}}_{P} \begin{pmatrix} A \\ B \end{pmatrix} \ .$$

Suppose (A, B) is an eigenvector of P with eigenvalue λ; then

$$A' = \lambda A \quad , \quad B' = \lambda B \ ,$$

and

$$y(t + 2\pi) = \lambda y(t) \quad \text{for all } t \ . \tag{5.3}$$

Let $\mu = \ln \lambda / 2\pi$ and define

$$\varphi(t) = e^{-\mu t} y(t) \ .$$

Then from (5.3)

$$\varphi(t + 2\pi) = \varphi(t) \quad \text{for all } t \ ,$$

and hence

$$y(t) = e^{\mu t} \varphi(t) \ ,$$

where $\varphi(t)$ is a 2π-*periodic* function. The solution is said to be

$$\begin{aligned} \text{unstable if} \quad & \Re(\mu) > 0 \ , \text{ and} \\ \text{stable if} \quad & \Re(\mu) \le 0 \ . \end{aligned}$$

In the case of the Mathieu equation, if $y(t)$ is a solution, so is $y(-t)$. Thus for stability we must have $Re(\mu) = 0$.

It is possible to show that there are regions of the (ω^2, ε) plane where solutions are stable, and other regions where solutions are unstable. We will attempt to find the 'stability boundaries' when $|\varepsilon| \ll 1$.

5.2.2 $\omega \neq n/2$.

Try the Poincaré expansion

$$y = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \dots .$$

From substitution into the Mathieu equation we obtain:

ε^0: $\quad \ddot{y}_0 + \omega^2 y_0 = 0 ,$

ε^1: $\quad \ddot{y}_1 + \omega^2 y_1 = -y_0 \cos t .$

If we seek a real solution, then

$$y_0 = A_0 \exp\bigl(\mathrm{i}\omega t\bigr) + \text{c.c.} ,$$

and

$$\ddot{y}_1 + \omega^2 y_1 = -\tfrac{1}{2} A_0 \exp\bigl(\mathrm{i}(\omega+1)t\bigr) - \tfrac{1}{2} A_0 \exp\bigl(\mathrm{i}(\omega-1)t\bigr) + \text{c.c} .$$

It follows that there are 'secular' terms if $\omega \pm 1 = \mp\omega$, i.e. if $\omega = \mp\frac{1}{2}$ (without loss of generality, henceforth assume $\omega > 0$). Further, it is possible to show that higher-order terms are secular only if $\omega = n/2$. Thus if $\omega \neq n/2$, we can solve at all orders to show that

$$y(t) = \exp\bigl(\mathrm{i}\omega t\bigr)\,\varphi(t) + \text{c.c.} ,$$

where φ is 2π-periodic.

We conclude that for $\varepsilon \ll 1$ and $\omega \neq n/2$, the solution is stable.

5.2.3 $|\omega^2 - 1| \ll 1$.

$$\omega^2 = 1 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots$$

However, from §5.2.2 it follows that resonance will only occur at second order. Hence if $a_1 \neq 0$, we expect there to be no instability; thus we set $a_1 = 0$.

ε^0 : 1^{st} harmonic
ε^1 : 0^{th} & 2^{nd} harmonics
ε^2 : 1^{st} & 3^{rd} harmonics
↑can force resonance

This suggests that we should consider an $\text{ord}(\varepsilon^{-2})$ slow time scale. Try

$$\tau = t \quad , \quad T = \varepsilon^2 t$$
$$y = y_0(\tau, T) + \varepsilon y_1(\tau, T) + \varepsilon^2 y_2(\tau, T) + \dots .$$

At leading order the governing equations is

$$\varepsilon^0 : \qquad y_{0\tau\tau} + y_0 = 0 \ ,$$

with solution

$$y_0 = A_0(T)e^{i\tau} + \text{c.c.} \ .$$

At next order

$$\varepsilon^1 : \qquad \begin{aligned} y_{1\tau\tau} + y_1 &= -y_0 \cos\tau \\ &= -\tfrac{1}{2}A_0(e^{2i\tau} + 1) + \text{c.c.} \ , \end{aligned}$$

with solution

$$y_1 = -\tfrac{1}{2}A_0 + \tfrac{1}{6}A_0 e^{2i\tau} + \text{c.c.} \ ,$$

where any homogeneous component can [usually] be absorbed by a suitable redefinition of A_0. At next order

$$\varepsilon^2 : \qquad \begin{aligned} y_{2\tau\tau} + y_2 &= -2y_{0\tau T} - a_2 y_0 - y_1 \cos\tau \\ &= \left(-2iA_{0T} + (\tfrac{1}{6} - a_2)A_0 + \tfrac{1}{4}\bar{A}_0\right) e^{i\tau} - \tfrac{1}{12}A_0 e^{3i\tau} + \text{c.c.} \ . \end{aligned}$$

For asymptoticness not to be lost when $T = \text{ord}(1)$, it follows from the secularity condition that

$$2\beta_T + \left(\tfrac{5}{12} - a_2\right)\alpha = 0 \quad , \quad 2\alpha_T + \left(\tfrac{1}{12} + a_2\right)\beta = 0 \ ,$$

where $A_0 = \alpha + i\beta$. Hence the oscillation is unstable on the slow time scale T if

$$\left(\tfrac{5}{12} - a_2\right)\left(\tfrac{1}{12} + a_2\right) > 0 \ ,$$

i.e. if

$$-\tfrac{1}{12} < a_2 < \tfrac{5}{12} \ .$$

5.3 WKBJLG† Theory.

This theory is concerned with asymptotic solutions to equations with slowly varying coefficients, e.g.

$$\ddot{x} + f(\varepsilon t)\, x = 0 \ . \qquad (5.3)$$

Its generalisation to two or more independent variables is called *ray theory*.

† Omit the J if not in Cambridge; omit the LG if a physicist.

5.3.1 Leading-order solution

Initially assume that $f = \omega^2 > 0$, and seek a multiple scales solution:

$$\tau = t\ , \quad T = \varepsilon t\ ,$$
$$x \equiv x(\tau, T) = x_0(\tau, T) + \varepsilon x_1(\tau, T) + \dots\ .$$

Then at leading order

$$x_{0\tau\tau} + \omega^2(T)\, x_0 = 0\ ,$$

with solution

$$x_0 = R_0(T) \cos\left(\omega(T)\, \tau + \theta_0(T)\right)\ .$$

At next order

$$\begin{aligned} x_{1\tau\tau} + \omega^2 x_1 &= -2x_{0\tau T} \\ &= 2(\omega R_0)_T \sin(\omega\tau + \theta_0) + 2\omega R_0\ (\theta_{0T} + \tau\omega_T) \cos(\omega\tau + \theta_0)\ . \end{aligned}$$

The secularity condition implies that

$$\theta_T(T) = -\tau\omega_T(T)\ ,$$

but this is 'impossible', because the fast variable appears in the 'drift' equation for the slow dependence. In some sense we want 'θ to be larger'. Instead we try

$$x_0(\tau, T) = R_0(T) \cos(\theta(T))\ ,$$

where

$$\theta = \tfrac{1}{\varepsilon}\Theta_0(T) + \Theta_1(T) + \dots\ ,$$

so that small variations in Θ_0 produce $\mathcal{O}(1)$ changes in θ. Since

$$\theta_T = \Theta_{0T} + \varepsilon\Theta_{1T} + \dots\ ,$$

it follows that

$$\begin{aligned} \dot{x}_0 &= -R_0\theta_{0T} \sin\theta + \varepsilon\big(R_{0T}\cos\theta - R_0\Theta_{1T}\sin\theta\big) + \dots\ , \\ \ddot{x}_0 &= -R_0\theta_{0T}^2 \cos\theta - \varepsilon\big((2R_{0T}\Theta_{0T} + R_0\Theta_{0TT})\sin\theta + 2R_0\Theta_{1T}\Theta_{0T}\cos\theta\big) + \dots\ . \end{aligned}$$

On substituting these expansions into (5.3) we find that

$$\theta_{0T}^2 = \omega^2\ , \quad \text{i.e.} \quad \theta_{0T} = \omega\ ,$$

where $\omega > 0$ wlog. On applying the secularity condition to the equation for x_1 we obtain

$$\left.\begin{aligned} 2R_0\Theta_{1T}\Theta_{0T} &= 0 \\ 2R_{0T}\Theta_{0T} + R_0\Theta_{0TT} &= 0 \end{aligned}\right\} \qquad \begin{aligned} \Theta_1 &= \text{const.} \\ R_0^2\omega &= \text{const.} \end{aligned}$$

Note that while the 'energy' $E = \frac{1}{2}R_0^2\omega^2$ is not conserved, the 'action' E/ω is conserved. Hence the multiple scales solution has the form

$$x \sim \frac{1}{[f(\varepsilon t)]^{1/4}} \left(a\cos\theta + b\sin\theta\right) , \tag{5.4a}$$

where a and b are constants, and

$$\theta = \int_0^t \left[f(\varepsilon q)\right]^{\frac{1}{2}} dq .$$

A similar analysis is possible if $f < 0$, except that exponentially growing/decaying solutions are found rather than harmonically oscillating ones. In particular

$$x \sim \frac{1}{[-f(\varepsilon t)]^{1/4}} \left(Ae^{\varphi} + Be^{-\varphi}\right) , \tag{5.4b}$$

where and A and B are constants, and

$$\varphi = \int_0^t \left[-f(\varepsilon q)\right]^{\frac{1}{2}} dq .$$

In order to obtain higher order approximations, at first sight it might appear that super slow time scales, $T_n = \varepsilon^n t$, are needed. However this is not so because of a dirty trick (see the last example sheet).

5.3.2 Turning points

What if $f = 0$ at some time? The solutions (5.4) are then singular. In order to investigate this case, we assume without loss of generality that $f(0) = 0$ and $f'(0) > 0$.

We recall that when $\varepsilon t = \text{ord}(1)$, we have (5.4$a$)as solution for $t > 0$,
(5.4b) as solution for $t < 0$.

In order to have a complete solution we need the relationship between (a, b) and (A, B). To this end we observe that when $|\varepsilon t| \ll 1$, then

$$\ddot{x} + \varepsilon t f'(0)\, x \approx 0 .$$

Therefore, all times are of a comparable scale when

$$\frac{x}{t^2} \sim \varepsilon f'(0)\, tx \quad \Rightarrow \quad t \sim \left(\varepsilon f'(0)\right)^{-\frac{1}{3}} .$$

Thus we introduce 'medium time', $\bar{\tau}$, defined by

$$\bar{\tau} = -t\big(\varepsilon f'(0)\big)^{\frac{1}{3}} \ ,$$

and scale x by

$$x = \frac{1}{\varepsilon^{\frac{1}{6}}}\bar{x}_0 + \dots \ .$$

The leading-order governing equation is then Airy's equation,

$$\bar{x}_{0\bar{\tau}\bar{\tau}} - \bar{\tau}\bar{x}_0 = 0 \ ,$$

with solution

$$\bar{x}_0 = \alpha\,\mathrm{Ai}(\bar{\tau}) + \beta\,\mathrm{Bi}(\bar{\tau}) \ , \tag{5.5}$$

where α and β are constants.

This solution must match with those valid when $t = \mathrm{ord}(1)$. First we match to (5.4b) as $\bar{\tau} \to \infty$ and $\varepsilon t \to 0-$. From the asymptotic expansions for the Airy function, etc.

(5.5) $$\bar{x}_0 \sim \frac{1}{\bar{\tau}^{1/4}\sqrt{\pi}}\left(\tfrac{1}{2}\alpha\exp\big(-\tfrac{2}{3}\bar{\tau}^{\frac{3}{2}}\big) + \beta\exp\big(\tfrac{2}{3}\bar{\tau}^{\frac{3}{2}}\big)\right) \ ,$$

(5.4b) $$x_0 \sim \frac{1}{\big[-\varepsilon t f'(0)\big]^{\frac{1}{4}}}\big(A\exp(\varphi) + B\exp(-\varphi)\big) \ ,$$

where

$$\varphi \sim -\tfrac{2}{3}\big[\varepsilon f'(0)\big]^{\frac{1}{2}}(-t)^{\frac{3}{2}} = +\tfrac{2}{3}\bar{\tau}^{\frac{2}{3}}.$$

Hence from matching

$$\frac{\alpha}{2\sqrt{\pi}} = \frac{A}{\big[f'(0)\big]^{1/6}} \ , \qquad \frac{\beta}{\sqrt{\pi}} = \frac{B}{\big[f'(0)\big]^{1/6}} \ .$$

Note that the determination of B this way is 'dangerous' since that part of the solution is exponentially small in (5.4b).

We can similarly match as $\bar{\tau} \to -\infty$, i.e. as $\varepsilon t \to 0+$. From above

(5.5) $$\bar{x}_0 \sim \frac{1}{\sqrt{\pi}(-\bar{\tau})^{1/4}}\big(\alpha\sin\bar{\Theta} + \beta\cos\bar{\Theta}\big) \ , \quad \bar{\Theta} = \tfrac{2}{3}(-\bar{\tau})^{\frac{3}{2}} + \tfrac{1}{4}\pi \ ,$$

(5.4a) $$x_0 \sim \frac{1}{\big[\varepsilon t f'(0)\big]^{1/4}}\big(a\cos\theta + b\sin\theta\big) \ , \quad \theta \sim \tfrac{2}{3}\big[\varepsilon f'(0)\big]^{\frac{1}{2}} t^{\frac{3}{2}} = \tfrac{2}{3}(-\bar{\tau})^{\frac{3}{2}} \ .$$

These two expansions match if:

$$\frac{a}{\big[f'(0)\big]^{1/6}} = \frac{(\alpha+\beta)}{(2\pi)^{1/2}} \ , \qquad \frac{b}{\big[f'(0)\big]^{1/6}} = \frac{(\alpha-\beta)}{(2\pi)^{1/2}} \ .$$

We therefore have the *connection formulae*

$$A = \frac{a+b}{2\sqrt{2}} \ , \quad B = \frac{a-b}{\sqrt{2}} \ .$$

5.4 Ray Theory.

Consider waves propagating through a slowly varying medium. Assume that they are governed by

$$L(\partial_t, \partial_x; \epsilon x, \epsilon t)Q = \epsilon N(\partial_t, \partial_x, Q; \epsilon x, \epsilon t, \epsilon) \ , \tag{5.6a}$$

where L is a linear operator,

N is a nonlinear operator,

and $X = \epsilon x$ and $T = \epsilon t$ represent the slowly varying nature of the medium. For instance

$$LQ \equiv \left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x}(c^2(X,T)\frac{\partial}{\partial x})\right)Q = 0 \ . \tag{5.6b}$$

Seek a solution of the form

$$Q = [A_0(X,T) + \epsilon A_1(X,T) + \ldots]\exp\left(i\frac{\theta}{\epsilon}(X,T)\right) + c.c. \ .$$

Then

$$Q_t = i\theta_T[A_0 + \epsilon A_1 + \ldots]e^{i\theta/\epsilon} + \epsilon[A_{0T} + \epsilon A_{1T} + \ldots]e^{i\theta/\epsilon} + c.c. \ ,$$

and the leading order approximation to (5.6) becomes

$$L(i\theta_T, i\theta_X; X, T) = 0 \ ,$$

or

$$L(-i\omega, ik; X, T) = 0 \ , \tag{5.7a}$$

Dispersion Relation

where $w = -\theta_T$ is the real frequency, and $k = \theta_X$ is the real wave number.

(5.7a) is often rewritten in the form

$$w = \Omega(k; X, T) \ . \tag{5.7b}$$

Also if $|\Delta T|, |\Delta X| \ll 1$, then

$$\exp\left(i\frac{\theta}{\epsilon}(X+\Delta X, T+\Delta T)\right) \approx \exp\left(\frac{i\theta(X,T)}{\epsilon}\right)\exp\left(\frac{i\theta_X}{\epsilon}\epsilon\Delta x + \ldots \frac{i\theta_T}{\epsilon}\epsilon\Delta t\right)$$
$$\approx \exp\left(\frac{i\theta(X,T)}{\epsilon}\right)\exp\left(ik\Delta x - i\omega\Delta t + \ldots\right) \ ,$$

hence the definitions of ω and k are consistent with convention. Further, because

$$\theta_{XT} - \theta_{TX} = 0 \ ,$$

it follows that

$$k_T + w_X = 0 \ ,$$

and hence from (5.7b) that

$$k_T + c_g k_X = -\frac{\partial\Omega}{\partial X} \ , \tag{5.8a}$$

where $c_g = \frac{\partial\Omega}{\partial k}$ is the group velocity.

In characteristic form

$$\frac{dk}{dT} = -\frac{\partial\Omega}{\partial X} \qquad \text{on} \qquad \frac{dX}{dT} = c_g \ . \tag{5.8b}$$

A **ray** is a path along the characteristic tranversed with speed c_g. In general rays are curved.

Exercise. Show that

$$\frac{d\omega}{dT} = \frac{\partial\Omega}{\partial T} \ . \tag{5.8c}$$

Hamilton's Equations.

Consider the transformations:

$$\begin{aligned} X &\rightarrow q \\ k(X,T) &\rightarrow p \\ \Omega(k;X,T) &\rightarrow H(q,p,T) \ , \end{aligned}$$

then (5.8b) becomes

$$\frac{dp}{dT} = -\frac{\partial H}{\partial q} \ , \quad \frac{dq}{dT} = \frac{\partial H}{\partial p} \ .$$

These are just Hamilton's equations; hence waves move like particles with speed c_g.

Further, from (5.7b)

$$\frac{\partial\theta}{\partial T} + H(q, \frac{\partial\theta}{\partial q}, T) = 0 \ .$$

This is the Hamilton-Jacobi equation with the phase, $\theta(q,T)$, as the action.

5.4.1 Example

Consider the equation

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x}\left(c^2(X,T)\frac{\partial}{\partial x}\right)\right)\phi = 0 \ . \tag{5.6b}$$

Substitute

$$\phi = (A_o(X,T) + \epsilon A_1(X,T) + \ldots)\exp\left(\frac{i\theta(X,T}{\epsilon}\right) + c.c. \ .$$

Then

$$\begin{aligned} \epsilon^0 : \quad & -\omega^2 A_0 + c^2k^2A_0 = 0 \ ; \\ \epsilon^1 : \quad & -\omega^2 A_1 - i(\omega_T A_0 + 2\omega A_{0T}) - 2cc_X ikA_0 \\ & \quad + c^2k^2A_1 - ic^2(k_X A_0 + 2kA_{0X}) = 0 \ . \end{aligned}$$

Hence

$$\omega = \pm ck \ , \qquad \textbf{Dispersion Relation}$$

and

$$(\omega A_0^2)_T + 2cc_X kA_0^2 + c^2(kA_0^2)_X = 0 \ ,$$

or

$$(\omega A_0^2)_T + (c_g \omega A_0^2)_X = 0 \ , \tag{5.9a}$$

where $c_g = \pm c = \frac{\omega}{k}$. In this case no further information comes from the complex conjugate equation. Write

$$A_0 = r_0 e^{i\psi_0} \ ,$$

then

$$\psi_{0T} + c_g \psi_{0X} = 0 \ ,$$

and

$$(\omega r_0^2)_T + (c_g \omega r_0^2)_X = 0 \ . \tag{5.9b}$$

The energy density of a wave satisfying (5.6*b*) is

$$E = \frac{1}{2}\omega^2 r_0^2,$$

hence (5.9*b*) represents conservation of **wave action** $\omega^{-1}E$.

[To see that E is the energy density show that

$$\frac{d}{dt}\int_0^{2\pi/k} \left(\frac{1}{2}\phi_t^2 + \frac{1}{2}c^2\phi_x^2\right) dx = 0 \ .]$$

Perturbation Methods 6. Asymptotics beyond all orders.†

Sometimes it is *not* sufficient to consider the asymptotic expansion of a solution. Indeed even a solution obtained to all orders can fail to give an accurate answer.

6.0 A model equation.

Consider the asymptotic solution to

$$f_{yy} + \lambda^3(1+iy)f = -\lambda^2, \qquad f \to 0 \text{ as } |y| \to \infty \ , \tag{6.A}$$

for large $|\lambda|$, and real y. Try

$$\lambda f = f_0 + \frac{f_1}{\lambda^3} + \frac{f_2}{\lambda^6} + \ldots = \sum_{n=0}^{\infty} \frac{f_n}{\lambda^{3n}} \ . \tag{6.B}$$

Then

$$f_0 = -\frac{1}{1+iy} \ , \qquad \text{and for } n = 0,1,2,\ldots \quad f_{n+1} = -\frac{f_n''}{(1+iy)} \ .$$

Hence

$$f_1 = -\frac{2}{(1+iy)^4} \ , \ \text{etc. .}$$

Thus an asymptotic expansion can be found to all orders, irrespective of the sign of λ. Further, the expansion satisfies the boundary conditions as $|y| \to \infty$. However the expansion (6.B) is only valid $\forall y$ if $\lambda \to -\infty$. To see this note that the exact solution is

$$f(y,\lambda) = \int_c \exp\left(\lambda(1+iy)z - \frac{1}{3}z^3\right) dz \ , \tag{6.C}$$

where c starts from $z=0$ and extends to $z=\infty$ in the sector $|\arg(z)| < \pi/6$.

If $\lambda \to -\infty$, (6.B) is recovered by Watson's Lemma.

If $\lambda \to +\infty$, and $|y| > \sqrt{3}$, then (6.B) is recovered, but if $\lambda \to +\infty$, and $|y| < \sqrt{3}$, then

$$f \sim \frac{\pi^{\frac{1}{2}}}{\lambda^{\frac{1}{4}}(1+iy)^{\frac{1}{4}}} \exp\left(\frac{2}{3}\lambda^{\frac{3}{2}}(1+iy)^{\frac{3}{2}}\right) \ , \tag{6.D}$$

which is exponentially large.

To understand this result, note that the equation has a turning point at

$$1 + iy = 0 \ .$$

Set

$$y = i + \left(\frac{i}{\lambda^3}\right)^{\frac{1}{3}} s \ ,$$

† Corrections and suggestions can be emailed to me at P.H.Haynes@damtp.cam.ac.uk.

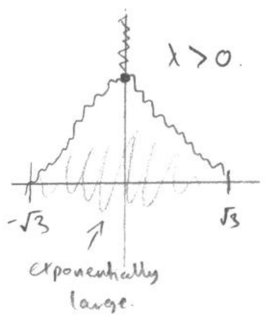
then

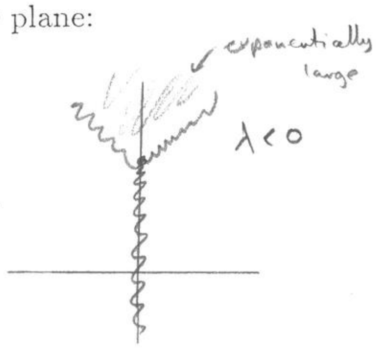
$$f_{ss} - sf = -i^{\frac{2}{3}} \ .$$

The complementary functions are $Ai(s)$ and $Bi(s)$, which have Stokes lines at

$$\arg s = -\frac{\pi}{3}, \frac{\pi}{3}, \pi \ .$$

We plot these Stokes lines in the complex y plane:





Hence when $\lambda > 0$, we see that since two Stokes lines cross the real y axis, the solution that decays as $|y| \to \infty$ can be exponentially small as $\lambda \to \infty$ for $|y| > \sqrt{3}$, but exponentially large for $|y| < \sqrt{3}$. This is not possible when $\lambda < 0$, since only one Stokes line crosses the real y-axis. Note that in the case when $\lambda > 0$, it is possible to get from $y = -\infty$ to $y = -\infty$ without seeing the exponentially large solution, by deforming into the complex y-plane.

6.1 A model of crystal growth

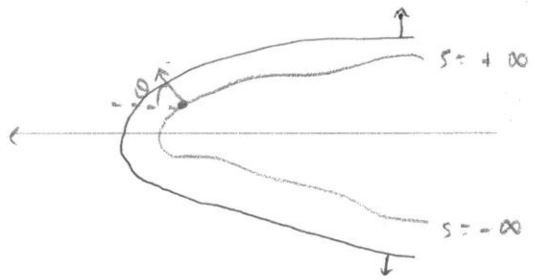
A simple geometric model of crystal growth is:

$$\varepsilon^2\theta''' + \theta' = \cos\theta \qquad -\infty < s < \infty\ ; \tag{6.1a}$$

ε	represents	surface tension;
s	—— " ——	arclength along the solid-liquid interface;
$\theta(s,\varepsilon)$	—— " ——	the angle between the local normal and the direction of propagation of the crystal.

A 'needle crystal' is a monotonic solution satisfying

$$\theta(s,\varepsilon) \to \pm\frac{\pi}{2} \quad \text{as} \quad s \to \pm\infty\ . \tag{6.1b}$$



6.1.1 *Regular perturbation*

Try

$$\theta = \theta_0 + \varepsilon^2\theta_1 + \varepsilon^4\theta_2 + \dots\ . \tag{6.2}$$

We fix the apex at $s = 0$ by requiring that $\theta_j(0) = 0$.

ε^0: $\quad \theta_0' - \cos\theta_0 = 0\ ,$

$$\theta_0 = -\frac{\pi}{2} + 2\tan^{-1}(e^s)\ ,$$

$$\theta_0 \to \pm\frac{\pi}{2} \quad \text{as} \quad s \to \pm\infty\ ,$$

θ_0 increases monotonically.

ε^2: $\quad \theta_0' + \sin\theta_0\,\theta_1 = -\theta_0'''\ ,$

$$\theta_1 = (2\tanh s - s)\operatorname{sech} s\ ,$$

$$\theta_1 \to 0 \quad \text{as} \quad s \to \pm\infty\ .$$

ε^4: $\quad \theta_2 = \left(-\tfrac{1}{2}s^2\tanh s + 5s - 4s\operatorname{sech}^2 s - \tfrac{32}{3}\tanh s + \tfrac{50}{3}\tanh s\operatorname{sech}^2 s\right)\operatorname{sech} s\ ,$

$$\theta_2 \to 0 \quad \text{as} \quad s \to \pm\infty\ .$$

It is possible to prove that: (a) $\theta_j(-s) = -\theta_j(s) \Rightarrow \theta_j''(0) = 0$,

(b) $\sum_0^N \varepsilon^{2n}\theta_j(s) \mp \pi/2 \to 0$ as $s \to \pm s\infty$,

(c) the solution is monotonic for small ε.

Hence we appear to have a solution correct to all orders!

6.1.2 *Too many boundary conditions*

How many boundary conditions are implied by (6.1*b*)?

Suppose we linearise about $s = -\infty$ by setting

$$\theta = -\frac{\pi}{2} + \alpha e^{ms} .$$

We find that

$$\varepsilon^2 m^3 + m = 1$$

$$m = \begin{cases} 1 - \varepsilon^2 + \dots & \text{decays as } s \to -\infty \\ \pm\frac{i}{\varepsilon} - \frac{1}{2} + \dots & \text{grow as } s \to -\infty. \end{cases}$$

Hence we have effectively imposed 2 boundary conditions as $s \to -\infty$. Similarly, we have imposed 2 boundary conditions as $s \to +\infty$.

Thus we have imposed 4 boundary conditions on a 3rd order ODE!

6.1.3 *A well posed problem*

Suppose that we just impose

$$\theta + \frac{\pi}{2} \to 0 \quad \text{as} \quad s \to -\infty . \tag{6.3a}$$

Then a one-parameter family of solutions will exist. We fix the solution by requiring that

$$\theta(0; \varepsilon) = 0 . \tag{6.3b}$$

The question is: 'Does this solution satisfy $(\theta - \frac{\pi}{2}) \to 0$ as $s \to +\infty$?'

Suppose that it does, then a second solution is

$$\Theta(s; \varepsilon) = -\theta(-s; \varepsilon) .$$

Θ and θ differ by at most a translation, hence θ is antisymmetric about *some* point. However, θ is monotonic, analytic and vanishes at $s = 0$, thus

$$\theta(s; \varepsilon) \qquad \text{is antisymmetric about } s = 0 .$$

We conclude that a needle crystal satisfies

$$\theta''(0; \varepsilon) = 0 . \tag{6.4}$$

6.1.4 *Analytical continuation into the complex plane*

We analytically continue solution into the complex s-plane; the continued solution still satisfies

$$\varepsilon^2 \theta''' + \theta' = \cos\theta .$$

For future reference we note that if $\theta(s; \varepsilon)$ is antisymmetric, then

$$\theta(s; \varepsilon) = \sum_0^\infty a_n s^{2n+1} ,$$

and hence $\Re(\theta) = 0$ if s is pure imaginary.

Next we analytically extend the asymptotic expansion (6.2) into the complex s-plane. We note that this asymptotic expansion breaks down near

$$s = \pm(2n+1)\frac{i\pi}{2} \qquad n = 0, 1, 2, \ldots ,$$

because $\operatorname{sech} s = \infty$ near such points. We seek an asymptotic expansion near to one of the points closest to the real axis, i.e. $s = \frac{i\pi}{2}$. In particular, if we let

$$s = \frac{i\pi}{2} + \sigma ,$$

then

$$\theta_0 = -\frac{\pi}{2} + 2i\tanh^{-1}(e^{\sigma}) ,$$

and

$$\theta_0 \sim i\ln\left(-\frac{2}{\sigma}\right) - \frac{\pi}{2} + \ldots \qquad \text{as } \sigma \to 0.$$

Further, from HOT (i.e. higher order terms),

$$\theta \sim -\frac{\pi}{2} + i\left[\ln\left(-\frac{2}{\sigma}\right) - 2\left(\frac{\varepsilon}{\sigma}\right)^2 + \frac{50}{3}\left(\frac{\varepsilon}{\sigma}\right)^4 + \ldots\right] \quad \text{as} \quad \sigma \to 0 .$$

This expansion becomes disordered for $\sigma = \mathcal{O}(\varepsilon)$.

When σ is this small we rescale:

$$s = \frac{i\pi}{2} + \varepsilon z ,$$

$$\theta = i\ln\left(\frac{2}{\varepsilon}\right) - \frac{\pi}{2} + i\varphi(z, \varepsilon) .$$

Then

$$\varphi''' + \varphi' = e^{\varphi} - \left(\frac{\varepsilon}{2}\right)^2 e^{-\varphi} , \qquad (6.5a)$$

and from matching we require that

$$\varphi \to -\ln(-z) - \frac{2}{z^2} + \ldots \quad \text{as} \quad \Re(z) \to -\infty.$$

We seek asymptotic solution to (6.5):

$$\varphi = \varphi_0 + \varepsilon^2\varphi_1 + \ldots ,$$

then

$$\varphi_0''' + \varphi_0' = e^{\varphi_0} \ , \tag{6.5b}$$

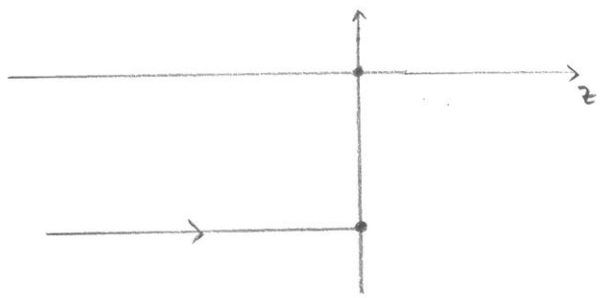
and

$$\varphi_0 \to -\ln(-z) - \frac{2}{z^2} \quad \text{as} \quad \Re(z) \to -\infty. \tag{6.5c}$$

It is possible to prove that $\exists$ a unique solution for φ_0 in $\Re(z) \leq 0$.

Strategy (a) Integrate (6.5*b*) from $\Re(z) = -\infty$ to $\Re(z) = 0$ along a line on which $\Im(z) = \text{constant} < 0$.

(b) Continue this solution down $\Re(z) = 0$ to $s = 0$ and compute $\theta''(0, \varepsilon)$.



Write

$$\varphi_0 = -\ln(-z) + \frac{2}{z^2} + \ldots + \tilde{\varphi} \ , \tag{6.6}$$

and linearise (6.5*b*) for large $|z|$. We find that

$$\tilde{\varphi} = \alpha\tilde{\varphi}_1 + \beta\tilde{\varphi}_2 + \gamma\tilde{\varphi}_3 \ ,$$

where

$$\tilde{\varphi}_1 \sim -\frac{1}{z} + \frac{4}{z^3} + \ldots \ ,$$

$$\tilde{\varphi}_2 \sim z^{\frac{1}{2}} e^{iz} \left(1 + \frac{3}{8}\frac{i}{z} + \ldots\right) \ ,$$

$$\tilde{\varphi}_3 \sim z^{\frac{1}{2}} e^{-iz} \left(1 - \frac{3}{8}\frac{i}{z} + \ldots\right) \ .$$

The matching condition (6.5*c*) implies that if we let $\Re(z) \to -\infty$ along $\Im(z) = \text{constant}$, then we deduce that in this 'direction'

$$\alpha = \beta = \gamma = 0 \ .$$

This does **not** mean that $\alpha = \beta = \gamma = 0$ in the direction specified by $\Im(z) \to -\infty$ with $\Re(z) = 0$. In particular, while we might expect that

$$\alpha = \beta = 0 \quad \text{for} \quad \Im(z) \to -\infty, \quad \Re(z) = 0 \ ,$$

it is possible that $\gamma \neq 0$ because $\varphi_3(z)$ is exponentially small.

Exponentially small terms often do not matter, but on $\Re(z) = 0$, the algebraic terms in (6.6) are real valued, hence as $\Im(z) \to -\infty$ with $\Re(z) = 0$

$$\Im(\varphi_0(z)) \sim -\frac{\pi}{2} + \Gamma |z|^{\frac{1}{2}} e^{-|z|} \left(1 + \mathcal{O}\left(|z|^{-1}\right)\right) ,$$

where $\Gamma = \Im\left(\gamma e^{-i\pi/4}\right)$.

Numerical solutions to (6.5) show that

$$\Gamma \approx 2.11 ;$$

a result that can also be obtained analytically using Borel summation. Hence

$$\Re\left(\theta(s,\varepsilon)\right) \sim -\Gamma\, |z|^{\frac{1}{2}}\, e^{-|z|} \left(1 + \mathcal{O}\left(|z|^{-1}\right)\right)$$

as $\Im(z) \to -\infty$ with $\Re(z) = 0 = \Re(s)$.

But this is non-zero! Hence $\theta(s,\varepsilon)$ is not antisymmetric, and hence it does not represent a needle crystal! Indeed, no needle crystal solutions exist for small ε.

Further analysis shows that by integrating along $\Re(s) = 0$ back to $s = 0$

$$\theta''(0,\varepsilon) \sim 2\Gamma\varepsilon^{-\frac{5}{2}} \exp(-\pi/2\varepsilon) ,$$

which is **exponentially** small.

Perturbation Methods 7. Magic: Summation of Series.†

How do we sum series? E.g. how do we find the value of

$$S_n = \sum_{r=0}^{n} a_r \quad \text{as} \quad n \to \infty \ .$$

For instance what are the sums of

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$,

(b) $1 - 1 + 1 - 1 + \dots$,

(c) $1 + 2 + 4 + 8 + \dots$.

Note that

$$\begin{aligned} S &= 1 - 1 + 1 - 1 + \dots \\ &= 1 - (1 - 1 + 1 - \dots) \\ &= 1 - S \ ; \end{aligned}$$

hence

$$S = \frac{1}{2} \ !$$

The value of the sum depends on the definition of the sum:

Cesàro Sum. $$\mathrm{S} = \lim_{n\to\infty} \frac{S_0 + S_1 + \dots + S_n}{n+1} \ .$$

Euler Sum. Define

$$f(x) = \sum_{r=0}^{\infty} a_r x^r \ .$$

Suppose that this series is convergent for $|x| < 1$; then define the Euler sum to be

$$S = \lim_{x \to 1-} f(x) \ .$$

For instance:

(b) $a_r = (-)^r$,

$f(x) = \sum_{r=0}^{\infty} (-)^r x^r = \frac{1}{1+x}$,

$f(1) = \frac{1}{2}$.

(c) $a_r = 2^r$,

$f(x) = \sum_{r=0}^{\infty} (2x)^r = \frac{1}{1-2x}$,

$f(1) = -1 = 1 + 2 + 4 + 8 + \dots$.

† Corrections and suggestions can be emailed to me at P.H.Haynes@damtp.cam.ac.uk.

(d) $\quad a_r = r ,$

$f(x) = \sum_{r=0}^{\infty} r x^r = \frac{x}{(1-x)^2}$.

Hence the Euler sum of $1 + 2 + 3 + 4 + \ldots$ is not defined.

Borel Sum. If the coefficients a_n grow too fast, then Euler summation is not applicable. However, the power series may still have meaning as an asymptotic series. Define

$$\phi(x) = \sum_{n=0}^{\infty} \frac{a_r x^r}{r!} ,$$
$$B(x) = \int_0^{\infty} e^{-t} \phi(xt) dt .$$

Define the Borel sum to be:

$$S = \lim_{x \to 1-} B(x) .$$

Inverse of Watson's lemma!

For instance consider the prototype Stieltjes series:

$$f(x) = \sum_{r=0}^{\infty} (-)^r r! x^r , \qquad a_r = (-)^r r! .$$

Then

$$\phi(x) = \sum_{r=0}^{\infty} (-)^r x^r = \frac{1}{1+x} ,$$
$$B(x) = \int_0^{\infty} \frac{e^{-t}}{1+xt} dt .$$

Hence

$$0! - 1! + 2! - 3! + \ldots \quad = \int_0^{\infty} \frac{e^{-t}}{1+t} dt.$$

Padé Approximants.

Suppose we only know partial sums. Let

$$\sum_{r=0}^{N+M} a_r x^r = \frac{\sum_{n=0}^{N} A_n x^n}{\sum_{m=0}^{M} B_m x^m} = P_M^N(x) .$$

Often if

$$f(x) = \sum_{r=0}^{\infty} a_r x^r ,$$

then

$$P_M^N(x) \to f(x) \quad \text{as} \quad N, M \to \infty ,$$

even if $\sum_{r=0}^{\infty} a_r x^r$ is divergent.

(a) If $a_r = 1$, then

$$P_N^N(x) = \frac{1}{1-x} \qquad \text{exact !}$$

(b) Stieltjes series, $a_r = (-)^r r!$

$$\begin{aligned} P_5^5(1) &= 0.59738\ldots && \text{11 terms} \\ P_{10}^{10}(1) &= 0.59638\ldots && \text{21 terms} \\ B(1) &= 0.59635\ldots . \end{aligned}$$

Padé Approximants work because they put

- poles near poles
- a cluster of poles at essential singularities
- sequences of poles and zeros along branch cuts.

Continued Fractions.

A variation of the Padé method of summing power series. Define

$$F_N(x) = \cfrac{c_0}{1 + \cfrac{c_1 x}{1 + c_2 x \cfrac{}{\ddots \cfrac{c_{N-1}x}{1 + c_N x}}}}$$

There are fast numerical methods for the evaluation of continued fractions.

Shanks' Transformation.

Suppose

$$S_n = \sum_{r=0}^{n} a_r = A + BC^n ,$$

then

$$S(S_n) = \frac{S_{n+1}S_{n-1} - S_n^2}{S_{n+1} - 2S_n + S_{n-1}} .$$

Can apply repeatedly, e.g. $S(S(S_n))$, to remove higher transients. For instance, consider

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots = 0.693147\ldots$$

Partial Sums	1-Shanks	2-Shanks	3-Shanks
1			
0.5			
0.833	0.7000		
0.583	0.6905		
0.783	0.6944	0.693277	
0.617	0.6924	0.693106	
0.760	0.6936	0.693163	0.693149

Richardson Extrapolation.

Suppose instead

$$S_n \sim Q_0 + \frac{Q_1}{n} + \frac{Q_2}{n^2} + \frac{Q_3}{n^3} + \ldots \quad \text{as} \quad n \to \infty \, .$$

Then if truncate at Q_N,

$$Q_0 = \sum_{k=0}^{N} \frac{S_{n+k}(n+k)^N(-)^{k+N}}{k!(N-k)!} \, .$$

Other Methods.

For instance: Neville tables;
Domb-Sykes plots (to find the nearest singularity);
Euler transformations, etc.