MODULAR FORMS 1

REB 1997

Reminder: the questions on the final exam will be based on example sheet questions.

Some of the questions need facts which have been stated but not yet proved in lectures.

1. Define the complex projective line $P^1(C)$ to be the set of pairs $(w_1 : w_2)$ with $w_1, w2$ complex numbers not both 0, modulo multiplication by nonzero complex numbers λ (so $(w_1 : w_2) = (\lambda w_1, \lambda w_2)$). Show that $P^1(C)$ can be identified with the set of lines through 0 in C^2. Show that the action of $SL_2(C)$ on C^2 gives an action on $P^1(C)$ (identified with a set of lines in C^2) by $\left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right)(w_1 : w_2) = (aw_1 + bw_2 : cw_1 + dw_2)$. Show that $P^1(C)$ can be identified with $C \cup \infty$ by mapping $(w_1 : w_2)$ to w_1/w_2. Show that this induces an action of $SL_2(C)$ on $C \cup \infty$ given by $\left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right)(\tau) = (a\tau + b)/(c\tau + d)$ for $\tau \in C \cup \infty$. Show that this restricts to an action of $SL_2(R)$ on the upper half plane H, given by the same formula.

2. Show that $\Im((a\tau+b)/(c\tau+d)) = \det \left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right)|c\tau+d|^{-2}\Im(\tau)$ for $\left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right) \in GL_2(R)$ (where $\Im(\tau)$ is the imaginary part of τ). (So $\Im(\tau)$ is some sort of "non holomorphic modular form".) Deduce that if w_1, w_2 is an oriented base for a lattice L then $aw_1 + bw_2, cw_1 + dw_2$ for $\left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right) \in GL_2(Z)$ is oriented if and only if $\left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right) \in SL_2(Z)$. Show that $\Im(\tau)$ is the area of a fundamental domain of the lattice spanned by 1 and τ. Show that (non holomorphic) functions f on H with $f((a\tau + b)/(c\tau + d)) = |c\tau + d|^k$ for $\left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right) \in SL_2(Z)$ can be identified with functions of lattices that are invariant under rotation and are real homogeneous of degree $-k$. Show that the area of a fundamental domain of a lattice is such a function (with $k = -2$).

3. Define an action $f \to f|_M$ of $SL_2(R)$ on functions on the upper half plane by $f|_{\left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right)}(\tau) = (c\tau + d)^{-k} f((a\tau + b)/(c\tau + d))$. Check that $f|_{MN} = (f|_M)|_N$. Show that modular forms of weight k are fixed by this action.

4. Given that the space of modular forms of weight 8 is one dimensional and that $E_4(\tau) = 1 + 240\sum_n \sigma_3(n)q^n$ and $E_8(\tau) = 1 + 480\sum_n \sigma_7(n)q^n$ are modular forms of weights 4 and 8, prove that $E_4(\tau)^2 = E_8$ and deduce that $\sigma_7(n) = \sigma_3(n) + 120\sum_{1\le i<n} \sigma_3(i)\sigma_3(n-i)$.

5. The E_8 lattice is defined to be the lattice of vectors $(x_1, \ldots, x_8)$ such that the sum of the x_i's is even, and either they are all integers or they are all integers $+1/2$. Check that this is a unimodular lattice such that the norm (v, v) of every vector v is even. (Unimodular means that the volume of a fundamental domain is 1.) Calculate the number of vectors of the E_8 lattice of norms 2 and 4 in two ways, either by counting them explicitly, or by writing the theta function of the lattice in terms of Eisenstein series E_4. (Recall that its theta function $\sum_{v\in E_8} q^{(v,v)/2}$ is a modular form of weight 4, and the space of modular forms of weight 4 is 1-dimensional and spanned by $E_4(\tau)$.)

6. If $f(\tau)$ is a modular function then show that $g(\tau) = f(2\tau) + f(\tau/2) + f((\tau+1)/2)$ is invariant under $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$ and hence is also a modular function. Find a generalization of this where 2 is replaced by 3. Assuming that any modular function which is holomorphic on the upper half plane and meromorphic at infinity is a polynomial in $j(\tau) = q^{-1}+744+196884q+\ldots$ prove that $j(2\tau)+j(\tau/2)+j((\tau+1)/2) = j(\tau)^2-1488j(\tau)+162000$. If $j(\tau) = \sum_n c(n)q^n$, use this to find some relations between the coefficients $c(n)$.

7. Prove that any modular form of odd weight is 0. (Hint: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL_2(Z)$.) Prove that any modular form of weight not divisible by 4 vanishes at i. (Hint: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(i) = i$.) Prove that any modular form of weight not divisible by 6 vanishes at $\omega = (-1+\sqrt{3}i)/2$. (Hint: $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}(\omega) = \omega$.) Prove that $j(\omega) = 0$ (where $j(\tau) = E_4(\tau)^3/\Delta(\tau)$). Prove that $j(i) = 1728$. (Hint: first find a linear relation between E_4^3, E_6^2, and Δ, given that they are all modular forms of weight 12 and the space of such forms is 2-dimensional.) Prove the stronger statements that $j(\tau)$ has a triple zero at ω, and $j(\tau) - 1728$ has a double zero at i. We will see later that it is possible to find an exact expression for $j(\tau)$ whenever τ satisfies a quadratic equation with integer coefficients.

8. Find a linear relation between E_4^3, E_{12}, and Δ, and use this to prove Ramanujan's congruence $\tau(n) \equiv \sigma_{11}(n) \bmod 691$ (where Ramanujan's function $\tau(n)$ is defined by $\Delta(\tau) = \sum_n \tau(n)q^n$; the two τ's in this equation have nothing to do with each other.)

9. Define the Bernoulli numbers B_n by $x/(e^x-1) = \sum_n B_n x^n/n!$. Prove that $B_n = 0$ if n is odd and greater than 2, and calculate B_n for $0 \le n \le 12$.

10. Let $E(\tau, s) = (1/2)\sum_{(c,d)\neq(0,0)} Im(\tau)^s/|c\tau+d|^{2s}$ (with $\tau \in H$). For which values of s does this converge absolutely? How does this function transform under $SL_2(Z)$? Show that for any fixed s, $E(\tau, s)$ is is an eigenvector of the operator $y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2)$, where $\tau = x+iy$. What is its eigenvalue? (This function is called a real analytic Eisenstein series.)

11. Prove that every element of H is conjugate under the action of $SL_2(Z)$ to an element τ with $|\tau| \ge 1$, $|Re(\tau)| \le 1/2$. When can two elements τ of this form be conjugate under $SL_2(Z)$?

MODULAR FORMS 2

REB 1997

1. Show that if k is an even positive integer then $\zeta(k) = (2\pi)^k|B_k|/k!2$ by comparing the coefficients of powers of z of both sides of $\frac{\pi}{\tan(\pi z)} = 1/z + \sum_{m>0}(1/(z-m)+1/(z+m))$.
Show that if k is a large even integer then $|B_k|$ is about $k!2/(2\pi)^k$.

2.
   a Show that for any integer m, $m^3 \equiv m^5 \bmod 24$ and deduce that $\sigma_3(m) \equiv \sigma_5(m) \bmod 24$.
   b Define $\Delta(\tau)$ to be $(E_4^3 - E_6^2)/1728$ (where $E_4(\tau) = 1 + 240\sum_{n>0}\sigma_3(n)q^n$ and $E_6(\tau) = 1 - 504\sum_{n>0}\sigma_5(n)q^n$). Show that Δ has integral coefficients in its q expansion.
   c Deduce that $1/\Delta$ has integral coefficients.
   d Show that every modular form with integral coefficients can be written as a polynomial in E_4, E_6, and Δ. (Hint: first show that if f is a modular form with integral coefficients which vanishes at $i\infty$ then f/Δ is a modular form with integral coefficients of smaller weight.)
   e Show that $j = E_4^3/\Delta$ has integral coefficients in its q expansion.

3. Define $E_2(\tau)$ to be $1 - \frac{4}{B_2}\sum_{m\geq 1}\sigma_1(m)q^m$ (where $B_2 = \frac{1}{6}$), and define $\Delta(\tau)$ to be $q\prod_{n>0}(1-q^n)^{24}$. Show that $2\pi i E_2(\tau) = \frac{d}{d\tau}\log(\Delta(\tau))$. Assuming that Δ is a modular form of weight 12, show that $E_2(-1/\tau) = \tau^2 E_2(\tau) + 12\tau/2\pi i$. More generally, show that

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{12c(c\tau+d)}{2\pi i}.$$

4.
   a Show that a formal Dirichlet series $\sum_{n>0} a(n)/n^s$ has multipliciative coefficients (i.e., $a(m)a(n) = a(mn)$ whenever m and n are coprime) if and only if it can be written as an Euler product $\prod_p(\sum_n a(p^n)/p^{ns})$.
   b If $a(n)$ and $b(n)$ are the coefficients of two Dirichlet series $f(s)$ and $g(s)$ then show that the coefficients of $f(s)g(s)$ are the numbers $\sum_{d|n} a(d)b(n/d)$. Deduce from part (a) that if a and b are multiplicative functions then so is $\sum_{d|n} a(d)b(n/d)$.
   c Show that the functions $\sigma_k(n) = \sum_{d|n} d^k$ are multiplicative functions of n.
   d Show that $\sum_{n>0}\sigma_k(n)/n^s = \zeta(s)\zeta(s-k)$ and write this Dirichlet series as an Euler product.
   e Show that the product decomposition

$$\sum_{n>0}\frac{\tau(n)}{n^s} = \prod_p \frac{1}{1-\tau(p)p^{-s}+p^{11-2s}}$$

   is equivalent to the statement that $\tau(n)$ is multiplicative and satisfies the relation

$$\tau(p^n) = \tau(p)\tau(p^{n-1}) - p^{11}\tau(p^{n-2})$$

for all primes p and integers $n \geq 2$.

5. A ring R with a bilinear map from $R \times R$ to R, denoted by $[r, s]$ for $r, s \in R$, is called a Poisson algebra if it satisfies the conditions $[a, a] = 0$, $[ab, c] = a[b, c] + [a, c]b$, and $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ for all a, b, c in R. (In particular it is a Lie algebra under $[,]$.)
   a If R is any ring and $[a, b]$ is defined to be $ab - ba$ show that R becomes a Poisson algebra. If R is the ring of smooth functions of 2 variables p and q and $[f, g]$ is defined to be $\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}$ show that R is a Poisson algebra.
   b If $f(\tau)$ is a modular function show that $f' = \frac{df}{d\tau}$ transforms as if it were a modular form of weight 2.
   c Show that if a and b are modular forms of weights m and n then $[a, b] = (na'b - mab')/4\pi i$ is a modular form of weight $m + n + 2$. (Hint: a^n/b^m is a modular function.)
   d Show that the ring of modular forms is a Poisson algebra under this operation.
   e Show that if a and b have integral coefficients in their q expansions then so does $[a, b]$. (Recall that $\frac{d}{d\tau} = 2\pi i q \frac{d}{dq}$).
   f Show that if a and b have zeros of order j and k at $i\infty$ then $[a, b]$ has a zero of order $j + k$ if $nj \neq mk$, and a zero of order at least $j + k + 1$ if $nj = mk$.
   g Show that $[E_4, E_6] = 1728\Delta$ by using the fact that Δ is the unique modular form of weight 12 whose q expansion starts off $q - \ldots$. Show that $[E_4, \Delta] = -2E_6\Delta$ and $[E_6, \Delta] = -3E_4^2\Delta$ by using the fact that $1728\Delta = E_4^3 - E_6^2$ and the formula $[ab, c] = a[b, c] + [a, c]b$.
   h Show that

$$12\tau(n) = 5n\sigma_3(n) + 7n\sigma_5(n) + 840 \sum_{1 \leq i < n} (2n - 5i)\sigma_3(i)\sigma_5(n - i)$$

   and deduce that $\tau(n) \equiv n\sigma_3(n) \bmod 7$ and $\tau(n) \equiv n\sigma_5(n) \bmod 5$.

6. A derivation ∂ of a Poisson algebra R is an additive map from R to R such that $\partial(ab) = a\partial b + (\partial a)b$ and $\partial[a, b] = [a, \partial b] + [\partial a, b]$.
   a If a is a modular form of weight k define ∂a to be $\frac{12}{2\pi i}\frac{da}{d\tau} - kE_2 a$ where E_2 is defined in question 3. Show that ∂ is a derivation on the Poisson algebra of modular forms. (Do not forget to check that ∂a is a modular form.)
   b Show that $\partial a = 2[a, \Delta]/\Delta$.
   c If a and b have weights m and n then show that $24[a, b] = nb\partial a - ma\partial b$.
   c Show that $\partial E_4 = -4E_6$, $\partial E_6 = -6E_4^2$, and $\partial\Delta = 0$.
   d Show that

$$21\sigma_5(n) = 10(3n - 1)\sigma_3(n) + \sigma_1(n) + 240 \sum_{1 \leq j < n} \sigma_1(j)\sigma_3(n - j).$$

   e Show that $(1-n)\tau(n) = 24\sum_{1 \leq i < n} \sigma_1(i)\tau(n-i)$ and use this to show that the first few coefficients of Δ are $q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 \ldots$.

MODULAR FORMS 3

REB 1997

1. Show that E_4 and E_6 are algebraically independent. (First show that modular forms of different weights are linearly independent, so it is sufficient to show that the forms $E_4^m E_6^n$ are linearly independent for $4m+6n=k$. There are several ways to show that these forms are linearly independent; you can either do this by using the fact that the only zeros of E_4 and E_6 in the fundamental domain are at ω, $-1/\omega$, and i, or by showing that the dimension of the space of modular forms of weight k is at least equal to the number of solutions of $4m+6n=k$, $m \geq 0$, $n \geq 0$.)

2. Let $\Delta_{16}(\tau) = E_4(\tau)\Delta(\tau)$ and define $\tau_{16}(n)$ by $\Delta_{16}(\tau) = \sum_n \tau_{16}(n) q^n$.
   a Show that $\Delta_{16}(\tau)$ spans the vector space of cusp forms of weight 16. (Recall that a cusp form is one that vanishes at $i\infty$.)
   b Show that $\tau_{16}(n)$ is an integer, and that $\tau_{16}(n) \equiv \tau(n) \bmod 240$.
   c Show that
   $$\tau_{16}(n) \equiv \sigma_{15}(n) \bmod 3617.$$
   (Hint: write Δ_{16} as a linear combination of E_{16} and something else. The Bernoulli number B_{16} is $-3617/510$.)

3. a If $f(\tau)$ is defined to be $\sum_{n>0} n^5/(e^{-2\pi i n\tau}+1)$ show that
   $$504 f(\tau) = -E_6(\tau) + 2E_6(2\tau) - 1.$$
   b Show that $\sum_{n>0, n \text{ odd}} n^5/(e^{-2\pi i n\tau}+1) = f(\tau) - 32 f(2\tau)$.
   c Show that
   $$\frac{1}{e^{\pi}+1} + \frac{3^5}{e^{3\pi}+1} + \frac{5^5}{e^{5\pi}+1} + \cdots = 31/504.$$

4. Draw a picture of the fundamental domain D of $SL_2(Z)$ and of some of its conjugates. Let $\Gamma(2)$ be the subgroup of $SL_2(Z)$ consisting of all matrices congruent to $\binom{10}{01}$ mod 2. Show that $SL_2(Z)/\Gamma(2)$ is a nonabelian group of order 6. Find a fundamental domain of $\Gamma(2)$. (One solution is the union of 6 copies $g_i D$ of D where the g_i's run through a set of 6 representatives of the cosets of $\Gamma(2)$ in $SL_2(Z)$. Another solution is the set of complex numbers τ with $Im(\tau) > 0$, $-1 \leq Re(\tau) \leq 1$, $|\tau \pm \frac{1}{2}| \geq \frac{1}{2}$.)

5. Use the Jacobi triple product identity
   $$\prod_{n>0}(1-q^{2n})(1-q^{2n-1}z)(1-q^{2n-1}/z) = \sum_{n\in Z}(-1)^n q^{n^2} z^n$$
   to prove the following formulas.
   $$\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n) = \sum_{n\in Z}(-1)^n q^{\frac{3}{2}(n+\frac{1}{6})^2}$$
   $$\prod_{n>0}(1-q^{2n})(1+q^{2n-1})^2 = \sum_{n\in Z} q^{n^2}$$
   $$\prod_{n>0}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)/2}$$
   $$= 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \cdots$$

(Hint: for the last one, replace z by z/q in Jacobi's identity, then divide both sides by $1-z$, then set z equal to 1, then replace q by $q^{\frac{1}{2}}$.)

6. Show that

$$\eta(\tau) = \sum_{n\in Z}(-1)^n q^{\frac{3}{2}(n+\frac{1}{6})^2} = \sum_{n>0, n\equiv \pm 1 \bmod 6} \pm q^{n^2/24}$$

where in the second sum the sign is $+1$ if $n \equiv \pm 1 \bmod 12$ and -1 otherwise. Apply the Poisson summation formula $\sum_{n\in Z} f(n) = \sum_{n\in Z} \hat{f}(n)$ to the function $f(x) = \exp(3\pi i\tau(x+\frac{1}{6})^2 + x\pi i)$ to show that $\eta(-1/\tau) = \sqrt{\tau/i}\eta(\tau)$.

7. a Show that

$$\prod_{n>0}\frac{1}{1-q^n} = \sum_{n\in Z} p(n)q^n$$

where $p(n)$ is the number of partitions of n, i.e., the number of ways of writing n as a sum of positive integers. (Hint: write $1/(1-q^n)$ as $\sum_{m\geq 0} q^{mn}$ and multiply everything together.)

b Use one of the identities from question 1 to show that if $n \geq 1$ then

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + \cdots.$$

c Use this to work out $p(n)$ for $n \leq 12$. (You should find that $p(12) = 77$.)

8. This question is part of Conway's proof that there is essentially only one even self dual lattice in 24 dimensions with no vectors of norm 2.

a If Λ is such a lattice and c_n is the number of vectors of norm $2n$ in Λ, use the fact that $c_0 = 1$, $c_1 = 0$ to express the theta function of Λ in terms of E_{12} and Δ.

b Show that

$$c_0 + \frac{c_2}{2} + \frac{c_3}{2} + \frac{c_4}{48} = 2^{24}.$$

c (This part may be rather harder than usual.) Use part b to show that if $2v \in \Lambda$ then exactly one of the following 4 possibilities occurs:

1. $v \in \Lambda$.
2. $v = \lambda + v_2/2$ where $\lambda \in \Lambda$, and v_2 is a norm 4 vector in Λ, and there are exactly 2 possible choices for v_2 (whose sum is 0).
3. $v = \lambda + v_3/2$ where $\lambda \in \Lambda$, and v_3 is a norm 6 vector in Λ, and there are exactly 2 possible choices for v_3 (whose sum is 0).
4. $v = \lambda + v_4/2$ where $\lambda \in \Lambda$, and v_4 is a norm 8 vector in Λ, and there are exactly $48 = 2dim(\Lambda)$ possible choices for v_4. Any 2 of the 48 possiblities for v_4 are either orthogonal or have sum 0.

(Hint: out of the 2^{24} elements of $\frac{1}{2}\Lambda/\Lambda$, show that exactly one has property 1, exactly $c_2/2$ have property 2, exactly $c_3/2$ have property 3, and at least $c_4/2dim(\Lambda)$ have property 4, and show that no element has more than one of these properties. You will need to use the fact that Λ has no vectors of norm 2. Then use part b to prove part c.)

d Show that every vector of Λ is congruent mod 2Λ to a vector of norm at most 8.

MODULAR FORMS 4

REB 1997

1. If f is a modular form of weight k, the Hecke operator $T_k(n)$ is defined by

$$(T_k(n)f)(\tau) = n^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_1 \backslash M_n} (c\tau + d)^{-k} f(\frac{a\tau + b}{c\tau + d}).$$

a. Show that $T_k(n)f$ is well defined, and is a modular form of weight k.
b. Show that a set of coset representatives for $M_1 \backslash M_n$ is the set of matrices $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ with $ad = n$, $0 \le b < d$.
c. If $f(\tau) = \sum_m q^m c(m)$ show that

$$(T_k(n)f)(\tau) = \sum_m q^m \sum_{a|(m,n), a>0} a^{k-1} c(mn/a^2).$$

d. Suppose that f is an eigenvector of $T_k(n)$, and is normalized so that $c(1) = 1$. Show that the eigenvalue of $T_k(n)$ is $c(n)$.
e. Show that the Eisenstein series E_k is an eigenvalue of $T_k(n)$ with eigenvalue $\sigma_{k-1}(n)$.

2. a. Prove that $T_k(m)T_k(n) = T_k(mn)$ whenever m and n are coprime.
b. Prove that $T_k(p^n)T_k(p) = T_k(p^{n+1}) + p^{k-1}T_k(p^{k-1})$ whenever p is prime and n is an integer.
c. Use parts a and b to show that $T_k(m)$ commutes with $T_k(n)$ for all m and n.
d. Suppose that $f = \sum c(n)q^n$ is a cusp form with $c(1) = 1$ which is an eigenvector of all the operators $T_k(n)$. Show that

$$\sum_{n>0} \frac{c(n)}{n^s} = \prod_{p\ prime} \frac{1}{1 - c(p)p^{-s} + p^{k-1-2s}}.$$

e. Show that $\Delta(\tau)$ satisfies the conditions of part d, by using the fact that $T_k(n)\Delta$ is a cusp form of weight 12 and the fact that Δ is a basis for the cusp forms of weight 12.
f. Show that $\tau(n)$ is multipliciative, and $\tau(p^{n+1}) = \tau(p)\tau(p^n) - p^{11}\tau(p^{n-1})$.

3. Define an inner product on the space of cusp forms of weight k by

$$(f, g) = \int_D f(\tau)\bar{g}(\tau)y^{k-2} dx\, dy$$

where $\tau = x + iy$ and D is a fundamental domain for $SL_2(Z)$.
a. Show that this is an hermitian inner product.
b. Show that $T_k(n)$ is a self adjoint operator, in other words, $(T_k(n)f, g) = (f, T_k(n)g)$.
c. Show that the space of cusp modular forms of weight k has a canonical basis of forms which are eigenvalues of all Hecke operators and whose coefficient of q^1 is 1. (Recall

that any finite dimensional Hilbert space acted on by a set of commuting self adjoint has a basis of eigenvectors for all these operators.)

d. Find this basis when $k = 24$. (Hint: find two cusp forms f and g of weight 24 and look for linear combinations of the form $\sum c(n)q^n$ with $c(1) = 1$ and $c(4) = c(2)^2 - 2^{23}$.)

e. Show that all the coefficients of the elements of this basis are totally real algebraic integers. (Hint: If T is a self adjoint operator on a finite dimensional Hilbert space H such that $H = L \otimes_Z C$ for some free abelian group L acted on by T, then the eigenvalues of T are all totally real algebraic integers. We can take L to be the cusp forms all of whose coefficients are integers.)

f. Show that if $f = \sum c(n)q^n$ is one of the elements of this basis, then there is a finite extension field K of the rationals such that all the coefficients of f are in K. If σ is any field homomorphism of K into C then show that $\sum \sigma(c(n))q^n$ is also an element of the canonical basis of cusp forms. (Hint: use the fact that there is a basis of forms with integral coefficients.)

4. Show that

$$T_k(m)T_k(n) = \sum_{d|(m,n)} d^{k-1}T_k(mn/d^2)$$

for all positive integers m, n. (Hint: show that this follows from the cases m, n coprime and the case when n is a power of the prime m.)

MODULAR FORMS 5

REB 1997

Warning: some of the problems on this sheet (2,5) are rather hard.

1. Find all Dirichlet series $L(s) = \sum_n c(n)/n^s$ with multiplicative coefficients such that $L(s)$ converges for $Re(s)$ sufficiently large and $L^*(s) = \Gamma(s)(2\pi)^{-s}L(s)$ extends to a holomorphic function on C which is rapidly decreasing in vertical strips and satisfies $L^*(24 - s) = L^*(s)$.

2. Let $L^*(s) = (2\pi)^{-s}\Gamma(s)\sum_n \tau(n)/n^s$. The (unproved) Riemann hypothesis for L^* says that all zeros of L^* have real part 6. In this question we will show that L^* has an infinite number of zeros with real part 6. It is based on Hardy's original proof of the corresponding fact for the Riemann zeta function.

a. Use the functional equation of L^* to show that $L^*(s)$ is real whenever $Re(s) = 6$.

b. Show that

$$\Delta(ie^{i\theta}) = \frac{e^{-6i\theta}}{2\pi}\int_{-\infty}^{\infty} L^*(6+iy)e^{y\theta}dy$$

whenever θ is real and $|\theta| < \pi/2$.

c. Show that $\Delta(ie^{i\theta})$ tends to 0 as θ tends to $\pi/2$.

d. Show that if $L^*(6+iy)$ is never 0 then it is always positive, and show that in this case the integral in part b does not tend to 0 as θ tends to $\pi/2$.

e. Show that $L^*(6+iy) = 0$ for some real y.

f. Show that all derivatives of $\Delta(ie^{i\theta})$ tend to 0 as θ tends to $\pi/2$.

g. Deduce that if p is any polynomial then

$$\int_{-\infty}^{\infty} p(y)L^*(6+iy)e^{y\theta}dy$$

tends to 0 as θ tends to $\pi/2$.

h. Show that if $L^*(6+iy)$ has only a finite number of zeros and p is a real polynomial with the same zeros, then the integral in part g does not tend to 0 as θ tends to $\pi/2$, and deduce that $L^*(s)$ has an infinite number of zeros s with $Re(s) = 6$.

3. Let $L(s) = \sum \tau(n)/n^s$.

a. Show that this series converges for $Re(s) > 7$, and can be extended to a holomorphic function on C which vanishes for $s = 0, -1, -2, \ldots$. (Recall that $\tau(n) = O(n^6)$.)

b. Show that the product $\prod_p 1/(1 - \tau(p)p^{-s} + p^{11-2s})$ converges for $Re(s) > 7$.

c. Show that the only zeros s of $L(s)$ other than $0, -1, -2, \ldots$ have $5 \le Re(s) \le 7$.

4. If a/c is a rational number or ∞ let $S_{a/c}$ be the image of the line $Im(\tau) = 1$ under the map $\tau \to \frac{a\tau+b}{c\tau+d}$ for some $\binom{ab}{cd} \in SL_2(Z)$.

a Show that $S_{a/c}$ is well defined and is a circle tangent to the real axis touching it at a/c.

b Show that $S_{a/c}$ touches $S_{b/d}$ if and only if $ad - bc = \pm 1$.

c Draw a picture of the circles $S_{a/c}$ for $0 \le a/c \le 1$, $0 \le c < 7$.

5.

a. Show that

$$\int_{S_{a/c}} \frac{e^{-2\pi i n\tau}}{\Delta(\tau)} d\tau = \frac{2\pi e^{2\pi i(d-na)/c}}{c} \sum_{j\ge 0} (2\pi/c)^{13+2j} n^j / j!(13+j)!$$

where d is any integer with $ad \equiv 1 \bmod c$, Hint: use the functional equation of Δ to convert the integral over each circle into an integral over a line, then work out the integral over the line as a residue.

b. Show that if $\Delta(\tau)^{-1} = \sum_n p_{24}(n+1)q^n$ then

$$p_{24}(1+n) = 2\pi n^{-13/2} \sum_{c>0} \frac{I_{13}(4\pi\sqrt{n}/c)}{c} \sum_{0\le a,d<c, ad\equiv 1 \bmod c} e^{2\pi i(d-na)/c}$$

where $I_{13}(z) = \sum_{j\ge 0} (z/2)^{13+2j}/j!(13+j)!$. Hint: show that the integral from i to $1+i$ is equal to the sum of the integrals over all the circles of question 4 which touch the real time in a rational point between 0 and 1. Then use part a.

6. We define numbers α_p and β_p by $x^2 - \tau(p)x + p^{11} = (x-\alpha_p)(x-\beta_p)$.

a. Show that

$$\sum \frac{\tau(n)}{n^s} = \prod_p \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}.$$

b. Show that

$$\tau(p^k)^2 = \frac{(\alpha_p^{k+1} - \beta_p^{k+1})^2}{(\alpha_p - \beta_p)^2}.$$

c. Show that

$$\sum_k \frac{\tau(p^k)^2}{p^{ks}} = \frac{1+p^{11-s}}{(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s})}.$$

d. Show that

$$\sum_n \frac{\tau(n)^2}{n^s} = \frac{\zeta(s-11)}{\zeta(2s-22)} \prod_p \frac{1}{(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s})}.$$