Knot Theory.

1. Introduction.

Definition: A link with n components consists of n piecewise linear simple closed curves. A 1-component link is a knot. A link $\subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. S^3 is considered to be oriented. Links are sometimes oriented, sometimes not.

Definition: Links L_1 and L_2 are equivalent, written $L_1 \sim L_2$ if $\exists$ orientation-preserving homeomorphism, piecewise linear, $h: S^3 \to S^3$, $h(L_1) = L_2$.
In fact, $L_1 \sim L_2 \Rightarrow \exists\, h_t : S^3 \to S^3$, $t \in [0,1]$, $h_0 = 1$, $h_1 = h$, continuous in t.

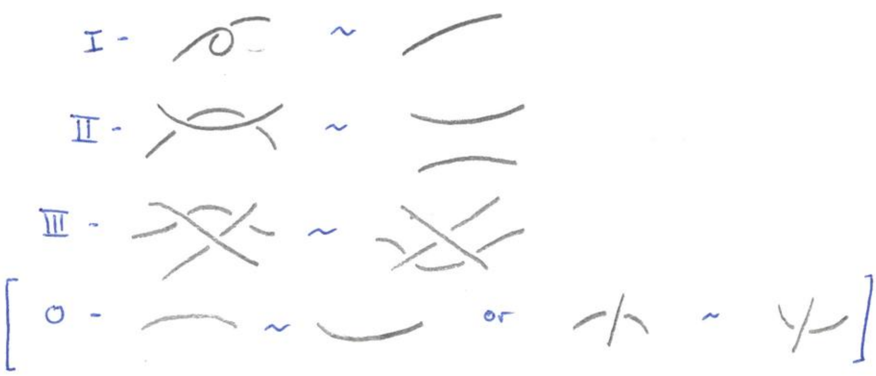
It can be shown that for L_1 we may produce $L_2 \sim L_1$ by triangle moves:

By means of these we can always change a given L to be in general position wrt $p: \mathbb{R}^3 \to \mathbb{R}^3$. We never have on the projection:

We only ever see:

Add under/over information and we have a link diagram.

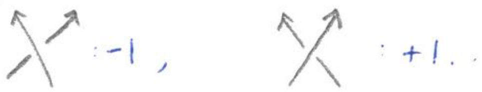
Any two diagrams of a link L are related by Reidemeister moves of the following 3 types:



Higher dimension knot theory concerns $S^n \subset S^{n+2}$
However, $S^1 \xrightarrow[\text{p.l.}]{} S^2$ - Theorem: S^1 bounds p.l. disc in S^2. (p.l. = piecewise linear)
$S^2 \xrightarrow[\text{p.l.}]{} S^3$ - S^2 divides S^3 into 2 components, the closure of each being a p.l. ball.
$S^3 \xrightarrow[\text{p.l.}]{} S^4$ - unknown.

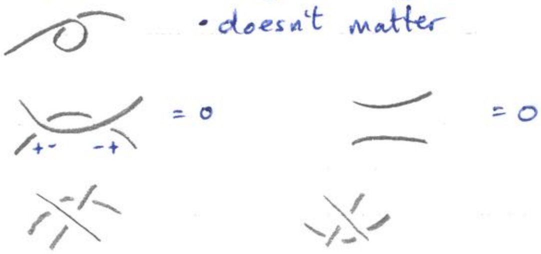
If k oriented, then rev k is the reverse of k.
If $\rho: S^3 \to S^3$ is orientation reversing, then ρk is the obverse/reflection of k

In a diagram of an oriented link, a crossing has a sign:



<u>Definition</u>: Suppose L is an oriented 2-component link, components L_1, L_2. Define the <u>linking number</u> of L_1, L_2 to be:
$\mathrm{lk}(L_1, L_2) := \frac{1}{2}\Sigma$(parity of crossings of L_1 with L_2) $= \mathrm{lk}(L_2, L_1) \in \mathbb{Z}$.

<u>Note</u>: This is well-defined by checking Reidemeister moves.



<u>Note</u>: Hopf link $\not\simeq$ unlink :

Suppose k is an oriented knot in S^3. k has a neighbourhood N which is a solid torus. The <u>exterior</u> of $k := S^3 \setminus N =: X \simeq S^3 \setminus k$. $\partial N = \partial X = N \cap X =$ torus.

<u>Theorem 1.1</u>: $H_1(X) \cong \mathbb{Z}$, canonically, generator $[\mu]$, where μ is a simple closed curve on ∂N that bounds a disc in N meeting k at one point. If C is an oriented simple closed curve in X, then $[C] = \mathrm{lk}(C, k)$.

<u>Proof</u>: $X \cup N = S^3$, $X \cap N =$ torus. We have Mayer-Vietoris sequence:

$$\dashrightarrow \underset{0}{H_3(X)} \oplus \underset{0}{H_3(N)} \longrightarrow \underset{\mathbb{Z}}{H_3(S^3)}$$
$$\therefore \cong$$
$$\hookrightarrow \underset{\mathbb{Z}}{H_2(X \cap N)} \xrightarrow{\therefore 0} \underset{0}{H_2(X)} \oplus \underset{0}{H_2(N)} \longrightarrow \underset{0}{H_2(S^3)}$$
$$\hookrightarrow \underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(X \cap N)} \xrightarrow{\therefore \cong} \underset{\mathbb{Z}}{H_1(X)} \underset{\oplus}{\oplus} \underset{\mathbb{Z}}{H_1(N)} \longrightarrow \underset{0}{H_1(S^3)}$$

$[\mu]$ is a generator of $H_1(X \cap N)$, $\therefore$ indivisible ($\neq nx$)
$\therefore [\mu] \to (1, 0)$, (or $(-1, 0)$, but we said "canonical")

in $H_1(X)$, $[C] = [C'] + [\mu]$.

So, $[C]$ = sum of times C goes under k = $\frac{1}{2}$ sum of times C crosses k.

μ is called the <u>meridian</u> of k. Let λ be a simple closed curve in ∂N.
See that $[\lambda] \mapsto (0, 1) \in \underset{\mathbb{Z}}{H_1(X)} \underset{\oplus}{\oplus} \underset{\mathbb{Z}}{H_1(N)}$ ← generator is $[k]$

Well-defined up to homotopy on ∂N.

2. Seifert Surface and Knot Addition.

Definition: A Seifert surface for an oriented link L in S^3 is a connected oriented surface embedded (piecewise linearly) in S^3 whose oriented boundary is L.

Example: The Seifert surface of the unknot is a disc: 

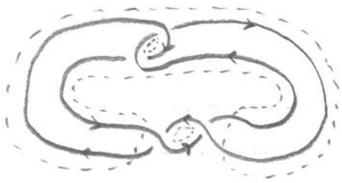
Theorem 2.1: Any oriented link has a (non-unique) Seifert surface.

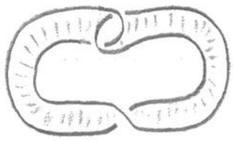
Proof: Let D be a diagram of L. Let $\hat{D}$ be D with each crossing replaced by a non-crossing, ie: 

$\hat{D}$ is the disjoint union of simple closed curves (Seifert circuits), in the plane. Circuits bound disjoint discs above the plane. Join the discs by twisted bands at the crossings:  - with orientation inherited from L.

If disconnected, join by a tube: 

Examples:



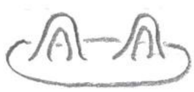


Seifert surfaces not always best!

Möbius band -NOT oriented.

From Algebraic Topology, know that if k is a knot in S^3, then $H^1(S^3 \setminus k) \cong \mathbb{Z}$. Represent a generator by a map $f: S^3 \setminus k \to K[\mathbb{Z}, 1] = S^1$ (Eilenberg-MacLane space). Make f transverse to a point. $f^{-1}(\text{point})$ is a Seifert surface.

Definition: The genus of a knot k is: $g(k) = \min\{\text{genus } F : F \text{ a Seifert surface for } k\}$

Examples:  $g = 1$,  $g = 2$. genus $F = \frac{1-\chi(F)}{2}$

Notes: (i) $g(k) = 0 \Leftrightarrow k$ spans disc $\Leftrightarrow k$ is unknot (see introduction to Massey).

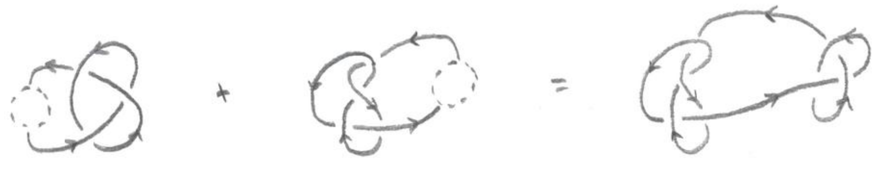
(ii) if $k \neq$ unknot and $\exists$ Seifert surface of genus 1, then $g(k) = 1$.

(iii) if D is a diagram of k, $g(k) \leq \frac{1}{2}(\#\text{crossings} - \#\text{Seifert circuits} + 1)$

One can consider knotted ball-arc pairs:   - unknot.

Definition: Let k_1, k_2 be oriented knots in (two copies of) S^3. Define $k_1 + k_2$ by: (i) remove an unknotted ball-arc pair from k_1, k_2
(ii) identify the resulting boundary spheres by orientation reversing homeomorphism, matching orientation of arcs.

Example:



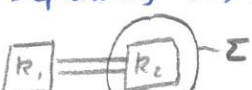
Need to check well-defined:
(i) we have an abelian semi-group, with zero = unknot.
(ii) important that knots are oriented: $8_{17} + \text{rev } 8_{17} \neq 8_{17} + 8_{17}$.
(iii) not well-defined for links.

Definition: A knot k is prime if $k \neq$ unknot and $k = k_1 + k_2 \Rightarrow k_1$ or k_2 is unknot.

Theorem 2.2 (Schubert): $g(k_1+k_2) = g(k_1) + g(k_2)$

Corollaries: (i) A knot $k \neq$ unknot has no additive inverse.
(ii) $g(k) = 1 \Rightarrow k$ is prime.
(iii) Assume Trefoil, k, is knotted, then $\exists$ infinitely many knots, $k + \cdots + k$.
(iv) any k can be expressed as $k = k_1 + \cdots + k_r$, k_i prime.

Proof of Theorem 2.2: Clear that $g(k_1+k_2) \leq g(k_1) + g(k_2)$. We can add the Seifert surfaces and genus adds.
Suppose F is a minimal genus Seifert surface for k_1+k_2. Let Σ be the 2-sphere meeting k_1+k_2 in two points, separating (S^3, k_1+k_2) into two ball pairs as in definition of k_1+k_2:



Wlog, $F \cap \Sigma$ is a 1-manifold (move simplexes by ε if necessary). Then $F \cap \Sigma$ is one arc, α, joining 2 points of $\Sigma \cap (k_1+k_2)$, $\cup$ finitely many simple closed curves.
Let C be a simple closed curve of $F \cap \Sigma$, innermost, (α is outermost).
C bounds disc $D \subset \Sigma$. $D \cap F = C = \partial D$. Use D to change F by "surgery".
Remove annulus neighbourhood of C, insert two parallel copies of D.

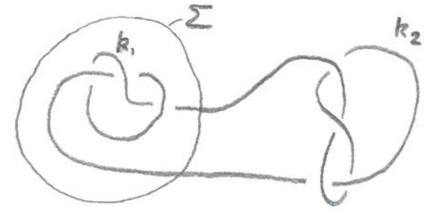
We create $\tilde{F}$ from F.

Let $\tilde{\tilde{F}}$ be the component of $\tilde{F}$ containing k_1+k_2.
Genus $\tilde{\tilde{F}} \leq$ genus F, so genus $\tilde{\tilde{F}} =$ genus F, as F has minimal genus.
But, the number of components of $\tilde{\tilde{F}} \cap \Sigma$ is fewer. Repeat.
We obtain F' such that $F' \cap \Sigma =$ one arc α.
Then, $g(k_1+k_2) =$ genus $F' \geq g(k_1) + g(k_2)$.

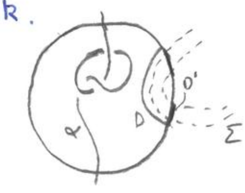
Theorem 2.3: Suppose $k = P + Q$ where P is prime and $k = k_1 + k_2$. Then either
(a) $k_1 = P + k_1'$, $Q = k_1' + k_2$, or
(b) $k_2 = P + k_2'$, $Q = k_1 + k_2'$.

Corollary: If P is prime and $P + Q = P + k_2$, then $Q = k_2$. So knots can be expressed uniquely as sums of prime knots.

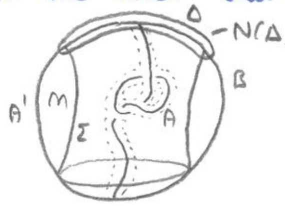
Proof of Theorem 2.3: Let Σ be the 2-sphere in S^3 meeting k transversely in 2 points, "demonstrating" $k = k_1 + k_2$.



$k = P + Q$ means $\exists$ 3-ball in S^3, B, with $B \cap k$ an arc α, (B, α) a ball-arc pair corresponding to P. Wlog $\Sigma \cap \partial B$ is a 1-manifold – disjoint simple closed curves. (Note that if $\Sigma \cap \partial B = \phi$ we are done). Try to reduce the number of components of $\Sigma \cap \partial B$.
Any simple closed curve in $\Sigma \setminus k$ has linking number $0, \pm 1$ with k. Select a component of $\Sigma \cap \partial B$ with linking number 0, innermost on Σ. ($\Sigma \cap k$ is "far"). It bounds a disc $D \subset \Sigma$. $D \cap \partial B = \partial D$. ∂D bounds a disc $D' \subset \partial B \setminus k$.
$D \cup D'$ bounds a ball.



Change B to a new position by adding (or subtracting) this ball and neighbourhood. Repeat till all components of $\Sigma \cap \partial B$ have linking number ± 1.
If $\Sigma \cap B$ has a component that is a disc D, then $D \cap k = \{1 \text{ point}\}$.
D divides (B, α) into two ball-arc pairs – one trivial as P is prime. Change B by removing the trivial pair. Repeat if necessary. Then, $\Sigma \cap B$ = disjoint union of annuli.
Let A be an annulus component of $\Sigma \cap B$ furthest from α in the sense that ∂A bounds an annulus A' in ∂B and $A' \cap \Sigma = \partial A'$.



Let M be the bit in between A and A', ie, the part of B bounded by $A \cup A'$. Note that A might be "hugging" the knot.
Let Δ be the closure of one component of $\partial B \setminus A$.
Δ is a disc, $\Delta \cap k = \{1 \text{ point}\}$. Let $N(\Delta)$ be a regular neighbourhood of Δ.
Then, $(N(\Delta), N(\Delta) \cap k)$ is a trivial ball-arc pair. $M \cup N(\Delta)$ is a ball (as its boundary is a sphere). $(M \cup N(\Delta), N(\Delta) \cap \alpha)$ is a ball-arc pair contained in (B, α), but P is prime, so this is trivial or a copy of (B, α), (when annulus hugs α). If trivial, remove by changing B.
Otherwise, M is a copy of the exterior of P, so $M \subset$ closure of one complementary domain of Σ, that containing k_1. Thus P is a summand of k_1 (wlog, could be k_2)
In this circumstance, remove M and replace with a solid torus (so that the boundary of a disc across the solid torus meets $\partial \Delta$ in one point).
(B, α) becomes (ball, unknotted arc). $(\overline{S^3 - B}, Q)$ is unchanged, hence (S^3, Q)
Thinking about Σ, get $(S^3, k_2 + k_1')$.

3. The Jones Polynomial.

Two diagrams are the same if homeomorphic in the plane.

Definition: The Kauffmann bracket is the function from (unoriented) link diagrams in (oriented) $\mathbb{R}^2$ to Laurent polynomials in A with coefficients in $\mathbb{Z}$, with $D \mapsto \langle D \rangle \in \mathbb{Z}[A, A^{-1}]$, given by:

(i) $\langle O \rangle = 1$,

(ii) $\langle D \cup O \rangle = (-A^{-2} - A^{2}) \langle D \rangle$,

(iii) $\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$, $\langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle)(\rangle$.

(iii) then refers to 3 diagrams identical save near a point where they differ as shown.
(ii) refers to D with an extra component not crossing D.

The rules (i), (ii), (iii) do define some $\langle D \rangle \in \mathbb{Z}[A, A^{-1}]$. Lose crossings by (iii), lose O by (ii), final O by (i).

Check that this is independent of the order of losing the crossings:

$\langle \times_1 \; \times_2 \rangle = A \langle)(\; \times \rangle + A^{-1} \langle \asymp \; \times \rangle = A^2 \langle)(\;)(\rangle + \langle)(\; \asymp \rangle + \langle \asymp \;)(\rangle + A^{-2} \langle \asymp \; \asymp \rangle$.

This is symmetric in 1 and 2, so this is the same if we reverse the order.

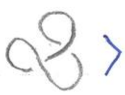
Lemma 3.1: The effects of Reidemeister type I moves: $\langle \text{—}\!\!\circlearrowleft\!\!\text{—} \rangle = -A^{3} \langle \frown \rangle$
$\langle \text{—}\!\!\circlearrowright\!\!\text{—} \rangle = -A^{-3} \langle \frown \rangle$

Proof: $\langle \text{—}\!\!\circlearrowleft\!\!\text{—} \rangle = A \langle \frown \; O \rangle + A^{-1} \langle \text{—}\!\!\circlearrowleft\!\!\text{—} \rangle = (A(-A^{-2} - A^{2}) + A^{-1}) \langle \frown \rangle = -A^{3} \langle \frown \rangle$.

Note: If $\bar{D}$ is the reflection of D (ie, change all the crossings),

eg:  , then $\langle \bar{D} \rangle = \overline{\langle D \rangle}$, where $\bar{\ } : A \mapsto A^{-1}$.

Example: $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle = -A^{4} - A^{-4}$.

$\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle = A(-A^{4} - A^{-4}) + A^{-7} = -A^{-3} - A^{5} + A^{-7}$

Lemma 3.2: (i) $\langle \asymp\!\!\!\times\!\!\!\times\!\!\!\asymp \rangle = \langle \asymp \rangle$, (ii) $\langle \text{-}\!\!\times\!\!\text{-} \rangle = \langle \text{-}\!\!\times\!\!\text{-} \rangle$

So, unchanged by moves of Reidemeister types II and III

Proof: (i) $\langle \asymp\!\!\!\times\!\!\!\times\!\!\!\asymp \rangle = A \langle \bowtie)(\rangle + A^{-1} \langle \asymp\!\!\times\!\!\asymp \rangle = -A^{-2} \langle)(\rangle + A^{-1} \{ A \langle \asymp \rangle + A^{-1} \langle)\,\subset\!\!\supset \rangle \}$
$= \langle \asymp \rangle$

(ii) $\langle \text{-}\!\!\times\!\!\text{-} \rangle = A \langle \text{-}\!\!\asymp\!\!\text{-} \rangle + A^{-1} \langle \text{-})(\text{-} \rangle = A \langle \text{-}\!\!\asymp\!\!\text{-} \rangle + A^{-1} \langle \text{-})(\text{-} \rangle = \langle \text{-}\!\!\times\!\!\text{-} \rangle$

Definition: The writhe, $w(D)$, of an oriented diagram D is: $w(D) = \Sigma$ signs of all crossings.

Note: $w(D)$ is not changed by Reidemeister moves of types II, III.
But, $w(\text{—}\!\!\circlearrowleft\!\!\text{—}) = w(\rightarrow) + 1$.

Theorem 3.3: The expression $(-A)^{-3w(D)}\langle D\rangle$ is an invarianted of an oriented link L (Independent of choice of diagram D for L).

Proof: Essentially done.

Definition: The Jones polynomial, $V(L)$, of an oriented link L is defined by

$$V(L) = (-A)^{-3w(D)}\langle D\rangle\big|_{t=A^{-4}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}].$$

$V(\text{unknot}) = 1$.

Does $\exists$ a knot $k \neq$ unknot with $V(k) = 1$? Unknown.

Proposition 3.4: $t^{-1}V(\times) - tV(\times) + (t^{-1/2} - t^{1/2})V(\uparrow\uparrow) = 0.$

Proof: $\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$

$\langle \times \rangle = A^{-1}\langle)(\rangle + A\langle \asymp \rangle$

$\therefore \underset{(-A^4)(-A^{-3})}{A\langle \times \rangle} - \underset{(A^{-4})(A^3)}{A^{-1}\langle \times \rangle} = (A^2 - A^{-2})\langle)(\rangle$

So, $-t^{-1}V(\times) + tV(\times) = (t^{-1/2} - t^{1/2})V(\uparrow\uparrow)$ - check sign!

Example: $V($  $) = (-A)^{-9}(A^{-7} - A^{-3} - A^{5}) = -t^4 + t^3 + t \neq V(\text{unknot})$

And, $V($  $) = -t^{-4} + t^{-3} + t^{-1}$, so right-hand trefoil $\neq$ left-hand trefoil.

However, changing the direction of the arrow on a knot does not change the Jones polynomial.

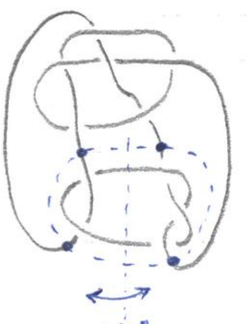
Exercise: $V(k_1 + k_2) = V(k_1)V(k_2)$

Sketch proof: $w(k_1 + k_2) = w(k_1) + w(k_2)$, of course. Now, $\langle$  $\rangle = \langle$  $\rangle\langle$  $\rangle$, and the result follows.

This also holds for links, although we do not have uniqueness.

Example: $V($  $+$  $)$ can be $V($  $)$ or $V($  $)$

The Conway and Kinoshita-Terasaka knots differ by a mutation:



180°

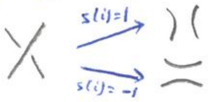
- Conway knot.

The Jones polynomial is not changed by mutation. Calculate the bracket before and after mutation. We may reduce the knot to $\alpha(A)\,)(+ \beta(A) \asymp$, and put back after rotation. However, the knots are distinct, demonstrated by the facts that $\pi_1(S^3 \setminus K)$ different, and different genus.

The Alexander polynomial is unaffected by mutation. In fact, both knots above have Alexander polynomial 1!

<u>Definition:</u> Let D be an n-crossing link diagram, with crossings labelled $1, \dots, n$. A <u>state</u> for D is a function $s: \{1, \dots, n\} \to \{-1, +1\}$

Let $s(D)$ be the diagram constructed from D by removing all crossings thus:



$s(D)$ is $|sD|$ disjoint simple closed curves.

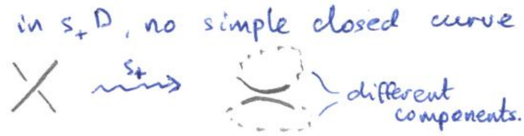
<u>Proposition 3.5:</u> $\langle D \rangle = \sum_s \langle D|s \rangle$, where $\langle D|s \rangle = A^{\sum_i s(i)} (-A^{-2} - A^2)^{|sD|-1}$

<u>Proof:</u> Clear - it satisfies the axioms for D.

Let s_+ be the state $s_+(i) = +1 \ \forall i$, so $\sum s_+(i) = +n$, and let s_- be the state $s_-(i) = -1 \ \forall i$, so that $\sum s_-(i) = -n$.

<u>Definition:</u> D is <u>+adequate</u> if $|s_+ D| > |sD| \ \forall s$ with $\sum s(i) = n-2$, and <u>-adequate</u> if $|s_- D| > |sD| \ \forall s$ with $\sum s(i) = 2-n$
D is adequate if it is both +adequate and -adequate.

The above means that in s_+D, no simple closed curve of s_+D abuts itself at a former crossing.



<u>Proposition 3.6:</u> A reduced alternating diagram is adequate.

<u>Definition:</u> Reduced means no crossing thus:

<u>Proof:</u> Colour regions of $S^2 \setminus D$ black/white in chessboard fashion. As D is alternating, components of s_+D are boundaries of black (say) regions, with corners rounded off.

Reduced $\Rightarrow$ adequate in this case.

Try pretzel knots. Some are adequate, some are not.

<u>Notation:</u> Write $M\langle D \rangle$ for the maximum degree of A in $\langle D \rangle$, and $m\langle D \rangle$ for the minimum.

Lemma 3.7: Let D be a knot diagram with n crossings. Then,

(i) $M\langle D\rangle \leq n + 2|s_+D| - 2$, with equality if D is +adequate.

(ii) $m\langle D\rangle \geq -n - 2|s_-D| + 2$, with equality if D is −adequate.

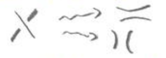
Proof: $\langle D|s\rangle = A^{\sum s(i)}(-A^{-2}-A^{2})^{|sD|-1}$. Results clear.

If s is any state, $\exists$ chain of states: $s_+ = s_0, s_1, \ldots, s_k = s$, with $s_r = s_{r-1}$, except at one crossing. $\sum s_r(i)$ reduces by 2 as $r \to r+1$. This changes $|s_r(D)|$ by ± 1. So, $M\langle D|s_r\rangle$ changes by 0 or -4.

s_+ to s_1 changes $|sD|$ by reducing it, by +adequacy. So, first step we go down by 4. After that, we cannot rise, so have equality in +adequate case. Others can and do fail.

Corollary: If D is adequate, $M\langle D\rangle - m\langle D\rangle = 2n + 2(|s_+D| + |s_-D|) - 4$.
(Remember: reduced, alternating $\Rightarrow$ adequate).

Lemma 3.8: Let D be a link diagram with n crossings that is connected (ie, not separable by a simple closed curve in the plane). Then, $|s_+D| + |s_-D| \leq n+2$.

Proof: Induction on n. True for $n=0$. Suppose true for $n-1$ crossings. Select a crossing c of D. Of the two changes of c, , one of the resulting diagrams is connected, wlog that via the positive change. Call this D'. Then, $|s_+D| = |s_+D'|$, $|s_-D| = |s_-D'| \pm 1$. So, $|s_+D| + |s_-D| \leq |s_+D'| + |s_-D'| + 1 \leq (n-1)+2+1 = n+2$.

Lemma 3.9: Let D be an alternating, connected, n-crossing link diagram. Then, $|s_+D| + |s_-D| = n+2$.

Proof: Colour regions of $\mathbb{R}^2 \setminus D$ in chessboard fashion. Alternating $\Rightarrow$ $|s_+D| = \#$ black regions, $|s_-D| = \#$ white regions. Suppose we have r regions, e edges in the diagram. We have n vertices. Euler characteristic of S^2: $n - e + r = 2$.
But $2e = 4n \Rightarrow r = n+2$, so $|s_+D| + |s_-D| = n+2$.

Lemma 3.10: Let D be a connected, n-crossing, non-alternating diagram that is diagrammatically prime (ie, if a simple closed curve meets D in 2 points, then one ball-arc pair is trivial). Then $|s_+D| + |s_-D| < n+2$.

Proof: Induction on n. Start with $n=2$, D:  $|s_+D| + |s_-D| = 3 < 4$, so okay.
So, take $n \geq 3$, and assume true for $n-1$. Now, $\exists$ two consecutive "over" (wlog) crossings. Let c be some other crossing. As in lemma 3.8, one of changes to c, wlog the positive change, is still connected. Call this D'. It is still diagrammatically prime (exercise), and non-alternating. By induction, $|s_+D'| + |s_-D'| \leq (n-1)+2 = n+1$.

Theorem 3.11: Let D be an n-crossing diagram of an oriented link L with Jones polynomial $V(L)$. Let $BV(L) = MV(L) - mV(L) =:$ t-breadth of $V(L)$. Then,

(i) $BV(L) \leq n$

(ii) If D is alternating and reduced, then $BV(L) = n$.

(iii) If D is non-alternating and diagrammatically prime, then $BV(L) < n$.

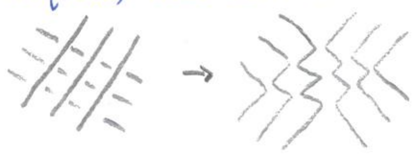
Proof: $V(L) = (-A)^{-3w(D)} \langle D \rangle$. $\therefore 4BV(L) = B\langle D \rangle$.

So, $\underset{\overset{\uparrow}{\text{Lemma 3.7}}}{4BV(L) \leq} 2n + 2|s_+D| + 2|s_-D| - 4 \leq 4n$, with $=$ if case (ii), $<$ if case (iii)

Corollary: If a link L has a connected reduced alternating diagram with n crossings, it has no diagram with $<n$ crossings.
(Solution to Tate conjecture, circa 1890).
And any non-alternating prime diagram of L has $>n$ crossings.

Suppose D is a link diagram. Let $r \geq 0$ be an integer. Let D' be D with every edge replaced by r parallel edges. Eg:

Lemma 3.12: If D is $\pm$adequate, then so is D^r.
Proof: $s_+(D^r) = (s_+D)^r$



Theorem 3.13: Let D, E be two diagrams of an oriented link L, with n_D and n_E crossings. Suppose D is $+$adequate. Then, $n_D - w(D) \leq n_E - w(E)$.

Proof: Let L have components $\{L_i\}$, and let $\{D_i\}, \{E_i\}$ be subdiagrams corresponding to the L_i. Choose integers μ_i, ν_i (≥ 0), such that $w(D_i) + \mu_i = w(E_i) + \nu_i$.
Now change D to D_* by adding μ_i positive kinks to each D_i, and E to E_* similarly (with ν_i) Note D_* is $+$adequate, as $s_+(\ ) = $.
Now, $w(D_{*i}) = w(E_{*i})$, so $w(D_*) = w(E_*)$.
Note that the sum of the signs of the crossings between any two components $= 2 \times$ their linking number.
D_*^r, E_*^r are diagrams of the same link L, with each component replaced by r parallel copies with mutual linking $w(D_{*i})$ on ith component.
They have the same Jones polynomial. But $wD_*^r = wE_*^r$, so $\langle D_*^r \rangle = \langle E_*^r \rangle$.

Now, $M\langle D_*^r \rangle = (n_D + \Sigma\mu_i)r^2 + (|s_+D| + \Sigma\mu_i)r - 2$

$ \overset{\shortparallel}{M}\langle E_*^r \rangle \leq (n_E + \Sigma\nu_i)r^2 + (|s_+E| + \Sigma\nu_i)r - 2$

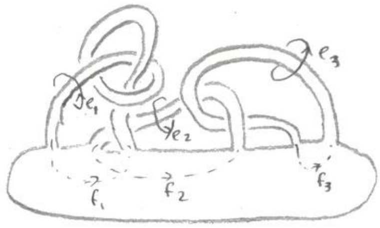
True $\forall r$, so $n_D + \Sigma\mu_i \leq n_E + \Sigma\nu_i \Rightarrow n_D - \sum_i w(D_i) \leq n_E - \sum_i w(E_i)$.
Subtract Σ signs of crossings of distinct components of both sides:
$n_D - w(D) \leq n_E - w(E)$.

Restate results: $n_D - w_D = 2.\#$negative crossings of D. So, $\#$-ve crossings of $D \leq \#$-ve crossings of E, where D $+$ adequate. Similarly, D $-$adequate $\Rightarrow \#$+ve crossings of $D \leq \#$+ve crossing in any E
$\therefore$ An adequate diagram of a link L has the minimal number of crossings, any type.
And: Any two adequate diagrams of a link L have the same writhe.

4. Seifert Form and Alexander Polynomial.

In the abstract, consider a compact orientable surface with boundary.

g pairs of bands, $n-1$ solitary bands.
$H_1(F;\mathbb{Z}) = \bigoplus_{2g+n-1} \mathbb{Z}$, $n = \#$ boundary components.
Generators, $\{[f_i]\}$, oriented simple closed curve.
Now embed $F \hookrightarrow S^3$. Consider $H_1(S^3 \setminus F; \mathbb{Z})$. Wave hands at Mayer-Vietoris sequence.



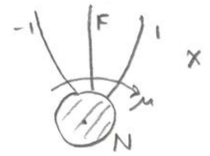
Suppose F, genus g, n-boundary component, orientable (piecewise linearly) in S^3.
$H_1(F) \cong \bigoplus_{2g+n-1} \mathbb{Z}$
Generators: $[f_1], \dots, [f_{2g+n-1}]$, where $\{f_i\}$ are simple closed curves.

F deformation retracts to $\bigvee_{2g+n-1} S^1$. If we repeat the Mayer-Vietoris calculation from chapter 1.*.. $S^3 = (\text{Nbhd of } F) \cup (S^3 \setminus \text{Nbhd of } F)$
But, $H_1(S^3 \setminus F) = \bigoplus_{2g+n-1} \mathbb{Z}$, generators $[e_1], \dots, [e_{2g+n-1}]$, where $\{e_i\}$ are in $S^3 \setminus F$, with $lk(e_i, f_j) = \delta_{ij}$.

Define a bilinear map, $\beta: H_1(S^3 \setminus F) \times H_1(F) \to \mathbb{Z}$ by $\beta([e_i],[f_j]) = \delta_{ij}$
Suppose c,d are oriented simple closed curves, c in $S^3 \setminus F$, d in F. So, $[c] = \sum \lambda_i [e_i]$, $[d] = \sum \mu_j [f_j]$, $\lambda_i, \mu_j \in \mathbb{Z}$. So, $\beta([c],[d]) = \sum \lambda_i \mu_i$
Now, $lk(c, f_j) = [c], \in H_1(S^3 \setminus f_j), = \sum \lambda_i [e_i], \in H_1(S^3 \setminus f_j), = \lambda_j$.
$lk(d,c) = [d], \in H_1(S^3 \setminus c), = \sum \mu_j [f_j], \in H_1(S^3 \setminus c) = \sum \mu_j \, lk(f_j, c) = \sum \mu_j \lambda_j = \beta([c],[d])$

Proposition 6.1: If F is a surface as above, then $\exists$ unique bilinear form $\beta: H_1(S^3 \setminus F) \times H_1(F) \to \mathbb{Z}$ such that $(*)$: $\beta([c],[d]) = lk(c,d)$ $\forall$ simple oriented closed curves c in $S^3 \setminus F$ and d in F

Suppose F as above, $\partial F = L$ an oriented link (so F a Seifert surface for L). Let N be a neighbourhood of L, so $N = \sqcup(\text{solid tori})$. Let $X = \overline{S^3 \setminus N}$
$F \cap X$ has a neighbourhood a copy of $F \times [-1,1]$, as F is oriented. $F \times 0 = F$ (nearly)
Direction of meridian is from $F \times \{-1\}$ to $F \times \{1\}$ via $F \times \{0\}$.
If $x \in F$, let $\iota^{\pm}(x) = x \times \{\pm 1\}$, so $\iota^{\pm}: F \hookrightarrow S^3 \setminus F$.

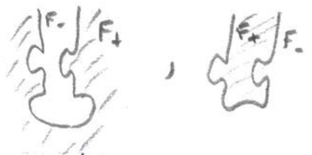


Definition: The Seifert Form of L (wrt F) is the bilinear form $\alpha: H_1 F \times H_1 F \to \mathbb{Z}$ defined by $\alpha(x,y) = \beta(\iota^-_* x, y) = \beta(\iota^+_* y, x)$

Note: If a,b are simple closed curves in F, $\alpha([a],[b]) = lk(\underbrace{\iota^- a}_{a_-}, b) = lk(a, \underbrace{\iota^+ b}_{b_+})$
α is not usually symmetric.

If $\{f_i\}$ are simple closed curves in F and $\{[f_i]\}$ are a basis for $H_1(F)$, let $\{[e_i]\}$ be the "dual" basis wrt β as before. Then α is represented by
$A_{ij} = lk(f_{i-}, f_j)$.
$H_1(S^3 \setminus F) \ni [f_{i-}] = \sum_j A_{ij}[e_j]$. $\quad A_{ij} = lk(f_i, f_{j+})$. So $[f_{j+}] = \sum_i A_{ij}[e_i] \in H_1(S^3 \setminus F)$.
A_{ij} is the <u>Seifert matrix</u> wrt a given basis $\{[f_i]\}$.

Let Y be X cut along F. ∂Y contains two copies of F: F_+, F_-.



$\exists$ homeomorphism $\varphi: F_- \to F_+$, $Y/\varphi \cong X$.
Take Y_i a copy of Y/φ for each $i \in \mathbb{Z}$.
$h_i: Y \to Y_i$ - homomorphism. Let $X_\infty = \sqcup Y_i/\sim$, where $h_i F_- \xrightarrow{h_{i+1}\varphi h_i^{-1}} h_{i+1} F_+$ defines $\sim$

Define $t: X_\infty \to X_\infty$; $h_i y \mapsto h_{i+1} y$
$\langle t \rangle$, the infinite cyclic group with generator t, acts on X_∞ as a group of homeomorphisms. $\therefore$ t (well, t_*...) acts on $H_1(X_\infty; \mathbb{Z})$, ie, $H_1(X_\infty; \mathbb{Z})$ is a module over the ring $\mathbb{Z}[t^{-1}, t]$
X_∞ is sometimes called the <u>infinite cyclic cover</u> of the complement of L.

<u>Theorem 6.2</u>: Let F be a (connected, oriented) Seifert surface of an oriented link L. Let A be the matrix of the corresponding Seifert form (wrt any base of $H_1(F)$). Then, $tA - A^T$ is a matrix that presents $H_1(X_\infty)$ as a $\mathbb{Z}[t^{-1}, t]$-module. - ie, $\exists$ exact sequence of f.g. modules

$$\text{free} \xrightarrow[\text{wrt basis}]{tA - A^T} \text{free} \longrightarrow H_1(X_\infty) \to 0.$$

The Alexander polynomial is $\det(tA - A^T)$.

<u>Proof:</u>