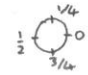
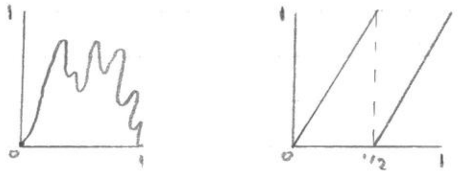
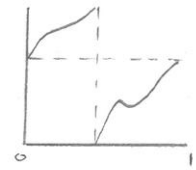
Dynamics of One-dimensional Maps.

Maps $f: X \to X$, $X = [0,1]$, or $X = S^1$, parametrized by $[0,1)$ 

f is usually continuous, often differentiable, or discontinuous in a nice way (finite number of discontinuities, restrictions on values on either side, etc.)

Examples: 

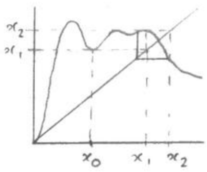
$x \mapsto 2x \bmod 1$.



If only one discontinuity, at c, with $f(c_-) = 1$, $f(c_+) = 0$ and $f(0) = f(1)$, then f is also a continuous map of the circle.

Often consider families of maps, f_r, depending continuously on a parameter r.
Eg, $f_r(x) = rx(1-x)$. (Drop subscript when context is clear.)

Questions: are about orbits. Take $x_0 \in X$. The sequence $x_0, x_1, x_2, \dots$ with $x_{n+1} = f(x_n)$ is the orbit of x_0 under f. (also called the trajectory, solution, etc.).
We are interested in the long-term behaviour of orbits $(n \to \infty)$.

Aside: geometric method of iterating. 

Take topological/geometric approach rather than measure-theoretic approach.

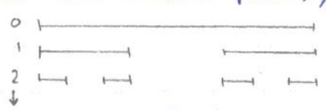
Definition: A point $x \in X$ wanders if $\exists$ a neighbourhood U of x such that
$U \cap f^n(U) = \emptyset \quad \forall\, n \geq 1$.

Definition: The non-wandering set Λ is the set of points which do not wander.

The non-wandering set includes all "recurrent" behaviour.

Exercise: Prove Λ is closed and f-invariant (ie, $f(\Lambda) = \Lambda$).

Examples: All fixed points $x^* = f(x^*)$ are in Λ.
All periodic points $x^* = f^p(x^*)$ are in Λ.
Can also have: Λ is a Cantor set

Aside: A Cantor Set is: closed, perfect (no isolated points), nowhere dense (no intervals).
Eg: Middle-third Cantor Set. 

Limit of this process is a Cantor Set.

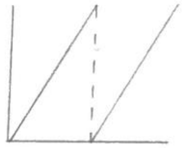
It is a Cantor set: nested sequence of closed sets $\Rightarrow$ Λ closed.

Λ contains all points whose expansion in base 3 has no 1's, and clearly has no intervals or isolated points.

Examples (cont.): Can also have: (a) Λ contains no periodic orbits
(b) periodic orbits are dense in Λ $\Big\}$ for Cantor set.

Or, Λ is an interval, or a collection of intervals, or even all of X.

Example: $x \mapsto 2x \bmod 1$.



$x_0 = 0 \cdot a_0 a_1 a_2 \cdots$, $a_i = 0, 1$.

$2x_0 \bmod 1 = 0 \cdot a_1 a_2 a_3 \cdots$

$f^n(x_0) = 0 \cdot a_n a_{n+1} a_{n+2} \cdots$

So any binary sequence that is periodic $\Rightarrow$ periodic x.
Arbitrarily close to any point $\exists$ periodic orbit (ie, y such that it's binary representation is periodic).
$\Rightarrow$ all x non-wandering.

Q.1: Decide topology of Λ and the dynamics of f on Λ.

Definition: A closed f-invariant set A is an attractor if $\exists$ neighbourhood $\mathcal{U}$ of A such that if $x \in \mathcal{U}$ then $f^n(x) \to A$ as $n \to \infty$.

Q.2: Understand minimal attractors of f. (ie, sets A which are attracting, but no proper subset of A is an attractor).
Note: $A \subseteq \Lambda$

Q.3: What is the behaviour for typical x_0? Usually "typical" means "in an open dense subset", or "in a subset of measure 1" (measure-theoretic sense).

Q.4: Complexity of f. Topological entropy, in 1-d, $h(f) = \limsup_{n\to\infty} \frac{1}{n} \log \left\{ \begin{array}{c} \#\text{ fixed} \\ \text{points of } f^n \end{array} \right\}$.
Eg: If f has only 1 fixed point, $h(f) = 0$.
If $f \equiv 2x \bmod 1$, then $h(f) = \log 2$.

Q.5: When we consider families f_r, want to know how answers to Q1-4 change. ("Bifurcation Theory").

Q.6: What is the behaviour for "typical" parameter value r. (Behaviour occurs for an open dense set of r-values).

Why bother?
(a) Answers don't depend on f very much.
(b) Universal features in behaviour of families.
(c) Same universal features occur in many more complicated dynamical systems (n-dimensional maps, differential equations, etc.)
(d) Simpler than the harder cases(!)

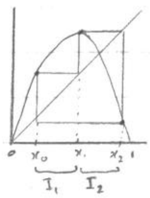
Sarkovskii's Theorem (1964): Suppose f is a continuous map of the interval I to itself. Suppose f has periodic orbit of least period m. Then it also has orbits of all periods $n \prec m$ in the order: $1 \prec 2 \prec 4 \prec \cdots \prec 2^n \prec \cdots \prec 2^n.5 \prec 2^n.3 \prec \cdots 2.5 \prec 2.3 \prec \cdots \prec 5 \prec 3$.

Lemma: If I is a closed interval and $f(I) \supset I$ then $\exists\, x \in I$ such that x is a fixed point, ie, $f(x) = x$.

Proof: Let $[a,b] = I$. Either a or b is a fixed point, or $\exists\, y_1 \in (a,b]$ such that $f(y_1) = a$ and $\exists\, y_2 \in [a,b)$ such that $f(y_2) = b$. So $y_1 > a$, $y_2 < b$.
So $f(y) - y < 0$ at y_1, >0 at y_2. So by IVT, $f(x) = x$ for some $x \in (y_1, y_2)$.

Lemma: Suppose $I_1, I_2, \ldots, I_n$ are closed intervals. We define a graph with n vertices, and a directed edge $i \to j$ iff $f(I_i) \supset I_j$. For any infinite path $i_0, i_1, i_2, \ldots$ of vertices on the graph (allowed by the edges) $\exists$ point $x \in I_0$ such that $f^n(x) \in I_{i_n}$.
Furthermore, if the sequence is periodic, $i_0, i_1, \ldots, i_{p-1}, i_0, \ldots$, then $\exists$ point $x \in I_{i_0}$, periodic of order p. ($I_1, \ldots, I_n$ are disjoint, except perhaps at their endpoints).

Example:



$f(I_1) \supset I_2$
$f(I_2) \supset I_1 \cup I_2$

G:


Paths look like: $12122212^{\alpha_1}12^{\alpha_2}\ldots$

Ie, can make with any given period. Lemma $\Rightarrow$ there are points periodic of any orbit.

Proof of Lemma: Given a sequence $i_0, i_1, \ldots, i_n$ define $I_{i_0 \ldots i_n} = \{x : f^j(x) \in I_{i_j}\ \forall\ 0 \le j \le n\}$.
This is a closed set and non-empty. Furthermore, $I_{i_0 \ldots i_{n+1}} \subseteq I_{i_0 \ldots i_n}$.
So, $\bigcap_{N=0}^{\infty} I_{i_0 i_1 \ldots i_N}$ is a closed non-empty set. So $\exists\, x \in \bigcap$ as above with desired behaviour. Furthermore, if sequence is periodic, $i_0, \ldots, i_{p-1}, i_0, \ldots$, define $K = I_{i_0 \ldots i_{p-1}}$.
Then $f^p(K) \supset K \Rightarrow \exists$ fixed point of f^p in $K \Rightarrow$ periodic point of period p.

Remarks:
- Beware of endpoints. Eg: $x \mapsto -2x$, $I_1 = [-1, 0]$, $I_2 = [0, 1]$. Then, $f(I_1) \supset I_2$, $f(I_2) \supset I_1$. ie: $\bullet \rightleftarrows \bullet$ Lemma $\Rightarrow$ points of period 2. Only periodic point is 0, of period 1. (Note Lemma didn't say least period p).
- if $f|_{I_j}$ is monotonic then the sets $I_{i_0 i_1 \ldots i_n}$ are closed intervals.
- if f' exists and is $> 1 + \varepsilon$ on intervals I_j, then corresponding to an infinite sequence $I_{i_0 i_1 i_2 \ldots}$ is a single point.
- There may be periodic points in addition to those given by the lemma.

Proof of Sarkovskii's Theorem: Assume f has a periodic orbit of period n maximal in ordering. So all orbits of f have period $m \prec n$. Label points of orbit: $p_1 < p_2 < \cdots < p_n$ (not in dynamical order).

(1). Prove $\exists$ a fixed point. $f(p_1) > p_1$ (as p_1 not fixed, and $f(p_1) = p_j$, some $j \neq 1$).
Similarly, $f(p_n) < p_n$. So $\exists$ some k minimal such that $f(p_{k-1}) > p_{k-1}$ and $f(p_k) < p_k$.
Label interval $[p_{k-1}, p_k] = J_1$. So $f(J_1) \supset J_1 \Rightarrow \exists$ fixed point $x \in J_1$, and since the p_i aren't fixed, $x \in (p_{k-1}, p_k)$.

So associated graph $\mathcal{G}$ includes: 

(2) Show J_1 maps, after some number of steps, over all intervals.

Consider $[p_{k-1}, x]$ under f^i, $1 \le i \le n$. For some i, $f^i(p_{k-1}) = p_1$, $f^i(x) = x$, so $[p_{k-1}, x]$ maps over all the intervals $[p_1, p_2], \dots, [p_{k-2}, p_{k-1}]$.

Similarly, $\exists\, j$ such that $f^j(p_{k-1}) = p_n$, so $[p_{k-1}, x]$ maps over all intervals to the right.

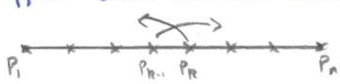
So on $\mathcal{G}$, $\exists$ paths (of some length) from J_1 to all other intervals.

(3) Do there exist paths back from other intervals to J_1? Answer: iff n is odd.

Proof: If $\exists$ paths back from J to J_1, then on the graph $\exists$ paths of odd length $J_1 \to J \to J_1$ (by repeating J_1 if necessary). 

$\Rightarrow \exists$ orbit of odd period. Since n was maximal in the ordering, n must be odd.

On the other hand, suppose there do not exist paths from any interval $J \neq J_1$ back to J_1.

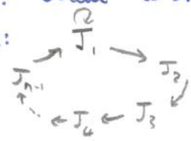


Then $f[p_1, p_{k-1}]$ cannot cover $J_1 \Rightarrow f[p_1, \dots, p_{k-1}] \subset [p_k, \dots, p_n]$.

Similarly, $f[p_k, \dots, p_n] \subseteq [p_1, \dots, p_{k-1}]$. But f is 1-1 on the p_i. So $n = 2(k-1)$ and $f\{p_1, \dots, p_{k-1}\} = \{p_k, \dots, p_n\}$ and $f\{p_k, \dots, p_n\} = \{p_1, \dots, p_{k-1}\}$. So n is even, and f^2 has an orbit of period $n/2$ which (for f^2) is maximal in the Sarkovskii ordering.

So treat the case when n is odd and $\exists$ paths back to J_1. If n is even, we have proved $\exists$ a point of period 1, and proceed by induction, considering orbits of f^2 acting on $[p_1, \dots, p_{k-1}]$. So assume n is odd.

So, $\exists$ circuit containing J_1. Want to show this implies minimal circuit including J_1 has length $n-1$, i.e. $\mathcal{G}$ will look like: (plus some other arcs, but none such that $\exists$ circuit of length $< n-1$ including J_1)

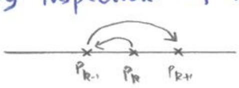


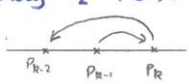
Proof: If $\exists$ circuit of length $m < n-1$, with m odd, then $\exists$ orbit of odd period $m < n$ which is $m \succ n$ in the Sarkovskii ordering. But n is maximal. $\#$.

If m even, $m < n-1$ then $\exists$ circuit of length $m+1$ (by repeating J_1 once). So $m+1 < n$, $m+1$ odd, so again $\exists$ orbit of odd period $m+1 \succ n$. $\#$

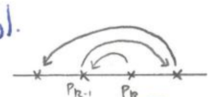
So shortest circuit containing J_1 is of length at least $n-1$. But $\exists$ only $n-1$ vertices, so this circuit visits each in turn. (Longer paths must visit some vertices more than once, and the piece of path between visits can be deleted).

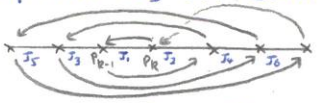
Proceed by inspection: J_1 maps over only J_2 (else $\exists$ shorter circuit).

Either: 

or: 

and we need only consider one of these (by symmetry).

Next step: $J_1 \cup J_2$ maps over only one extra interval J_3: 

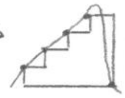
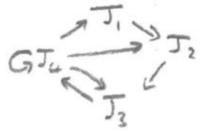
By induction, get: 

and J_{n-1} maps over $J_1, J_3, J_5, \dots J_{n-2}$.

So graph $\mathcal{G}$ has only arcs: $J_1 \circlearrowleft$, $J_1 \to J_2 \to \dots \to J_{n-1} \to J_1$, and $J_{n-1} \to J_{odd}$.

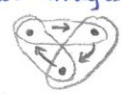
Now, trivially by inspecting the graph, orbits of all periods $m \prec n$ can occur (and none with $m \succ n$).

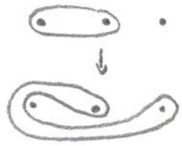
Remark: 1. Can extend this to produce an ordering on permutations (period of orbit plus a specification of how f permutes them).

Example:  $\Rightarrow$  Graph: 

So $\exists$ orbits of all periods. (Here, 5 was not a maximal period in the Sarkovskii ordering).

Remark 2: For continuous maps in higher dimension, results are more complicated - simple extension not possible. Eg: rotation through $\frac{2\pi}{n} \Rightarrow$ all points have period n and there are no other periods. But can do something.

Eg: if you have period 3 in $\mathbb{R}^2$, look at images of curves surrounding pairs of points.  Eg: rotation: 

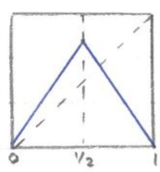
- but can have maps  $\Rightarrow$ orbits of all periods.

Remark 3: Proof was based on an article by Bloch, Guckenheimer, Misiurewicz, Young - 1980.

2. Unimodal Maps. I

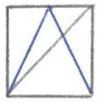
(a) Tent maps.

$$F_s(x) = \begin{cases} sx & 0 \le x \le 1/2 \\ s(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$



Qn: What is the non-wandering set? What are the dynamics of it?

We take $1 < s \le 2$. If $s < 1$ then $F_s(x) < x$ and all orbits $\to 0$.
If $s = 1$ then all points in $[0, \frac{1}{2}]$ are fixed points.

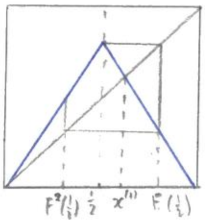
Note also $s = 2$:  This has two intervals: 
Has orbits of all periods.

Definition: $f|_I$ is transitive on I if $\exists\, x \in I$ such that $\{f^n(x)\}$ is dense in I.

Remark: This would imply that the whole of I is non-wandering.

Lemma: For these maps, f is transitive on I if for every open set $U \subset I$, $f^n(U)$ expands as $n \to \infty$, until it covers all of I. (so for $n \ge N$, $f^n(U) = I$).

Proof: Exercise.



For $s>1$, $\exists$ fixed point $x^{(1)}$ of F_s.
Define interval $J_0 = [F^2(\frac{1}{2}), F(\frac{1}{2})]$

<u>Lemma:</u> $F_s(J_0) \subset J_0$. For $x \neq 0$, $F^n(x) \in J_0$ for n large enough.

Let Ω_s be the non-wandering set of F_s.

<u>Lemma:</u> For $s>1$, $\Omega_s = \{0\} \cup \Omega'$, where $\Omega' \subseteq J_0$ and is the non-wandering set of $F_s|_{J_0}$.

<u>Lemma:</u> If $s > \sqrt{2}$ then $\Omega' = J_0$.
<u>Proof:</u> Will show that any small interval $J \subset J_0$ expands to cover J_0. Take an interval J of length $|J|$.

Case (i): If $\frac{1}{2} \notin J$ then $|F(J)| = s|J|$

Case (ii): If $\frac{1}{2} \in J$, write $J = J_l \cup \{\frac{1}{2}\} \cup J_r$. Get $|F(J)| = \max\{s|J_l|, s|J_r|\} \geq \frac{s}{2}|J|$.

Consider what happens if case (ii) occurs twice in a row. ie $\frac{1}{2} \in J$ and $\frac{1}{2} \in F(J)$.

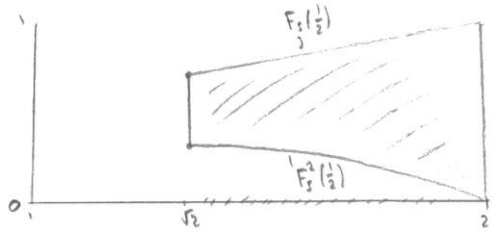
Then, $\frac{1}{2}, F(\frac{1}{2}) \in F(J)$. $\Rightarrow [\frac{1}{2}, F(\frac{1}{2})] \subseteq F(J)$. $\Rightarrow [F^2(\frac{1}{2}), F(\frac{1}{2})] \subseteq F^2(J)$. $\Rightarrow J_0 \subseteq F^2(J)$

So J expands in at most two steps.

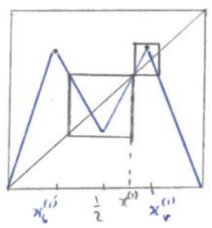
Put this all together, get that the sequence $|F^n(J)|$ increases by a factor of s in case (i) and by at least $\frac{s}{2}$ in case (ii), so provided case (ii) does not occur twice in succession, $|F^{n+2}(J)| \geq \frac{s^2}{2}|F^n(J)| \geq (1+\varepsilon)^2|F^n(J)|$ (as $s > \sqrt{2}$). So it expands - but $|F^n(J)| \leq |J_0|$, so eventually case (ii) occurs twice in succession

$\Rightarrow$ in at least two more steps $F^n(J) \supseteq J_0 \Rightarrow F$ is transitive and $\Omega' = J_0$.

For the non-wandering set, we have:



For $1 < s \leq \sqrt{2}$, look at F_s^2:



Slope $\pm s^2$. 4 linear segments.
Symmetric.
$\exists$ two fixed points of F^2 not fixed points of F
$\Rightarrow$ orbit of period 2 for F.

Fixed point of F_s is $x^{(1)}$. Consider two closest pre-images of $x^{(1)}$ under F_s^2, $x_l^{(1)}, x_r^{(1)}$, where $F_s^2(x_{l,r}^{(1)}) = x^{(1)}$. Define intervals $J_l^{(1)} = [x_l^{(1)}, x^{(1)}]$, $J_r^{(1)} = [x^{(1)}, x_r^{(1)}]$

<u>Lemma:</u> If $1 < s \leq \sqrt{2}$, F_s^2 maps each of $J_{l,r}^{(1)}$ on to itself, and after appropriate rescaling, is a tent map of slope s^2 on each interval.

<u>Proof:</u> Consider $J_l^{(1)}$. $x = \frac{1}{2}$ is midpoint. If we can show $F_s^2(\frac{1}{2}) \geq x_l^{(1)}$, then clearly $F^2(J_l^{(1)}) \subseteq J_l^{(1)}$. But $F^2(\frac{1}{2}) = x^{(1)} - \frac{1}{2}s^2(x^{(1)} - x_l^{(1)})$. So $F^2(\frac{1}{2}) \geq x_l^{(1)}$ iff $x^{(1)}(1 - \frac{1}{2}s^2) \geq x_l^{(1)}(1 - \frac{1}{2}s^2)$ iff $1 - \frac{1}{2}s^2 \geq 0$ iff $s \leq \sqrt{2}$.

Proceed inductively:

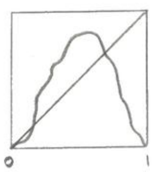
<u>Theorem</u>: $2^{2^{-(n+1)}} < s \leq 2^{2^{-n}}$. Then $\Omega(F_s) = \{0\} \cup P_1 \cup P_2 \cup \dots \cup P_n \cup \left\{ \bigcup_{i=0}^{2^n-1} J_{(i)}^{(n)} \right\}$, where P_i is a periodic orbit of period 2^i, and where the J_i are intervals, cyclically permuted by F, and such that $F^{2^n}(J_i^n)$ is transitive.

<u>Proof</u>: Induction.

<u>Note</u>: The orbits P_i are disjoint from the intervals, except in the case $s = 2^{2^{-n}}$, when points on P_n are endpoints of J_i^n which abut in pairs.
For a sketch of the non-wandering set of F_s, see end of notes.

3. Unimodal Maps II

$f \in C^1$. $f: [0,1] \circlearrowleft$. $f(0) = f(1) = 0$. $\exists\, c \in (0,1)$ such that on $[0,c]$, f is increasing, and on $[c,1]$, f is decreasing.
c is called the <u>critical point</u> (usually take $c = 1/2$..)



Sketch: f

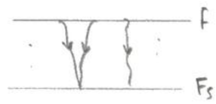
$h(f) = 0$
- all orbits have period 2^n. Ω consists of these, and perhaps a Cantor set, "infinite register shift".

$h(f) > 0$
- $\exists\, \varphi$, semi-conjugacy, between f and tent-map F_s with $\log s = h(f)$. (So any dynamics of F_s occur in f)

$\frac{1}{2}$ is periodic for F_s → renormalise

$\frac{1}{2}$ isn't periodic for F_s → φ is a conjugacy (1-1)

Semi-conjugacy:



<u>Aside on Entropy.</u>
Recall $h(f) = \lim_{n\to\infty} \frac{1}{n} \log(\#\text{ fixed points of } f^n)$

<u>Definition</u>: An interval J <u>f-covers K n-times</u> if $\exists$ n subintervals $J_1, \dots, J_n \subset J$ such that $f(J_i) = K$.

if we have disjoint intervals $I_1, \dots, I_n$, can make a <u>transition matrix</u> $A = (a_{ij})$, where I_i f-covers I_j a_{ij} times.

<u>Lemma</u>: Suppose f has transition matrix A, $B = A^n$. Then b_{ij} gives the number of independent paths (on graph) from I_i to I_j in n steps.

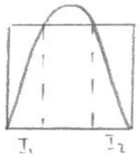
<u>Proof</u>: Exercise. (by induction).

In particular, b_{ii} is a lower bound on the number of fixed points of f^n in I_i.
So, $T_r(B)$ gives a lower bound on the number of fixed points of f^n.

<u>Lemma</u>: If A has largest eigenvalue λ, then $h(f) \geq \log \lambda$.

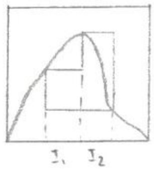
<u>Proof</u>: Exercise.

Examples: (i)



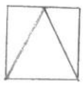
$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. $h(f) \geq \log 2$.

(ii)



$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.
$h(f) \geq \log\left(\frac{1+\sqrt{5}}{2}\right)$

(iii) Tent map. In fact, $h(F_s) = \log s$.
We can show it when $s = 2^{2^{-n}}$

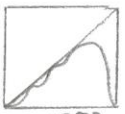
Eg: $s = 2$  cf: example (i). $h(F_2) = \log 2$.

$s = \sqrt{2}$ $(F^2)^n$ has $2^n + 1$ fixed points, ie $\sim 2^n$, so F^n has $\sim 2^{n/2}$ fixed points.
$\lim \frac{1}{n} \log(2^{n/2}) = \frac{1}{2}\log 2 = \log\sqrt{2}$.

Decomposition of Non-wandering Set for Unimodal Maps.

Case (i): $h(f) = 0$.

Either f has an orientation reversing fixed point, or it doesn't. (ORFP: x fixed, $f'(x) < 0$).



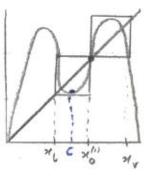
no ORFP

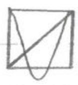
has an ORFP

(a) No ORFP $\Rightarrow$ $\Omega = \{$fixed points in $[0,c]\}$

Proof: $f(c) < c$, so $f[0,c] \subset [0,c]$ and f is monotonic. $f[1,c] \subset [0,c]$.

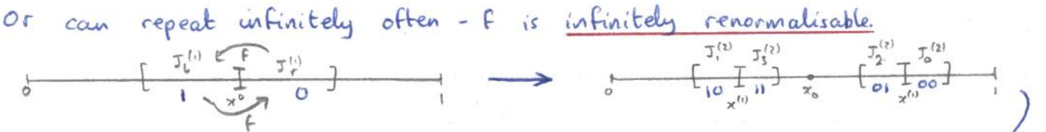
(b) $\exists\, x_0$, an ORFP, then it is unique (as f is unimodal). Consider f^2.
Define $x_{l,r}^{(1)}$ as with tent maps. As in that case, we get two intervals $J_l^{(1)} = [x_l^{(1)}, x_0]$ and $J_r^{(1)} = [x_0, x_r^{(1)}]$ which are mapped to themselves by f^2 providing $f^2(c) \geq x_l^{(1)}$



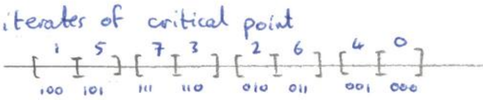
But if $f^2(c) < x_l^{(1)}$, have:  , and $h(f^2) > \log 2 \Rightarrow h(f) > 0$ - #.

So, in case of ORFP, $\Omega(f) = \{$fixed points in $[0,c]\} \cup \{x^0\} \cup \Omega(f^2|_{J_l^{(1)}}) \cup \Omega(f^2|_{J_r^{(1)}})$, and $f^2|_{J_{l,r}^{(1)}}$ is unimodal, $h(f^2|_{J_{l,r}^{(1)}}) = 0$, and f maps $J_l^{(1)} \to J_r^{(1)} \to J_l^{(1)}$.

Proceed by induction.
Can either do it n times - f^{2^n} has no ORFP. $\Omega(f) = \{$fixed points$\} \cup P_2 \cup P_4 \cup \dots \cup P_{2^n}$,
where $P_{2^i} := \bigcup($periodic points of least period $2^i)$ - non-empty.
Or can repeat infinitely often - f is infinitely renormalisable.



Note endpoints of intervals are iterates of critical point
Label points as shown. Then:



In this case $\Omega(f) = \bigcup_{i=0}^{\infty} P_{2^i} \cup \bigcap_{i=1}^{\infty}\left(\bigcup_{j=0}^{2^i-1} J_j^{(i)}\right)$, where P_{2^i} = union of orbits of period 2^i, and for each $J_j^{(i)}$, $f^{2^i}|_{J_j^{(i)}}$ is unimodal, $f(J_j^{(i)}) \subseteq J^{(i)}_{j+1 \pmod{2^i}}$, and $J_j^{(i+1)} \subset J^{(i)}_{j \bmod 2^i}$

Note: Let $\Lambda = \bigcap_{i=1}^{\infty}\left(\bigcup_{j=0}^{2^i-1} J_j^{(i)}\right)$. Λ is intersection of nested set of closed intervals - non-empty.
Often a Cantor set - provided length of intervals $J_j^{(i)} \to 0$ as $i \to \infty$. (If not, it is a Cantor set with some points replaced by intervals).
$f|_\Lambda$ is called an "infinite register shift".
Each point in Λ can be labelled by a sequence $\varphi(x) = s_0 s_1 s_2 \cdots$, $s_i = 0, 1$, where $x \in J_j^{(i)}$,
$\varphi(x) = \underbrace{j}_{\text{backwards in binary}} \cdots$
$\varphi(f(x)) = 1 + \varphi(x)$ (add 1, starting at the left).

So $f|_\Lambda$ has following properties:
(i) For every $x \in \Lambda$, $f^n(x)$ is dense in Λ. (Every orbit cycles round all 2^i intervals at level i, or adding $1 \Rightarrow$ first n symbols cycle every 2^n iterations).
(ii) No sensitive dependence on initial conditions. SDIC means $\exists\, \varepsilon > 0$ such that $\forall x, \delta$, $\exists\, y, N$ such that $|x-y| < \delta$ and $|f^N(x) - f^N(y)| > \varepsilon$.
(Orbits started in some $J_j^{(i)}$ cycle round together).
(iii) No periodic orbits. (Symbol sequence never repeats).

Remarkable Fact: $\lim_{i\to\infty} f^{2^i}|_{J_0^{(i)}} = f^*$, independent of f (with $h(f)=0$ and infinitely renormalisable), in some big class of unimodal maps.

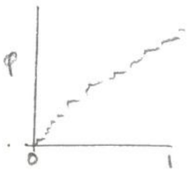
Theorem (Rand, Feigenbaum, Sullivan,...). More later. For now: f^* determines details of geometry of set Λ, independent of family f. (to a large extent...)

Aside: wandering intervals.
Definition: Suppose $J \subset I$ such that $f^n(J) \cap f^m(J) = \phi$, $\forall\, n \neq m$. Suppose also that J is not in basin of attraction of a stable periodic orbit. Then J is a wandering interval.

Facts: (i) If J is a wandering interval, then $f^n(J)$ accumulates on critical point.
(ii) If f has negative Schwarzian derivative, then $\nexists$ wandering intervals. (Eg: $rx(1-x)$ has none).
(iii) If f is C^3 and $f''(c) \neq 0$, then $\nexists$ wandering intervals. (1987).

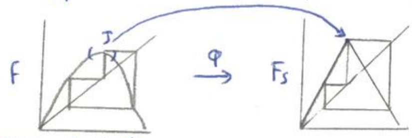
In the case of the decomposition from before, if Λ is not a Cantor Set (so it contains intervals), then the intervals are wandering. So, "normally", Λ is a Cantor Set.

Case (ii): $h(f) > 0$
We will use:
Theorem (Milnor & Thurston, 1976): f is unimodal, $h(f) = \log s > 0$, then $\exists\, \varphi$, a semiconjugacy, ie $\varphi f = F_s \varphi$, between f and the tent map F_s, that preserves the critical point. [$\varphi(c) = 1/2$].
Proof: Later.



φ is increasing, but may map intervals to points. Sometimes does - since unimodal maps can have stable periodic orbits and F_s can't.

Example: Suppose f has a superstable point of period 3. $((f^3)' = 0)$.



J is a basin of attraction of c.

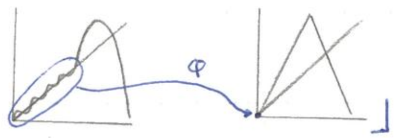
Lemma: (a) Suppose $F_s^n(\frac{1}{2}) = \frac{1}{2}$ for some n. Then $\exists\, J \ni c$, non-trivial interval such that $\varphi(J) = 1/2$, $f^n|_J$ is a unimodal map, and $h(f^n) > h(f^n|_J)$ $\quad (\Rightarrow h(f) > h(f|_{J \cup f(J) \cup \dots \cup f^{n-1}(J)})$.

(b) $F_s^n(\frac{1}{2}) \neq \frac{1}{2}$ for any n. Then φ only collapses wandering intervals and basins of attraction of stable periodic orbits to points.

So, "normally", φ is a conjugacy in case (b), since

(i) no wandering intervals in nice families.

(ii) In families with negative Schwarzian derivative, all stable orbits have a critical point in their basin of attraction, and so this case is excluded too - see later.



Proof of lemma: (a) $F_s^n(\frac{1}{2}) = \frac{1}{2}$. Let $J = \varphi^{-1}(\frac{1}{2})$. Then J is either an interval or the single point c. Suppose $J = \{c\}$. So $\varphi f^n(c) = F_s^n \varphi(c) = F_s^n(\frac{1}{2}) = 1/2 \Rightarrow f^n(c) = c$. Since f is C^1, $f'(c) = 0$ ($f^n(c) = c$), so $(f^n)'(c) = 0 \Rightarrow \exists$ interval $J' \ni c$ with $f^n(J') \subset J'$. By semiconjugacy, $\varphi f = F\varphi \Rightarrow \varphi f^n = F^n \varphi$, so $\varphi f^n(J') = F^n \varphi(J')$. But $\varphi(J')$ is an interval containing $\frac{1}{2}$. LHS is contracting, RHS is expanding #.

So J is an interval.

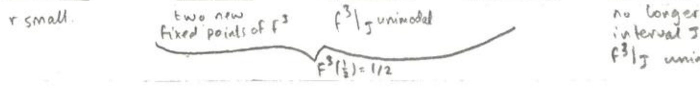
Since the points $F^i(\frac{1}{2})$, $0 \leq i \leq n-1$ are distinct, then the $f^i(J)$, $0 \leq i \leq n-1$ are disjoint. Notice that $c \in J$ and $c \notin f^i(J)$, $0 < i < n$. So $f^n|_J$ has exactly one turning point (As f unimodal on J, and strictly monotonic on each $f^i(J)$, $i \neq 0$).

Also, J is maximal with the property that $f^n(J) \subset J$. (If $K \supsetneq J$ and $f^n K \subset K$, then $K \subset \varphi^{-1}(c)$ #). $\Rightarrow f^n$ (endpoints of J) $\subset$ {endpoints of J}, and since unimodal, f^n maps both endpoints to one.

For the entropy, note that $F_s^n(\frac{1}{2}) = \frac{1}{2}$ only if $s^n > 2$. (Exercise - Look at decomposition for tent maps). $h(f) = \log s$ (by Milner & Thurston) $> \frac{1}{n} \log 2$. But $h(f^n|_J) \leq \log 2$, so $h(f|_{J \cup \dots \cup f^{n-1}(J)}) \leq \frac{1}{n} \log 2 < h(f)$.

Example: For which maps f in family $rx(1-x)$ is f semi-conjugate to F_s, $F_s^3(\frac{1}{2}) = \frac{1}{2}$.

Think about f^3:



r small. | two new fixed points of f^3 | $f^3|_J$ unimodal | no longer an interval J with $f^3|_J$ unimodal

$\underbrace{\qquad\qquad}_{F^3(\frac{1}{2}) = 1/2}$

Proof of lemma: (b) $F_s^n(\frac{1}{2}) \neq \frac{1}{2}$. Suppose $\varphi^{-1}(y) = J$, non-trivial, for some point y. Either:

(i) $F_s(y)$ is (pre)periodic, or

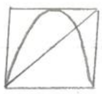
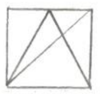
(ii) not.

If (i), $\exists\, y'$ such that $F_s^i(y) = y'$, some $i \geq 0$ and $F_s^n(y') = y'$. Now, none of $F_s^i(y') = 1/2$

$\Rightarrow c \notin f^{i+j}(J)\ \forall j$. So $f^n|_{f^i(J)}$ is a homeomorphism (no turning points)

$\Rightarrow f^n$:  or:  and J consists of basins of fixed points (not including c). (since $f^i(J)$ does).

If (ii), $F^i(y) \neq F^j(y)$ for $i \neq j$. $\Rightarrow$ J is a wandering interval for f.

Example: $r = 4$.  φ. conjugacy  $s = 2$

$F_s^n(\frac{1}{2}) = 0$, $n \geq 2$

Theorem (Structure of Non-wandering set): If f is a C^1 unimodal map, $\exists\ 0 \leq r \leq \infty$ (r an integer, the number of times you can renormalise), and intervals $J_0 \supset J_1 \supset \dots \supset J_n$ (n finite, $n \leq r$), with $c \in J_i$, and integers $k(n)$ ("period" of nth renormalisation) such that $f^{k(n)}|_{J_n}$ is unimodal.

For $n < r$:

($<+$): $f^{k(n)}|_{J_n}$ has positive topological entropy and is semi-conjugate to a tent map with $F^{k(n)/k(n-1)}(\frac{1}{2}) = \frac{1}{2}$.

(<0): $f^{k(n)}|_{J_n}$ has zero entropy, and an ORFP.

For $n = r$:

($=+$): $f^{k(n)}|_{J_n}$ has positive entropy and is semiconjugate to F_s, $F_s^m(\frac{1}{2}) \neq (\frac{1}{2})$ for any m.

($=0$): - has zero entropy and only has fixed points which preserve orientation.

Furthermore: Define $K_n = J_n \cup f(J_n) \cup \dots \cup f^{k(n)-1}(J_n)$, $\Omega_n = \Omega(f|_{\text{closure}[K_{n-1} \setminus K_n]})$.

If $r = \infty$, write $\Lambda = \bigcap_{i \geq 0} K_i$. (If Λ contains wandering intervals, want to ignore the interior of these intervals, so take $\Omega_\infty = \Lambda \cap \Omega(f)$. Otherwise $\Omega_\infty = \Lambda$). Then $\Omega(f) = \bigcup_{0 \leq n \leq r} \Omega_n$.

In cases (≤ 0): $f: \Omega_n \to \Omega_n$ has only periodic points of period $k(n)$.

($=+$): $f: \Omega_n \to \Omega_n$. Ω_n is a collection of $k(n)$ intervals, $f^{k(n)}$ on each one conjugate to tent map.

($<+$): $f: \Omega_n \to \Omega_n$ is conjugate to a subshift of finite type.

See end of notes for a "flow diagram" (!)

Kneading Theory.

Definition: For each x in $[0,1]$, define kneading sequence $k_f(x) = \sum_{i=0}^{\infty} \theta_i t^i$, where $\theta_i = \left\{ \begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix} \right\}$ according as $\frac{d}{dx} f^{i+1}(x) \left\{ \begin{smallmatrix} < \\ = \\ > \end{smallmatrix} \right\} 0$.

Note: The chain rule $\Rightarrow \frac{d}{dx} f^{i+1}(x) = \prod_{j=0}^{i} \frac{d}{dx} f(x_j)$, where $x_j = f^j(x)$.

$= f'(x) . f'(f(x)) . f'(f^2(x)) \dots$

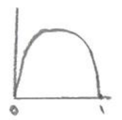
$\theta_i = 0$ iff $f^j(x) = c$, some $j \leq i$.

$\theta_i = +1$ if an even number of iterates lie in $[c, 1]$

$\theta_i = -1$ ------ odd ------------------------.

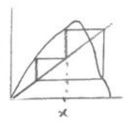
$\theta_i / \theta_{i-1} = 1$ if $f^i(x) \in [0, c)$, $= -1$ if $f^i(x) \in (c, 1]$.

Examples:



$k_f(0,t) = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}$

$k_f(1,t) = -1 - t - t^2 - t^3 - \cdots = -k_f(0,t).$



$k_f(x,t) = 1 - t - t^2 - t^3 + t^4 + t^5 + \cdots = (1-t-t^2)(1-t^3+t^6-t^9+\cdots)$

Define the lexicographical order on $k(x)$: $\sum \theta_i t^i < \sum \varphi_i t^i$ if $\theta_i = \varphi_i$, $0 \le i \le n$, and $\theta_{n+1} < \varphi_{n+1}$.
(This is the same as the ordinary order on $\mathbb{R}$ if $t < 1/2$).

Lemma: $x \mapsto k(x,t)$ is monotonically decreasing (not necessarily strictly).
Proof: Take $y < z$. Let $k(y) = \sum \theta_i t^i$, $k(z) = \sum \varphi_i t^i$. Let n be the smallest integer such that $\theta_n \neq \varphi_n$. If $n = 0$ then $y \le c \le z$ (at most one $=$) – trivial.
If $n > 0$, f^n maps $[y,z]$ homeomorphically to $[f^n(y), f^n(z)] \ni c$.
Either, $\theta_{n-1} = \varphi_{n-1} = \left\{ \begin{smallmatrix} +1 \\ -1 \\ 0 \end{smallmatrix} \right\}$
Eg, $= -1$. Then f^n is orientation reversing, $f^n(z) < c < f^n(y)$. So $\theta_n = \theta_{n-1} \cdot \operatorname{sgn} f'(f^n(y)) = -1 \times -1 = 1$.
$\varphi_n = \varphi_{n-1} \cdot \operatorname{sgn} f'(f^n(z)) = -1 \times 1 = -1$. So $\theta_n > \varphi_n$. (Other cases similarly).

Define a metric on kneading sequences: $d(\sum \theta_i t^i, \sum \varphi_i t^i) = \sum_{i=0}^{\infty} \frac{|\theta_i - \varphi_i|}{2^i}$.
Sequences are close if they agree on their first symbols.

Lemma: $x \mapsto k(x,t)$ is continuous at points x such that $f^n(x) \neq c$ for any n.
Proof: Exercise.

To deal with case where x is a pre-image of c, we introduce
$k(x_-, t) = \lim_{y \uparrow x} k(y,t)$ (y not a pre-image of c). (This limit exists)
$k(x_+, t) = \lim_{y \downarrow x} k(y,t)$.
In particular, $k(c_+, t)$ and $k(c_-, t)$ are important. $k(c_+, t) = -k(c_-, t)$.
And for pre-images of c, $f^n(x) = c$, have $\begin{cases} k(x_-, t) = \theta_0 + \cdots + \theta_{n-1} t^{n-1} + \theta_n t^n k(c_\pm, t) \\ k(x_+, t) = \theta_0 + \cdots + \theta_{n-1} t^{n-1} + \theta_n t^n k(c_\mp, t) \end{cases}$

Define the kneading invariant of f, $k_f = k_f(c_-, t)$.
"Basically," k_f determines the dynamics completely.
k_f will determine which sequences $k_f(x,t)$ occur for some point x under f.

Definition: A sequence k is k_f-admissable if for all n, $\sigma^n(k) \begin{cases} \ge k_f \\ \le -k_f \\ = 0 \end{cases}$, where $\sigma^n\left(\sum_{i=0}^{\infty} \theta_i t^i\right) = \sum_{i=0}^{\infty} \theta_{i+n} t^i$.
$\sigma(k(x)) = \theta_0(x)\, k(f(x))$.

Lemma: For each x, $k(x)$ is k_f-admissable. For each k that is k_f-admissible $\exists\, x$ such that $k(x)$ or $k(x_+)$ or $k(x_-)$ equals k.
Proof: $x < c \Rightarrow k(x) > k(c_-) = k_f$. $x > c \Rightarrow k(x) < k(c_+) = -k(c_-) = -k_f$. $x = c \Rightarrow k(x) = 0$.
Also true for all iterates of f, and therefore for all $\sigma^n(k(x))$, so $k(x)$ is k_f-admissible.
For second part, suppose k is k_f-admissible. Let $x = \inf\{y : k(y) \le k\}$.

From monotonicity, $k(x_-) \geqslant k \geqslant k(x_+)$.

Either: $x \mapsto k(x)$ is continuous at x, in which case $k(x_-) = k(x_+)$, and so $= k$, so $k(x) = k$.

Or: it's not continuous, so $f^n(x) = c$, some n. $\Rightarrow$ $k(x_-)$, $k(x_+)$ agree up to $(n-1)^{th}$ symbol, and $\sigma^n(k(x_-))$, $\sigma^n(k(x_+)) = \pm k_f$. But $\sigma^n(k)$ is between $\sigma^n(k(x_-))$ and $\sigma^n(k(x_+))$, and it is also admissible $\Rightarrow$ $\sigma^n(k) = \pm k_f$, 0. So either $k = k(x)$, $k(x_-)$, or $k(x_+)$.

Lemma: If $x \mapsto k(x)$ is constant on some interval J, then $\exists\, n, k, L$ such that $f^n(J) \subset L$ and $f^k(L) \subset L$, or J is a wandering interval.

Proof: If $k(x)$ is constant on J, then $c \notin f^n(J)$, any n. So $f^n|_J$ is a homeomorphism for all n. Either J is a wandering interval, or $\exists\, n, k$ such that $f^n(J) \cap f^{n+k}(J) \neq \phi$. Let $L = \bigcup_{p \geqslant 0} f^{n+pk}(J)$, then L is an interval and $f^k(L) \subset L$.

Remarks: Same cases as we had when considering whether semi-conjugacy collapsed intervals - will see that semi-conjugacy collapses intervals on which k is constant.

Families of maps, f_r.

Which k_f's can occur?

Definition: A sequence k is *admissible* iff $|\sigma^n(k)| \geqslant k$ for every n. $\left(|k| = \begin{cases} k & \text{if } k = +1+\cdots \\ -k & \text{if } k = -1+\cdots \end{cases} \right)$.

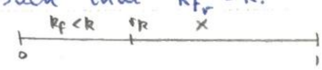
Eg: $k = 1 - t - t^2 + t^3 + \cdots + t^n - t^{n+1} - t^{n+2} - t^{n+3} + \cdots$. Then $|\sigma^n(k)| = 1 - t - t^2 - t^3 + \cdots < k$, so not admissible.

Note: given a map f, k_f is admissible, by the previous lemma.

Suppose f_r is a family of C^1 unimodal maps, $r \in [0,1]$, and f_r depends continuously on r.

Theorem: Suppose $k_{f_0} < k < k_{f_1}$, where k is admissible. Then $\exists\, r \in (0,1)$ such that $k_{f_r} = k$.

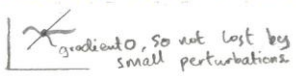
Proof: Let $S = \{ r \in [0,1] : k_{f_\beta} < k \ \forall \beta \leq r \}$. Set $r_k =$ least upper bound of S.



Write now $R = r_k$. Will show either $k_{f_R} = k$ or $k_{f_{R+\varepsilon}} = k$.

If $r \mapsto k_{f_r}$ is continuous at R then $k_{f_R} = k$.

But if not continuous, critical point c is periodic for f_R. $\Rightarrow$ c lies on a superstable periodic orbit with a basin of attraction. $\Rightarrow$ for r near R, f_r has a stable periodic orbit which attracts critical point.



Let x_0 be a point on this stable periodic orbit, close to c. $k(x_0) = \begin{cases} p + t^n p + t^{2n} p + \cdots \\ p - t^n p + t^{2n} p - \cdots \end{cases}$ if $\exists \begin{cases} \text{even} \\ \text{odd} \end{cases}$ number of points on orbit in $(c, 1]$, with p a polynomial of degree $n-1$.

And since orbit attracts the critical point, $k_{f_r} = \begin{cases} |p| . (1 + t^n + t^{2n} + \cdots) =: \beta \\ |p| . (1 - t^n + t^{2n} - \cdots) =: \alpha. \end{cases}$

In every neighbourhood of R, $k_{f_r} =$ either α or β, and in every neighbourhood it takes both values, since R is $\sup S$.

Claim: $\nexists$ admissible k, $\alpha < k < \beta$.

If we accept this, $\Rightarrow$ k admissible, $k \leqslant \alpha$ or $k \geqslant \beta$. But R is a l.u.b., so $k = \alpha$ or $k = \beta$.

In either case, k_{f_r} takes both values in a neighbourhood of R, so $\exists\, r^*$ such that $k_{f_{r^*}} = k$.

(Either $r^* = R$, or r^* can be chosen from interval $(R, R+\varepsilon)$)

Proof of claim: Let $\alpha < k < \beta$. Suppose $k = p_0 + t^n p_1 + t^{2n} p_2 + \cdots$, where each p_i is a polynomial of degree $n-1$. Since α, β agree on first n terms, $p_0 = |p|$.

Suppose $p_1 = +1 \pm \cdots$. Will show $k = \beta$. (Other case: $p_1 = -1 \pm \cdots \Rightarrow k = \alpha$ similarly).

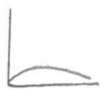
Now, $p_1 \leq |p|$ since $k < \beta$. Also, $\sigma^n(k) = p_1 + t^n p_2 + \cdots \geq k$, since k is admissable.

So $p_1 + p_2 t^n + \cdots \geq p_0 + p_1 t^n + \cdots \Rightarrow p_1 \geq p_0 = |p|$. So $p_1 = |p|$.

Continue by induction, (consider $\sigma^{2n}(k), \ldots$) $\Rightarrow p_0 = p_1 = p_2 = \cdots = |p| \Rightarrow k = \beta$.

Definition: A full family of unimodal maps is a family with parameter $0 \leq r \leq 1$ such that

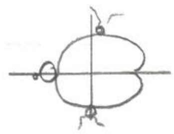
$k_{f_0} = \frac{1}{1-t} = 1 + t + t^2 + \cdots$, $k_{f_1} = \frac{-1}{1-t} = -1 - t - t^2 - \cdots$

Example: Family $F_r(x) = 4rx(1-x)$. $r = 0 + \varepsilon$:  $r = 1$: 

Also, any interval of r-values in which map is renormalisable. $f^n|_J$ is a full family.

Note: No requirement that k_{f_r} varies monotonically with r - often doesn't.

Eg, for $rx(1-x)$, can prove monotonicity by looking at complex map $z \mapsto z^2 + c$.



Semi-conjugacy, Milner & Thurston.

Theorem: Suppose $k_f(t)$ has a zero in $[0,1]$. Let r be the smallest zero in $[0,1]$. Then $\exists$ a semi-conjugacy φ to tent map F_s such that $\varphi(c) = \frac{1}{2}$, $s = \frac{1}{r}$.

Proof: $\varphi(x,t) = \dfrac{k(0,t) - k(x,t)}{k(0,t) - k(1,t)}$ Claim that $\varphi(x,t)$ is the required semi-conjugacy.

Need to show: (i) $\varphi(c) = 1/2$

(ii) φ is continuous.

(iii) $F_s \varphi = \varphi f$

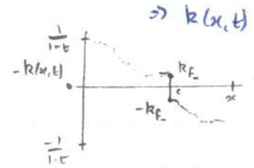
(iv) φ is monotonic in x. (not necessarily strictly).

$t < 1 \Rightarrow k(x,t)$ is a continuous function of t, series converges, etc.

(i) $\left.\begin{aligned} k(0,t) &= +1 + t + t^2 + \cdots = \tfrac{1}{1-t} \\ k(1,t) &= -1 - t - t^2 \cdots = \tfrac{-1}{1-t} \\ k(c,t) &= 0. \end{aligned}\right\} \Rightarrow \varphi(c,t) = 1/2.$

(ii) [Fix $t < 1/2$. $\Rightarrow$ normal order $\equiv$ lexicographical order.

$\Rightarrow k(x,t)$ is decreasing as a function of x.



For x not a pre-image of c, $k(x)$ as a function $x \mapsto \{\text{polynomials}\}$ is continuous, so $k(x,t)$ as a function $x \mapsto \mathbb{R}$ is continuous ($t < 1$).

For $f^n(x) = c$, have $k(x_-) = p \pm t^n k_{f_-}$

$k(x) = p + 0$

$k(x_+) = p \mp t^n k_{f_-}$

But $k_f(r) = k(c_-, r) = k_{f_-}(r) = 0$ by assumption.

So $\varphi(x, r)$ is continuous for all x.

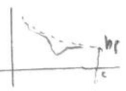
(iii) $\underline{x<c}$: $k(x) = 1 + \sum_{i\geq 1} \theta_i t^i$. $k(f(x)) = \sum_{i=1}^{\infty} \theta_i t^{i-1}$

$$\varphi(f(x)) = \frac{k(0,t) - \sum_{i=1}^{\infty}\theta_i t^{i-1}}{k(0,t) - k(1,t)} = \frac{1+t+t^2+\cdots - \sum_{i=1}\theta_i t^{i-1}}{k(0,t)-k(1,t)} = \frac{(1-\theta_1) + t(1-\theta_2) + t^2(1-\theta_3)+\cdots}{k(0,t) - k(1,t)}$$

$$= \frac{1}{t}\cdot\frac{(1-1) + (1-\theta_1)t + (1-\theta_2)t^2 + \cdots}{k(0,t) - k(1,t)} = \frac{1}{t}\cdot\frac{(1+t+t^2+\dots) - (1+\sum_{i=1}^{\infty}\theta_i t^i)}{k(0,t) - k(1,t)} = F_{\frac{1}{t}}\,\varphi(x).$$

$\underline{x>c}$: $k(x) = -1 + \sum_{i=1}^{\infty}\theta_i t^i$. $kf(x) = -\sum_{i=1}^{\infty}\theta_i t^{i-1}$. Get $\varphi f(x) = \frac{1}{t}(1-\varphi(x)) = F_{\frac{1}{t}}\varphi(x)$.

(iv) We know that for $t<\frac{1}{2}$, $k(x,t)$ is monotonic decreasing. Also, for $t<r$, $k(c_-,t)>0$. At $t=r$, $k(c_-,r)=0$. Increase t towards r until $k(x,t)$ is not monotonic. Note that for each x, $k(x,t)$ is continuous in t $(t<1)$.



So if it becomes non-monotonic in $t<r$ then it will do so at some t where $k(x,t)>0$ for $x<c$. Will show:

Lemma: If $k(x,t)>0$ for $x<c$, $k(x,t)<0$ for $x>c$, then $k(x,t)$ is monotonic in x.

Proof: Suppose not. Then $\exists$ points x,y with $x<y$, $k(x)<k(y)$. Since $k(x)\neq k(y)$, $\exists$ smallest n such that $[f^n(x), f^n(y)] \ni c$. If f is orientation preserving (other case similar) then $f^n(x) < c < f^n(y)$. But also $k(f^n(x)) < k(f^n(y))$. (From $k(x)<k(y)$, sequences agree on $n-1$ terms, f^n is orientation preserving). - #.

So, at $t=r$, $k(x,t)$ is monotonic.

We now want to investigate the relationship between r (smallest positive root of $k_f\cdot(1-t)$), $s=1/r$, and entropy of f $(h(f) = \log s)$.

Entropy, Critical Points, Laps.

Let $\gamma_n(J) = \#\{x\in J: f^n(x) = c,\ f^i(x)\neq c,\ i<n\}$. $\gamma_n(I) =: \gamma_n$.
Define $\mathrm{Lap}_n(J) = \#\{$monotonic pieces of $f^n|_J\}$.
Clearly, $\mathrm{lap}_n(J) = \sum_{0\leq i<n}\gamma_i(J) + 1$. As with kneading sequences, write $\gamma_f(J) = \sum\gamma_i(J)t^i$, $\mathrm{lap}_f(J) = \sum \mathrm{lap}_i(J)t^i$.

Lemma: $J = (a,b) \Rightarrow k_f\cdot\gamma_f(J) = \frac{1}{2}(k_f(a^+) - k_f(b^-))$.

Corollary: Take $J = I$. Then $k_f\gamma_f = \frac{1}{1-t}$.

Examples: $rx(1-x)$. $r=4 \Rightarrow k_f = 1-t-t^2-t^3-\cdots$, $\gamma_f = 1+2t+4t^2+8t^3+\cdots$

At $\frac{1}{2}$, $k_f(\frac{1}{2}) = 0$. γ_f does not converge for $t\geq 1/2$. Entropy $= \log 2 = \log\frac{1}{1/2}$.

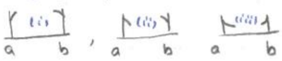
$r<1$, say $\frac{1}{2}x(1-x)$. $k_f = 1+t+t^2+\cdots$, $\gamma_f = 1$. "Least zero" $=1$. $h(f) = \log 1 = 0$.

Proof of Lemma: $k_f = \sum_{i=0}^{\infty}\theta_i t^i$. $k_f\gamma_f(J) = \sum_{n\geq 0}\left(\sum_{0\leq i\leq n}\theta_i\gamma_{n-i}(J)\right)t^n$

$\sum_{0\leq i\leq n}\theta_i\gamma_{n-i}(J) = \#\max. f^{n+1}|_J - \#\min. f^{n+1}|_J$ $(=-1, 0, \text{or } 1)$.

[If $f^i(x) = c$ for $i\leq n$ then f^{n+1} has a local max/min if $\theta_i = 1/-1$]

Now, f^{n+1} looks like:

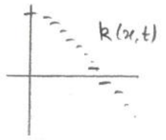


In (i), $\#\max = 1 + \#\min$, (ii): $\#\max - \#\min = 0$, in (iii), $\#\max - \#\min = -1$.

So, $\#\max - \#\min = \frac{1}{2}\left(\begin{smallmatrix}\text{sign of}\\ f^{n+1}\text{ at } a_+\end{smallmatrix} - \begin{smallmatrix}\text{sign of}\\ f^{n+1}\text{ at } b_-\end{smallmatrix}\right)$. Hence result.

Proof of Corollary: Trivial.

<u>Remark:</u>



- makes a jump at each pre-image of c.

Size of jump is $t^i \cdot 2k_f$

Σ jumps $= \sum_{i=0}^{\infty} 2k_f . t^i \times \#$ of pre-images of c $(f^i(x) = c)$

$= \sum 2k_f . \gamma_i t^i$.

$k(1) - k(0) = \frac{2}{1-t}$. So $k_f . \sum 2\gamma_i t^i = \frac{2}{1-t}$, so $k_f \gamma_f = \frac{1}{1-t}$. (This is not rigorous!)

<u>Sketch.</u>

Want $\lim_{n\to\infty} (\gamma_n)^{1/n}$ to exist and $= s$ -(*) since Misieurewicz & Szlenk: $h(f) = \lim_{n\to\infty} \frac{1}{n} \log \gamma_n$ (1977).

Easiest limit to prove: $\lim_{n\to\infty} (\text{lap}_n)^{1/n}$ exists.

Since $\text{lap}(f\circ g) \le \text{lap} f . \text{lap}\, g \Rightarrow \log \text{lap} f^{n+m} \le \log \text{lap} f^n + \log \text{lap} f^m$

By subadditivity $\Rightarrow \lim \frac{1}{n} \log \text{lap} f^n$ exists.

Since $\text{lap}_n = \sum_{0\le k\le n-1} \gamma_k + 1 \Rightarrow$ required limit for γ_n exists and is the same.

Given (*), radius of convergence of $\gamma(t)$ is $\frac{1}{s} = r$. Also all coefficients of $\gamma(t)$ are positive, so first pole of $\gamma(t)$ is on positive real axis, at $t = r$. So from $k(t)\gamma(t) = \frac{1}{1-t} \Rightarrow r$ is smallest positive zero of $k(t).(1-t)$.

<u>Remark (van Strien):</u> Can set up semi-conjugacy in terms of laps instead of kneading invariants.

Define $\Lambda(J) = \lim_{t\to r^-} \frac{\text{lap}(J,t)}{\text{lap}(t)}$ = proportion of complexity in interval J.

Then map $x \mapsto \Lambda(0,x)$ gives the semi-conjugacy, same as before.

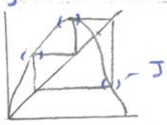
<u>Remark:</u> Milner & Thurston proved $f \to h(f)$ is continuous. (1977) (f in the class of C^1 unimodal maps).

<u>"Proof":</u> Need to show the smallest zero of k_f varies continuously with f. Easy if c is not periodic.

If c is periodic for f, we had $k_f = \begin{cases} p(1+t^n+t^{2n}+\dots) \\ p(1-t^n+t^{2n}-\dots) \end{cases}$ for all f in a neighbourhood of f_r, and so r_f is constant (smallest root of p) in this neighbourhood.

<u>Examples:</u> In general if f is renormalisable, $f^n|_J$ is unimodal, $k_f = p_n(t)\{k(t^n)\}$, where p_n is a polynomial of degree n and k is kneading invariant for $f^n|_J$.

<u>Eg (i):</u> $f^3|_J$ unimodal. $k_f = (1-t-t^2)k(t^3)$



& entropy is $\log$ (smallest root of $1-t-t^2)^{-1}$.

(ii) f is renormalisable of period 2. $k_f = (1-t)k(t^2)$. (ie $f^2|_J$ is unimodal).

By induction, if f has just orbits of periods 1, 2, 4, 8, ..., $k_f = (1-t)(1-t^2)(1-t^4)\dots$

Note that this has no roots <1. Entropy $= 0$.

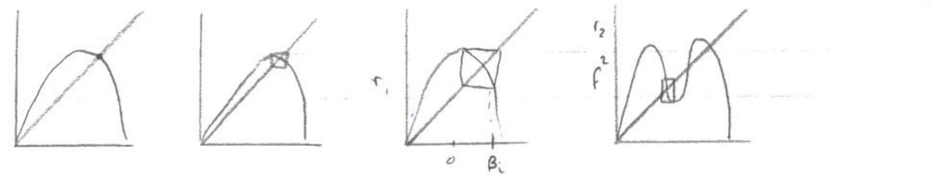
<u>Remark:</u> Note we know $h(f) > h(f^n|_J)$ - see correction (overleaf) when f is renormalisable. $\Rightarrow$ when $k_f = p . k'(t^n)$, p has a smaller root than any root of $k'(t^n)$.

<u>Exercise:</u> Can you prove this using only kneading theory - k_f is admissable.

<u>Metric Universality.</u>

f_r a family of C^1-unimodal maps. f_0 is at accumulation of period-doubling, f_{r_1} has superstable orbit of period 2, f_{r_2} ... of period 4, ..., f_{r_n} has 2^n - ie c is periodic. So $r_n \to 0$ as $n \to \infty$.

As r increases:



At each parameter value r_i, scale up "circulation box"

Observe that $F_i \to F^*$

f_{r_i} has a superstable orbit of period 2^i.
Define β_i by $\frac{1}{\beta_i} = |f_{r_i}^{2^{i-1}}(0)|$. So $\beta_i \to \infty$. $\quad f_{r_{n+1}}^{2^n}(0) = {(-1)^n}/{\beta_{n+1}}$, $f_{r_{n+1}}^{2^n}\left(\frac{(-1)^n}{\beta_{n+1}}\right) = 0$.
$F_{n+1}(x) = (-1)^n \beta_{n+1} f_{r_{n+1}}^{2^n}\left({x}/{(-1)^n\beta_{n+1}}\right)$ - map on $[0,1] \circlearrowleft$

Observe $F_n \to F^* \simeq 1 - 1.52763x^2 + 0.10481x^4 + 0.62670x^6 - 0.00352x^8 + \ldots$

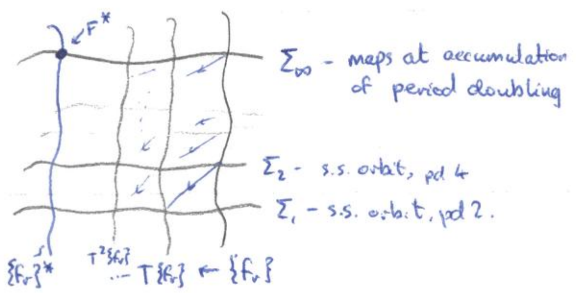
$\alpha_i = \frac{\beta_{i+1}}{\beta_i} \to 2.5029\ldots$, $\quad \delta_i = \frac{r_i - r_{i-1}}{r_{i+1} - r_i} \to \delta = 4.6692\ldots$ (Feigenbaum's δ)

Consider $T: f \to f$ - also takes families $\to$ families.
$T(f_r) = -\alpha f_{r/\delta}^2[{x}/{-\alpha}]$. $\quad$ Fix α, δ - later we choose them in a good way.

Claim: $\{f_r\}$ has a superstable orbit of period 2^{n+1} at $r \Rightarrow T\{f_r\}$ has a superstable orbit of period 2^n at δr.
Proof: Later.

T takes families with superstable orbits of periods $2, 4, \ldots 2^n$ to families with superstable orbits of periods $2, 4, \ldots, 2^{n-1}$.
Our family has superstable orbits of all periods 2^i, as $r \to 0$, $i \to \infty$. So does $T\{f_i\}$.

Space of unimodal maps:



Question: Is there a map F^* such that $F^* = -\alpha F^{*2}[{x}/{-\alpha}]$?
Answer: (Landford, 1980). Yes, if $\alpha \approx 2.5029$. Furthermore, F^* is a fixed point of operator T at which spectrum of T has one positive eigenvalue δ and all other eigenvalues are inside the unit disc. ($\delta \simeq 4.6692\ldots$)

So under $T^i\{f_r\}$, $i \to \infty$, tends to a nice family $\{f_r^*\}$. $T\{f_r^*\} = \{f_r^*\}$. (So $\{f_r^*\}$ is unstable manifold - 1-dimensional - of F^*). $\quad Tf_r^* = f_{\delta r}^*$.
For this family, $\frac{r_i - r_{i-1}}{r_{i+1} - r_i} = \delta$ exactly, but for other families obtain same ratio asymptotically.

Proof of earlier claim: Step (i): $T(f^{2^n}) = (Tf)^{2^n}$

Step (ii): Write f^T for $T(f)$. $[f^T_{\delta r}]^{2^{n-1}}(0) = T(f_{\delta r}^{2^{n-1}})(0)$

$$= -\alpha f_r^{2^n}(0) = -\alpha / (-1)^n . \beta_{n+1}$$

$$[f^T_{\delta r}]^{2^{n-1}}\left(\frac{-\alpha}{(-1)^n.\beta_{n+1}}\right) = T(f_r^{2^{n-1}})\left(\frac{-\alpha}{(-1)^n.\beta_{n+1}}\right) = -\alpha f_r^{2^n}\left(\frac{(-1)^n}{\beta_{n+1}}\right) = 0.$$

This proves the claim.

Correction: Earlier we claimed that $h(f) > h(f^n|J)$. This is often false! Meant to say

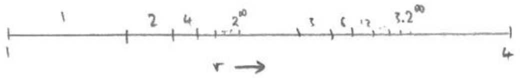
$h(f) > h(f|_{J \cup f(J) \cdots \cup f^{n-1}(J)})$

Kneading sequence in renormalisable case: $k_f = P_n(t)\, k'(t^n)$, then if smallest zero of k_f is $r<1$ then r is a root of $P_n(t)$. [$k'(t)$ may have smaller roots, but $k'(t^n)$ does not].

If $P_n(t)$ has a root $r<1$, then $r < \sqrt[n]{1/2}$. (cf: $s^n > 2$).

Milner & Thurston Paper - See Springer LNM 1342 (1988)

What do we see in families like $rx(1-x)$?



Theorem: For logistic family there are only 3 possibilities:

(i) Attracting periodic orbit

(ii) Attracting Cantor set.

(iii) Attracting finite collection of cyclically permuted intervals, $f^n|_J$ is conjugate to F_s.

In (i): attracts almost all initial conditions. Unique.

In (ii): Lebesgue measure zero. Attracts almost all initial conditions.

In (iii): Intervals attract almost all initial conditions - they have orbits dense in intervals.

Quite a lot of this follows from $Sf<0$ for this family. $Sf = \frac{f'''}{f'} - \frac{3}{2}\left[\frac{f''}{f'}\right]^2$

$Sf<0 \Rightarrow S(f^n)<0$. Also, every attracting periodic orbit attracts a critical point.

$\Rightarrow$ uniqueness of attracting orbits for unimodal maps.

With more work, $Sf<0 \Rightarrow \nexists$ wandering intervals (Guckenheimer).

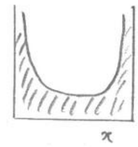
In more general families... Note Sf isn't invariant under smooth coordinate transformations, so rather unnatural for topological results - now use "bounded distortion."

Attracting periodic behaviour for Logistic family...
- occurs for a dense set of parameter values (Proof hard: structure of Mandelbrot set for $z \mapsto z^2 + c$)
- but measure of set of r-values in case (iii) is > 0.

Jacobson (or Collet-Eckmann) Theorem: $\lim_{t \to 4} \left(\frac{\text{Leb. meas. } \{ r \in [t, 4] : \text{ in case (iii)} \}}{(4-t)} \right) = 1.$

Example: For each rational p/q, put an interval of length ε/q^3 around it.
Total length of intervals around $\frac{1}{q}, \frac{2}{q}, \dots, \frac{q-1}{q}$ is ε/q^2.
Total measure of whole set $\le \sum_{q=1}^{\infty} \varepsilon/q^2$, which is as small as we want.

In case (iii):
Example: $r = 4$. (a) Invariant measure:



density $\frac{c}{\sqrt{x(1-x)}}$

- is invariant under f. Absolutely continuous invariant measure. (w.r.t. to Lebesgue - $\pi(A) = 0$ if $\text{Leb}(A) = 0$). Also $r = 4$ satisfies the Collet-Eckmann condition (positive Liapunov exponent):
$\lim_{i \to \infty} \inf \left| \frac{\log Df^i(f(x))}{i} \right| > 0$. (Slope of f^i grows exponentially fast - definition of chaos, strange attractors,...)

Question: does case (iii) imply (i) $\exists$ an absolutely continuous invariant measure, or (ii) Collet-Eckmann condition?

Answers are hard:
- Yes, if in a case where orbit of critical point is bounded away from critical point (Misiurewicz maps)
- No, $\exists$ counterexamples (even for Logistic map) when orbit of c returns too often and too close.
- Hard cases in between - still active research.

'Yes' is the usual answer, in some appropriate sense...