Finite Dimensional Lie Algebras.

1. Basics.

1.1 Algebras and Modules.

k - any field.
A k-algebra A is a k-vector space with a k-bilinear multiplication $A \times A \to A$.
Usually, assume in addition that multiplication is associative and unital, ie $\exists\, 1_A \in A$ such that $1_A \cdot a = a \cdot 1_A = a$.
An A-module is a k-vector space M with a k-bilinear multiplication $A \times M \to M$.
If A is associative, assume in addition that $(a \cdot b)m = a(b \cdot m)$ $\forall\, a, b \in A, m \in M$ and $1_A \cdot m = m$ $\forall\, m \in M$.
Giving an A-module M for A associative is equivalent to giving a representation, ie a map ρ of associative algebras, $\rho: A \to \mathrm{End}_k(M)$, the associative algebra of all linear maps $M \to M$. Conversely, given $\rho: A \to \mathrm{End}_k M$, we can make M an A-module.

1.2 Lie Algebras.

A k-algebra L is a Lie Algebra if the bilinear map $[\ ]: L \times L \to L$ satisfies the extra conditions:

(L1) : $[x\,x] = 0$ $\forall\, x \in L$

(L2) : $[x[y\,z]] + [y[z\,x]] + [z[x\,y]] = 0$ $\forall\, x, y, z \in L$

(L2) is the Jacobi identity.

Lie algebras are usually not associative. They are never unital.

Exercise: Show L1 $\Rightarrow [x\,y] = -[y\,x]$ $\forall\, x, y \in L$.

A homomorphism $\theta: L \to L'$ between two Lie algebras L, L' is a k-linear map such that $\theta[x\,y] = [\theta x\ \theta y]$ $\forall\, x, y \in L$.

Examples: (i) Let M be a k-vector space. $\mathrm{End}_k(M)$ is an associative algebra.
Define $\mathfrak{gl}(M)$ to be $\mathrm{End}_k(M)$ as a k-space with new multiplication
$[\cdot]: \mathfrak{gl}(M) \times \mathfrak{gl}(M) \to \mathfrak{gl}(M)$
$$[x\ \ y] = xy - yx \quad \forall\, x, y \in \mathfrak{gl}(M)$$
Exercise: Check this $[\cdot]$ satisfies L1, L2.

(ii) Suppose M is finite dimensional over k, with basis $e_1, \dots, e_n$.
So $\mathrm{End}_k(M) \equiv$ all $n \times n$ matrices with entries in k via this choice of basis.
So let $\mathfrak{gl}_n(k)$ = all $n \times n$ matrices, with multiplication: $[M\,N] = MN - NM$.
This is the general linear Lie algebra.

(iii) Any k-subspace L of $\mathfrak{gl}_n(k)$ is also a Lie algebra, provided $[x\,y] \in L$ $\forall\, x, y \in L$. This is a Lie subalgebra of $\mathfrak{gl}_n(k)$.

(iv) Given any associative k-algebra A, it becomes a Lie algebra if we define $[x\, y] = xy - yx \quad \forall\, x y \in M$.

Given a Lie algebra L, an L-module is a k-space M with a k-bilinear map $L \times M \to M$ such that $[xy]\cdot m = x\cdot(y\cdot m) - y\cdot(x\cdot m) \quad \forall\, x, y \in L,\ m \in M$.
Given an L-module M for the Lie algebra L, we obtain a homomorphism $\rho: L \to \mathfrak{gl}(M)$ defined by $\rho(l)\cdot m = l\cdot m \quad \forall\, l \in L,\ m \in M$.
Any Lie homomorphism $L \to \mathfrak{gl}(M)$ is called a representation of L.
The two notions, "module" and "representation" are easily seen to be equivalent.

1.3. Ideals and Homomorphisms.

A homomorphism $\theta: L \to L'$ (always Lie) is injective/surjective/$\cong$ if it is so as a linear map.
An automorphism θ of L is an isomorphism $L \hookrightarrow\!\!\!\rightarrow L$.

Some important subalgebras of L:
(i) Given any homomorphism $\theta: L \to L'$, $\operatorname{Im}\theta$ is a Lie subalgebra of L'.
(ii) The centre $Z(L) := \{x \in L : [xy] = 0 \ \forall\, y \in L\}$ is a Lie subalgebra of L.
(iii) More generally, given any subalgebra $K < L$,
its centraliser is $C_L(K) := \{x \in L : [xy] = 0 \ \forall\, y \in K\}$, and
its normaliser is $N_L(K) := \{x \in L : [xy] \in K \ \forall\, y \in K\}$. Both are subalgebras.
Note $C_L(L) = Z(L)$.

Given subsets M, N of L, let $[M, N]$ denote the k-span of all products $[m\, n]$ $\forall\, m \in M,\ n \in N$.
An ideal $I \triangleleft L$ is a k-subspace of L such that $[I, L] \subset I$.
So an ideal is certainly a subalgebra.
Given $I \triangleleft L$, the quotient $L/I = \{x + I : x \in L\}$ becomes a Lie algebra in the natural way.
The map $\theta: L \to L/I$ is a surjection with kernel I.
$x \mapsto x + I$
Conversely, given $\theta: L \to L'$, $\ker\theta = \{x \in L : \theta(x) = 0\}$ is an ideal of L, and the usual isomorphism theorems hold:

Theorem: (a) $\theta: L \to L'$ a homomorphism. Then $L/\ker\theta \cong \operatorname{im}\theta$. If I is any ideal of L contained in $\ker\theta$, then $\exists$ a unique map $\psi: L/I \to L'$ such that

$$\begin{tikzcd} L \arrow[r, "\theta"] \arrow[dr, "\text{canon.}"'] & L' \\ & L/I \arrow[u, "\psi"'] \end{tikzcd}$$

commutes.

(b) If $I \subset J \subset L$ are both ideals of L, then J/I is an ideal of L/I and $\dfrac{L/I}{J/I} \overset{\text{nat.}}{\cong} L/J$.

(c) If I, J are ideals of L, then $\underbrace{(I+J)}_{=\text{ the } k\text{-span of } \{i+j \,:\, i \in I,\ j \in J\}}/J \cong I/I\cap J$

Exercise: Show, given Lie algebras $K < L$, then K is an ideal of $N_L(K)$.

The Lie algebra L is simple if: (i) it has no non-zero proper ideals.
(ii) it is not abelian, ie $Z(L) \neq L$.

In this course, will only consider finite dimensional Lie algebras.
From section 2 onwards, $k = \mathbb{C}$.

Highlights:
- We classify all (finite dimensional) simple Lie algebras over $\mathbb{C}$. They fall into families: A_n, B_n, C_n, D_n - infinite families, "classical". E_6, E_7, E_8, F_4, G_2 - "exceptional."
- Given any simple L over $\mathbb{C}$, we will classify all its finite dimensional simple modules. Moreover, we will describe their structure explicitly by proving Weyl's character formula.

Example (v): $sl_n(k)$, subspace of $gl_n(k)$ consisting of matrices of trace 0. $\mathrm{Tr}[xy] = \mathrm{tr}(xy - yx) = 0$.

1.4 The Adjoint Representation and Derivations.

A representation of L is a homomorphism $L \to gl(V)$ for some V. Say V is faithful if the kernel of $L \to gl(V)$ is 0. In this case, we can identify L with a subalgebra of $gl(V)$
A linear Lie algebra is any Lie algebra L with a faithful, finite-dimensional representation V.
Any Lie algebra has an adjoint representation: $ad : L \to gl(L)$, defined by: for $x \in L$, $ad\,x \in gl(L)$ is the map $y \mapsto [x\,y]$ $\forall y \in L$.
Check: $ad[xy]z = (ad x\, ad y)z - (ad y\, ad x)z$ $\forall x, y, z \in L$
ie, $[[xy]z] = [x[yz]] - [y[xz]]$ - which is just the Jacobi identity.

Lemma: $\ker ad = Z(L)$
Proof: Suppose $x \in \ker ad$, ie $ad x\, y = 0$ $\forall y \in L$
ie $[xy] = 0$ $\forall y \in L$
ie $x \in Z(L)$, and conversely.

In particular, if L is centreless, ie $Z(L) = 0$, then ad is faithful, so L is a linear Lie algebra.
(In fact, any finite-dimensional Lie algebra is linear - Ada - Iwasawa - see Jacobson).

Take any k-algebra A, with multiplication written as $\cdot$. A derivation of A is a k-linear map $\delta : A \to A$ such that $\delta(a.b) = a.\delta(b) + \delta(a).b$ $\forall a, b \in A$.
Let $\mathrm{Der}(A) = \{$all derivations $\delta : A \to A\}$

Exercise: Show that if $\delta, \delta' \in \mathrm{Der}A$, then $[\delta\,\delta'] = \delta\delta' - \delta'\delta$ is also a derivation.

Hence $\mathrm{Der}\,L$ becomes a Lie algebra with $[\,,\,]$.
- a very important source of examples of Lie algebras.

Now take $A = L$, a Lie algebra. Then $\mathrm{Der}(L) \subset \mathfrak{gl}(L)$ is a Lie algebra. Now, the Jacobi identity implies that for any $x \in L$, $\mathrm{ad}\,x \in \mathfrak{gl}(L)$ is in fact a derivation. So $\mathrm{ad}\,L \subset \mathrm{der}\,L$. So the adjoint representation is a map $\mathrm{ad}: L \to \mathrm{Der}\,L$. Call any derivation δ of L of the form $\mathrm{ad}\,x$ for some $x \in L$ an <u>inner derivation</u>.

2. The Universal Enveloping Algebra

<u>Motivation:</u> Aim to show any Lie algebra L can be embedded into an associative algebra $U(L)$ in some "universal way".

Suppose G is a finite group. Representation theory of G/k $\equiv$ representation theory of group algebra kG. kG has the following universal property: given any associative algebra A and any map $j: G \to A$ such that $j(xy) = j(x)j(y)$, there exists a unique map φ such that

$$\begin{tikzcd} G \arrow[r, hook, "i"] \arrow[dr, "j"'] & kG \arrow[d, dashed, "\varphi"] \\ & A \end{tikzcd}$$

commutes.

2.1 Definition.

Fix a finite dimensional k-space V, $\dim V = n$. Let $T(V)$ be the <u>tensor algebra</u>. So $T(V) = \bigoplus_{r \geq 0} \otimes^r V$, and multiplication in $T(V)$ is induced from $(x, y) \mapsto x \otimes y$.

Recall $T(V)$ has the following universal property: given any associative algebra A and any map $j: V \to A$, then there is a unique map φ such that

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "j"'] & T(V) \arrow[d, dashed, "\varphi"] \\ & A \end{tikzcd}$$

commutes.

Given $T(V)$, the <u>symmetric algebra</u> $S(V) = T(V)/I$, where I is the two-sided ideal of $T(V)$ generated by $\{x \otimes y - y \otimes x : x, y \in V\}$.

Note the generators of I all lie in $\otimes^2 V$, but $V = \otimes^1 V$. So the natural map $V \to S(V)$, is injective, so V is a subspace of $S(V)$.

Let L be a Lie algebra. A <u>universal enveloping algebra</u> of L is a pair (U, i), where U is an associative (unital) k-algebra and $i: L \to U$ is such that $i([xy]) = i(x)i(y) - i(y)i(x)$ and the following universal property: given any other such pair (V, j), there exists a unique map φ of associative algebras such that

$$\begin{tikzcd} L \arrow[r, "i"] \arrow[dr, "j"'] & U \arrow[d, dashed, "\varphi"] \\ & V \end{tikzcd}$$

commutes.

Given existence, uniqueness is easy and routine.

For existence, construct (U, i) as follows:

Start with $T(L)$. Set $U(L) = T(L)/J$, where J is the two-sided ideal generated by $\{xy - yx - [xy] : x, y \in L\}$. Let $\pi: T(L) \to U(L)$ be the quotient map.

Let $i = \pi|_L$, the restriction of π to L. This gives a pair $(U(L), i)$ such that $i([xy]) = i(x)i(y) - i(y)i(x)$.

<u>Exercise:</u> Show $(U(L), i)$ actually is a universal enveloping algebra.

Problem: The generators of J are of mixed degrees, 1 and 2. So, unlike symmetric algebra construction, for which $i: V \to S(V)$ was obviously injective, this is far from clear for $i: L \to U(L)$. We will prove that it is, so that L can be identified with a subspace of $U(L)$.
For now, assume we have proved this. Some applications...

2.2 Universal Enveloping Algebras and Representations.

Let $L \hookrightarrow U(L)$ be a Lie algebra embedded in its universal algebra. Let V be an L-module.
So $[xy]v = xyv - yxv \quad \forall\, x, y \in L,\ v \in V \quad - (*)$.
By the universal property of the tensor algebra, V is a $T(L)$-module, as an associative algebra. Now, by $(*)$, the generators of the ideal J of $T(L)$ act as zero on V. So V is a $T(L)/J = U(L)$-module as an associative algebra.
Conversely, given a $U(L)$-module V, as L is a subspace of $U(L)$, restriction of multiplication gives a k-bilinear map $L \times V \to V$. Moreover, as $xy - yx - [xy] = 0$ in $U(L)$ $\forall\, x, y \in L$, $(xy - yx - [xy]).v = 0 \ \forall\, v \in V$, ie $[xy]v = xyv - yxv$. So V is an L-module.
In fact, this is an equivalence of categories:

$$\{L\text{-modules}\} \cong \{U(L)\text{-modules}\}.$$

2.3 Generators and Relations.

Need to define a free Lie algebra. Let L be a Lie algebra generated by a set $X \subset L$. Say L is free on X if given any map $\theta: X \to M$, M a Lie algebra, there exists a unique Lie homomorphism $\psi: L \to M$, extending φ.
As usual, free Lie algebras on X are unique by the universal property.

Existence: Let V be a k-space having X as basis. Then, $T(V)$ is associated algebra, so a Lie algebra by $[t, s] = ts - st$. Define L to be the Lie subalgebra of $T(V)$ generated by X.
Exercise: Use "injectivity" result to show L is free on X.

So we can define Lie algebras by generators and relations.

Lemma: The above L is free on X.
Proof: Take any map $X \to M$. It extends to a linear map $V \to M \to U(M)$. By universal property of $T(V)$, it extends to a map $T(V) \to U(M)$. Let i be the restriction of this to L. It has image lying inside M as L is generated by X and the image of X lies in M. - uses injectivity. Rest is routine.

$$\begin{tikzcd} T(V) \arrow[r] & U(M) \\ L \arrow[u, hook] \arrow[r] & M \arrow[u, hook] \end{tikzcd}$$

Now say L is generated by a set X subject to restrictions $R \subset F(X)$ if L is the quotient of the free Lie algebra on X by the ideal of $F(X)$ generated by R.
So we may define Lie algebras by generators and relations.

2.4 The Poincaré-Burkhoff-Witt Theorem.

<u>Reminder:</u> L is a finite dimensional Lie algebra over k.

$T(L) = \bigoplus_{r \geq 0} T_r = \bigoplus_{r \geq 0} \otimes^r L$ - tensor algebra.

$S(L) = T(L)/I$, I ideal generated by $\{x \otimes y - y \otimes x : x, y \in L\}$

$U(L) = T(L)/J$, J ideal generated by $\{x \otimes y - y \otimes x - [xy] : x, y \in L\}$.

Let $\left.\begin{array}{l} S_r = \text{image of } T_r \\ U_r = \text{image of } T_r \end{array}\right\}$ in the quotients. Let $\pi: T(L) \to U(L)$ be the quotient.

Both $T(L)$ and $S(L)$ are <u>graded algebras</u>. ie, $T(L) = \bigoplus T_r$, $T_r T_s \subset T_{r+s}$

$S(L) = \bigoplus S_r$, $S_r S_s \subset S_{r+s}$.

But J is not a homogeneous ideal so $U(L)$ is not a graded algebra.
Instead, $U(L)$ is a <u>filtered algebra</u>, ie $U(L)$ has a filtration. Let $U^i = \bigoplus_{j \leq i} U_j$.
Then, $0 < U^0 < U^1 < U^2 < \cdots$ is a filtration such that $U^i U^j \subset U^{i+j}$.

Let $\operatorname{gr} U$ be the <u>associated graded algebra</u>. By definition, $\operatorname{gr}(U) = \bigoplus_{r \geq 0} U(r)$, where $U(r) = U^r/U^{r-1}$, and the multiplication $U(r) \times U(s) \to U(r+s)$ is inherited naturally from the multiplication $U^r \times U^s \to U^{r+s}$.

Define a map $\varphi: T(L) \to \operatorname{gr} U$ as follows. $\pi: T(L) \to U$ maps T_r to U_r and so to U^r. Now compose with the quotient $U^r \to U^r/U^{r-1} = U(r)$. Glue all these together for all r, to get a map $\varphi: T(L) \to \operatorname{gr}(U)$.

<u>Claim:</u> $\varphi(I) = 0$.

<u>Proof:</u> $\varphi(x \otimes y - y \otimes x) = \pi(x \otimes y - y \otimes x) + U^1$

But $\pi(x \otimes y - y \otimes x) = \pi([xy])$. So $\varphi(x \otimes y - y \otimes x) = \pi([xy]) + U^1 = U^1 = 0$ in $\operatorname{gr} U$.

Consequently, $\varphi: T(L) \to \operatorname{gr} U$ factors to give a map $S(L) \to \operatorname{gr}(U)$. It is easy to see that it's surjective.

<u>Theorem (PBW):</u> $\theta: S(L) \to \operatorname{gr} U$ is injective.

<u>Proof:</u> Non-examinable - see handout.

<u>Corollary A:</u> Let W be a subspace of $\otimes^r L = T_r$ which is isomorphic to $S^r L = S_r$ as vector spaces. Then $\pi(W)$ is a complement to U^{r-1} in U^r.

<u>Proof:</u>

$$\begin{tikzcd} & U^r \arrow[dr, "\text{can.}"] & \\ \otimes^r L \arrow[ur, "\pi"] \arrow[dr, "\text{can.}"'] & & U(r) = U^r/U^{r-1} \\ & S^r L \arrow[ur, "\cong"'] & \end{tikzcd}$$

restriction of θ - an isomorphism by the PBW theorem.

So $W < \otimes^r L$ maps to $U(r)$ along the bottom two maps. Using the top two maps, $\pi(W)$ is the required complement.

<u>Corollary B:</u> $i: L \to U(L)$ is injective.

<u>Proof:</u> Apply corollary A to $W = L$ to deduce $i(L)$ is a complement to $U^0 = k$ in U^1.
Now, $\dim U^1/U^0 = \dim S^1 L = \dim L$ using PBW Theorem. So $\dim i(L) = \dim U^1/U^0 = \dim L$.
So $\dim i(L) = \dim L$, so i is injective.

Corollary C: Let $x_1, .., x_n$ be any basis for L. Then, $\{x_1^{i_1}, .., x_n^{i_n} : i_1, .., i_n \geq 0\}$ is a basis for $U(L)$. Equivalently, $\{x_{j_1} \cdots x_{j_m} : 1 \leq j_1 \leq j_2 \leq \cdots \leq j_m \leq n\}$ is a basis.

Proof: The second set maps to the usual basis of $S^m L$, so by corollary A, its image gives a basis for the complement to U^{m-1} in U^m. So by induction on m, the set of all monomials for all $m \geq 0$ is a basis of $U = \Sigma U^m$.

3. Soluble and Nilpotent Lie Algebras.

3.1 Solubility.

Let $L' = [L, L]$ = k-span of all $[xy]$, $\forall x, y \in L$. L' is the derived algebra.
L' is an ideal of L; in fact, it's the smallest ideal such that L/L' is abelian.

Example: $(\mathfrak{gl}(n,k))' = \mathfrak{sl}(n,k)$.

More generally, let $L^{(0)} = L$, $L^{(1)} = [L^{(0)}, L^{(0)}] = L'$, .., $L^{(i)} = [L^{(i-1)}, L^{(i-1)}] = (L^{(i-1)})'$.
Get a chain $L = L^{(0)} \geq L^{(1)} \geq \cdots \geq L^{(n)} \geq \cdots$ - the derived series.
Say L is soluble if $L^{(n)} = 0$ for some n.

Basic Properties.

Lemma: (a) If L is soluble, so are all subalgebras and quotients of L.
(b) If $I \trianglelefteq L$ and I, L/I are soluble, then so is L.
(c) If I, J are soluble ideals of L, so is $I + J$.

Proof: As for soluble groups.

In particular, (c) implies that any Lie algebra L (finite dimensional) has a unique largest soluble ideal, called the radical, $\operatorname{rad} L$. If $\operatorname{rad} L = 0$, say L is semisimple.
Example: Any simple L is semisimple, as there are no non-trivial (soluble) ideals.

Let $\underset{\sim}{t}(n,k)$ = all upper triangular $n \times n$ matrices under $[,]$ is a Lie subalgebra of $\mathfrak{gl}(n,k)$. In fact, it is soluble. Let $\underset{\sim}{n}(n,k)$ be all upper triangular matrices with zeroes on the diagonal. Then $\underset{\sim}{n}$ is an ideal of $\underset{\sim}{t}$, and $t(n,k)/n(n,k) \cong \partial(n,k)$, where $\partial(n,k)$ is all diagonal $n \times n$ matrices. This is abelian, so $t(n,k)' \subset n(n,k)$. We will show that $n(n,k)$ is soluble, in fact nilpotent. So $t(n,k)$ is soluble.

3.2. Nilpotency.

Let $L' = [L, L]$, $L^{i+1} = [L, L^i]$. Check $L^i \trianglelefteq L \ \forall i$.
This defines a chain of ideals: $L \geq L' \geq L^2 \geq \cdots$ - Lower central series.
Say L is nilpotent if this series eventually reaches zero, ie if $L^n = 0$ for some $n \geq 1$.
By induction on i it is genuinely easy to show $L^i > L^{(i)}$. So if $L^n = 0$ for some n,

then $L^{(n)} = 0$ for some n. So L is nilpotent $\Rightarrow$ L is soluble.

$t_n(k)$ = upper triangular $n \times n$ matrices
$\partial_n(k)$ = diagonal $n \times n$ matrices.
$n_n(k)$ = upper-zero triangular matrices.
$t_n(k)/n_n(k) \cong \partial_n(k)$. Claimed $n_n(k)$ was nilpotent, hence proved $t_n(k)$ was soluble.
if $k = \mathbb{C}$, $n > 1$, $t_n(k)$ is soluble but not nilpotent.

Lemma: (a) If L is nilpotent, so are all subalgebras and quotients.
(b) If $L/Z(L)$ is nilpotent, so is L
(c) If L is nilpotent, $Z(L) \neq 0$.
(d) If I, J are two nilpotent ideals of L, so is $I + J$.

Proof: (b): If $L/Z(L)$ is nilpotent, then $L^n \subset Z(L)$, some n.
So $L^{n+1} = [L, L^n] \subset [L, Z(L)] = 0$. So L is nilpotent.
(c) The last non-zero term L^{n-1} in the lower central series satisfies $[L^{n-1}, L] = 0$.
So $0 \neq L^{n-1} \subset Z(L)$

Define the nilpotent radical of L to be the (unique) largest nilpotent ideal of L, $\mathrm{rad}_n(L)$. Say L is reductive if $\mathrm{rad}_n L = 0$

Recall: A linear map $\theta \in \mathrm{End}\, V$ is nilpotent if $\theta^n = 0$ for some n.

Theorem: Let L be a subalgebra of $gl(V)$, V finite dimensional. If every $x \in L$ is a nilpotent endomorphism of V, then there exists $0 \neq v \in V$ such that $Lv = 0$.

Proof: Use induction on $\dim L$.

Step 1: For every $x \in L$, $\mathrm{ad}\,x$ is a nilpotent endomorphism of $gl(V)$.
(Remember: $x \in L$, $\mathrm{ad}\,x \in \mathrm{End}(L)$, $\mathrm{ad}x.y = [x,y]$ $\forall x, y \in L$).

Proof: As x is a nilpotent endomorphism of V, $\exists\, n$ such that $x^n = 0$.
Now, for $y \in gl(V)$, $(\mathrm{ad}x)^{2n}.y = [x[\cdots[x[x,y]]\cdots]$
$= x^{2n}y + \binom{2n}{1} x^{2n-1}.y.(-x) + \cdots + \binom{2n}{2n-1} x.y.(-x)^{2n-1} + \binom{2n}{2n} y(-x)^{2n}$.
Each term involves $x^a.y.x^b$, $a+b = 2n$. Hence, one of a or $b \geq n$.
So $x^a y x^b = 0$. So each term vanishes, and $(\mathrm{ad}x)^{2n} y = 0$.

Step 2: If $0 \neq K \neq L$ is any subalgebra, then $N_L(K) \neq K$.

Proof: By Step 1, for any $x \in K$, $\mathrm{ad}x$ is a nilpotent endomorphism of L, hence a nilpotent endomorphism of L/K. So by induction, there is a vector $0 \neq x + K \in L/K$ with $\mathrm{ad}y\,(x+K) = K$ $\forall y \in K$. This shows $x \in L$, $x \notin K$, and x normalises K as required. So $N_L(K) \neq K$.

Step 3: L has an ideal K of codimension 1.

Proof: Let K be a maximal proper subalgebra of L. By Step 2 and maximality, $N_L(K) = L$, so $K \trianglelefteq L$. Now, if $\dim L/K > 1$, the pre-image of a 1-dimensional subalgebra of L/K would be a proper subalgebra of L, contradicting maximality. So $K \trianglelefteq L$ of codimension 1.

<u>Step 4</u>: By step 3, $L = K + \langle z \rangle$ for some $z \in L \setminus K$. By induction, $W = \{v \in V : Kv = 0\}$ is non-zero. As $K \trianglelefteq L$, W is L-stable. Now, z is a nilpotent endomorphism of W, so it has at least one non-zero eigenvector $v \in W$ of eigenvalue 0. So $Lv = 0$.

<u>Corollary (Engel's Theorem)</u>: L is nilpotent iff $\operatorname{ad} x$ is a nilpotent endomorphism of L $\forall x \in L$.
<u>Proof</u>: <u>($\Leftarrow$)</u>: The algebra $\operatorname{ad} L < gl(L)$ satisfies the conditions of the theorem. So $\exists\, 0 \neq x \in L$ such that $[L, x] = 0$. So $Z(L) \neq 0$. Now, by induction on $\dim L$, $L/Z(L)$ is nilpotent. So L is nilpotent, by lemma 3.2 (b).
<u>($\Rightarrow$)</u>: By definition of nilpotent algebra, $\exists\, n$ such that $\operatorname{ad} x_1 \cdots \operatorname{ad} x_n\, y = 0$ $\forall x_i, y \in L$. In particular, this shows $(\operatorname{ad} x)^n y = 0$ $\forall x \in L$.

<u>3.3. Lie's Theorem.</u>

Theorem 3.2 shows if $L < gl(V)$ consisting of nilpotent endomorphisms, then $\exists$ a common eigenvector $v \in V$ for all of L

From now on, $\underline{k = \mathbb{C}}$.

<u>Theorem (Lie's Theorem)</u>: $L < gl(V)$ a linear Lie algebra, L soluble, then V contains a common eigenvector $0 \neq v$ for all of L. Ie $x.v \in \langle v \rangle$ $\forall x \in L$.
<u>Proof</u>: Use induction on $\dim L$. If $\dim L = 1$, then any eigenvector is a common one.
<u>Step 1</u>: L has an ideal K of codimension 1.
<u>Proof</u>: L/L' is abelian and non-zero. So any codimension 1 subspace of L/L' is an ideal. Its pre-image in L is what we want.
<u>Step 2</u>: V contains a common eigenvector for all $x \in K$
<u>Proof</u>: K is soluble by lemma 3.1 (b). So by induction, $\exists\, v \in V$ such that $x.v = \lambda(x) v$ $\forall x \in K$, where $\lambda : K \to k$ is a linear map.
Let V_λ = "generalised" eigenspace. $V_\lambda = \{v \in V : xv = \lambda(x) v \ \forall x \in K\}$. So $V_\lambda \neq 0$ by step 2.
<u>Step 3</u>: L stabilises V_λ. Need to prove that for any $x \in L$, $w \in V_\lambda$, any $y \in K$,
$y.x.w = \lambda(y).x.w$. $\quad y.x.w = x.y.w - [xy].w = \lambda(y).x.w - \lambda([xy])w$.
So, we need to show $\lambda([xy]) = 0$ $\forall x \in L$, $y \in K$. Fix $w \in V_\lambda$, $x \in L$. Let $n > 0$ be minimal such that $w, xw, \dots, x^n w$ are linearly dependent.
Let V_i = k-span of $\{w, \dots, x^{i-1} w\}$. $V_0 = 0$, $\dim V_n = n$, $V_{n+i} = V_n$, so x maps V_n into V_n. As $K \trianglelefteq L$, each $y \in K$ stabilises each V_i.
<u>Claim</u>: $y x^i w = \lambda(y)\, x^i w$ modulo V_i - induction exercise.

$i=0$ - obvious. Write $yx^i w = yxx^{i-1} w = xyx^{i-1} w - [xy]x^{i-1} w$. By induction, $yx^{i-1} w = \lambda(y) x^{i-1} w + w'$ ($w' \in V_{i-1}$). Since $x : V_{i-1} \to V_i$, get claim.

The claim shows that each $y \in K$ has upper triangular matrix with $\lambda(y)$ on the diagonal, when y is written wrt the basis $w, xw, \dots, x^{n-1} w$, of V_n.
Consequently, $\operatorname{trace}_{V_n}(y) = n\lambda(y)$ $\forall y \in K$.
Now, fix $x \in L$, $y \in K$. x, y stabilise V_n, so $\operatorname{trace}[xy] = \operatorname{trace}(xy) - \operatorname{trace}(yx) = 0$, but $\operatorname{trace}[xy] = n\lambda([xy])$. So, $\operatorname{char} k = 0 \Rightarrow \lambda([xy]) = 0$.
<u>Step 4</u>: Write $L = K + \langle z \rangle$ for some $z \in L \setminus K$. As $\mathbb{C}$ is algebraically closed we can find an eigenvector $v_0 \in V_\lambda$ for z. Now v_0 is an eigenvector for K and z, hence for L.

Example: Consider $\mathfrak{d}_n(k)$, $\mathfrak{t}_n(k)$, $\mathfrak{n}_n(k)$. We still have to show that $\mathfrak{n}_n(k)$ is nilpotent.
Each $x \in \mathfrak{n}_n(k)$ is a nilpotent matrix. Hence $\operatorname{ad} x$ is a nilpotent endomorphism of $\mathfrak{n}_n(k)$. So $\mathfrak{n}_n(k)$ is nilpotent, by Engel.
$\mathfrak{t}_n(k)' = \mathfrak{n}_n(k)$, so in particular, $\mathfrak{t}_n(k)$ is soluble.

Theorem (also Lie's Theorem): If $L < \mathfrak{gl}(V)$, V finite dimensional, dimension $= n$, then L is soluble iff V has a basis wrt which L lies in $\mathfrak{t}_n(k)$.

Proof: ($\Leftarrow$). We have just shown that $\mathfrak{t}_n(k)$ is soluble, so any subalgebra $< \mathfrak{t}_n(k)$ is too.
($\Rightarrow$) Use induction on $\dim V$. By Lie's Theorem, $\exists\, v \in V$ such that $xv = \lambda(x)v \quad \forall x \in L$.
So $V/\langle v\rangle$ is L-stable and by induction, $V/\langle v\rangle$ has a basis $v_2 + \langle v\rangle, \dots, v_n + \langle v\rangle$ wrt which the matrix of the image of L in $\mathfrak{gl}(V/\langle v\rangle)$ is upper triangular.
$\left(\begin{array}{c|c} * & * \cdots * \\ \hline 0 & \begin{matrix} * \\ 0 \end{matrix} \end{array}\right)$. So any $x \in L$ has upper triangular matrix wrt basis $v, v_2, \dots, v_n$ for V.

3.4. Cartan's Criterion.

- criterion for L to be soluble.

Linear Algebra Lemma (See Humphreys, lemma 4.3): Let $A \subset B$ be two subspaces of $\operatorname{End} V$ for some finite dimensional k-space V. Let $M = \{x \in \operatorname{End} V : [x, B] < A\}$.
Suppose $x \in M$ satisfies $\operatorname{Tr}_V(xy) = 0 \quad \forall y \in M$. Then, x is nilpotent.

Theorem (Cartan's Criterion): Let $L < \mathfrak{gl}(V)$. Suppose $\operatorname{tr}(xy) = 0 \quad \forall x \in L', y \in L$.
Then L is soluble.

Proof: It suffices to show that L' nilpotent. So, by Engel, it suffices to show that $\forall x \in L'$ $\operatorname{ad} x$ is nilpotent. To show $\operatorname{ad} x$ is nilpotent, it suffices to show that x is a nilpotent endomorphism of V. Apply linear algebra lemma with $A = L'$, $B = L$, $M = \{x \in \mathfrak{gl}(V) : [x, L] \subset [L, L]\}$. Note M contains L. By the lemma, we just need to show that for $x \in L'$, $y \in M$ we have $\operatorname{tr}(xy) = 0$.
Take a generator $[xy]$ of L' and $z \in M$. $\operatorname{tr}([xy]z) = \operatorname{tr}(x[yz]) = \operatorname{tr}([yz]x)$
Now, by definition on M, $[yz] \in L'$. So $\operatorname{tr}([yz]x) = 0$, by hypothesis.

Corollary (Cartan's Criterion): Let L be a Lie algebra such that $\operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0 \quad \forall x \in L', y \in L$.
Then L is soluble.

Proof: Apply theorem to adjoint representation of V to deduce $\operatorname{ad} L = L/Z(L)$ is soluble.
So L is also soluble.

3.5. The Killing Form.

Let L be any Lie algebra. Recall the adjoint representation $\operatorname{ad} : L \to \operatorname{Der} L < \mathfrak{gl}(L)$, $\operatorname{ad} x.y = [xy]$. If $M < L$ is a subalgebra, we will write $\operatorname{ad}_M : M \to \operatorname{Der} M$, and $\operatorname{ad}_L|_M : M \to \operatorname{Der} L$, to avoid ambiguity.

The Killing Form, $K: L \times L \to k$, is the symmetric bilinear form on L defined by $K(x,y) = \mathrm{tr}_L\,(\mathrm{ad}x \cdot \mathrm{ad}y)$.

Exercise: K is an associative bilinear form. $K([xy], z) = K(x, [yz])$.

Recall that K is non-degenerate if $\mathrm{rad}\, K = \{x \in L : K(x,y) = 0 \ \forall\, y \in L\} = 0$.
Recall that L is semisimple if $\mathrm{rad}\, L$ (= largest soluble ideal) $= 0$.

Theorem: L is semisimple iff the Killing Form K is non-degenerate.

Proof: ($\Rightarrow$): Let $S = \mathrm{rad}\, K$. By definition, $\mathrm{Tr}\,(\mathrm{ad}x \cdot \mathrm{ad}y) = 0 \ \forall\, x \in S, y \in L$. So, by Cartan's Criterion, $\mathrm{ad}_L S$ is soluble. So S is soluble. As K is associative, S is an ideal of L. Hence S is a soluble ideal, so $S < \mathrm{rad}\, L = 0$. Hence $\mathrm{rad}\, L = 0 \Rightarrow \mathrm{rad}\, K = 0$.

($\Leftarrow$): Let $S = \mathrm{rad}\, K$. First, without any assumption on S, we show that every abelian ideal of L lies in S. Let I be an abelian ideal of L, $x \in I$, $y \in L$.
Then $\mathrm{ad}x \cdot \mathrm{ad}y$ maps $L \to L \to I$. So $(\mathrm{ad}x\, \mathrm{ad}y)^2$ maps L into $[II] = I' = 0$.
So $\mathrm{ad}x \cdot \mathrm{ad}y$ is a nilpotent endomorphism of L. So $\mathrm{tr}_L(\mathrm{ad}x\, \mathrm{ad}y) = 0 = K(x,y)$.
So $x \in S$, so $I < S$.
Now, assume $S = 0$. Then L has no non-trivial abelian ideals. Hence $\mathrm{rad}\, L = 0$, as otherwise the last term in the derived series of $\mathrm{rad}\, L$ would be a non-zero abelian ideal.

First application.

Lemma: Let $I \triangleleft L$, with Killing Forms $K_I = \mathrm{tr}_I(\mathrm{ad}x\, \mathrm{ad}y)$, $K_L = \mathrm{tr}_L(\mathrm{ad}x\, \mathrm{ad}y)$ respectively. Then K_I is equal to K_L restricted to $I \times I$.

Proof: If W is a subspace of V, and $\varphi: V \to W$ is a linear map, then $\mathrm{tr}_V\, \varphi = \mathrm{tr}_W(\varphi|_W)$ - just extend a basis of W to V and look at matrix of φ wrt this basis $\begin{array}{cc} W & V \\ \multicolumn{2}{c}{\begin{pmatrix} \boxed{*} & * \\ 0 & \end{pmatrix}} \end{array}$
Now apply this to $\mathrm{ad}x\, \mathrm{ad}y : L \to I$ for $x, y \in I$.

Theorem: Let L be semisimple. Then $\exists$ unique simple ideals $L_1, \ldots, L_t$ such that $L = L_1 \oplus \cdots \oplus L_t$. Every simple ideal of L equals some L_i, and the Killing Form of L_i is just the restriction of the Killing Form of L to L_i.

Note: The lemma proves the final statement.

Proof: Step 1: Let $I \triangleleft L$. Let $I^{\perp} = \{x \in L : (x,y) = 0 \ \forall\, y \in I\}$. This is also an ideal of L, by the associativity of K. Moreover, $I \cap I^{\perp}$, which is an ideal, is soluble by Cartan's criterion.
So $I \cap I^{\perp} = 0$. So $L = I \oplus I^{\perp}$.

Step 2: Pick any ideal $I \triangleleft L$, write $L = I \oplus I^{\perp}$. Observe $I, I^{\perp}$ are semisimple as Lie Algebras. So result follows by induction on $\dim L$.

4. Representations of Semisimple Lie Algebras I

4.1. Generalities.

Remember, given any Lie Algebra L, a representation is just a homomorphism $L \to \mathfrak{gl}(V)$.
We now always assume L is finite dimensional.
Equivalently, think of V as an L-module (bilinear map $L \times V \to V$, $[xy]v = (xy - yx)v$).
Or, V is a $U(L)$-module, where $U(L)$ is the associative unital universal enveloping algebra.
Just need to remember that $L \hookrightarrow U(L)$ and $[xy] = xy - yx$.

A homomorphism of L-modules $\varphi: V \to W$ is a k-linear map such that $\varphi(x.v) = x.\varphi(v)$, $\forall x \in L, v \in V$.
An isomorphism of L-modules is just a homomorphism that is an isomorphism as a linear map.
An L-module is irreducible or simple if it has no non-zero proper L-submodules.
For example, k itself is an irreducible L-module, called the trivial module, with L acting as 0.
V is completely reducible or semisimple if V can be written as a direct sum of irreducible L-modules.

Schur's Lemma: If V is an irreducible L-module, then the only L-homomorphisms $\theta: V \to V$ are the scalars.
Proof: $\ker\theta$ is either 0 or V, and the image is either V or 0.

Since L-modules are also $U(L)$-modules, all the usual results – eg. Jordan-Hölder – hold for L-modules, as they hold for modules of associative algebras.

New L-modules from old.
1. Direct sums. $V \oplus W$ is an L-module, with action $x(v,w) = (xv, xw)$.
2. Submodules, quotients.
3. Duals. If V is an L-module, $V^* = \mathrm{Hom}_k(V,k)$ becomes an L-module if we define the action to be: $(x.f)(v) = -f(x.v)$ $\forall x \in L, f \in V^*, v \in V$.
   Check: $([xy]f)(v) = -f([xy]v) = -f(xyv - yxv) = f(yxv) - f(xyv) = -(yf)(xv) + (xf)(yv)$
   $= +(xyf)(v) - (yxf)(v)$, so $[xy]f = (xy)f - (yx)f$, as required.

4. Tensor Products. V, W are L-modules. Make $V \otimes W$ into an L-module by defining action of $x \in L$ on a typical generator $v \otimes w \in V \otimes W$ by $x(v \otimes w) = (xv) \otimes w + v \otimes (xw)$
   Check that $[xy] - (xy - yx)$ acts as zero.
5. Hom (V,W) for L-modules V, W. $\mathrm{Hom}(V,W) \cong V^* \otimes W$. (Explicitly, a typical generator $f \otimes w \in V^* \otimes W$ maps to $(v \mapsto f(v)w)$).
   So we can define L-action on $\mathrm{Hom}(V,W)$ via this identification with $V^* \otimes W$.
   Explicitly, for $x \in L$, $\theta: V \to W$, $(x.\theta)(v) = x(\theta(v)) - \theta(x.v)$ $\forall v \in V$.
   In particular, this makes $\mathrm{End}\,V = \mathrm{Hom}(V,V)$ into an L-module.

Back to usual assumptions. L is a semisimple Lie algebra over $\mathbb{C}$. We know the Killing form $(x,y) := \mathrm{tr}_L(\mathrm{ad}x\,\mathrm{ad}y)$ is non-degenerate. This is a symmetric, associative (ie $([xy],z) = (x,[yz])$) bilinear form.

Lemma: Let $\langle , \rangle$ be any other non-degenerate, symmetric, associative, bilinear form on L, with L simple. Then, $\exists\, a \in k^\times$ such that $\langle x,y \rangle = a(x,y)$ $\forall\, x,y \in L$.

Proof: The two forms $\langle , \rangle$ and $(,)$ set up two vector space isomorphisms $L \to L^*$. Associativity of the forms ensures that these are L-module maps. Now compose one with the inverse of the other, to get an L-module isomorphism $L \to L$. L simple, so it is an irreducible L-module. So by Schur's Lemma, this isomorphism from L to L is just a non-zero scalar a. This gives result that $\langle , \rangle = a(,)$.

4.2. The Casimir Operator.

Let L be a simple Lie algebra over $\mathbb{C}$, killing form $(,)$. Let $x_1, \dots, x_n$ be any basis for L. Let $y_1, \dots, y_n$ be dual basis wrt $(,)$, so $(x_i, y_j) = \delta_{ij}$.

Definition: The Casimir operator $C_L = \sum_{i=1}^{n} x_i y_i \in U(L)$.

Exercise: Show that the definition of C_L is independent of the choice of basis.

Lemma: C_L lies in the centre of $U(L)$.

Proof: As $U(L)$ is generated as an associative algebra by L, we need to show that $[x, C_L] = xC_L - C_L x = 0$ $\forall\, x \in L$. Since $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are bases for L, we may write $[x, x_i] = \sum_j a_{ij} x_j$, $[x\, y_i] = \sum_j b_{ij} y_j$.

$a_{ik} = ([x x_i], y_k) = -([x_i x], y_k) = -(x_i, [x y_k]) = -(x_i, \sum_j b_{kj} y_j) = -b_{ki}$. So $a_{ik} = -b_{ki}$.

Need to show $[x, C_L] = 0$.

$[x, C_L] = \sum_i [x, x_i y_i] = \sum_i ([x x_i] y_i + x_i [x y_i]) = \sum_{i,j} a_{ij} x_j y_i + \sum_{i,j} x_j b_{ji} y_i = \sum_{i,j} (a_{ij} + b_{ji}) x_j y_i = 0.$

4.3 Weyl's Theorem on Complete Reducibility.

Theorem (Weyl): Let L be a semisimple Lie algebra over $\mathbb{C}$. Every finite dimensional L-module V is completely reducible.

To prove it, we just need to show every short exact sequence $0 \to W \to V \to V/W \to 0$ splits for V, W finite dimensional L-modules. (equivalent to saying every L-submodule W of V has an L-stable complement).

Lemma 1: Let L be simple. Any short exact sequence $0 \to W \to V \to V/W \to 0$ splits where W is an irreducible L-module and $\dim V/W = 1$. (Assume $W \neq k$, so L acts on V faithfully).

Proof: We first claim that C_L acts on W as a non-zero scalar. To prove this, note that the map $W \to W$, $w \mapsto C_L w$ is an L-module homomorphism as C_L commutes with the action of L. So by Schur's lemma, C_L acts by some scalar $c \in k$. We need to show $c \neq 0$.

To compute c, note $tr_V(c_L) = \dim V . c = tr_V(\sum_i x_i y_i)$. So we just need to show that $tr_V(\sum x_i y_i) \neq 0$. Define a form $\langle , \rangle$ on L by $\langle x, y \rangle = tr_V(xy)$.
This is an associative, symmetric bilinear form on L. As W is faithful, Cartan's criterion implies rad $\langle , \rangle$ (which is an ideal of L by associativity) is soluble, hence zero as L is simple. So $\langle , \rangle$ is a non-degenerate form on L, hence a non-zero scalar multiple of the Killing form by lemma 4.1.
So it suffices to show $\underbrace{\sum (x_i, y_i)}_{= \dim L} \neq 0$. This is clear, proving the claim.
So c_L acts on W as a non-zero scalar, and on V/W ($\cong k$) as 0. So $\ker c_L : V \to V$ is a 1-dimensional L-stable submodule of V, giving the required complement to W.

Lemma 2: Let L be semisimple. Any short exact sequence $0 \to W \to V \to V/W \to 0$ splits, where W is any finite dimensional L-module and $\dim V/W = 1$.

Proof: Step 1: Suppose that L is simple and use induction on $\dim V$.
If W is irreducible, done by lemma 1. So, we can pick a non-zero $W' \subsetneq W$ and obtain a s.e.s. $0 \to W/W' \to V/W' \to V/W \to 0$ which splits by induction.
So there is an L-submodule $W'' < V$ such that $V/W' = W/W' \oplus W''/W'$
Now we have reduced to the s.e.s. $0 \to W' \to W'' \to W''/W' \to 0$.
By induction again, this splits to prove step 1. Ie. $\exists\, \tilde{W} \subset W''$ s.t. $W'' \overset{\dim!}{=} \tilde{W} \oplus W'$. $\dim \tilde{W} = 1$
So $V = W \oplus \tilde{W}$, since $\sum$ dims $= \dim V$ & $W \cap \tilde{W} = \phi$

Step 2: Suppose L is semisimple and W is irreducible.
We may assume that W is faithful. Let $L = L_1 \oplus L_2$, with L_1 a simple ideal. Then by step 1, we can write $V = W \oplus W'$ for some L_1-stable submodule W'.
Suppose W' is not L-stable. Then we can find $x \in L_2$, $w \in W'$ such that $xw \notin W'$.
Since W' is 1-d and is L_1-stable, $yw = 0$
So $xw = u + v$, some $u \in W$, $v \in W'$, $u \neq 0$. Then, take $y \in L_1$. $yxw = xyw = 0$
So $y(u+v) = 0$. So $yu = 0$. Hence $\{u \in W : L_1 u = 0\} \neq 0$ But this is an L-submodule of W. Hence $W = \{u \in W : L_1 u = 0\}$ as W is irreducible.
Hence L_1 acts trivially on all of W, contradicting faithfulness.

Step 3: Now if L is semisimple, W arbitrary, repeat argument of step one, using step two to start the induction.

Lemma 3: L semisimple. Any s.e.s. $0 \to W \to V \to V/W \to 0$, with V, W arbitrary finite dimensional L-modules, splits.

Proof: Recall $Hom(V, W)$ is an L-module. Let $\begin{Bmatrix} \mathcal{V} \\ \mathcal{W} \end{Bmatrix}$ = subspace of $Hom(V,W)$ all of whose elements act as $\begin{Bmatrix} \text{scalars} \\ \text{zero} \end{Bmatrix}$ on restriction to W. Now, $\mathcal{V}, \mathcal{W}$ are L-submodules of $Hom(V,W)$ and $\dim(\mathcal{V}/\mathcal{W}) = 1$. since each $f \in \mathcal{V}$ is determined (modulo $\mathcal{W}$) by the scalar $f|_W$
So have s.e.s. $0 \to \mathcal{W} \to \mathcal{V} \to \mathcal{V}/\mathcal{W} \to 0$, which splits by lemma 2.
So we can find $f \in \mathcal{V} \setminus \mathcal{W}$ such that $\langle f \rangle$ is L-stable complement to $\mathcal{W}$ in $\mathcal{V}$.
So $x.f = 0 \ \forall x \in L$. So $0 = (xf)(v) = xf(v) - f(x.v) \ \forall v \in V$. ie, f is an L-homomorphism
Then $\ker f < V$ is L-stable and gives the required complement to W in V.

This proves Weyl's Theorem.

4.4. Representations of $sl_2(\mathbb{C})$.

Let $L = sl_2(k)$ where $k = \mathbb{C}$. L has standard basis $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with $[ef] = h$, $[he] = 2e$, $[hf] = -2f$. By example sheet 1, L is a simple Lie algebra. So Weyl's Theorem applies - finite dimensional L-modules are completely reducible. So suffices to describe all the irreducible finite dimensional L-modules.

Problem: Let $\theta : L \to gl(V)$ be a finite dimensional representation. We need to know that $\theta(h)$ is a diagonalisable matrix in $gl(V)$. Assume this until later.

Assuming this, take any finite dimensional irreducible L-module V. $\theta : L \to gl(V)$. $\theta(h)$ is diagonalisable, so V decomposes as a direct sum of h-eigenspaces, $V = \bigoplus_{\lambda \in k} V_\lambda$, where $V_\lambda = \{ v \in V : hv = \lambda v \}$. Whenever $V_\lambda \neq 0$, call λ a weight of V, and V_λ a weight space.

Lemma: If $v \in V_\lambda$, then $e.v \in V_{\lambda+2}$, $fv \in V_{\lambda-2}$

Proof: $h.e.v = [he]v + ehv = 2ev + \lambda ev = (\lambda+2)ev$. fv similarly.

Since V is finite dimensional, there are only finitely many weight spaces. So there is a maximal weight λ such that $V_\lambda \neq 0$, $V_\mu = 0$ $\forall \mu = \lambda + z$, $z \in \mathbb{R}_{>0}$.
Pick such a λ, call λ a high weight of V. Pick any $0 \neq v_0 \in V_\lambda$. Let $v_{-1} = 0$, $v_i = \frac{1}{i!} f^i v_0$, $i \geq 0$.

Lemma: (a) $hv_i = (\lambda - 2i)v_i$
(b) $fv_i = (i+1)v_{i+1}$
(c) $ev_i = (\lambda - i + 1)v_{i-1}$

Proof: (a), (b) are easy.
(c) Induction exercise.

(a) $hv_i = \frac{1}{i!} hf^i v = \frac{1}{i!}(fhf^{i-1}v + [hf]f^{i-1}v) = \frac{1}{i!}(f^2hf^{i-2}v + f[hf]f^{i-2}v - 2f^i v_i)$
$= \cdots = \frac{1}{i!}(\lambda - 2i) f^i v_i$

(b) definition.

(c) $i e v_i = efv_{i-1} = [ef]v_{i-1} + fev_{i-1} = hv_{i-1} + fev_{i-1}$
$= (\lambda - 2(i-1))v_{i-1} + (\lambda - i + 2) f v_{i-2}$ (by (a) and induction)
$= (\lambda - 2i + 2)v_{i-1} + (i-1)(\lambda - i + 2)v_{i-1} = i(\lambda - i + 1)v_{i-1}$.

By (a), each non-zero v_i lies in a different eigenspace, so the non-zero v_i's are linearly independent. So, as V is finite dimensional, $\exists$ m minimal such that $v_m \neq 0$, $v_{m+1} = 0$. Then, the lemma implies $\langle v_0, .., v_m \rangle$ is an L-submodule of V. But V is irreducible, so $V = \langle v_0, .., v_m \rangle$ and we have constructed a basis of V. Moreover, by (c), $ev_{m+1} = 0 = (\lambda - m)v_m$ and $v_m \neq 0$, so $m = \lambda$.
So m and the module structure of V are uniquely determined by the highest weight λ.
In particular, λ is a positive integer. Finally note that $\dim V = m+1 = \lambda + 1$.

Theorem: Let V be an irreducible $sl_2(\mathbb{C})$-module of dimension $m+1$ ($m \geq 0$). Then, the unique maximal weight of V is m, the weight spaces of V are each 1-dimensional with weights $m, m-2, .., -(m-2), -m$. V is uniquely determined by m, and the module structure is given by (a) - (c).

Let $V(m)$ be the irreducible sl_2-module of dimension $m+1$. Eg: $V(0)$ = trivial representation $\mathbb{C}$.
$V(1)$ = natural sl_2-module of all 2×1 column vectors.
$V(2)$ = adjoint representation of sl_2 on itself, basis f, h, e
wt: $-2, 0, 2$

Example: $V(3) \otimes V(3)$ is a 16-dimensional $sl_2(\mathbb{C})$-module.

By Weyl's Theorem, it splits as a direct sum of irreducibles. What are they?

$V(3)$ has weights $\underset{v_0}{3}, \underset{v_1}{1}, \underset{v_2}{-1}, \underset{v_3}{-3}$, each eigenvalue having multiplicity 1.

So $V(3) \otimes V(3)$, basis $\{ v_i \otimes v_j \, ; \, 0 \le i, j \le 3 \}$, has weights:

	3	1	-1	-3	
3	6	4	2	0	V_0
1	4	2	0	-2	V_2
-1	2	0	-2	-4	V_4
-3	0	-2	-4	-6	V_6

The highest weight is 6, so by Theorem, $V(3) \otimes V(3)$ contains a copy of $V(6)$. In what is left, highest weight is 4, so must have a $V(4)$. Next get $V(2)$, leaving just $V(0)$. So $V(3) \otimes V(3) = V(0) \oplus V(2) \oplus V(4) \oplus V(6)$.

4.5 The Jordan Decomposition.

Need to overcome the problem in 4.4 - show that h is diagonalisable in any representation. Now, L is any semisimple Lie algebra over $\mathbb{C}$.

Recall: $\theta \in \mathrm{End}(V)$ is semisimple if θ is diagonalisable. Equivalently, roots of the minimal polynomial are all distinct.

$\theta \in \mathrm{End}(V)$ is nilpotent if $\theta^n = 0$ for some n.

Result from Linear algebra: Let V be a finite dimensional k-space, $\theta \in \mathrm{End}\, V$. Then, there exist unique $\theta_s, \theta_n \in \mathrm{End}(V)$ such that (i) $\theta = \theta_s + \theta_n$

(ii) θ_s is semisimple, θ_n is nilpotent

(iii) θ_s and θ_n commute.

Moreover, θ_s, θ_n can be written as polynomials in θ without constant term.

Eg: if the Jordan normal form of θ is $\begin{pmatrix} \boxed{\begin{matrix} \lambda & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda \end{matrix}} & 0 \\ 0 & \boxed{\begin{matrix} \mu & 1 & 0 \\ 0 & & \mu \end{matrix}} \ddots \end{pmatrix}$, then

$$\theta_s = \begin{pmatrix} \boxed{\begin{matrix} \lambda & 0 \\ 0 & \lambda \end{matrix}} & 0 \\ 0 & \boxed{\begin{matrix} \mu & 0 \\ 0 & \mu \end{matrix}} \ddots \end{pmatrix} \quad \text{and} \quad \theta_n = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 & 0 \\ 0 & & 1 \\ & & 0\end{matrix}} & \\ & \boxed{\begin{matrix} 0 & 1 & \\ & & 0 \end{matrix}} \end{pmatrix}$$

The decomposition $\theta = \theta_s + \theta_n$ is called the Jordan decomposition of θ. We want to show that if L is a semisimple Lie algebra, then for every $x \in L$, $\exists$ unique $x_s, x_n \in L$ such that

(i) $x = x_s + x_n$

(ii) For any finite dimensional representation $\rho : L \to gl(V)$, $\rho(x_s) = \rho(x)_s$ and $\rho(x_n) = \rho(x)_n$.

(iii) $[x_s \, x_n] = 0$.

Warning: This is false if L is not semisimple or $k \neq \mathbb{C}$.

Lemma: Let $L < gl(V)$ be a semisimple Lie algebra, V finite dimensional. For any $x \in L$, Let $x = x_s + x_n$ be the Jordan decomposition of x as an element of $\mathrm{End}(V)$. So $x_s, x_n \in gl(V)$. In fact, x_s, x_n are elements of L.

Proof: As L is semisimple, $tr_V(x) = 0 \ \forall x \in L$, so $tr_V(x_s) = 0 = tr_V(x_n)$.
For any submodule W of V, let $L_W = \{y \in gl(V) : yW \subset W \text{ and } tr_W(y|_W) = 0\}$.
So $L \leq L_W$ and x_s, x_n are elements of L_W for each W.
Let $N = N_{gl(V)}(L) = \{x \in gl(V) : [xL] \subset L\}$. Then $L < N$, so $x \in N$.
Now, x_s, x_n are polynomials in x without constant term, so $x_s, x_n \in N$.
Let $\tilde{L} = N \cap (\bigcap_{W < V} L_W)$. So $x_s, x_n \in \tilde{L}$. Note $L \triangleleft \tilde{L}$.
Suffices to show $L = \tilde{L}$, so that $x_s, x_n \in L$. By Weyl's Theorem, $\tilde{L} = L \oplus M$ for some L-module M. Since $[L, \tilde{L}] \subset L$, $[L, M] = 0$. So elements of M commute with L.
Take $y \in M$. So y commutes with L and stabilises every L-submodule W of V as $y \in L_W$.
V splits as a direct sum of irreducible L-submodules W. To show $y = 0$, we need to show y acts as zero on each such irreducible W.
Multiplication by y is an L-module map $W \to W$, so Schur's Lemma implies y acts on W as a scalar. But $tr_W(y|_W) = 0$ as $y \in L_W$, so this scalar is zero.
So $yW = 0$, as required.

Now apply this to the adjoint representation of L. For any $x \in L$, the lemma implies $\exists$ unique $x_s, x_n \in L$ such that $(\mathrm{ad}\, x_s) = (\mathrm{ad}\, x)_s$, $(\mathrm{ad}\, x_n) = (\mathrm{ad}\, x)_n$, and $x = x_s + x_n$, $[x_s\, x_n] = 0$
Call the decomposition $x = x_s + x_n$ the <u>abstract Jordan decomposition</u>.
If $x = x_s$, say x is <u>semisimple</u> (ie, $\mathrm{ad}\, x$ is diagonalisable)
If $x = x_n$, say x is <u>nilpotent</u> (ie, $\mathrm{ad}\, x$ is nilpotent).

<u>Theorem (Jordan decomposition)</u>: The abstract Jordan decomposition $x = x_s + x_n$ just defined satisfies
(i) $x = x_s + x_n$
(ii) For any finite dimensional representation $\rho : L \to gl(V)$, $\rho(x_s) = \rho(x)_s$, $\rho(x_n) = \rho(x)_n$.
(iii) $[x_s\, x_n] = 0$.

<u>Proof</u>: Just need to check (ii). By the definition, $\rho(x) = \rho(x_s) + \rho(x_n)$, and this gives the abstract Jordan decomposition of the element $\rho(x) \in \rho(L) = L'$. We need to show it agrees with the usual Jordan decomposition of $\rho(x) \in \mathrm{End}(V)$.
Well, this is $\rho(x) = \rho(x)_s + \rho(x)_n$. $\rho(x)_s$ is semisimple implies $\mathrm{ad}\, \rho(x)_s$ is semisimple; $\rho(x)_n$ is nilpotent implies $\mathrm{ad}\, \rho(x)_n$ is nilpotent. So by the lemma, $\rho(x)_s \in L'$, $\rho(x)_n \in L'$, and now uniqueness in the abstract Jordan decomposition implies $\rho(x_s) = \rho(x)_s$, $\rho(x_n) = \rho(x)_n$.

Return to sl_2. $h = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$. $\mathrm{ad}\, h$ is diagonalisable, so h is semisimple.
So, theorem $\Rightarrow$ $\rho(h) = \rho(h)_s$ as $h_n = 0$. So $\rho(h)$ is diagonalisable.

5. The Structure Of a Semisimple Lie Algebra.

<u>Ingredients</u>: 1. The existence of non-degenerate Killing form on L.
2. Weyl's theorem, and the classification of irreducible $sl_2(\mathbb{C})$-modules.
3. The definition (from the Jordan decomposition) of a semisimple element of L. Recall $x \in L$ is semisimple if $\mathrm{ad}_L x$ is diagonalisable. The image in any representation of a semisimple x is always diagonalisable

L is always a semisimple Lie algebra over $\mathbb{C}$.

5.1. The Cartan decomposition.

Call a subalgebra T of L toral if every element of T is semisimple.

Lemma: Every toral subalgebra T of L is abelian.

Proof: Take $x \in T$. We need to show $ad_T x = 0$. Equivalently, as $ad_T x$ is diagonalisable, we need to show every eigenvalue of x is zero. Take an eigenvector $y \in T$, eigenvalue a. So $ad_T x \cdot y = [xy] = ay$. Now, y is semisimple, so we can write x as $\sum x_i$ with $x_i \in T$ such that the x_i are linearly independent eigenvectors for y, of eigenvalues a_i respectively. $[yx] = ad_T y \cdot x = \sum a_i x_i = -ay$. $-(*)$. Apply $ad_T y$ again to deduce $\sum a_i^2 x_i = 0$. The x_i's are linearly independent, so $a_i = 0 \ \forall y$, and $(*)$ implies $ay = 0$, so $a = 0$ as required.

As L is semisimple, not every element of L is nilpotent (Engel's Theorem). So at least one semisimple element exists in L. (Jordan decomposition). So toral subalgebras of L exist.
So let H be any maximal toral subalgebra of L (ie, if $H \leq H'$ with H' toral, then $H = H'$).
So H is abelian.

Example: $sl_2(\mathbb{C})$, e, f, h. Here $\langle h \rangle = H$ is a maximal toral subalgebra.

Define a weight to be any element $\lambda \in H^*$, ie a function $H \to k$. A vector v in an L-module V is of weight λ if $hv = \lambda(h) v \ \ \forall h \in H$.
Now, for any representation V, the image of h is diagonalisable $\forall h \in H$. Moreover, h is abelian, so we can "simultaneously" diagonalise V for all of H at once. In other words, V has a basis $v_1, \dots, v_n$ such that $hv_i = \lambda_i(h) v_i \ \ \forall h \in H$.
That is, $V = \bigoplus_{\lambda \in H^*} V_\lambda$ where $V_\lambda = \{ v \in V : hv = \lambda(h) v \ \ \forall h \in H \}$.
This is called the weight space decomposition of V. The λ for which $V_\lambda \neq 0$ are called the weights of V.

Apply this to ad. Get $L = L_0 \oplus \bigoplus_{\alpha \in \Phi \subset H^*} L_\alpha$, where Φ is the set of all non-zero weights for the adjoint representation. This is called the Cartan decomposition. Φ is called the root system, and the elements of Φ are called roots.
Note $L_0 = \{ x \in L : ad_L h \, x = 0 \ \forall h \in H \} = C_L(H)$. So $H \leq C_L(H)$.
Warning: all of this depends on our initial fixed choice of H.

Lemma: For $\alpha, \beta \in H^*$, $[L_\alpha L_\beta] \subset L_{\alpha+\beta}$

Proof: Take $x \in L_\alpha$, $y \in L_\beta$. $[h[xy]] = [[hx]y] + [x[hy]] = \alpha(h)[xy] + \beta(h)[xy] = (\alpha+\beta)(h)[xy]$, so $[xy] \in L_{\alpha+\beta}$.

One consequence - if $x \in L_\alpha$ and $\alpha \neq 0$, then $ad_L x$ is nilpotent. For, there are only finitely many roots, and $(ad_L x)^n L_\beta \subset L_{\beta + n\alpha}$, so is zero for large enough n.

Lemma: If $\alpha, \beta \in H^*$ and $\alpha+\beta \neq 0$, then L_α is orthogonal to L_β wrt the Killing form.
Proof: As $\alpha+\beta \neq 0$, $\exists\, h \in H$ such that $(\alpha+\beta)(h) \neq 0$. Take $x \in L_\alpha$, $y \in L_\beta$. Then,
$\alpha(h)(x,y) = ([hx],y) = -([xh],y) = -(x,[hy]) = -\beta(h)(x,y)$. So $(\alpha+\beta)(h)(x,y) = 0$, so $(x,y)=0$.

We have $L = \underset{\substack{\Vert \\ H}}{C_L(H)} \oplus \underbrace{\bigoplus_{\alpha \in \Phi \subset H^*} L_\alpha}$, as before

- all elements of H are semisimple. — all elements here are nilpotent.

Wrt the Killing form, $C_L(H)$ and all $(L_\alpha + L_{-\alpha})$ for $\alpha \in \Phi$ are pairwise orthogonal. In particular, the restriction of the Killing form to $C_L(H)$ and each $(L_\alpha + L_{-\alpha})$ is non-degenerate.

5.2 An Important Example.

$L = sl_n(\mathbb{C})$ - all $n \times n$ trace zero matrices.

Lemma: L is semisimple
Proof: Let V be the natural n-dimensional L-module. Note V is an irreducible L-module, as L acts transitively on 1-dimensional subspaces of V. Let $S = \text{rad}\, L$. By Lie's Theorem, S has a common eigenvector in V, v, ie $sv = \lambda(s)v \;\forall s \in S$, $\lambda: S \to \mathbb{C}$.
For $x \in L$, $[sx] \in S$, so $sxv = \lambda([sx])v + \lambda(s)xv$ — $(*)$. Pick a basis for V
$\{v, x_1v, \ldots, x_mv : \text{certain } x_i \in L\}$. Then for $s \in S$, its matrix looks like
$$\begin{array}{c} \begin{array}{cccc} v & x_1v & \cdots & x_mv \end{array} \\ \begin{pmatrix} \lambda(s) & * & \cdots & * \\ & \lambda(s) & & 0 \\ & & \ddots & \\ 0 & & & \lambda(s) \end{pmatrix} \end{array}$$
, by $(*)$.

So $tr(s) = 0 = (m+1)\lambda(s)$. So $\lambda(s) = 0 \;\forall s \in S$, so $\lambda = 0$.
So $(*)$ shows $sw = 0 \;\forall w \in V$. So $s = 0$, so $S = 0$.

What does the Cartan decomposition of $L = sl_n(\mathbb{C})$ look like?
Let $H = \{\text{diagonal matrices}\}$. For $i \neq j$, let x_{ij} be the matrix with a 1 as (i,j)th entry and 0 elsewhere. So $L = H \oplus \bigoplus_{i \neq j} \langle x_{ij} \rangle$
H is toral - ie $ad_L h$ is diagonalisable $\forall h \in H$ (easy exercise).

Compute $ad\,h . x_{ij}$ to see $ad\,h . x_{ij} = (h_i - h_j)x_{ij}$, where $h = diag(h_1, \ldots, h_n)$.
Let $\varepsilon_i \in H^*$ be the map $h \mapsto h_i$. So x_{ij} is a weight vector for H of weight $\varepsilon_i - \varepsilon_j$ and $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$, where $\Phi = \{\varepsilon_i - \varepsilon_j : 1 \le i, j \le n,\ i \neq j\}$ and $L_{\varepsilon_i - \varepsilon_j} = \langle x_{ij} \rangle$.

This shows: (i) H is a maximal toral subalgebra.
(ii) This is the Cartan decomposition.
(iii) $C_L(H) = H$.
(iv) Each L_α, $\alpha \in \Phi$, is 1-dimensional, and $\alpha \in \Phi$ iff $-\alpha \in \Phi$.

Finally, can also show (example sheet 2) that: $(x,y) = 2n.\, tr_V(xy)$ for $x, y \in L$. Can check that the restriction of the Killing to H and each $(L_\alpha + L_{-\alpha})$ is non-degenerate - (v).

5.3 $C_L(H) = H$

Theorem: $H = C_L(H)$.

Proof: Let $C = C_L(H)$

Step 1: C contains the semisimple and nilpotent parts of all of its elements, and all the semisimple ones are in H.

Proof: Note $x \in C$ iff $\operatorname{ad} x \cdot H = 0$. Write $x \in C$ as $x_s + x_n$ with $x_s, x_n \in L$. $\operatorname{ad} x_s$, $\operatorname{ad} x_n$ are polynomials in $\operatorname{ad} x$ without constant term. $\operatorname{ad} x \cdot H = 0 \Rightarrow \operatorname{ad} x_s \cdot H = \operatorname{ad} x_n \cdot H = 0$.
So indeed $x_s, x_n \in C$. Take $x \in C$ semisimple. It centralises H, so $\langle H, x \rangle$ is a bigger toral algebra. So $x \in H$ as H is maximal.

Step 2: The restriction of the Killing form on L to H is nondegenerate.

Proof: Suppose $(h, H) = 0$ for some $h \in H$. We know $(h, C) = 0$ $\Rightarrow h = 0$ as the restriction to C is non-degenerate. Any element of C is of form $x_s + x_n$ with $x_s \in C$, semisimple, $x_n \in C$, nilpotent. So suffices to show $(h, x_s) = 0$, $(h, x_n) = 0$.
$x_s \in H$, so $(h, x_s) = 0$ as $(h, H) = 0$. So consider (h, x_n). $\operatorname{ad} h$ and $\operatorname{ad} x_n$ commute, and $\operatorname{ad} x_n$ is nilpotent. So $\operatorname{tr}_L(\operatorname{ad} h \operatorname{ad} x_n) = 0$ (taking simultaneous Jordan decomposition), ie $(h, x_n) = 0$, as required.

Step 3: C is a nilpotent Lie algebra.

Proof: By Engel's theorem, we need to show $\operatorname{ad}_C x$ is nilpotent for all $x \in C$. If x is nilpotent (ie $\operatorname{ad}_L x$ is nilpotent) this is obvious. So we may consider x as semisimple (by Step 1). So $x \in H$. Then $\operatorname{ad}_C x$ is actually zero as C centralises H, so is obviously nilpotent.

Step 4: C is abelian.

Proof: Suppose not. So $[CC] \neq 0$. As $K := [CC]$ is an ideal of C, K is a C-module (via ad).
C is nilpotent, so Engel's Theorem $\Rightarrow \exists\, z \in K$ such that $[xz] = 0$ $\forall x \in C$.
So $\exists\, 0 \neq z \in [CC] \cap Z(C)$. Suppose first that z is semisimple, so $z \in H \cap [CC]$.
But $(H, [CC]) = ([\overset{=0}{HC}], C) = 0$. Hence $H \cap [CC]$ is zero as the Killing form is non-degenerate on H. So this cannot happen.
So z has a non-zero nilpotent part, $0 \neq z_n \in C$, hence $\in Z(C)$ (as $\operatorname{ad} z_n$ is a polynomial in $\operatorname{ad} z$ without constant term, as in Step 1). But then z_n commutes with all $c \in C$ and is nilpotent, so $(z, C) = 0$. This contradicts the fact that the Killing form is non-degenerate on C.

Step 5: $C = H$.

Proof: Otherwise, $\exists\, 0 \neq x \in C \setminus H$ that is nilpotent. $\operatorname{ad} x$ is nilpotent, and commutes with $\operatorname{ad} c$ $\forall c \in C \Rightarrow (x, c) = 0$ $\forall c \in C$, contradicting the non-degeneracy of the Killing form on C.

5.4 Basic Properties of Root System Φ

The restriction of the Killing form to H is non-degenerate. So have $(\,,\,): H \times H \to \mathbb{C}$ a non-degenerate symmetric bilinear form. So we may identify H and H^*. In other words, for all $\lambda \in H^*$, $\exists$ a unique $t_\lambda \in H$ such that $(t_\lambda, h) = \lambda(h)$ $\forall h \in H$.
So we can lift the form $(\,,\,)$ on H to H^* by defining $(\lambda, \mu) = (t_\lambda, t_\mu)$ for $\lambda, \mu \in H^*$.
(restriction of Killing form on H.)

Note next that $\Phi \subset H^*$ spans H^*. [For if not, $\exists\ 0 \neq h \in H$ such that $\alpha(h) = 0\ \forall \alpha \in \Phi$. But then, $(h, L_\alpha) = 0\ \forall \alpha \in \Phi$, so $[h, L] = 0$, so $h \in Z(L)$ - #].

Take $\alpha \in \Phi$. The restriction of the Killing form to $L_\alpha + L_{-\alpha}$ is non-degenerate, but its restriction to L_α is zero. So $L_{-\alpha} \neq 0$. So $-\alpha \in \Phi$.

Moreover, given any $e_\alpha \in L_\alpha$, we can pick $f_\alpha \in L_{-\alpha}$ such that $(e_\alpha, f_\alpha) \neq 0$.

Lemma: (i) $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)\, t_\alpha \neq 0$.

(ii) $(t_\alpha, t_\alpha) \neq 0$, so we may rescale f_α if necessary such that $(e_\alpha, f_\alpha) = \frac{2}{(t_\alpha, t_\alpha)}$.

(iii) Having done this, let $h_\alpha = \frac{2t_\alpha}{(t_\alpha, t_\alpha)}$. Then, $[e_\alpha, f_\alpha] = h_\alpha$, $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, f_\alpha] = -2f_\alpha$.

So $\langle e_\alpha, h_\alpha, f_\alpha \rangle$ span a 3-dimensional subalgebra of L isomorphic to $sl_2(\mathbb{C})$. Call it S_α.

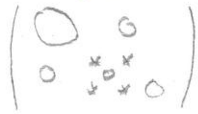
Proof: (i) We know $(e_\alpha, f_\alpha) \neq 0$. Take $h \in H$ arbitrary. Then, $(h, [e_\alpha, f_\alpha]) = ([h\, e_\alpha], f_\alpha) = \alpha(h)(e_\alpha, f_\alpha)$
$= (t_\alpha, h)(e_\alpha, f_\alpha) = ((e_\alpha, f_\alpha) t_\alpha, h)$. So $([e_\alpha, f_\alpha] - (e_\alpha, f_\alpha)t_\alpha, h) = 0\ \forall h \in H$.
So $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)\, t_\alpha$, as required.

See sheets at end (ii) Suppose $(t_\alpha, t_\alpha) = 0$. Then $\mathrm{ad}_L t_\alpha L_\beta = (\alpha, \beta) L_\beta$ which is zero unless $\beta = \pm\alpha$, and $(\alpha, \pm\alpha)$ is 0 by assumption. So $[t_\alpha, L] = 0$. So $t_\alpha \in Z(L)$ - #.

(iii) Now have $e_\alpha, f_\alpha, h_\alpha$, where $(e_\alpha, f_\alpha) = \frac{2}{(t_\alpha, t_\alpha)}$, and $h_\alpha = \frac{2t_\alpha}{(t_\alpha, t_\alpha)}$.
So $[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)\, t_\alpha = h_\alpha$, clearly.
$[h_\alpha, e_\alpha] = \frac{2}{(t_\alpha, t_\alpha)}[t_\alpha, e_\alpha] = \frac{2\alpha(t_\alpha)}{(t_\alpha, t_\alpha)} e_\alpha = 2e_\alpha$. Similarly, $[h_\alpha, f_\alpha] = -2f_\alpha$.

We have shown, for any $0 \neq e_\alpha \in L_\alpha$, there exist $f_\alpha \in L_{-\alpha}$, $h_\alpha = [e_\alpha, f_\alpha]$ such that $\langle e_\alpha, h_\alpha, f_\alpha \rangle = S_\alpha \cong sl_2(\mathbb{C})$. But it is not yet clear that S_α is independent of the choice of α.

For $sl_n(\mathbb{C})$, we may choose $\alpha = \varepsilon_i - \varepsilon_j$ for $1 \le i, j \le n$, $i \neq j$. ie, $\alpha(\mathrm{diag}(h_1, \dots, h_n)) = h_i - h_j$
And we showed $L_\alpha = \langle x_{ij} \rangle$, where x_{ij} is the matrix with (i,j) th entry 1, and 0 elsewhere.
$e_\alpha = x_{ij}$, $f_\alpha = x_{ji}$, $h_\alpha = [x_{ij}, x_{ji}] = \mathrm{diag}(\dots, \overset{i\text{th}}{1}, \dots, \overset{j\text{th}}{-1}, \dots 0)$



Lemma: Given $\alpha \in \Phi$, the only other multiples of α in Φ are $\pm\alpha$, and $\dim L_\alpha = 1$.

Proof: Consider the representation $\mathrm{ad}_L : S_\alpha \to gl(L)$. Look at the S_α-module of L spanned by H and all root spaces $L_{c\alpha}$ $(c \in \mathbb{C})$. Call it $M < L$. Consider weights of h_α on M.
By representation theory of sl_2, they are all integers. The weight of $L_{c\alpha}$ for h_α is $2c$.
So c is an integer multiple of $\frac{1}{2}$. Now S_α acts trivially on $\ker\alpha$ $(\alpha : H \to \mathbb{C})$. This is a codimension 1 subspace of H. The only other 0 weight space comes from $\langle h_\alpha \rangle$ which lies in S_α. So all zero weight spaces of h_α lie in $\ker\alpha \oplus S_\alpha$.
Now, by representation theory of sl_2, the only even weights in M are $0, \pm 2$.
So twice a root is not a root. So (repeating argument with $\frac{1}{2}\alpha$ instead) $\frac{1}{2}$ a root is not a root either. So 1 is not a weight in M. So no odd weights occur in M. This shows $M = H \oplus S_\alpha$.
Hence $\dim L_\alpha = 1$ as required, and only $\pm\alpha \in \Phi$.

$L = \underset{\text{semisimple}}{H} \oplus \underset{\text{nilpotent}}{\bigoplus_{\alpha \in \Phi} L_\alpha}$.

$\dim L_\alpha = 1$. The Lie subalgebra generated by $L_\alpha, L_{-\alpha}$, $= S_\alpha = sl_2$.
Also know Φ spans H^*, so $\dim H = \dim H^*$ is determined by Φ.
Call $\dim H$ the rank of L. So $\dim L = \mathrm{rank}\, L + |\Phi|$.

Lemma: Let $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Let r, q respectively be the largest integers such that $\beta - r\alpha \in \Phi$, $\beta + q\alpha \in \Phi$. Then all $\beta + i\alpha$ $(-r \leq i \leq q)$ are roots in Φ, and $\beta(h_\alpha) = r - q$. In particular, $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$

$\beta(h_\alpha)$ called a Cartan integer.

Proof: Second statement follows easily. Now consider how S_α acts on $K = \sum_{i \in \mathbb{Z}} L_{\beta + i\alpha}$.

The weight of $L_{\beta+i\alpha}$ for h_α is $\beta(h_\alpha) + 2i$. So all the weight spaces of K for h_α are 1-dimensional; the biggest is $\beta(h_\alpha) + 2q$, the smallest is $\beta(h_\alpha) - 2r$.

Representation theory for $sl_2 \Rightarrow K$ is an irreducible S_α-module (weights $0, 1$ each occur at most once, but not both).

The weights of K must be all $\beta(h_\alpha) + 2i$ for all $-r \leq i \leq q$. So $\beta + i\alpha \in \Phi$ for all $-r \leq i \leq q$. Moreover, $\beta(h_\alpha) + 2q = -(\beta(h_\alpha) - 2r)$. So $\beta(h_\alpha) = r - q \in \mathbb{Z}$.

5.5 Root Systems.

Now want to show Φ is an abstract root system. Now let E be a Euclidean space (a finite-dimensional $\mathbb{R}$-space, with an inner product $(,)$). Given $\alpha \in E$, $S_\alpha : E \to E$ given by reflection in the hyperplane perpendicular to α. Ie: $S_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$

Abbreviate: $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ by $\langle \beta, \alpha \rangle$ - note this is not linear in the second variable.

Say a subset Φ of E is an abstract root system if:

(R1): Φ is finite, spans E, and $0 \notin \Phi$.

(R2): If $\alpha \in \Phi$, the only multiples of α in Φ are $\pm\alpha$.

(R3): If $\alpha \in \Phi$, then $S_\alpha(\Phi) = \Phi$.

(R4): If $\alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

We want to make the root system Φ of L into an abstract root system.

Problem: Φ lies in H^*, which is a $\mathbb{C}$-space; the bilinear form $(,)$ on H^* is a complex form.

Let E be the vector subspace of H^* over $\mathbb{R}$ spanned by $\Phi \subset H^*$.

Lemma: $\dim_{\mathbb{R}} E = \dim_{\mathbb{C}} H^* = \operatorname{rank} L$. (So $H^* = E \otimes_{\mathbb{R}} \mathbb{C}$ and E is an $\mathbb{R}$-lattice in H^*).

Proof: Let $\alpha_1, .., \alpha_l$ be a basis for H^* over $\mathbb{C}$, $\alpha_i \in \Phi$. Then, $\alpha_1, .., \alpha_l$ are linearly independent over $\mathbb{R}$. We need to show $\alpha_1, .., \alpha_l$ span E over $\mathbb{R}$.

So take any $\beta \in \Phi$, and let $\beta = \sum_{i=1}^{l} c_i \alpha_i$ for $c_i \in \mathbb{C}$. Must show each $c_i \in \mathbb{R}$.

Now, $\frac{2(\beta,\alpha_j)}{(\alpha_j,\alpha_j)} = \sum c_i \cdot \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$, so $\beta(h_{\alpha_j}) = \sum c_i \alpha_i(h_{\alpha_j})$.

So we have a system of linear equations for unknowns c_i with integer coefficients.

They have a unique solution over $\mathbb{C}$, so in fact solution is over $\mathbb{Q}$, so each $c_i \in \mathbb{Q} \subset \mathbb{R}$.

So we've now constructed $\Phi \subset E$, E a real vector space of dimension rank L.

Still need a real inner product on E.

Lemma: The restriction of the form $(,)$ on H^* gives a real, symmetric, non-degenerate, positive definite, bilinear form $E \times E \to \mathbb{R}$. Ie, E is a Euclidean space.

Proof: We need to show $(\alpha,\beta) \in \mathbb{R}$ $\forall\ \alpha,\beta \in \Phi$, and that for $\lambda \in E \setminus \{0\}$, $(\lambda,\lambda) > 0$.

Given $\lambda, \mu \in H^*$, $(\lambda,\mu) = (t_\lambda, t_\mu) = \sum_{\alpha\in\Phi} \alpha(t_\lambda)\alpha(t_\mu) = \sum_{\alpha\in\Phi} (\alpha,\lambda)(\alpha,\mu)$ $\quad -(*)$.

Now take $\beta \in \Phi$. Then $(\beta,\beta) = \sum_{\alpha\in\Phi} (\alpha,\beta)^2$. Divide by $(\beta,\beta)^2$.

So $\frac{1}{(\beta,\beta)} = \sum_{\alpha\in\Phi} \frac{(\alpha,\beta)^2}{(\beta,\beta)^2} = \sum \frac{1}{4}\left(\frac{2(\alpha,\beta)}{(\beta,\beta)}\right)^2 = \sum \frac{1}{4}\alpha(h_\beta) \in \mathbb{Q}$, as $\alpha(h_\beta) \in \mathbb{Z}$.

So for $\alpha,\beta \in \Phi$, $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, $(\beta,\beta) \in \mathbb{Q}$. This shows $(\alpha,\beta) \in \mathbb{Q} \subset \mathbb{R}$.

Now take $\lambda \in E \setminus \{0\}$. $(*)$ shows $(\lambda,\lambda) = \sum_{\alpha\in\Phi} (\alpha,\lambda)^2 > 0$ (non-zero as $(\,,\,)$ is non-degenerate).

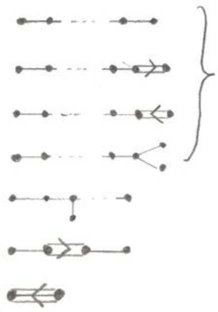
So we've made $\Phi \subset H^*$ into an subset of a Euclidean space, E. (R1), (R2) - from earlier lemma.
(R3) - we showed $\beta - \beta(h_\alpha)\alpha \in \Phi$ if $\alpha,\beta \in \Phi$, ie $\beta - \langle\beta,\alpha\rangle\alpha = s_\alpha(\beta) \in \Phi$.
(R4) - $\langle\beta,\alpha\rangle = \beta(h_\alpha) \in \mathbb{Z}$ $\rightarrow$ the Cartan integer.

Aim: (1) Classify abstract root systems $\Phi \subset E$.
(2) Show the semisimple L is determined up to isomorphism by $\Phi \subset E$, ie every $\Phi \subset E$ gives at most one semisimple L.
(3) For every $\Phi \subset E$ there is a semisimple L having $\Phi \subset E$ as its root system.

$\{\text{semisimple } L\} \xleftrightarrow{1-1} \{\text{abstract root systems}\}$

We will show that an abstract root system is a sum of irreducible root systems.
Irreducible root systems are classified by Dynkin diagrams.
The Dynkin diagrams are:

A_l, $(l \geq 1)$:
B_l, $(l \geq 2)$:
C_l, $(l \geq 3)$:
D_l, $(l \geq 4)$:
E_l $(l = 6,7,8)$:
F_4 :
G_2 :



l nodes in all.

6. A Survey of Abstract Root Systems.

Recall E is a Euclidean space, inner product $(\,,\,)$.
$\langle\alpha,\beta\rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)}$ for $\alpha,\beta \in E$. $\quad s_\alpha$ = reflection in hyperplane perpendicular to α, $s_\alpha(\beta) = \beta - \langle\alpha,\beta\rangle\alpha$.
$\Phi \subset E$ will always be an abstract root system satisfying (R1)-(R4). rank $\Phi := \dim E$ ($=$ rank L).

6.1 First Properties.

Lemma: Take $\alpha,\beta \in \Phi$, $\alpha \neq \pm\beta$. If $(\alpha,\beta) \left\{\substack{> \\ <}\right\} 0$ (ie the angle between α,β is $\left\{\substack{< \\ >}\right\} 0$), then $\alpha \left\{\substack{- \\ +}\right\} \beta \in \Phi$
Proof: Second follows from first. Now, $\langle\beta,\alpha\rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = \frac{2|\beta|.|\alpha|.\cos\theta}{|\alpha|^2}$. So $\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle = 4\cos^2\theta \in \mathbb{Z}$.
So $4\cos^2\theta \in \{0,1,2,3,4\}$. There are very few possible θ: (assume $|\langle\beta,\alpha\rangle| \geq |\langle\alpha,\beta\rangle|$).

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\lvert\alpha\rvert^2/\lvert\beta\rvert^2$
0	0	$\pi/2$	?
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

So if $(\alpha,\beta) > 0$, $\langle\alpha,\beta\rangle > 0$ and table shows one of $\langle\alpha,\beta\rangle$ and $\langle\beta,\alpha\rangle$ is equal to 1.
Suppose $\langle\alpha,\beta\rangle = 1$. $s_\beta(\alpha) = \alpha - \langle\alpha,\beta\rangle\beta = \alpha - \beta$.
Suppose $\langle\beta,\alpha\rangle = 1$. $s_\alpha(\beta) = \beta - \langle\beta,\alpha\rangle\alpha = \beta - \alpha$. So $\alpha - \beta \in \Phi$.

Take $\alpha, \beta \in \Phi$, $\alpha \neq \pm\beta$. Let r, q respectively be the largest integers such that $\beta - r\alpha, \beta + q\alpha \in \Phi$.

Lemma: (i) For all $-r \le i \le q$, $\beta + i\alpha \in \Phi$.
(ii) $\langle\beta,\alpha\rangle = r - q$. So $|r-q|$ is either 0, 1, 2 or 3 (by table).

Proof: (i) Suppose $\beta + i\alpha \notin \Phi$ for some $-r < i < q$, but $\beta + (i-1)\alpha \in \Phi$. By Lemma, $(\beta + (i-1)\alpha, \alpha) \ge 0$.
We can find $j \geq i$ such that $\beta + j\alpha \notin \Phi$, but $\beta + (j+1)\alpha \in \Phi$. Lemma $\Rightarrow$ $(\beta + (j+1)\alpha, \alpha) \le 0$.
$i - 1 < j+1$ and $(\alpha,\alpha) > 0$, so this is a contradiction.
(ii) $s_\alpha(\beta + i\alpha) = \beta + i\alpha - \langle\beta + i\alpha, \alpha\rangle\alpha$, which is of the form $\beta - j\alpha$ for some j.
In fact, s_α must invert the chain $\beta - r\alpha, \ldots, \beta + q\alpha$, ie, $s_\alpha(\beta + q\alpha) = \beta - r\alpha$.
So $\langle\beta,\alpha\rangle = r - q$, as required.

Call the "connected" chain $\beta - r\alpha, \ldots, \beta + q\alpha$ the α-chain through β. The lemma shows that α-chains have length at most 4.

6.2 Bases.

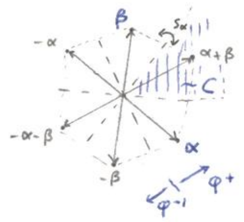
A subset Δ of Φ is called a base if:
(B1): Δ is a basis for E.
(B2): Whenever $\beta \in \Phi$ is written as $\sum_{\alpha\in\Delta} c_\alpha \alpha$ then either all $c_\alpha \ge 0$ or all $c_\alpha \le 0$.

Theorem: Any abstract root system Φ has a base Δ.
Proof: See Carter, "Simple Groups of Lie Type."

Roughly speaking, the proof depends on the following construction:
Given $\alpha \in \Phi$, let $H_\alpha = \{\lambda \in E : (\lambda, \alpha) = 0\}$ - hyperplane perpendicular to α. Then $\bigcup_{\alpha\in\Phi} H_\alpha$ cuts E into finitely many connected components. Call the connected components of $E \setminus \bigcup_{\alpha\in\Phi} H_\alpha$ chambers.
Given a chamber C, define $\Delta_C = \{\alpha \in \Phi : H_\alpha$ bounds C; the angle between α and any vector in C is $< \pi/2\}$.
Then Δ_C is a base for Φ, and all bases for Φ arise in this way.

Given a fixed base Δ, call elements of Δ simple roots. Note $|\Delta| = \text{rank}\,\Phi$.
Any $\beta \in \Phi$ can be written as $\sum_{\alpha \in \Delta_C} c_\alpha \alpha$. If all $c_\alpha \geq 0$, say β is positive.
If all $c_\alpha \leq 0$, say β is negative.
Let $\Phi^+ = \{\text{positive roots}\}$, $\Phi^- = -\Phi^+ = \{\text{negative roots}\}$. Then $\Phi = \Phi^+ \dot\cup \Phi^-$.

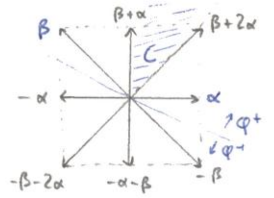
Examples: A_2:



$\Phi =$ the 6 points •
$H_\alpha =$ - - - = hyperplanes.
6 chambers C.
$\Delta_C = \{\alpha, \beta\}$.

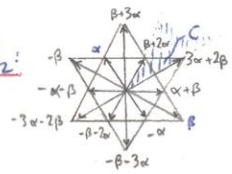
All root chains have length 2. All roots have same length. $\frac{|\alpha|^2}{|\beta|^2} = 1$

B_2:



$\Phi =$ 8 roots.
2 different root lengths. $\frac{|\beta|^2}{|\alpha|^2} = 1$ or 2
$\Delta_C = \{\alpha, \beta\}$.
Chains have length 2 or 3.

G_2:



$\Phi =$ 12 roots. 12 chambers C.
$\Delta_C = \{\alpha, \beta\}$.
Chains have length 2 or 4.
2 different root lengths. $\frac{|\beta|^2}{|\alpha|^2} = 1$ or 3.

6.3. The Weyl Group.

Let W be the subgroup of $GL(E)$ generated by the reflections s_α ($\alpha \in \Phi$).
By (R3), W acts on Φ, which is finite and spans E by (R1). So W acts faithfully on E, so we may embed $W \hookrightarrow \text{Sym}(\Phi)$. So W is finite. Call E the reflection representation of W.

Eg: A_2: $W \cong S_3$ of order 6.
B_2: $W \cong D_8$ of order 8 (symmetries of □)
G_2: $W \cong D_{12}$ of order 12 (symmetries of ⬡).

Two basic facts about W. (See Carter or Humphreys)
- W is Coxeter group, generated by $\{s_\alpha : \alpha \in \Delta\}$ for any base Δ of Φ.
- W acts faithfully and transitively on the set of all chambers of E. So $W \overset{1-1}{\longleftrightarrow} \{\text{chambers in } E\}$

Recall that every chamber C determines a base Δ_C, and all bases arise in this way.
Ie, $\{\text{chambers}\} \longleftrightarrow \{\text{bases of } \Phi\}$
$C \longleftrightarrow \Delta_C$.
So W acts faithfully and transitively on $\{\text{all bases of } \Phi\}$. So given any bases $\Delta\ \Delta'$, $\exists\, w \in W$ such that $w\Delta = \Delta'$.
So it doesn't matter (up to isometry) which base Δ we work with. Fix some Δ from now on.
Hence we've also fixed Φ^+, Φ^-, and generators $\{s_\alpha : \alpha \in \Delta\}$ for W.

Lemma: For each $\alpha \in \Phi$, $\exists\, w \in W$ such that $w\alpha \in \Delta$.

Proof: $\exists$ a chamber C such that $\alpha \in \Delta_C$. Now, $\Delta = w\Delta_C$ for some $w \in W$. So $w\alpha \in \Delta$.

6.4 Irreducible Root Systems

Lemma: The following are equivalent:

(i) Φ cannot be partitioned as $\Phi = \Phi_1 \cup \Phi_2$ with $\Phi_1, \Phi_2 \neq \phi$ and $(\Phi_1, \Phi_2) = 0$.

(ii) Δ cannot be partitioned as $\Delta = \Delta_1 \cup \Delta_2$ with $\Delta_1, \Delta_2 \neq \phi$ and $(\Delta_1, \Delta_2) = 0$.

(iii) E is an irreducible representation of W.

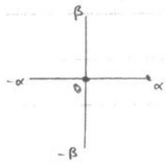
Call Φ *irreducible* if (i) - (iii) hold.

Proof: (ii) $\Rightarrow$ (i): Suppose $\Phi = \Phi_1 \cup \Phi_2$, $(\Phi_1, \Phi_2) = 0$. Let $\Delta_i = \Delta \cap \Phi_i$. So $\Delta = \Delta_1 \cup \Delta_2$, $(\Delta_1, \Delta_2) = 0$.
So one of $\Delta_1, \Delta_2 = \phi$. Assume $\Delta_2 = \phi$, then $\Delta_1 = \Delta \subset \Phi_1$.
Then $(\Delta, \Phi_2) = (E, \Phi_2) = 0$. So $\Phi_2 = \phi$ as $(,)$ is non-degenerate.

(iii) $\Rightarrow$ (ii): Suppose $\Delta = \Delta_1 \cup \Delta_2$, $(\Delta_1, \Delta_2) = 0$. Note $\Delta_1 \cap \Delta_2 = \phi$ as $(,)$ is non-degenerate.
Let $E_i = \langle \Delta_i \rangle$, so $E = E_1 \oplus E_2$, $(E_1, E_2) = 0$.
Suffices to show each E_i is W-stable, as E is irreducible. W is generated by $s_\alpha, \alpha \in \Delta$.
If $\alpha \in \Delta_1$, then s_α fixes all of Δ_2 as $(\alpha, \Delta_2) = 0$. So s_α fixes E_2.
If $\alpha \in \Delta_2$, consider $s_\alpha(\beta)$ for $\beta \in \Delta_2$, $= \beta - \langle \beta, \alpha \rangle \alpha \in E_2$. So E_2 is W-stable.
Similarly for E_1.

(i) $\Rightarrow$ (iii): Suppose E has a W-submodule E_1. Let $E_2 = E_1^\perp$, so $E = E_1 \oplus E_2$.
Must show: for $\alpha \in \Phi$, either $\alpha \in E_1$ or $\alpha \in E_2$. Suppose $\alpha \notin E_1$. s_α stabilises E_1, so for any $\beta \in E_1$, $E_1 \ni s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in E_1$. So $\langle \beta, \alpha \rangle = 0$, so $\alpha \in E_2$.
Now let $\Phi_i = E_i \cap \Phi$. $\Phi = \Phi_1 \cup \Phi_2$, $(\Phi_1, \Phi_2) = 0$. So, wlog, $\Phi_2 = 0$. So $E_2 = 0$ ※.

Eg: $A_1 A_1$:



$\Phi = \{\pm\alpha\} \cup \{\pm\beta\}$, is a partition.
- so this is not an irreducible root system.

Let $\Phi \subset E$ be arbitrary. Let $\Delta = \Delta_1 \cup \dots \cup \Delta_t$ be the unique partition of Δ into mutually orthogonal subsets. Let $E_i = \langle \Delta_i \rangle$. $E = E_1 \oplus \dots \oplus E_t$, orthogonal direct sum.
Let $\Phi_i = \Phi \cap E_i$. Then as in (i) $\Rightarrow$ (iii) in the lemma, $\Phi = \Phi_1 \cup \dots \cup \Phi_t$.
If $\alpha \in \Delta_i$, s_α fixes each E_j, $j \neq i$. So s_α, s_β commute for $\alpha \in \Delta_i$, $\beta \in \Delta_j$ $(j \neq i)$.
Each $\Phi_i \subset E_i$ is an abstract root system, so $W_i := \langle s_\alpha : \alpha \in \Delta_i \rangle$ is its Weyl group, and $W \cong W_1 \times \dots \times W_t$. We've shown:

Proposition: Φ decomposes uniquely as $\Phi = \Phi_1 \cup \dots \cup \Phi_t$, $(\Phi_i, \Phi_j) = 0$. Each Φ_i is irreducible.
Moreover, $\Delta = \Delta_1 \cup \dots \cup \Delta_t$, $E = E_1 \oplus \dots \oplus E_t$, $W = W_1 \times \dots \times W_t$

Call the components $\Phi_i \subset E_i$ the *irreducible components* of Φ.

Lemma: Let Φ be irreducible. There are at most two distinct root lengths in Φ. All roots of the same length in Φ are W-conjugate.

Proof: If $\alpha, \beta \in \Phi$, the $w\alpha$ $(w \in W)$ span E. So not all the $w\alpha$ $(w\in W)$ are orthogonal to β.
So replacing α by $w\alpha$ (which has the same length), we may assume $(\alpha,\beta) \neq 0$.
Then by the table in Lemma 6.1, the possible ratios of squared root lengths $|\alpha|^2/|\beta|^2$ are $1, 2, 3, \frac{1}{2}, \frac{1}{3}$. Suppose more than two root lengths occur, so have all of $1, 2, 3$ for these squared ratios. Then also have $3/2$ as a squared ratio – $\divideontimes$.
Now let α, β have the same length. We may assume $(\alpha,\beta)\neq 0$. The table in lemma 6.1 again shows that $\langle\alpha,\beta\rangle = 1$ or $\langle \alpha,\beta\rangle = -1$. Replacing β by $s_\beta(\beta) = -\beta$ if necessary, we may assume $\langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle = +1$. Then $(s_\alpha s_\beta s_\alpha)(\beta) = (s_\alpha s_\beta)(\beta-\alpha) = s_\alpha(-\beta-\alpha+\beta) = \alpha$.
So α,β are W-conjugate.

6.5. Classification.

Let Δ be a base for Φ, $\Delta = \{\alpha_1, \dots, \alpha_l\}$, with arbitrary but fixed ordering. $l = \dim E = \operatorname{rank}\Phi$.
The $l \times l$ matrix $(\langle \alpha_i, \alpha_j\rangle)_{1\le i,j\le l}$ is called the Cartan matrix. Its entries are the Cartan integers.
It is a non-singular matrix, and is independent (up to reordering of Δ) of the choice of base Δ.
Two root systems Φ, Φ' in E, E' respectively are isomorphic if there exists a vector space isomorphism $\theta: E \to E'$ inducing a bijection $\Phi \to \Phi'$ such that $\langle \theta(\alpha), \theta(\beta)\rangle = \langle\alpha,\beta\rangle$ $\forall\, \alpha,\beta\in\Phi$.

Lemma: If two root systems $\Phi \subset E$, $\Phi'\subset E'$ have the same Cartan matrix, they are isomorphic.

Proof: Let $\alpha_1, \dots, \alpha_l$ and $\alpha_1', \dots, \alpha_l'$ be simple roots such that $\langle\alpha_i,\alpha_j\rangle = \langle \alpha_i', \alpha_j'\rangle$ $\forall\, i,j$.
Define $\theta: E\to E'$ by $\theta(\alpha_i) = \alpha_i'$. So θ is a vector space isomorphism $E \to E'$.
Need to show $\langle\theta(\alpha),\theta(\beta)\rangle = \langle\alpha,\beta\rangle$ $\forall\,\alpha,\beta\in\Phi$, and that θ induces a bijection $\Phi\to\Phi'$.
Note $s_{\theta(\alpha)}(\theta(\beta)) = \theta(\beta) - \langle\theta(\beta),\theta(\alpha)\rangle\theta(\alpha) = \theta(\beta - \langle\beta,\alpha\rangle\alpha)$ if $\alpha,\beta\in\Delta$
$= \theta(s_\alpha(\beta))$.
Since $\begin{Bmatrix}\Delta\\ \Delta'\end{Bmatrix}$ is a basis for $\begin{Bmatrix}E\\ E'\end{Bmatrix}$ and $\begin{Bmatrix}W\\ W'\end{Bmatrix}$ is generated by the s_α $\left(\begin{Bmatrix}\alpha\in\Delta\\ \alpha\in\Delta'\end{Bmatrix}\right)$,
this shows the map $W\to W'$ induced by $s_\alpha \mapsto s_{\theta(\alpha)}$ $(\alpha\in\Delta)$ is an isomorphism of Weyl groups.
Each $\beta\in\Phi$ is conjugate to a simple root (§6.4), so $\beta = w\alpha$, some $\alpha\in\Delta$.
Then $\theta(\beta) = \underbrace{(\theta\circ w\circ\theta^{-1})}_{\in W'}(\underbrace{\theta\alpha}_{\in \Phi'})$. So $\theta(\beta)\in W'\Delta'\subset\Phi'$.
So θ maps Φ onto Φ', hence is a bijection $\Phi\to\Phi'$.
Finally, $\theta(s_\alpha(\beta)) = \theta(\beta) - \langle\beta,\alpha\rangle\theta(\alpha) = s_{\theta(\alpha)}(\theta(\beta))$. So $\langle\beta,\alpha\rangle = \langle\theta(\beta),\theta(\alpha)\rangle$, so θ does indeed preserve all Cartan integers.

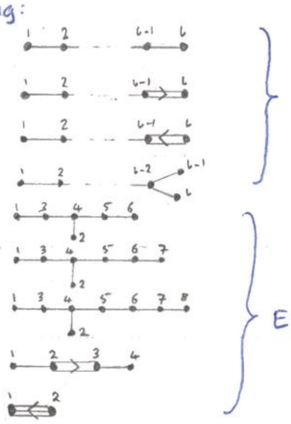
Now, we know $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle$ is either $0, 1, 2$ or 3.
Define the Dynkin diagram of Φ to be the graph with vertices labelled by Δ, and $\langle\alpha_i,\alpha_j\rangle\langle\alpha_j,\alpha_i\rangle$ edges between vertex α_i and vertex α_j. If α_i is longer than α_j, put a $>$ sign on the edge: $\alpha_i \rightarrow\!\!- \alpha_j$.

Exercise: Show that the Cartan matrix can be recovered from the Dynkin diagram.

So, root systems are determined up to isomorphism by their Dynkin diagram.
The root system Φ is irreducible iff its Dynkin diagram is connected.

Theorem: If Φ is an irreducible root system of rank l, its Dynkin diagram is one of the following:

A_l $(l \geq 1)$:
B_l $(l \geq 1)$:
C_l $(l \geq 1)$:
D_l $(l \geq 1)$:
E_6 :
E_7 :
E_8 :
F_4 :
G_2 :



Classical.

Exceptional.

6.6 The Classical Root Systems.

There is also an existence theorem showing that for each Dynkin diagram in §6.5, there exists a root system Φ having that diagram. The proof of this is just to construct them explicitly. We will do this for types $A_l, \dots, D_l$.
Work in $\mathbb{R}^n$ with orthonormal basis $\varepsilon_1, \dots, \varepsilon_n$, $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$.

A_l: Let E be the l-dimensional subspace of $\mathbb{R}^{l+1}$ orthogonal to the vector $\varepsilon_1 + \dots + \varepsilon_{l+1}$.
Let $\Phi = \{\varepsilon_i - \varepsilon_j : 1 \leq i, j \leq l+1, \ i \neq j\}$. $|\Phi| = l(l+1)$.
$\overset{\cup}{\Delta} = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_l - \varepsilon_{l+1}\}$ - a basis for E.
Now Φ is finite, spans E, $0 \notin \Phi$ - (R1) holds. (R2) is clear. (R4) - easy to check.
For (R3): $s_{\varepsilon_i - \varepsilon_j}(\varepsilon_k) = \varepsilon_k - \dfrac{2(\varepsilon_k, \varepsilon_i - \varepsilon_j)}{2} = \varepsilon_k - \delta_{ik}(\varepsilon_i - \varepsilon_j) + \delta_{kj}(\varepsilon_i - \varepsilon_j)$.
So $s_{\varepsilon_i - \varepsilon_j}(\varepsilon_k) = \begin{cases} \varepsilon_k & k \neq i \text{ or } j \\ \varepsilon_i & k = j \\ \varepsilon_j & k = i \end{cases}$ So $s_{\varepsilon_i - \varepsilon_j}$ permutes $\{\varepsilon_1, \dots, \varepsilon_{l+1}\}$ as transposition $(\varepsilon_i \ \varepsilon_j)$
Now (R3) is easy to check.
So $\Phi \subset E$ is a root system. In fact, we've shown that the Weyl group, $W \cong \mathrm{Sym}_{l+1}$.
Next, note Δ is a base for Φ. The reflections s_α $(\alpha \in \Delta)$ are just transpositions $(1\,2), (2\,3), \dots, (l\ l+1)$.

Cartan matrix:

$$\begin{array}{cccc} \varepsilon_1-\varepsilon_2 & \varepsilon_2-\varepsilon_3 & \cdots & \varepsilon_l-\varepsilon_{l+1} \end{array}$$

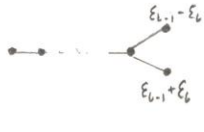
$$\begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

Get diagram:  - type A_l.

$\langle \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3 \rangle^2 = 1$.

B_l: $E = \mathbb{R}^l$. $\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i < j \leq l\} \cup \{\pm \varepsilon_i : 1 \leq i \leq l\}$.
(length)$^2 = 2$ - "long" length $= 1$ - "short"
$\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_l\}$. Get:  - type B_l.

C_l: As B_l, except replace $\{\pm \varepsilon_i\}$ by $\{\pm 2\varepsilon_i\}$.

D_l: $E = \mathbb{R}^l$. $\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i, j \leq l\}$. $\Delta = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{l-1} - \varepsilon_l, \varepsilon_{l-1} + \varepsilon_l\}$.  - type D_l.

E, F, G: Look them up - See Humphreys - "Reflection Groups and Coxeter Groups."

7. Classification of Semisimple Lie Algebras.

Let L be a semi-simple Lie algebra over $\mathbb{C}$, H fixed maximal toral subalgebra. $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$
- Cartan decomposition. $\Phi \subset H^*$ root system.
Let $E = \mathbb{R}$-span of Φ, then $\Phi \subset E$ is an abstract root system. $\operatorname{rank} \Phi = \dim_{\mathbb{R}} E = \dim_{\mathbb{C}} H^* = \operatorname{rank} L$.

7.1. Simple Lie Algebras.

Lemma: (i) If L is simple then Φ is irreducible.
(ii) Conversely, if $L = L_1 \oplus \cdots \oplus L_t$ is not simple, each L_i simple. Let $H_i = H \cap L_i$, so $H = H_1 \oplus \cdots \oplus H_t$ and $H^* = H_1^* \oplus \cdots \oplus H_t^*$. Let $\Phi_i = \Phi \cap H_i^*$. Then $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$ is the decomposition of Φ into its irreducible components.

Proof: (i) Suppose L is simple, $\Phi = \Phi_1 \cup \Phi_2$ where $(\Phi_1, \Phi_2) = 0$. Suppose $\Phi_1 \neq \phi$.
if $\alpha \in \Phi_1$, $\beta \in \Phi_2$ then $(\alpha+\beta, \alpha) \neq 0$, so $\alpha+\beta \notin \Phi_2$, and $(\alpha+\beta, \beta) \neq 0$, so $\alpha+\beta \notin \Phi_1$.
So $\alpha+\beta \notin \Phi$, so is not a root. $[L_\alpha L_\beta] = 0$. So the subalgebra K of L generated by all L_α ($\alpha \in \Phi_1$) is centralised by all L_β ($\beta \in \Phi_2$). So K is an ideal of L, so as $K \neq 0$, L simple $\Rightarrow K = L$. So all L_β ($\beta \in \Phi_2$) lie in $Z(L) = 0$. So $\Phi_2 = \phi$.
(ii) Note each H_i is a maximal toral subalgebra of L_i, and H_i is orthogonal to H_j, $i \neq j$.
For $\alpha \in \Phi$, suppose $L_\alpha \subset L_i$, then $h_\alpha \in [L_\alpha, L_{-\alpha}]$ also lies in L_i, hence in H_i.
But the h_α's span H, so $H = H_1 \oplus \cdots \oplus H_t$. Hence $H^* = H_1^* \oplus \cdots \oplus H_t^*$ and so $\Phi = \Phi_1 \cup \cdots \cup \Phi_t$, and $(\Phi_i, \Phi_j) = 0$. Each Φ_i is the root system of L_i, so each Φ_i is irreducible, by (i).

7.2. Generators for L.

Fix now a base Δ of Φ, hence a set Φ^+ of positive roots and $\Phi^- = -\Phi^+$ of negative roots.
So $\Phi = \Phi^+ \cup \Phi^-$. For each $\alpha \in \Phi^+$, fix $0 \neq e_\alpha \in L_\alpha$. This determines unique elements $0 \neq f_\alpha \in L_{-\alpha}$, $h_\alpha \in H$ such that $[e_\alpha f_\alpha] = h_\alpha$, $[h_\alpha e_\alpha] = 2e_\alpha$, $[h_\alpha f_\alpha] = -2f_\alpha$.
So $\langle e_\alpha, f_\alpha, h_\alpha \rangle = S_\alpha \cong sl_2(\mathbb{C})$.

Let $N^+ = \langle e_\alpha : \alpha \in \Phi^+ \rangle$, $N^- = \langle f_\alpha : \alpha \in \Phi^+ \rangle$.
So $L = H \oplus N^+ \oplus N^-$ - the triangular decomposition of L.

Lemma: N^+ (and N^-) is a nilpotent Lie algebra.
$H + N^+$ (and $H + N^-$) is a soluble Lie algebra.

Proof: The first is immediate from Engel's Theorem.
For the second part, note $N^+ \trianglelefteq N^+ + H$, $N^+ + H / N^+ \cong H / H \cap N^+ \cong H$, which is abelian.
So $N^+ \geq (N^+ + H)'$. N^+ is nilpotent, so $N^+ + H$ is soluble.

Now Δ is a basis for H^*, so $\{t_\alpha : \alpha \in \Delta\}$ is a basis for H. So $\{h_\alpha : \alpha \in \Delta\}$ is a basis for H.
So $\underbrace{\{h_\alpha : \alpha \in \Delta\}}_{H} \cup \underbrace{\{e_\alpha : \alpha \in \Phi^+\}}_{N^+} \cup \underbrace{\{f_\alpha : \alpha \in \Phi^+\}}_{N^-}$ is a basis for L. Such is called the standard basis for L.

Lemma: L is generated (as a Lie algebra) by $\{e_\alpha, f_\alpha : \alpha \in \Delta\}$.

Proof: Will show that N^+ is generated by $\{e_\alpha : \alpha \in \Delta\}$. Similarly, N^- by $\{f_\alpha : \alpha \in \Delta\}$.
So $L = \langle N^+, H, N^- \rangle$ is generated by given set.
Take $\beta \in \Phi^+ \setminus \Delta$. We need to show e_β lies in the subalgebra generated by $\{e_\alpha : \alpha \in \Delta\}$.
Claim: There exists $\alpha_1 \in \Delta$ such that $\beta - \alpha_1 \in \Phi^+$.
Proof: Suppose $(\beta, \alpha) \leq 0 \ \forall \alpha \in \Delta$. Then the set $\Delta \cup \{\beta\}$ is a set of vectors all lying on the same side of a hyperplane in E, such that the angle between any pair of the vectors is obtuse. Hence $\Delta \cup \{\beta\}$ is a linearly independent set $- \#$.
So $\exists \alpha_1 \in \Delta$ such that $(\beta, \alpha_1) > 0$. Lemma 6.1 shows $\beta - \alpha_1 \in \Phi$.
Now write $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$, $k_\alpha \geq 0$. Then $\beta - \alpha_1 = \sum_{\alpha \in \Delta \setminus \{\alpha_1\}} k_\alpha \alpha + (k_{\alpha_1} - 1)\alpha_1 \in \Phi$.
Now $\beta \notin \Delta$, so one of these coefficients is > 0. Hence all are positive, and $\beta - \alpha_1 \in \Phi^+$.
Repeating claim, $\exists \alpha_1, \dots, \alpha_s \in \Delta$ such that each partial sum $\alpha_i + \dots + \alpha_s \in \Phi^+ \ \forall 1 \leq i \leq s$, and $\alpha_1 + \dots + \alpha_s = \beta$. Now, we prove e_β lies in subalgebra generated by $\{e_\alpha : \alpha \in \Delta\}$ by induction on s. By induction, $e_{\beta - \alpha_1}$ lies in this subalgebra, as does e_{α_1}.
So suffices to show $[e_{\alpha_1}, e_{\beta - \alpha_1}] \neq 0$, since then it spans $L_\beta \ni e_\beta$.
Claim: Whenever $\alpha, \beta \in \Phi$, $\alpha + \beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.
Proof: Let $M = \langle L_{\beta + k\alpha} : k \in \mathbb{Z} \rangle$. We showed in §5.4 that M was a simple S_α-module.
So $\operatorname{ad} e_\alpha L_\beta = L_{\beta+\alpha}$ by representation theory of sl_2.

7.3. Relations for L.

Fix Δ, Φ, L, H and a standard basis $\{h_\alpha : \alpha \in \Delta\} \cup \{e_\alpha, f_\alpha : \alpha \in \Phi^+\}$ as usual.
Let $\Delta = \{\alpha_1, \dots, \alpha_l\}$. Let $e_i = e_{\alpha_i}$, $f_i = f_{\alpha_i}$, $h_i = h_{\alpha_i}$ for $1 \leq i \leq l$.

Lemma: The following ("Serre") relations hold in L:
(S1): $[h_i h_j] = 0 \quad \forall\ 1 \leq i, j \leq l$.
(S2): $[e_i f_i] = h_i$, $[e_i f_j] = 0$ if $i \neq j$.
(S3): $[h_i e_j] = \langle \alpha_j, \alpha_i \rangle e_j$. $[h_i f_j] = -\langle \alpha_j, \alpha_i \rangle f_j$.
$$\left.\begin{aligned} (S_{ij}^+)&: (\operatorname{ad} e_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(e_j) = 0 \\ (S_{ij}^-)&: (\operatorname{ad} f_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(f_j) = 0 \end{aligned}\right\} (i \neq j).$$

Proof: (S1), (S3) obvious. $(\langle \alpha_j, \alpha_i \rangle = \alpha_j(h_i))$. $[e_i f_i] = h_i$ in (S2) is definition.
Consider $[e_i f_j]$, $i \neq j$. $\alpha_i - \alpha_j$ is not a root, so $L_{\alpha_i - \alpha_j} = \{0\}$ and contains $[e_i f_j]$.
Look at (S_{ij}^+). Since $i \neq j$, $\alpha_j - \alpha_i$ is not a root. So the α_i-chain through α_j looks like: $\alpha_j, \alpha_j + \alpha_i, \dots, \alpha_j + q\alpha_i$, where $-q = \langle \alpha_j, \alpha_i \rangle$. So $\alpha_j + (q+1)\alpha_i \notin \Phi$
So $(\operatorname{ad} e_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(e_j) = 0$.

Theorem (Serre): Fix a root system Φ with base $\Delta = \{\alpha_1, \dots, \alpha_l\}$. Let L_Φ be the Lie algebra generated by $\{e_i, f_i, h_i : 1 \leq i \leq l\}$ subject to Serre relations. Then L_Φ is a semisimple Lie algebra, with maximal toral subalgebra H_Φ, generated by the h_i's, and the corresponding root system is Φ.

7.4. Applications of Serre's Theorem.

Corollary 1 (The existence theorem): Given an abstract root system Φ, there exists a semisimple Lie algebra with root system Φ (relative to some maximal toral H).

Proof: L_φ, H_φ in Serre's Theorem do.

Corollary 2 (The uniqueness theorem): Let L be a semisimple Lie algebra with a fixed maximal toral H, and corresponding root system Φ. Then L is determined up to isomorphism by the isomorphism type of Φ.

Proof: Let $\Delta = \{\alpha_1, \dots, \alpha_l\}$, and fix standard generators $\{e_i, f_i, h_i\}$ for L. Let e_i', f_i', h_i' be the generators of L_φ in Serre's Theorem. By lemma 7.3, e_i, f_i and h_i satisfy the Serre relations.

So, there is a homomorphism $\theta: L_\varphi \to L$, $e_i' \mapsto e_i$
$f_i' \mapsto f_i$
$h_i' \mapsto h_i$, by Serre's Theorem.

θ is onto as the e_i, f_i generate L.

$\dim L_\varphi = \operatorname{rank}\Phi + |\Phi| = \dim L$. (They have the same Cartan decomposition).

So θ is an isomorphism.

Lemma: Let Φ, Φ' be two non-isomorphic abstract root systems. Then, $L_\varphi \ncong L_{\varphi'}$.

Proof: By 7.1, it suffices to prove this when Φ, Φ' are irreducible root systems, or equivalently, when $L_\varphi, L_{\varphi'}$ are simple. Suppose $L_\varphi \cong L_{\varphi'}$, both simple. So $\dim L_\varphi = \dim L_{\varphi'}$

So $\operatorname{rank}\Phi + |\Phi| = \operatorname{rank}\Phi' + |\Phi'|$

Φ	$\operatorname{rank}\Phi + \lvert\Phi\rvert$	dim (*)
A_l	$l^2 + 2l$	$l+1$
B_l	$2l^2 + l$	$2l+1$
C_l	$2l^2 + l$	$2l$
D_l	$2l^2 - l$	$2l$
E_6	78	27
E_7	133	56
E_8	248	248
F_4	52	26
G_2	14	7

We will show in the next chapter (independent of this lemma), that the dimension of the smallest faithful irreducible L_φ-module depends only on Φ, and are in column (*)

Now observe that these two numbers are the same for L_φ and $L_{\varphi'}$. Moreover, by the table, if the numbers are equal, then $\Phi \cong \Phi'$ ✱. Hence $L_\varphi \ncong L_{\varphi'}$ as required.

Corollary 3 (The conjugacy theorem): Let L be a semisimple Lie algebra, and let H, H' be two maximal toral subalgebras. Then $\exists$ automorphism $\theta: L \to L$ such that $\theta(H) = H'$.

Proof: Lemma shows root systems Φ, Φ' of L relative to H, H' respectively, are isomorphic.

So, by corollary 2, $\exists$ isomorphism $\theta_1: L_\varphi \to L$ sending H_φ onto H
and $\theta_2: \overset{\shortparallel}{L_{\varphi'}} \to L$ sending $H_{\varphi'}$ onto H'

Then $\theta_2 \circ \theta_1^{-1}: L \to L$, and this sends $H \to H_\varphi = H_{\varphi'} \to H'$.

Corollary 3 shows that (up to isomorphism) the root system Φ of L is independent of the choice of H. So by corollary 2, L is determined uniquely (up to isomorphism) by its root system. Moreover, by corollary 1, for every Φ, $\exists$ an L having Φ as its root system. So...

Classification Theorem: The map $L \to \Phi$ is an isomorphism from:

$$\left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{semisimple Lie algebras over } \mathbb{C} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{isomorphism types of} \\ \text{abstract root systems} \end{array} \right\}.$$

The inverse is the map $\Phi \to L_\Phi$.

L is a simple Lie algebra iff Φ is an irreducible root system.

Remarks: (i) This theorem is due to Dynkin.

(ii) The proof of the lemma before corollary 3 is a bit hopeless! There is a much better way of proving this: it's an easy corollary of the conjugacy theorem.

To prove corollary 3 directly:

A Cartan subalgebra of an arbitrary Lie algebra L is any nilpotent, self-normalising subalgebra of L (ie, $C = N_L(C)$).

In case L is semisimple, Cartan subalgebra $\equiv$ maximal toral subalgebra.

One can prove:

Theorem: If L is an arbitrary (not necessarily semisimple) Lie algebra over $\mathbb{C}$, then all Cartan subalgebras of L are conjugate under (inner) automorphisms of L.

7.5. The Classical Lie Algebras.

$A_l, \ldots, D_l$. Will give explicit constructions of Lie algebras of classical type.

We already know what their root systems look like. (§6.6).

A_l: This is just $sl_n(\mathbb{C})$, $n = l+1$.

We've already shown (§5.2) that $sl_n(\mathbb{C})$ is a semisimple Lie algebra with maximal toral $H = \{$diagonal trace 0 matrices$\}$. We constructed the Cartan decomposition, and saw that $\Phi = \{\varepsilon_i - \varepsilon_j : 1 \le i \ne j \le n\}$, where $\varepsilon_i \in H^*$ was the function sending $\mathrm{diag}(h_1, \ldots, h_n) \mapsto h_i$.

The killing form on $sl_n(\mathbb{C})$ is given by $\mathrm{tr}_L(\mathrm{ad}x\, \mathrm{ad}y) = 2n\, \mathrm{tr}(xy)$ (example sheet 2).

Using this, it is easy to check that $\langle \varepsilon_i - \varepsilon_j, \varepsilon_h - \varepsilon_k \rangle = \delta_{ih} + \delta_{jk} - \delta_{jh} - \delta_{ik}$.

Hence Φ is the root system of type A_l.

B_l: This is $SO_{2l+1}(\mathbb{C})$ = all $(2l+1)\times(2l+1)$ matrices x such that $sx = -x^T s$ where s is the matrix $\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 0 & I_l \\ \hline 0 & I_l & 0 \end{array}\right)$.

D_l: is $SO_{2l}(\mathbb{C}) = (2l)\times(2l)$ matrices x such that $sx = -x^T s$ where $s = \left(\begin{array}{c|c} 0 & I_l \\ \hline I_l & 0 \end{array}\right)$

C_l: is $Sp_{2l}(\mathbb{C})$ = $(2l)\times(2l)$ matrices x such that $sx = -x^T s$, where $s = \left(\begin{array}{c|c} 0 & I_l \\ \hline -I_l & 0 \end{array}\right)$

One can check (as for type A) that the diagonal matrices in each case is a maximal toral subalgebra, hence by constructing the Cartan decomposition explicitly that the root systems are B_l, D_l, C_l as claimed.

7.6. Some notes on the proof of Serre's Theorem. (see handout).

The most important idea in the proof is the proof of Claim 2. - constructing an action of the Weyl group W of L on the set of weight spaces of L wrt H, using exponential operators.

8. Representation Theory of Semisimple Lie Algebras II.

- all finite-dimensional L-modules split as a direct sum of irreducibles (Weyl's Theorem).
- sl_2 - theory: irreducible, finite dimensional sl_2-modules are parametrised by their "highest weight" - we worked out the h-eigenspace, all weights were in $\mathbb{Z}$.

From now on, fix notation: L = semisimple Lie algebra over $\mathbb{C}$, H = maximal toral subalgebra, $\Phi \subset H^*$ be root system, $E = \mathbb{R}\text{-span}(\Phi)$, $\Delta = \{\alpha_1, .., \alpha_l\}$ be base for Φ, W be the Weyl group, $W = \langle s_{\alpha_i} : 1 \le i \le l \rangle$.

8.1. Weights.

Put a partial order on H^*, called the dominance order: say $\lambda \le \mu$ $(\lambda, \mu \in H^*)$ if $\mu - \lambda$ is a linear combination of positive roots in Φ, with coefficients in $\mathbb{Z}_{\ge 0}$.
Eg: for any root $\alpha \in \Phi^+$, $\alpha - 0$ is a linear combination of Φ^+ with non-negative coefficients. So $\alpha \ge 0$ in dominance order.

Let X be the set of all $\lambda \in H^*$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ $\forall \alpha \in \Phi^+$ (equivalently, $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}$ $\forall 1 \le i \le l$).
Call elements of X integral weights.
Let X^+ be the set of all $\lambda \in X$ such that $\langle \lambda, \alpha \rangle \ge 0$ $\forall \alpha \in \Phi^+$ (or $\langle \lambda, \alpha_i \rangle \ge 0$ $\forall 1 \le i \le l$).
Call elements of X^+ dominant integral weights.

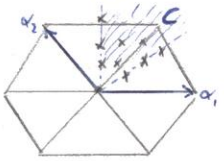
If $\Delta = \{\alpha_1, .., \alpha_l\}$, let $\{\omega_1, .., \omega_l\}$ be the new basis for E satisfying $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$ $\forall 1 \le i, j \le l$.
The ω_i are called fundamental dominant weights.

Now, any $\lambda \in H^*$ can be written $\sum_{i=1}^{l} \lambda_i \omega_i$. $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}$ (resp. $\mathbb{Z}_{\ge 0}$) iff $\lambda_i \in \mathbb{Z}$ (resp. $\mathbb{Z}_{\ge 0}$).
So $X = \{ \sum \lambda_i \omega_i : \lambda_i \in \mathbb{Z} \}$, $X^+ = \{ \sum \lambda_i \omega_i : \lambda_i \in \mathbb{Z}_{\ge 0} \}$.

Examples: $\Phi = A_1$. So $\Phi = \{\pm\alpha_1\}$, $\Delta = \{\alpha_1\}$. Then $\omega_1 = \frac{1}{2}\alpha_1$.

$\langle \omega_1, \alpha_1 \rangle = \frac{2(\omega_1, \alpha_1)}{(\alpha_1, \alpha_1)} = \frac{2 \cdot 2 \cdot \frac{1}{2}}{2} = 1.$

$\Phi = A_2$.



$\Delta = \{\alpha_1, \alpha_2\}$, with $\alpha_1 = \varepsilon_1 - \varepsilon_2$, $\alpha_2 = \varepsilon_2 - \varepsilon_3$.

$\omega_1 = \frac{1}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3)$

$\omega_2 = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3)$

Here, X = lattice points.

X^+ = lattice points lying in $\bar{C} = X \cap \bar{C}$.

In fact this is true in general, $X^+ = X \cap \bar{C}$. On one of the handouts: lists all ω_i (λ_i!) in terms of α_i.

Recall that the Weyl group W is the subgroup of $GL(E)$ generated by $\{s_\alpha : \alpha \in \Delta\}$.
As $s_\alpha^2 = 1$, $s_\alpha = s_\alpha^{-1}$, so any word $w \in W$ can be written $w = s_{\beta_1} \cdots s_{\beta_m}$, $\beta_i \in \Delta$.
Call the minimal m for which such a word exists the length of w, and in that case the expression $w = s_{\beta_1} \cdots s_{\beta_m}$ is called a reduced expression for w. Let $l(w) = m$ be length.
Define the sign representation
$$\varepsilon : W \to \{\pm 1\}, \quad w \mapsto (-1)^{l(w)}$$

This is a homomorphism of W onto $\{\pm 1\}$.

[Note each $s_\alpha \in GL(E)$ is a reflection of E, so has matrix $\begin{pmatrix} -1 & 0 \\ 0 & I \end{pmatrix}$ in a suitable basis.
So $\det s = -1$, so $\det w = (-1)^m$. Now, $\det : GL(E) \to \mathbb{R}$ is a homomorphism, and ε is just the restriction of $\det$ to $W < GL(E)$]

Eg: if $\Phi = A_l$, $W = \mathrm{Sym}_{l+1}$, ε = signature of symmetric group.

The Weyl group W acts on Φ and preserves $(\,,\,)$. So W acts on X.
Every $\lambda \in X$ is conjugate under W to a unique element of $X^+ = X \cap \bar{C}$.

8.2 Weight Spaces.

Let V be a finite dimensional L-module. By the Jordan decomposition, each $h \in H$ is diagonalisable on V. H is abelian, so we can simultaneously diagonalise: $V = \bigoplus_{\lambda \in H^*} V_\lambda$, where $V_\lambda = \{v \in V : hv = \lambda(h)v \ \forall h \in H\}$. So V is a direct sum of its weight spaces.
Call the λ for which $V_\lambda \neq 0$ the weights on V.

What if V is infinite dimensional? Then V need no longer be a direct sum of its weight spaces. We can still define $V_\lambda = \{v \in V : hv = \lambda(h)v \ \forall h \in H\}$. Let $V' = \bigoplus_{\lambda \in H^*} V_\lambda$.
This is a subspace of V, but need not be equal to V in general.

Lemma: L_α ($\alpha \in \Phi$) maps V_λ to $V_{\lambda+\alpha}$.
Proof: $v \in V_\lambda$, $h \in H$, $x \in L$. Then $h \cdot xv = x \cdot hv + [hx]v = \lambda(h)xv + \alpha(h)xv = (\lambda+\alpha)(h)xv$, so $xv \in V_{\lambda+\alpha}$.

So, by the lemma, each L_α stabilises V'. So $V' < V$ is an L-submodule of V.

Key Definition: a highest weight vector is a vector $0 \neq v \in V_\lambda$ for some $\lambda \in H^*$ such that $L_\alpha v = 0 \quad \forall \alpha \in \Phi^+$. Call λ the highest weight of v.

If V is infinite dimensional, highest weight vectors need not exist. But if V is finite dimensional, there are finitely many non-zero weight spaces. So we can pick one such V_λ such that $\lambda + \alpha$ is not a weight of V for any $\alpha \in \Phi^+$. So $L_\alpha V_\lambda = 0 \quad \forall \alpha \in \Phi^+$. So any $0 \neq v \in V_\lambda$ is a highest weight vector in V of weight λ.

Say V is a highest weight module if it is generated as an L-module by a single highest weight vector $v \in V_\lambda$.

Note: any finite dimensional irreducible L-module is an example of a highest weight module.

Reminder: L a Lie algebra, $U(L)$ is its universal enveloping algebra.

L-modules $\longleftrightarrow$ $U(L)$-modules.

For example, saying V is generated by the vector v as an L-module is the same as saying $V = \{uv : u \in U(L)\} = U(L).v$

Theorem: Let V be a highest weight module generated by a highest weight (h/w) vector $0 \neq v \in V_\lambda$. Let $\{e_\alpha, f_\alpha : \alpha \in \Phi\} \cup \{h_i : 1 \le i \le l\}$ be a standard basis for L. Then,

(a) V is spanned by $\{ f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} . v : \Phi^+ = \{\beta_1, \dots, \beta_m\},\ \forall i_j \geq 0\}$.

Hence V is the direct sum of its weight spaces.

(b) All weights of V are $\leq \lambda$ in the dominance order. Each weight space is finite dimensional, and $\dim V_\lambda = 1$.

(c) Each submodule W of V is also the direct sum of its weight spaces.

(d) V has a unique maximal (proper) submodule.

Proof: (a) Let $L = \underset{\{f_{\beta_1},\dots,f_{\beta_m}\}}{\underset{\text{basis}}{N^-}} \oplus \underset{\{h_1,\dots,h_l\}}{\underset{\text{basis}}{H}} \oplus \underset{\{e_{\beta_1},\dots,e_{\beta_m}\}}{\underset{\text{basis}}{N^+}}$ be the triangular decomposition of L.

By Theorem 2.2, corollary C, $U(L)$ has basis $\{ f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} h_1^{j_1} \cdots h_l^{j_l} e_{\beta_1}^{k_1} \cdots e_{\beta_m}^{k_m} : \forall i_1, \dots, i_m, j_1, \dots, j_l, k_1, \dots, k_m \geq 0\}$.

Also, $\{f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m}\}$, $\{h_1^{j_1} \cdots h_l^{j_l}\}$, $\{e_{\beta_1}^{k_1} \cdots e_{\beta_m}^{k_m}\}$ are bases for $U(N^-)$, $U(H)$, $U(N^+)$ respectively.

So $U(L) = U(N^-).U(H).U(N^+) \quad (\cong U(N^-) \otimes_{\mathbb{C}} U(H) \otimes_{\mathbb{C}} U(N^+))$.

So, as $V = U(L).v$, $\{mv : \forall \text{ monomials } m \text{ in basis for } U(L)\}$ is a spanning set for V.

But $U(H)U(N^+)v \in \langle v \rangle$. So $V = U(L)v = U(N^-)v$

So actually $\{f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} : \forall i_1, \dots, i_m \geq 0\}$ is a spanning set for V.

And each $f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} v \in V'$, so $V = V'$ and is a direct sum of its weight spaces.

(b) $wt(f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m}) = \lambda - \sum i_j \beta_j$ by lemma 8.2

$\hat{\wedge}$
λ in dominance order.

Each weight space is finite dimensional as only finitely many tuples $(i_1, \dots, i_m)$ satisfy $\mu = \lambda - \sum i_j \beta_j$. We can get $\lambda = \mu$ only if each $i_j = 0$, so precisely one tuple $(i_1, \dots, i_m) = (0, \dots, 0)$ spans V_λ, so $\dim V_\lambda = 1$.

(c) Take any $W < V$ and some $w \in W$. We can write $w = \sum_{\mu \in H^*} w_\mu$ for unique $w_\mu \in V_\lambda$.

Must show: each w_μ lies in W. Note only finitely many w_μ are non-zero in this sum. Prove this by induction on the number of non-zero w_μ's in the sum.

Suppose $w_\mu \neq 0 \neq w_\nu$ for $\mu \neq \nu$. Then we can find $h \in H$ such that $\mu(h) \neq \nu(h)$. Consider $\frac{h-\nu(h)}{\mu(h)-\nu(h)} \cdot w \in W$. But $(h-\nu(h)) \cdot w_\nu = 0$, while $\left(\frac{h-\nu(h)}{\mu(h)-\nu(h)}\right) w_\mu = w_\mu$.
So $\frac{h-\nu(h)}{\mu(h)-\nu(h)} w$ can be written as $\sum_{\mu'} w_{\mu'}$, with each $w_{\mu'} \in V_\mu$, and with strictly fewer non-zero terms in the summation, and $w_{\mu'} = w_\mu$.
So by induction, $w_\mu \in W$. So each $w_\mu \in W$ which is what we wanted.

(d) As $\dim V_\lambda = 1$ and any non-zero $v' \in V_\lambda$ generates all of V. Every proper submodule W - which is a direct sum of weight spaces by (c) - cannot have λ as a weight. So the sum of all proper submodules of V - which is a direct sum of weight spaces by (c) - does not have λ as a weight. So the sum of all proper submodules is a proper submodule, hence the unique maximal proper submodule of V.

Corollary: Let V be an irreducible h/w module. Suppose λ, λ' are two weights such that V is generated by $0 \neq v \in V_\lambda$ and by $0 \neq v' \in V_{\lambda'}$, where v, v' are h/w vectors. Then $\lambda = \lambda'$ and v, v' are scalar multiples.

Proof: $\dim V_\lambda = 1 = \dim V_{\lambda'}$. $\left.\begin{matrix}\lambda' \leq \lambda \\ \lambda \leq \lambda'\end{matrix}\right\} \Rightarrow \lambda = \lambda'$, by part (b) above.

8.3. Verma Modules.

Problem: Which $\lambda \in H^*$ have h/w modules of h/w λ?
Fix $\lambda \in H^*$. Let $L = N^- \oplus H \oplus N^+$ and let $B = H \oplus N^+$. (B is called a Borel subalgebra).
As $N^+ \triangleleft B$, we can define a 1-dimensional B-module $\underset{\sim}{\lambda}$ by $\underset{\sim}{\lambda} = \langle 1_\lambda \rangle$ and $h \cdot 1_\lambda = \lambda(h) \cdot 1_\lambda$, $e \cdot 1_\lambda = 0 \ \forall e \in N^+$.
Now, $U(L) \cong U(N^-) \otimes U(B)$, by Corollary 2.2(c). So $U(L)$ is a right $U(B)$-module, and $\underset{\sim}{\lambda}$ is a left $U(B)$-module.
Set $M(\lambda) = U(L) \otimes_{U(B)} \underset{\sim}{\lambda}$ - call this the Verma module of h/w λ.

Remark: If $H < G$ are two groups and M is an H-module, then $\mathrm{ind}_H^G M$ from group theory is just the module $kG \otimes_{kH} M$. - induced module.

Recall: Frobenius Reciprocity. If $B < A$ are two associative algebras, then
$$\mathrm{Hom}_B(N, \mathrm{res}_B^A M) \cong \mathrm{Hom}_A(\underbrace{A \otimes_B N}_{= \mathrm{ind}_B^A N}, M) \quad \forall \ A\text{-modules } M, B\text{-modules } N.$$
Proof is identical to proof for finite groups.

Theorem: The Verma module $M(\lambda)$ is a h/w module of h/w λ, and any other h/w module of h/w λ is a quotient of $M(\lambda)$. [Note: some books write $Z(\lambda)$ for $M(\lambda)$].

Proof: $M(\lambda) = U(L) \otimes_{U(B)} \underset{\sim}{\lambda}$. $1 \otimes 1_\lambda$ is a highest weight vector of weight λ. (h/w only depends on action of B!) More, $U(L) \otimes_{U(B)} \underset{\sim}{\lambda} = U(N^-) U(B) \otimes_{U(B)} \underset{\sim}{\lambda} = U(N^-) . 1 \otimes_{U(B)} \underset{\sim}{\lambda}$.
So $M(\lambda)$ is generated as a $U(N^-)$-module by the vector $1 \otimes 1_\lambda$.
So it is a h/w module of h/w λ.
Now, let V be any other h/w module of h/w λ. Take $0 \neq v \in V_\lambda$.
$$\mathrm{Hom}_{U(L)}(\underbrace{U(L) \otimes_{U(B)} \underset{\sim}{\lambda}}_{= M(\lambda)}, V) \cong \mathrm{Hom}_{U(B)}(\underset{\sim}{\lambda}, \mathrm{res}_{U(B)}^{U(L)} V)$$

Now, $U(B)$ acts on v just as it acts on $1_\lambda \in \underset{\sim}{\lambda}$. So the map $\begin{matrix} \underset{\sim}{\lambda} \to V \\ 1_\lambda \mapsto v \end{matrix}$ gives a homomorphism on the RHS as $U(B)$-modules. By Frobenius reciprocity, get a map of $U(L)$-modules on LHS, $\begin{matrix} M(\lambda) \to V \\ 1\otimes 1_\lambda \mapsto v \end{matrix}$. This is onto as v generates V.

So V is a quotient of $M(\lambda)$.

We constructed the Verma module, $M(\lambda)$. For each $\lambda \in H^*$, there exists a module $M(\lambda)$ which is a h/w module of h/w λ, such that all other h/w modules of h/w λ are quotients of $M(\lambda)$.

Corollary: For each $\lambda \in H^*$, $\exists$ a unique (up to isomorphism) irreducible h/w module of h/w λ.

Proof: $M(\lambda)$ is a h/w module of h/w λ. By Theorem 8.2, $M(\lambda)$ has a unique proper maximal submodule - call it $\operatorname{rad} M(\lambda)$. Let $V(\lambda) = M(\lambda)/\operatorname{rad} M(\lambda)$. Then $V(\lambda)$ is an irreducible h/w module of h/w λ.

Given any irreducible h/w module W of h/w λ, W is a quotient of $M(\lambda)$, by the universal property of Verma modules. So $W \cong M(\lambda)/N$, for some submodule N.

W irreducible $\Rightarrow N = \operatorname{rad} M(\lambda)$, so $W \cong V(\lambda) = M(\lambda)/\operatorname{rad} M(\lambda)$.

8.4 Finite dimensional Irreducible L-modules

We've shown
- 8.3 → For each $\lambda \in H^*$, $\exists$ a unique irreducible h/w module $V(\lambda)$ of h/w λ.
- 8.2 → if V is a finite dimensional irreducible L-module, then V is a h/w module with a unique h/w λ. So $V \cong V(\lambda)$.

So $\{V(\lambda) : \lambda \in H^*, \dim V(\lambda) < \infty\}$ is a complete set of non-isomorphic, irreducible, finite dimensional L-modules. Which $V(\lambda)$ satisfy $\dim V(\lambda) < \infty$?

Lemma: Suppose $\dim V(\lambda) < \infty$. Then $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}_{\geq 0}$ $\forall \alpha_i \in \Delta$, ie λ is a dominant integral weight, so $\lambda \in X^+$, and all other weights in $V(\lambda)$ are integral weights in X.

Proof: $\langle \lambda, \alpha_i \rangle = \lambda(h_i)$. We need to show $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ $\forall\ 1 \leq i \leq L$.

Fix i, let $S_i = \langle e_i, f_i, h_i \rangle \cong sl_2(\mathbb{C})$. Let v^+ be a h/w vector in $V(\lambda)$ of h/w λ.

v^+ lies in a finite dimensional S_i-submodule of $V(\lambda)$, and it's an S_i-h/w vector.

By sl_2-theory, $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$. True $\forall i$ so done.

For convenience, fix $\lambda \in X^+$, let $V = V(\lambda)$, let $\Pi(\lambda)$ = set of weights of $V = \{\mu \in H^* : V_\mu \neq 0\}$.
Aim is to show V is finite dimensional.

Theorem: If $\lambda \in X^+$, then $V = V(\lambda)$ is finite dimensional, and the Weyl group W - which acts on X - stabilises $\Pi(\lambda)$, and $\dim V_\mu = \dim V_{w\mu}$ $\forall \mu \in \Pi(\lambda), w \in W$.

The proof is similar in spirit to the proof of Serre's theorem.

Proof: Fix generators $\{e_i, f_i, h_i : 1 \le i \le l\}$ for L (satisfying Serre relations).

So e_i, f_i generate $U(L)$ as an associative algebra. By induction, check the following identities in $U(L)$:

(I1): $[e_j, f_i^{k+1}] = 0$ when $i \neq j$, $k \geq 0$.

(I2): $[h_j, f_i^{k+1}] = -(k+1)\langle \alpha_i, \alpha_j \rangle f_i^{k+1}$, $k \geq 0$.

(I3): $[e_i, f_i^{k+1}] = -(k+1) f_i^k (k - h_i)$, $k \geq 0$.

Let v^+ be the h/w vector in V of h/w λ. Let $\varphi : L \to gl(V)$ be the representation.

Let $m_i = \langle \lambda, \alpha_i \rangle \in \mathbb{Z}_{\geq 0}$, by assumption.

Lemma 1: Fix $1 \le i \le l$. Then V is the sum of all its finite dimensional S_i-submodules.

Proof: Let $w = f_i^{m_i+1} v^+$. By (I1), if $i \neq j$, $e_j w = 0$.

Also $e_i w = e_i f_i^{m_i+1} v^+ = f_i^{m_i+1} \underbrace{e_i v^+}_{=0} - (m_i+1) f_i^{m_i} \underbrace{(m_i - h_i) v^+}_{=0} = 0$.

So w is a h/w vector of weight $\neq \lambda$. So $w = 0$.

So $\langle v^+, f_i v^+, \dots, f_i^{m_i} v^+ \rangle$ is f_i-stable.

It's also e_i-stable by (I1)-(I3), so we have constructed a finite dimensional S_i-submodule of V.

Let V' = sum of all finite dimensional S_i-submodules of V. We've shown $V' \neq 0$.

Take any finite dimensional S_i-submodule U of V. Then the span of all $\{e_\alpha U, f_\alpha U : \alpha \in \Phi^+\}$ is finite dimensional. It is easily seen to be S_i-stable.

So LU is S_i-stable. So $LU \subset V'$. So V' is L-stable. V is irreducible, so $V = V'$.

By lemma 1, for any $v \in V$, $\exists$ finite dimensional S_i-submodule U containing v.

Now $\varphi(e_i)|_U$ and $\varphi(f_i)|_U$ are nilpotent endomorphisms of U. So V is annihilated by sufficiently large power of $\varphi(e_i)$, $\varphi(f_i)$, $\forall i$. This means that

$\exp(\varphi(e_i)) = 1 + \varphi(e_i) + \frac{\varphi(e_i)^2}{2!} + \dots$ acts in a well-defined way on any vector $v \in V$.

So $\exp(\varphi(e_i))$, $\exp(\varphi(f_i))$ are endomorphisms $V \to V$.

Lemma 2: Let $s_i = \exp(\varphi(e_i)) \exp(\varphi(-f_i)) \exp(\varphi(e_i)) : V \to V$. Then s_i is an automorphism of V, and for $\mu \in \Pi(\lambda)$, $s_i(V_\mu) = V_{s_{\alpha_i}\mu}$.

Proof: s_i is an automorphism as $\exp(\varphi(e_i))$ is invertible, with inverse $\exp(\varphi(-e_i))$, by formal properties of exp.

So we just need to show that s_i sends V_μ to $V_{s_{\alpha_i}\mu}$.

V_μ lies in a finite dimensional S_i-submodule of V. So it suffices to show that s_i acts on the weight spaces of a finite dimensional irreducible S_i-submodule of V meeting V_μ as s_{α_i} does.

The weights of such an irreducible S_i-module will be $\mu - r\alpha_i, \dots, \mu + q\alpha_i$ - the α_i-chain through μ. s_{α_i} acts on these weights be reversing the chain.

So we need to show that given an irreducible S_i-module V' with weights $-m, -m+2, \dots, m-2, m$, the automorphism s_i acts by reversing the weight spaces.

Exercise: Show $s_i^{-1} \varphi(h_i) s_i = \varphi(\underbrace{\exp \operatorname{ad}(e_i) \exp \operatorname{ad}(-f_i) \exp \operatorname{ad}(e_i)}_{=:\tau,\text{ an automorphism of}} . h_i)$

So $\varphi(h_i) s_i v = s_i (\underbrace{s_i^{-1} \varphi(h_i) s_i}_{=\tau(h_i)}) v = s_i \tau(h_i) v$.

Exercise: Compute $\tau(e_i) = -f_i$, $\tau(f_i) = -e_i$, $\tau(h_i) = -h_i$.

(back to lemma): So $\varphi(h_i) s_i v = s_i \varphi(-h_i) v$.

So s_i does indeed reverse the weight chain.

By lemma 2, as each $s_{\alpha_i}: 1 \le i \le l$ generates W, and $s_i \in \mathrm{Aut}(V)$ permutes the weights of $\Pi(\lambda)$. as s_i does, we obtain an action of W on $\Pi(\lambda)$ such that $\dim V_\mu = \dim V_{w\mu}$.
Now we can show that V is finite dimensional. Sufficient to show that W has finitely many orbits on $\Pi(\lambda)$. Each W-orbit in X contains a unique element of X^+, so sufficient to show that $\Pi(\lambda) \cap X^+$ is finite, $\subseteq \{\mu \in X^+ : \mu \le \lambda\}$. So indeed the set is finite, proving the theorem.

8.6 Characters.

We have shown that $\{V(\lambda) : \lambda \in X^+\}$ is a complete set of non-isomorphic finite dimensional irreducible L-modules. Any finite dimensional L-module splits as a direct sum of irreducibles.
We know $V(\lambda)$ is a direct sum of its weight spaces, where
(1) $\dim V(\lambda)_\mu \neq 0 \Rightarrow \mu \le \lambda$ in the dominance order.
(2) $\dim V(\lambda)_\lambda = 1$.
(3) W acts on the weights of $V(\lambda)$, and $\dim V(\lambda)_\mu = \dim V(\lambda)_{w\mu}$ $\forall w \in W$.
All weights lie in $X < E < H^*$. In fact, $X = \{\sum a_i \omega_i : a_i \in \mathbb{Z}\} \cong \mathbb{Z}^l$.
So X is an abelian group of rank l.
We write group multiplication in X additively. We want to work in the group ring of X, $\mathbb{Z}[X]$.
$\mathbb{Z}[X]$ will be free $\mathbb{Z}$-module with basis $\{e(\lambda) : \lambda \in X\}$ and multiplication induced by $e(\lambda)e(\mu) = e(\lambda+\mu)$.

Given an L-module V, which (i) equals the direct sum of its weight spaces,
(ii) every weight space V_μ is finite dimensional,
(iii) every weight of V lies in X,
define $\mathrm{ch}\, V = \sum_{\mu \in X} \dim V_\mu \cdot e(\mu) \in \mathbb{Z}[X]$.
Call $\mathrm{ch}\, V$ the formal character.
So $\mathbb{Z}[X]$ is just a tool for keeping track of $\dim V_\mu$ $\forall \mu$. Note we've defined $\mathrm{ch}\, V$ even if V has infinitely many weights! Should be worried about such infinite sums. To get round this, regard elements of $\mathbb{Z}[X]$ as functions $X \to \mathbb{Z}$ where $e(\mu)$ is the function sending $\lambda \in X$ to $\delta_{\lambda\mu} \in \mathbb{Z}$.
$\mathbb{Z}[X]$ = "infinite sum" over $\mathbb{Z}$ of basis $\{e(\mu) : \mu \in X\}$.

The Weyl group W acts on X, so it acts on $\mathbb{Z}[X]$ by $w.e(\mu) = e(w\mu)$ $\forall \mu \in X, w \in W$.
Let $\mathbb{Z}[X]^W = \{f \in \mathbb{Z}[X] : wf = f \ \forall w \in W\} = \{W\text{-invariant characters}\}$.
If V is finite dimensional, then $\mathrm{ch}\, V$ is a nice finite sum, and it splits as a direct sum of $V(\lambda)$'s for $\lambda \in X^+$ by Weyl's Theorem. Moreover, (3) above says $\mathrm{ch}\, V(\lambda) \in \mathbb{Z}[X]^W$ $\forall \lambda \in X^+$.
So also $\mathrm{ch}\, V \in \mathbb{Z}[X]^W$.

Lemma: $\{\mathrm{ch}\, V(\lambda) : \lambda \in X^+\}$ is a basis for $\mathbb{Z}[X]^W$. (In particular, $\mathbb{Z}[X]^W$ is free).
Proof: $\mathbb{Z}[X]^W$ clearly has basis $\{\sum_{w \in W} w e(\lambda) : \lambda \in X^+\}$ as each $\lambda \in X$ has a unique element

of X^+ in its W-orbit. Now, $\mathrm{ch}\, V(\lambda) = \sum_{w \in W} w e(\lambda) + \sum_{\substack{\mu \prec \lambda \\ \mu \in X^+}} c_\mu \left(\sum_{w \in W} w e(\mu) \right)$ for $\lambda \in X^+$, by (1), (2).

Now use induction on the dominance order.

Corollary: If V is a finite dimensional L-module, then V is determined up to isomorphism by its character.

Proof: By Weyl's Theorem, $V \cong \bigoplus_{\lambda \in X^+} n_\lambda V(\lambda)$. So $\mathrm{ch}\, V = \sum_{\lambda \in X^+} n_\lambda \cdot \mathrm{ch}\, V(\lambda)$.

But the $\mathrm{ch}\, V(\lambda)$ are a basis for $\mathbb{Z}[X]^W$ and $\mathrm{ch}\, V \in \mathbb{Z}[X]^W$, so these coefficients n_λ are uniquely determined by $\mathrm{ch}\, V$.

Consequently, to understand finite dimensional L-modules, it suffices to know $\mathrm{ch}\, V(\lambda)$ for $\lambda \in X^+$. In particular, this will tell us what $\dim V(\lambda)$ is.

Aim: Weyl's Character Formula!

Now, compute $\mathrm{ch}\, M(\lambda)$ for $\lambda \in X^+$.

Consider $\mathbb{Z}[X] \ni 1 + e(-\alpha) + e(-2\alpha) + e(-3\alpha) + \cdots$ for $\alpha \in \Phi^+$

$$= 1 + e(-\alpha) + e(-\alpha)^2 + e(-\alpha)^3 + \cdots$$

$$= \frac{1}{1 - e(-\alpha)}$$

Lemma: For $\lambda \in X$, $\mathrm{ch}\, M(\lambda) = \dfrac{e(\lambda)}{\prod_{\alpha \in \Phi^+} (1 - e(-\alpha))} = \dfrac{e(\lambda + \rho)}{\prod_{\alpha \in \Phi^+} (e(\frac{1}{2}\alpha) - e(-\frac{1}{2}\alpha))}$, where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$

so $e(\rho) = \prod_{\alpha \in \Phi^+} e(\frac{1}{2}\alpha)$.

[For now, write $\alpha > 0$ instead of $\alpha \in \Phi^+$].

Proof: Recall $M(\lambda) = U(L) \otimes_{U(B)} \underset{\sim}{\lambda}$.

Now $U(L)$ has basis $\{ f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} \underbrace{h_1^{j_1} \cdots h_l^{j_l} e_{\beta_1}^{k_1} \cdots e_{\beta_m}^{k_m}}_{\text{basis for } U(B)} : i, j, k \geqslant 0 \}$, where $\Phi^+ = \{\beta_1, \ldots, \beta_m\}$,

while $U(B)$ has basis $\{ h \cdots e \cdots \}$.

So $M(\lambda)$ has basis $\{ f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} \otimes 1_\lambda : i_1, \ldots, i_m \geqslant 0 \}$. $\mathrm{wt}\,(f_{\beta_1}^{i_1} \cdots f_{\beta_m}^{i_m} \otimes 1_\lambda) = \lambda - \sum i_j \beta_j$

So the dimension of $M(\lambda)_\mu$ is $\# \{ (i_1, \ldots, i_m) \in \mathbb{Z}_{\geqslant 0}^m : \lambda - \sum i_j \beta_j = \mu \}$.

This is precisely the $e(\mu)$-coefficient of $\dfrac{e(\lambda)}{\prod_{\alpha > 0} (1 - e(-\alpha))}$.

8.7. Weyl's Character Formula.

We have shown that for $\lambda \in X$, $\mathrm{ch}\, M(\lambda) = \dfrac{e(\lambda + \rho)}{\prod_{\alpha > 0} (e(\frac{\alpha}{2}) - e(-\frac{\alpha}{2}))}$ where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \left(= \sum_{\alpha \in \Delta} \alpha . \right)$

Now recall the Casimir operator, $c_L \in U(L)$ from §4.2. This was $\sum_{i=1}^N x_i y_i$ where x_i, y_i are any two bases for L such that $(x_i, y_j) = \delta_{ij}$. c_L lies in $Z(U(L))$, it acts on any h/w L-module as a scalar because it acts as a scalar on any L-module generated by a single vector.

Lemma: c_L acts on $M(\lambda)$ by the scalar $(\lambda + \rho, \lambda + \rho) - (\rho, \rho)$.

Proof: Let $\{ e_\alpha, f_\alpha : \alpha \in \Phi^+ \} \cup \{ h_i : 1 \leq i \leq l \}$ be a standard basis of L. Let $\{ \bar{e}_\alpha, \bar{f}_\alpha : \alpha \in \Phi^+ \} \cup \{ \bar{h}_i : 1 \leq i \leq l \}$ be the dual basis. So $c_L = \sum_{\alpha \in \Phi^+} (e_\alpha \bar{e}_\alpha + f_\alpha \bar{f}_\alpha) + \sum_i h_i \bar{h}_i = \sum_{\alpha \in \Phi^+} (\bar{e}_\alpha e_\alpha + f_\alpha \bar{f}_\alpha + t_\alpha) + \sum_i h_i \bar{h}_i$

Let v^+ be a h/w vector in $M(\lambda)$. Then $c_L(v^+) = \left(\sum_{\alpha\in\Phi^+} t_\alpha + \sum_i h_i \bar{h}_i\right) v^+$, since $e_\alpha, f_\alpha \in$ positive root space.

So the scalar is $\sum_{\alpha\in\Phi^+} \lambda(t_\alpha) + \sum_i \lambda(h_i)\lambda(\bar{h}_i) = \sum_{\alpha\in\Phi^+} (\lambda,\alpha) + \sum_i \langle\lambda,\alpha_i\rangle(\lambda,\omega_i)$, $-(*)$.
as $h_i \leftrightarrow \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$ and $\bar{h}_i \leftrightarrow \omega_i$ under $H \leftrightarrow H^*$.
Write $\lambda = \sum a_i\omega_i$.
Then $(*) = 2(\lambda,\rho) + \sum_i a_i(\lambda,\omega_i) = 2(\lambda,\rho) + \sum_{i,j} a_i a_j(\omega_j,\omega_i) = 2(\lambda,\rho) + (\lambda,\lambda) = (\lambda+\rho,\lambda+\rho) - (\rho,\rho)$.

Proposition: Let V be any h/w module of h/w $\lambda\in X$. Then V has a finite composition series with composition factors of the form $V(\mu)$, $\mu\in X$, where $\mu\le\lambda$ and $(\mu+\rho,\mu+\rho) = (\lambda+\rho,\lambda+\rho)$.

Proof: $\{\mu\in X : (\mu+\rho,\mu+\rho) = (\lambda+\rho,\lambda+\rho)\}$ is a finite set as X is discrete. So only finitely many $\mu\in X$ satisfy $(\mu+\rho,\mu+\rho) = (\lambda+\rho,\lambda+\rho)$. So $d := \sum_{\text{all such }\mu} \dim V(\mu)$ is finite.
We will prove the result by induction on d.
If V is irreducible, we are done. So suppose V is reducible. It has a proper h/w submodule W of h/w $\mu\in X$ for some $\mu \lneq \lambda$. $(\lambda+\rho,\lambda+\rho) - (\rho,\rho) = (\mu+\rho,\mu+\rho) - (\rho,\rho)$ (via $c_L|_W$)
So $(\lambda+\rho,\lambda+\rho) = (\mu+\rho,\mu+\rho)$.
Now use induction for W and V/W, both of which have smaller d.

By the proposition, we can write: $\operatorname{ch} M(\lambda) = \sum_{\mu\in X} a_{\mu\lambda} \operatorname{ch}(V_\mu)$, some $a_{\mu\lambda}\in\mathbb{Z}_{\ge 0}$, where only finitely many of the $a_{\mu\lambda}$ are non-zero.
Moreover, $a_{\lambda\lambda} = 1$, and $a_{\mu\lambda} = 0$ unless $\mu\le\lambda$ and $(\mu+\rho,\mu+\rho) = (\lambda+\rho,\lambda+\rho)$.
So, the matrix $(a_{\mu\lambda})_{\mu,\lambda\in X}$ is unitriangular (when rows and columns are ordered refining $\le$), and each column has only finitely many non-zero entries. So $(a_{\mu\lambda})$ is invertible over $\mathbb{Z}$.
So for fixed $\lambda\in X^+$, we can write: $\operatorname{ch} V(\lambda) = \sum_{\substack{\mu\le\lambda \\ (\mu+\rho,\mu+\rho) = (\lambda+\rho,\lambda+\rho)}} c_\mu \operatorname{ch} M(\mu)$, where $c_\mu\in\mathbb{Z}$, $c_\lambda = 1$.

Recall: (i) $\operatorname{ch} V(\lambda)$ is W-stable as $\lambda\in X^+$.
(ii) $\operatorname{ch} M(\mu) = \dfrac{e(\mu+\rho)}{\prod_{\alpha>0}\left(e(\frac{\alpha}{2}) - e(-\frac{\alpha}{2})\right)}$

Let $\chi = \prod_{\alpha>0}\left(e(\frac{\alpha}{2}) - e(-\frac{\alpha}{2})\right)$. So, $\chi \operatorname{ch} V(\lambda) = \sum_{\substack{\mu\le\lambda \\ (\mu+\rho,\mu+\rho) = (\lambda+\rho,\lambda+\rho)}} c_\mu e(\mu+\rho)$ $\quad -(*)$.

Theorem (Weyl's Character Formula): Fix $\lambda\in X^+$.
$$\operatorname{ch} V(\lambda) = \frac{\sum_{w\in W} \varepsilon(w) e(w(\lambda+\rho)-\rho)}{\prod_{\alpha>0}(1 - e(-\alpha))} = \frac{\sum_{w\in W}\varepsilon(w) e(w(\lambda+\rho))}{\sum_{w\in W}\varepsilon(w) e(w\rho)}.$$

where $\varepsilon : W \to \{\pm 1\}$ is the sign representation, $\varepsilon(s_\alpha) = -1$ for $\alpha\in\Delta$.

Proof: Second form is equivalent to the first as $\chi = \prod_{\alpha>0}\left(e(\frac{\alpha}{2}) - e(\frac{-\alpha}{2})\right) = \sum_{w\in W}\varepsilon(w) e(w\rho)$ (Ex. sheet 4).
This also implies: $w\chi = \varepsilon(w)\chi$ $(w\in W)$.
Apply $w\in W$ to both sides of $(*)$ to deduce
$$\varepsilon(w)\,\chi \operatorname{ch} V(\lambda) = \sum_{\substack{\mu\le\lambda \\ (\mu+\rho,\mu+\rho) = (\lambda+\rho,\lambda+\rho)}} c_\mu e(w(\mu+\rho)) = \varepsilon(w)\sum c_\mu e(\mu+\rho).$$

Hence, by equating coefficients, $\{\mu+\rho : c_\mu \ne 0\}$ is W-stable and the coefficients c_μ within an orbit of W on this set differ only by $\varepsilon(w)$.
Rewrite the equation $(*)$ as a sum over W-orbits and use the fact that $c_\lambda = 1$ to deduce:
$\chi \operatorname{ch} V(\lambda) = \sum_{w\in W}\varepsilon(w) e(w(\lambda+\rho)) + S$, S = sum over all orbits not containing $\lambda+\rho$.

So to prove the theorem, just need to show $S = 0$.

If $S \neq 0$, $\exists\, \mu \lneq \lambda$, $\mu + \rho \in X^+$, $(\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho)$. So $\lambda - \mu = \pi$ is a non-negative sum of positive roots, $\pi \neq 0$.

Then, $0 = (\lambda+\rho, \lambda+\rho) - (\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho) - (\lambda+\rho-\pi, \lambda+\rho-\pi)$

$$= (\lambda+\rho, \pi) + (\mu+\rho, \pi)$$

$$= (\lambda, \pi) + (\rho, \pi) + (\mu+\rho, \pi) \geq 0 + (\rho, \pi)$$

But $(\rho, \pi) = \sum_{\alpha \in \Delta} (\alpha, \pi) > 0$ – $\#$. So $S = 0$.

<u>Example:</u> $L = sl_2(\mathbb{C})$. Let $\Delta = \{\alpha_1\}$ be a base. Let $\omega_1 = \frac{1}{2}\alpha_1$ be corresponding fundamental dominant weight. So $X^+ = \{c\omega_1 : c \in \mathbb{Z}_{\geq 0}\}$. Write $V(c)$ for $V(c\omega_1)$.

We already know that $\operatorname{ch} V(c) = e(c\omega_1) + e((c-2)\omega_1) + \cdots + e(-c\omega_1)$.

Here $W = \{\pm 1\}$ of order 2. $\rho = \omega_1$.

Weyl $\Rightarrow$ $\operatorname{ch} V(c\omega_1) = \dfrac{e((c+1)\omega_1) - e(-(c+1)\omega_1)}{e(\omega_1) - e(-\omega_1)} = \dfrac{e(c\omega_1) - e(-(c+2)\omega_1)}{1 - e(-2\omega_1)}$

$$= \left(e(c\omega_1) - e(-(c+2)\omega_1)\right) \times \left(1 + e(-2\omega_1) + e(-4\omega_1) + \cdots\right)$$

$$= \left(e(c\omega_1) + e((c-2)\omega_1) + e((c-4)\omega_1) + \cdots\right) - \left(e(-(c+2)\omega_1) + e(-(c+4)\omega_1) + \cdots\right)$$

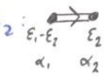
$$= e(c\omega_1) + e((c-2)\omega_1) + \cdots + e(-c\omega_1).$$

<u>Note:</u> $\rho = \frac{1}{2}\sum_{\alpha > 0} \alpha \overset{?}{=} \sum_{i=1}^{l} \omega_i$. $\quad \langle \rho, \alpha_j \rangle = 1$ (RHS)

So need to check: $\langle \sum_{\alpha>0} \alpha, \alpha_j \rangle = 2$ (LHS)

Apply s_{α_j}, which preserves $\langle , \rangle$ $\quad \langle -2\alpha_j + \sum_{\alpha>0} \alpha, -\alpha_j \rangle = 2$.

<u>Example</u>: (of applying Weyl's character formula).

B_2 

$\Phi^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1, \varepsilon_2\}$ $\quad \omega_1 = \varepsilon_1$, $\omega_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$, $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$

$\rho = \omega_1 + \omega_2 = \frac{3}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2$.

Compute $\operatorname{ch} V(\omega_1)$ using Weyl's character formula.

We know W is of order 8. Explicitly it's all permutations of ε_1 and ε_2, and all sign changes ie, all permutations:

$\{1, (\varepsilon_1 \varepsilon_2), (\varepsilon_1\ {-\varepsilon_1}), (\varepsilon_2\ {-\varepsilon_2}), (\varepsilon_1 \varepsilon_2)(\varepsilon_1\ {-\varepsilon_1}), (\varepsilon_1 \varepsilon_2)(\varepsilon_2\ {-\varepsilon_2}), (\varepsilon_1\ {-\varepsilon_1})(\varepsilon_2\ {-\varepsilon_2}), (\varepsilon_1 \varepsilon_2)(\varepsilon_1\ {-\varepsilon_1})(\varepsilon_2\ {-\varepsilon_2})\}$.

Then $\operatorname{ch} V(\omega_1) = \dfrac{\sum_{w \in W} \varepsilon(w)\, e(w(\frac{5}{2}\varepsilon_1 + \frac{1}{2}\varepsilon_2) - \frac{3}{2}\varepsilon_1 - \frac{1}{2}\varepsilon_2}{\prod_{\alpha>0} (1 - \varepsilon(-\alpha))}$

$\vdots$

$$= \frac{(e(\varepsilon_1) - e(-4\varepsilon_1))(1 - e(-\varepsilon_2)) + (e(-\varepsilon_1 - 3\varepsilon_2) - e(-\varepsilon_1 + 2\varepsilon_2))(1 - e(-\varepsilon_1))}{(1 - e(-\varepsilon_2))(1 - e(-\varepsilon_1))(1 - e(-\varepsilon_1 - \varepsilon_2))(1 - e(-\varepsilon_1 + \varepsilon_2))}$$

$\vdots$

$$= e(\varepsilon_2) + e(\varepsilon_1) + e(0) + e(-\varepsilon_1) + e(-\varepsilon_2).$$

8.8. Weyl's Dimension Formula.

<u>Theorem:</u> $\dim V(\lambda) = \prod_{\alpha > 0} \dfrac{(\lambda+\rho, \alpha)}{(\rho, \alpha)} = \prod_{\alpha>0} \dfrac{\langle \lambda+\rho, \alpha \rangle}{\langle \rho, \alpha \rangle}$

<u>Proof:</u> If $\operatorname{ch} V(\lambda) = \sum c_\mu e(\mu)$ then $\dim V(\lambda) = \sum c_\mu$. We want to compute the image of $\operatorname{ch} V(\lambda) \in \mathbb{Z}[X]$ under the homomorphism $\mathbb{Z}[X] \to \mathbb{C}$ which sends each $e(\mu)$ to 1.

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w \in W} \varepsilon(w)\, e(w(\lambda+\rho))}{\sum_{w \in W} \varepsilon(w)\, e(w\rho)}$$

Problem: Each $\sum_{w\in W} \varepsilon(w) e(w\mu)$ maps to 0 under this homomorphism, so we have to be more cunning.

Factor the homomorphism through the ring $\mathbb{C}[[t]]$ of formal power series:

$\mathbb{Z}[X] \xrightarrow{\Psi} \mathbb{C}[[t]] \xrightarrow{f} \mathbb{C}$, where $f(t)=0$, so f just picks out the constant term

and $\Psi : e(\alpha) \mapsto e^{(\rho,\alpha)t}$.

Check: $f\Psi(\mathrm{ch}\, V(\lambda)) = \sum c_\mu = \dim V(\lambda)$.

So we need to compute the lowest order terms of $\Psi(\sum_{w\in W} \varepsilon(w) e(w\mu))$, where $\mu = \lambda+\rho$ or ρ.

$\dim V(\lambda)$ will be their quotient.

So the theorem will follow from:

Claim: $\Psi(\sum_{w\in W} \varepsilon(w) e(w\mu)) = \left[\prod_{\alpha>0} (\mu,\alpha)\right] t^{|\Phi^+|}$ + higher order terms.

Proof: Write Ψ_μ for the map $e(\alpha) \mapsto e^{(\mu,\alpha)t}$.

Then $\Psi(\sum_{w\in W} \varepsilon(w) e(w\mu)) = \sum_{w\in W} \varepsilon(w)\, e^{(\rho, w\mu)t} = \sum_{w\in W} \varepsilon(w)\, e^{(w^{-1}\rho,\mu)t} = \sum_{w\in W} \varepsilon(w)\, e^{(w\rho,\mu)t}$

$$= \Psi_\mu\left(\sum_{w\in W} \varepsilon(w) e(w\rho)\right) = \Psi_\mu\left(\prod_{\alpha>0}\left(e(\tfrac{\alpha}{2}) - e(-\tfrac{\alpha}{2})\right)\right) = \prod_{\alpha>0}\left(e^{(\mu,\alpha)t/2} - e^{-(\mu,\alpha)t/2}\right)$$

$$= \prod_{\alpha>0}\left[\left(1 + (\mu,\alpha)\tfrac{t}{2} + \text{higher}\right) - \left(1 - (\mu,\alpha)\tfrac{t}{2} + \text{higher}\right)\right] = \prod_{\alpha>0}\left((\mu,\alpha)t + \text{higher}\right)$$

$$= \left[\prod_{\alpha>0}(\mu,\alpha)\right] t^{|\Phi^+|} + \text{higher.}$$

More examples.

1. $L = A_2 = \mathfrak{sl}_3(\mathbb{C})$. ω_1, ω_2 fundamental dominant weights, $\rho = \omega_1+\omega_2$. $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$.

$$\dim V(m_1\omega_1 + m_2\omega_2) = \frac{\prod_{\alpha>0} \langle (m_1+1)\omega_1 + (m_2+1)\omega_2, \alpha\rangle}{\prod_{\alpha>0} \langle \omega_1+\omega_2, \alpha\rangle} = \frac{(m_1+1)(m_2+1)(m_1+m_2+2)}{2}$$

$m_1, m_2 \in \mathbb{Z}_{\geq 0}$

Similar formulas for B_2, G_2.

2. $L = A_l = \mathfrak{sl}_{l+1}(\mathbb{C})$. Let $\omega_1, \dots, \omega_l$ be the fundamental dominant weights. $\rho = \omega_1 + \dots + \omega_l$.

(Let $n = l+1$) Let $\alpha_1, \dots, \alpha_l$ be the corresponding simple roots. $(\alpha_i = \varepsilon_i - \varepsilon_{i+1})$

$\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\} = \{\alpha_i + \alpha_j : 1 \leq i \leq j \leq l\}$.

Compute $\dim V(\omega_1)$.

$$\dim V(\omega_1) = \prod_{\alpha>0} \frac{\langle 2\omega_1 + \omega_2 + \dots + \omega_l, \alpha\rangle}{\langle \omega_1 + \dots + \omega_l, \alpha\rangle}$$ – terms cancel, unless $\alpha = \alpha_1 + \dots + \alpha_j$, $\forall\, 1 \leq j \leq l$.

$$= \prod_{j=1}^{l} \frac{j+1}{j} = \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{l+1}{l} = l+1 = n.$$

So $V(\omega_1)$ is just the natural n-dimensional module for $L = \mathfrak{sl}_n(\mathbb{C})$.

The same calculation shows $V(\omega_l)$ has dimension n. In fact, $V(\omega_l) \cong V(\omega_1)^*$.

For $1 \leq r \leq l$, look at $\Lambda^r V$, $S^r V$, where $V = V(\omega_1)$.

The highest weights of these modules are ω_r and $r\omega_1$ respectively.

So $\Lambda^r V$ contains $V(\omega_r)$ as a submodule, and $S^r V$ contains $V(r\omega_1)$.

Now check using Weyl's dimension formula that $\Lambda^r V = V(\omega_r)$

$S^r V = V(r\omega_1)$.

Similarly, $S^r V^* = V(r\omega_l)$, $L = V(\omega_1 + \omega_l) = V\otimes V^* / \langle \text{a 1-space}\rangle$.

3. Other classical Lie algebras.

$B_l = so_{2l+1}(\mathbb{C})$, $C_l = sp_{2l}(\mathbb{C})$, $D_l = so_{2l}(\mathbb{C})$.

In each case, $V(\omega_1)$ is the natural module.

For orthogonal groups, $\Lambda^r V(\omega_1) = V(\omega_r)$, so Λ's are irreducible. But $S^r V$ are not irreducible for $r>1$.

For $sp_{2l}(\mathbb{C})$, $S^r V(\omega_1) = V(r\omega_1)$, so S's are irreducible, but $\Lambda^r V$ are not irreducible for $r>1$.