III Commutative Algebra   Michaelmas Term 1996

EXAMPLE SHEET 1

All rings are commutative with a 1.

1. Find an example of a unique factorisation domain which is not Noetherian.

2. Prove that the direct product of finitely many Noetherian rings is Noetherian.

3. By considering trailing coefficient ideals, prove that a ring R is Noetherian if and only if the power series ring $R[[X]]$ is Noetherian.

4. Show that an integral domain is a unique factorisation domain if and only if all its non-zero prime ideals contain a non-zero principal prime ideal. Use this to show that if R is a principal ideal domain then $R[[X]]$ is a unique factorisation domain.

5. Let M be the subset of a free Abelian group A of finite rank consisting of elements a satisfying a finite set of inequalities of the form $f_i(a) \geq 0$ where each f_i is a group homomorphism of A to the additive group of the integers $\mathbb{Z}$. Show that the subset $\mathbb{Z}M$ of $\mathbb{Z}A$ is a Noetherian ring. Does this remain true if we use defining maps f_i to the additive group of the real numbers?

6. Show that r lies in the Jacobson radical of R if and only if $1 - rs$ is a unit for all s in R.

7. Show, using Zorn's lemma, that every ring has a maximal ideal. Now assume that the ring is countable and prove this result without appealing to Zorn.

8. Show that the set of prime ideals in a ring possesses a minimal member (with respect to inclusion).

9. Let R be a Noetherian ring and θ be a ring homomorphism from R to R. Prove that if θ is surjective then it is also injective.

10. Let $R = k[X_1, X_2, \ldots]$ be the polynomial ring with countably infinite indeterminates and I be the ideal generated by all the elements X_i^i. Show that R/I is not Noetherian and that its nilradical is not nilpotent.

11. Let R be a Noetherian ring and f be a power series in $R[[X]]$. Prove that f is nilpotent

if and only if all its coefficients are nilpotent.

12. Let N be a submodule of a module M. Show that M is Artinian if and only if both N and M/N are Artinian.

13. A local ring is one which has a unique maximal ideal. Show that a ring is Artinian if and only if it is the direct product of finitely many Artinian local rings.

14. Let R be an Artinian ring and θ be an R-module map from R to R. Show that if θ is injective then it is also surjective.

15. Let $E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} : \alpha = (r/p^n) + \mathbb{Z} \text{ for some } r \in \mathbb{Z}, n \in \mathbb{N}_0\}$ for a rational prime p. Show that $E(p)$ is an Artinian, non-Noetherian $\mathbb{Z}$-module.

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EXAMPLE SHEET 2

All rings R are commutative with a 1.

1. Let S be a multiplicatively closed subset of a ring R, and M be a finitely generated R-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

2. Let N_1 and N_2 be submodules of the R-module M and let S be a multiplicatively closed subset of R. Show that $S^{-1}(N_1 + N_2) = S^{-1}N_1 + S^{-1}N_2$ and $S^{-1}(N_1 \cap N_2) = S^{-1}N_1 \cap S^{-1}N_2$ as submodules of $S^{-1}M$.

3. Let I be an ideal of a ring R, and let $S = 1 + I$. Show that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}R$.

4. Let R be a ring. Suppose that for each prime ideal P the local ring R_P has no non-zero nilpotent element. Show that R has no non-zero nilpotent element. If each R_P is an integral domain, is R necessarily an integral domain?

5. A multiplicatively closed subset S of a ring R is *saturated* when $xy \in S$ if and only if both x and y are in S. Prove that (i) S is saturated if and only if $R \setminus S$ is a union of prime ideals. (ii) If S is an multiplicatively closed subset of R, there is a unique smallest saturated multiplicatively closed subset S' containing S, and that S' is the complement in R of the union of the prime ideals which do not meet S. If $S = 1 + I$ for some ideal I, find S'.

6. Let $\phi : M \longrightarrow N$ be an R-module map. Show that the following are equivalent: (i) ϕ is surjective; (ii) $\phi_P : M_P \longrightarrow N_P$ is surjective for each prime ideal P; (iii) $\phi_Q : M_Q \longrightarrow N_Q$ is surjective for each maximal ideal Q.

7. Construct universal $\mathbb{Z}$ -bilinear maps

$$(\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z}) \longrightarrow (\mathbb{Z}/3\mathbb{Z})$$

$$(\mathbb{Z}/6\mathbb{Z}) \times (\mathbb{Z}/10\mathbb{Z}) \longrightarrow (\mathbb{Z}/2\mathbb{Z})$$

and show that, if r and s are coprime integers, then any $\mathbb{Z}$-bilinear map on $(\mathbb{Z}/r\mathbb{Z}) \times (\mathbb{Z}/s\mathbb{Z})$ is zero.

8. Prove that for R-modules M, N and L

$$M \otimes (N \otimes L) \cong (M \otimes N) \otimes L.$$

9. Show that there can be an element in a tensor product $M \otimes N$ which cannot be written as a single term $m \otimes n$ for any elements $m \in M$ and $n \in N$.

10. Show that the universality of $\otimes$ implies that $M \otimes N$ is spanned by the elements $m \otimes n$.

11. Let I be an ideal of a ring R. Show that $(R/I) \otimes M$ is isomorphic to M/IM.

12. Let R be a local ring, and M and N be R-modules. Prove that if $M \otimes N = 0$ then $M = 0$ or $N = 0$.

13. Let $R = \mathbb{C}[X]$, and I and J be the ideals of R generated by $X - \alpha$ and $X - \beta$ respectively. Show that $(R/I) \otimes_R (R/J)$ is a cyclic R-module and identify its annihilator. Show that $(R/I) \otimes_{\mathbb{C}} (R/J)$ is a cyclic R-module when using the diagonal action and identify its annihilator.

14. Show that any unique factorisation domain is integrally closed.

15. Let $R \leq T$ be rings with $T \setminus R$ closed under multiplication. Show that R is integrally closed in T.

16. Let $R \leq T$ be rings with T generated by n elements as an R-module. Show that over every maximal ideal of R there lies at most n maximal ideals of T.

17. Let T be of finite type and integral over R and P be a prime ideal of R. Show that T has only finitely many primes lying over P.

18. Let R be an integrally closed integral domain with fraction field K, and let $f(X) \in R[X]$ be a monic polynomial. Show that if $f(X)$ is reducible in $K[X]$ then it is also reducible in $R[X]$.

19. Let m be a square-free integer and R be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}[\sqrt{m}]$. Show that $R = \mathbb{Z}[(1+\sqrt{m})/2]$ if $m \equiv 1 \bmod 4$ and $R = \mathbb{Z}[\sqrt{m}]$ otherwise.

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EXAMPLE SHEET 3

All rings R are commutative with a 1.

1. Let $0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2$ be a sequence of R-modules. Then the sequence is exact if and only if for all R-modules M the sequence $0 \longrightarrow Hom(M, N_1) \longrightarrow Hom(M, N) \longrightarrow Hom(M, N_2)$ is exact.

2. A *projective* R-module M is an R-module for which any R-module map to an R-module N/N_1 lifts to a map to N. Show that M is projective if and only if it is a direct summand of a free R-module.

3. An R-module M is *injective* if any R-module map from an R-submodule N_1 (of an R-module N) to M extends to an R-module map from N to M. Show that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective $\mathbb{Z}$-modules.

4. An R-module M is *flat* if tensoring any short exact sequence of R-modules with it yields a short exact sequence. Show that $\mathbb{Z}/2\mathbb{Z}$ is not a flat $\mathbb{Z}$-module.

5. A ring R is *absolutelyflat* if every R-module is flat. Show that a Noetherian local ring is absolutely flat only if it is a field.

6. A chain of prime ideals is maximal if it is not a proper subset of another chain of primes. Prove that all maximal chains of prime ideals in an affine algebra which is an integral domain are of the same length.

7. Give an example of a Noetherian integral domain which has maximal ideals of different heights.

8. Give an example of an affine algebra T with a prime ideal P for which $\text{ht}P + \dim T/P < \dim T$.

9. Let k be a field. Show that every k-subalgebra R of $k[X]$ is of finite type over k and is of dimension 1 if $R \neq k$.

10. Let $R \leq T$ be affine domains over the field k. Prove that $\dim R \leq \dim T$.

11. Prove that any field which is finitely generated as a ring is finite.

12. Let $R = k[X_1, \dots, X_n]$ where k is a field, and M be a non-zero R-module. Consider the set of all ideals which are annihilator ideals of non-zero elements of M. Show that every maximal member of this set is prime. A module N is *residually simple* if it is non-zero and the intersection of all its maximal submodules is zero. Show that M contains a residually simple submodule.

13. Let R be a Noetherian regular local ring. Show that $R[[X]]$ is a regular local ring of dimension $\dim R + 1$. Deduce that if k is a field then $k[[X_1, \dots, X_n]]$ of formal power series in n indeterminates is a regular local ring of dimension n.

14. Let R be a Noetherian ring and $P_1 < P_2$ be prime ideals of R. Suppose there is some other prime Q lying strictly between P_1 and P_2. Show that there are infinitely many such Q.

15. Let I be an ideal contained in the Jacobson radical of R, and let M be an R-module and N be a finitely generated R-module. Let θ be an R-module map from M to N. Show that if the induced map from M/IM to N/IN is surjective then θ is surjective.

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EXAMPLE SHEET 4

All rings are commutative with a $1 \neq 0$.

1. Show that in a valuation ring any finitely generated ideal is principal.

2. Let $A \leq B$ be valuation rings with fraction field K, and let P and Q be the maximal ideals of A and B respectively. Show that if $A \neq B$ then $Q < P$ and that A/Q is a valuation ring of B/Q.

3. Show that if A is a valuation ring of Krull dimension 1 with fraction field K then there do not exist any rings intermediate between A and K. (In other words A is maximal among proper subrings of K.) Conversely show that if a ring R, not a field, is a maximal proper subring of a field K then R is a valuation ring of Krull dimension 1.

4. Let A be a valuation ring of a field K. The group U of units of A is a subgroup of the multiplicative group $K^\times$ of K. Let $\Gamma = K^\times/U$. If α and β are represented by x and $y \in K$ define $\alpha \geq \beta$ to mean $xy^{-1} \in K$. Show that this defines a total ordering on Γ which is compatible with the group structure (i.e. $\alpha \geq \beta$ implies $\alpha\gamma \geq \beta\gamma$ for all $\gamma \in \Gamma$). (In other words Γ is a totally ordered Abelian group. It is called the value group of A.) Let $v : K^\times \longrightarrow \Gamma$ be the canonical homomorphism. Show that $v(x+y) \geq \min(v(x), v(y))$ for all $x, y \in K^\times$.

5. Conversely, let Γ be a totally ordered Abelian group written additively, and let K be a field. Let $v : K^\times \longrightarrow \Gamma$ be a non-Archimedean valuation. Show that the set of elements $x \in K^\times$ such that $v(x) \geq 0$ is a valuation ring of K.

6. Let R be a discrete valuation ring with field of fractions K, and let L be an extension field of K of finite degree. Show that a valuation ring of L containing R is a discrete valuation ring.

7. Show that any ideal in a Dedekind domain can be generated by at most 2 elements.

8. Let R be the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{10})$. Show that R is a Dedekind domain but

not a principal ideal domain.

9. Let R be a discrete valuation ring with maximal ideal P. Show that the P-adic completion of R is again a discrete valuation ring.

10. Show that an R-module M is Hausdorff with respect to the I-adic topology if and only if $\cap_n I^n M = 0$. (A topological space is *Hausdorff* if given distinct x and y there are disjoint open sets U and V containing x and y respectively.)

11. Show that the additive group of an R-module M is a topological group with respect to the I-adic topology. (You have to show that the maps $M \times M \longrightarrow M$ $(x, y) \longrightarrow x + y$ and $M \longrightarrow M$ $m \longrightarrow -m$ are continuous.)

12. Show that the ring of p-adic integers $\mathbb{Z}_p$ is compact.

13. Show that in the I-adic completion $\hat{R}$ the ideal $\hat{I}$ is contained in the Jacobson radical of $\hat{R}$.

14. Let k be a field and f be a homogeneous polynomial in $R = k[X_1, \ldots, X_n]$. Calculate the Hilbert polynomial for $R/(f)$ and hence show that $d(R/(f)) = n - 1$.

15. Let R be a Noetherian local domain. Show for non-zero x that $d(R/(x)) \leq d(R) - 1$.

16. Show that the composition length of an Artinian module is independent of the composition series.

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