Chevalley Groups.

Lots of groups: D_{2n}, $F = C_{p_1}{}^{n_1} \times \cdots \times C_{p_k}{}^{n_k}$. S_n. A_n, $n \geq 5$ simple.
$GL_n(\mathbb{C})$, $GL_n(k)$ - Example 1.
$SL_n(\mathbb{C})$, $SL_n(k)$.
Monster - largest of 26 sporadic simple groups.
Finite Reflection Groups.
$PSL_n(\mathbb{C})$, $PSL_n(\mathbb{F}_q)$ - finite simple groups.

Examples: V, vector space with a bilinear form $\varphi(x,y)$ over a field k.

$$V \times V \ni (x,y) \mapsto \varphi(x,y).$$
$$GO(\varphi) = \{ T \in GL_n(V) : \varphi(Tx, Ty) = \lambda \varphi(x,y),\ \lambda \in k \}$$
$$SO(\varphi) = \{ T \in GL_n(V) : \varphi(Tx, Ty) = \varphi(x,y) \}.$$

Examples:

2. If $\varphi(x,y) = -\varphi(y,x)$ then V is a symplectic space.

$$GL_n(k) \geq Sp_n(k) = \{ T : T^t J T = J \}, \text{ where } J = \begin{pmatrix} 0 & & 1 \\ & \iddots\ {-1} & \\ -1 & & 0 \end{pmatrix}$$

$$PSp_n(k) = Sp_n(k)/Z, \quad \text{where the centre } Z = \{ \pm I \}.$$

3. If φ is symmetric, $\varphi(y,x) = \varphi(x,y)$, char $k \neq 2$. Quadratic form $f(x) = \varphi(x,x)$.

Then we get $GO(\varphi)$, $O(\varphi)$, orthogonal groups.

Over the algebraic closure $\bar{k}$ of k, symmetric matrix is equivalent to:

$$n = 2m: \quad Q_1 = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}, \qquad n = 2m+1: \quad Q_2 = \begin{pmatrix} 0 & & 1 & 0 \\ & \iddots & & \vdots \\ 1 & & 0 & \\ 0 & \cdots & & 1 \end{pmatrix}$$

For example, $k = GL(q)$, finite field, and ε a non-square in k.

$n = 2m+1$: symmetric matrix equivalent to Q_2 or εQ_2 $\longrightarrow O_{2m+1}(q)$

$n = 2m$: symmetric matrix equivalent to Q_1 or $\begin{pmatrix} 0 & & 1 & 0 & 0 \\ & \iddots & & \vdots & \vdots \\ 1 & & 0 & & \\ 0 & \cdots & & 1 & \\ 0 & \cdots & & & \varepsilon \end{pmatrix} \longrightarrow O_{2m}^-(q)$

$\downarrow$
$O_{2m}^+(q)$

4. Suppose that k has an automorphism τ of order 2 (eg $\mathbb{C}$ with complex conjugation).

V, with a non-singular hermitian form: $h(y,x) = h(x,y)^\tau$.

Give rise to unitary groups, $U_n(k,h) \geq SU_n(k,h)$.

Plan.

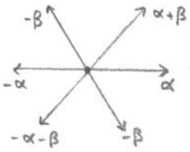
1. Introduction to Weyl Groups and root systems.
2. Semisimple Lie Algebras.
3. Automorphisms of Lie Algebras: Chevalley Groups.
4. Combinatorial structures: Bruhat decomposition and building.
5. Chevalley Groups are (almost always) simple.
6. Twisted Chevalley Groups.

1. Weyl Groups.

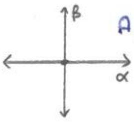
To define reflections we need a Euclidean space endowed with a positive definite symmetric bilinear form, V, $\dim V = l$, and $(,) : V \times V \to \mathbb{R}$.
If $\alpha \in V$, denote by w_α reflection in hyperplane orthogonal to α, $H_\alpha = \{v \in V : (v, \alpha) = 0\}$.
$w_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$.
For which sets of vectors do the reflections generate a finite group?

Definition 1.1: $\Phi \subseteq V$ is a system of roots in V if:

1. Φ is a finite set of non-zero vectors.
2. Φ spans V.
3. If $\alpha, \beta \in \Phi$, then $w_\alpha(\beta) \in \Phi$.
4. If $\alpha, \beta \in \Phi$, then $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$
5. If $\alpha, \lambda\alpha \in \Phi$, then $\lambda = \pm 1$.

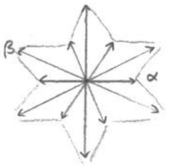


A_2
$W(\Phi) \cong D_6 \cong S_3$



$A_1 \times A_1$

A_1



G_2 $\Pi = \{\alpha, \beta\}$
$W(\Phi) = D_{12}$

Remarks 1.2: (i) $\Phi = -\Phi$, from (3).

(ii) (5) is optional - such a system is called a reduced root system.

(iii) Φ is not necessarily linearly independent. Φ contains a subset Π such that:

(a): Π is linearly independent.

(b): Every root in Φ is a linear combination of roots in Π with coefficients either all non-negative, or all non-positive.

Π is called a set of simple roots.

Exercise: (i) Prove that such a Π exists.

(ii) Prove that every root α is a linear integer combination of roots in Π.
(Hint: induction on height of root: $ht(\alpha) = \lambda_1 + \dots + \lambda_l$ where $\alpha = \lambda_1\alpha_1 + \dots + \lambda_l\alpha_l$)

Remarks: (iv) The rank of $\Phi = l$.

(v) Π will partition Φ into a set of positive roots Φ^+ and negative roots Φ^-.

Definition 1.4: $W(\Phi) = \langle w_\alpha \mid \alpha \in \Phi \rangle$ is called the Weyl group of Φ

Why is $W(\Phi)$ a finite group? W acts faithfully on Φ by (2) and (3). So W "$\leq$" $S(\Phi)$.
A_2: $W(\Phi) \cong S_3$. B_2: $W(\Phi) \cong D_8$.

Proposition 1.5: (i) Every root in Φ is the image of some simple root under W.

(ii) $W = \langle w_\alpha \mid \alpha \in \Pi \rangle$.

Exercise: Prove this.

Define $m(\alpha,\beta)$ = order of the element $w_\alpha w_\beta \in W$. Then W can be realised as an abstract group by generators $w_\alpha : \alpha \in \Pi$ and relations $(w_\alpha w_\beta)^{m(\alpha,\beta)} = 1$.
Such a group (ie, a group with such a presentation) is called a Coxeter group.

Example: $W = \langle w_1, w_2, w_3 : (w_1 w_2)^3 = (w_2 w_3)^5 = (w_1 w_3)^3 = 1,\ w_i^3 = 1 \rangle$ is a finite Coxeter group.
(Exercise: which finite group is W?)

If $w \in W$, define $l(w)$ = minimum length of an expression for w in terms of w_α, $\alpha \in \Pi$
$= |\Phi^+ \cap w^{-1}(\Phi^-)|$

More groups: $GL_n(k)$ has subgroups $SL_n(k)$, $Tr_0(k) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, $Tr_1(k) = \begin{pmatrix} k^* & & * \\ & \ddots & \\ 0 & & k^* \end{pmatrix}$, $D(k) = \begin{pmatrix} k^* & & 0 \\ & \ddots & \\ 0 & & k^* \end{pmatrix}$

$$\underset{\overset{\uparrow}{n_1 + \cdots + n_k = n}}{P_{[n]}(k)} = \begin{pmatrix} \boxed{GL_{n_1}(k)} & & * \\ & \ddots & \\ 0 & & \boxed{GL_{n_k}(k)} \end{pmatrix}$$ - parabolic subgroup.

$Tr_0(k)$ is a nilpotent group.
$\gamma_0 = G$. $\gamma_i = [\gamma_{i-1}, G]$ - Lower central series.
$Tr_1(k)$ is soluble. $D_n(G) = [D_{n-1}, D_{n-1}]$, $D_0 = G$ - derived series. It is called a Borel subgroup.

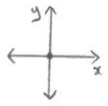
$V = \mathbb{R}^2$, $(x,y) = |x|.|y| \cos\theta$, $|x|^2 = (x,x)$. $\frac{2(x,y)}{(x,x)} \in \mathbb{Z} \Rightarrow \frac{4(x,y)^2}{(x,x)(y,y)} \in \mathbb{Z} \Rightarrow 4\cos^2\theta \in \mathbb{Z}$.

$4\cos^2\theta$:	0	1	2	3	4
$\cos\theta$:	0	$\pm\frac{1}{2}$	$\pm 1/\sqrt{2}$	$\pm\frac{\sqrt{3}}{2}$	± 1
θ :	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$0, \pi$
$\frac{2(x,y)}{(x,x)}, \frac{2(x,y)}{(y,y)}$:	0, 0	±1, ±1	±1, ±2	±1, ±3	

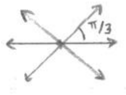
$\rightarrow x = \lambda y \Rightarrow \lambda = \pm 1$
$x = -y$.

? $|x| = |y|$ $\sqrt{2}|x| = |y|$ $\sqrt{3}|x| = |y|$

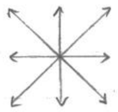
$A_1 \times A_1$:



A_2 :



G_2 :



A_1 :

The existence of a set of simple roots.

- put a total ordering on V, eg $\sum \lambda_i v_i > 0$ if $\lambda_1, \ldots, \lambda_k = 0$, $\lambda_{k+1} > 0$. $\Phi^+ = V^+ \cap \Phi$.

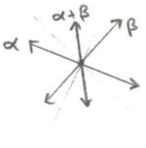
Π: (i) Every root in Φ^+ should be a positive linear combination of roots in Π.
(ii) No subset of Π satisfies (i).

Exercise: Π is a linearly independent set.
First prove that $\alpha, \beta \in \Pi$ then $(\alpha,\beta) \leq 0$, ie, oblique.

Corollary: if $\alpha, \beta \in \Pi$ then $(\alpha,\beta) \leq 0$.

Recall $W(\Phi) = \langle w_\alpha : \alpha \in \Phi \rangle$.

Proposition 1.5: (i) Every root of Φ is the image of a root of Π under $W_0 = \langle w_\alpha : \alpha \in \Pi \rangle$.
(ii) $W = \langle w_\alpha : \alpha \in \Pi \rangle$.



Proof: (i) Use induction on $ht(\alpha)$.
(ii) w_β, $\beta \in \Phi$. $w_{w(\alpha)} = w w_\alpha w^{-1}$ for some $w \in W$, $\beta = w(\alpha)$, $\alpha \in \Pi$.

Proposition 1.7: $l(w)$ = minimal length of an expression for w in terms of $w_\alpha : \alpha \in \Pi$.
$= |\Phi^+ \cap w^{-1}(\Phi^-)| = n(w)$.

Exercise: (i) $w_\alpha : \alpha \in \Pi$, only changes the sign of α and $-\alpha$. $n(w_\alpha) = 1$, and moreover:

(a) $n(w_\alpha w) = n(w)+1$ if $w^{-1}(\alpha) \in \Phi^+$.

(b) $n(w_\alpha w) = n(w) - 1$ if $w^{-1}(\alpha) \in \Phi^-$,

(ii) $l(w) \geqslant n(w)$. If $l(w) > n(w)$ then (b) is happening at some point. Deduce you can write a shorter expression for w, in terms of w_α's.

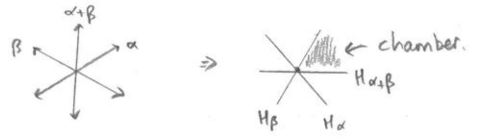
Corollary 1.8: If $w \in W$ and $w(\Pi) = \Pi$ then $w = 1$.

Proposition 1.9: If Π is a set of simple roots, then $w(\Pi)$ is also a set of simple roots.
If Π_1, Π_2 are two sets of simple roots, then $\exists\, w \in W$ such that $w(\Pi_1) = \Pi_2$.

Proof: Do an induction on $|\Phi_1^+ \cap \Phi_2^-|$. $\exists\, \alpha \in \Pi_1 \cap \Phi_2^-$ with $|w_\alpha(\Phi_1^+) \cap \Phi_2^-| = n-1$.

Definition 1.10: The connected components of $V - \bigcup_{\alpha \in \Phi} H_\alpha$ are called chambers.

A_2:



A hyperplane H_α is called a wall of a chamber C if $H_\alpha \cap \bar{C}$ is not contained in any proper subspace of H_α.

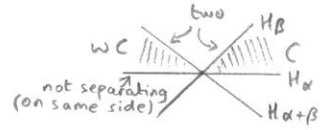
If $\Pi = \{\alpha_1, .., \alpha_l\}$ is a set of simple roots, let $C = \{x \in V : (\alpha_i, x) > 0 \ \ \forall\, i = 1, .., l\}$.
If we take $\alpha \in \Phi^+$ then $H_\alpha = \{x : (\alpha, x) = 0\}$. Let $H_\alpha^+ = \{x : (\alpha, x) > 0\}$
$H_\alpha^- = \{x : (\alpha, x) < 0\}$.

If $x \in C$ then $(\alpha_i, x) > 0 \Rightarrow (\alpha, x) > 0 \Rightarrow C$ is a chamber.
Show that hyperplanes H_α ($\alpha \in \Pi$) are walls of C, and if $\alpha \in \Phi^+ \setminus \Pi$ then H_α is not a wall of C.

Proposition 1.11: The roots orthogonal to walls of a chamber and "pointing into" the chamber C form a set of simple roots. Every set of simple roots arises in this way.
(A choice of simple roots gives rise to a choice of chamber, which we call the fundamental chamber).

Proof: If C is a chamber defined by a set of simple roots Π, we know $w(\Pi)$ is also a set of simple roots. $w(C)$ is the chamber defined by $w(H_\alpha). = H_{w(\alpha)}$
(For, $w(H_\alpha) = w\{x : (\alpha, x) = 0\} = \{w(x) : (w(\alpha), w(x)) = 0\} = \{y : (w(\alpha), y) = 0\} = H_{w(\alpha)}$).
Choose another chamber C'. We need to prove that $\exists\, w$ such that $w(C') = C$.
Let $v \in C'$. Let v' = greatest transform $w(v)$ with respect to the ordering on V defined by Π. Must check $v' \in C$, ie, $(\alpha_i, v') > 0 \ \ \forall\, \alpha_i \in \Pi$.
$w_{\alpha_i}(v') = v' - 2\frac{(\alpha_i, v')}{(\alpha_i, \alpha_i)} \alpha_i$. But $w_{\alpha_i}(v') = w_{\alpha_i} w(v) \leqslant v \Rightarrow \frac{2(\alpha_i, v')}{(\alpha_i, \alpha_i)} > 0 \Rightarrow (\alpha_i, v') > 0$.
So $v' \in C$. Check that $w(C') = C$.

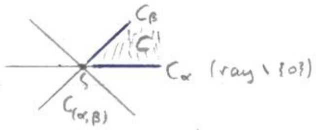
Proposition 1.12: $l(w)$ = number of hyperplanes H_α, $\alpha \in \Phi$, separating C from $w(C)$.



For each subset $I \subseteq \Pi$ define $C_I = \{v \in \bar{C} : (\alpha, v) = 0 \text{ for } \alpha \in I, (\alpha, v) > 0 \text{ for } \alpha \in \Pi \setminus I\}$.

1. C_I is intersections of H_α and H_β's.
2. The sets C_I partition $\bar{C}$ ($C = C_\phi$, $C_\Pi = \{0\}$).
3. Dimension of the linear span of $C_I = n - |I|$.
4. $\mathcal{C} = \{w C_I : w \in W, I \subseteq \Pi\}$ partitions V.

Definition 1.13: $\mathcal{C}$ is called a Coxeter complex of W.
A set $w C_I$ is called a facet of type I.



How many elements in the Coxeter complex of A_2? - 13.

Order $\mathcal{C}$ by saying: if $A_1, A_2 \in \mathcal{C}$, $A_1 \preceq A_2$ if $A_1 \subseteq \bar{A_2}$.
Then $(\mathcal{C}, \preceq)$ is a simplicial complex. (with vertices given by rays).

Definition 1.14: A simplicial complex consists of a vertex set V and a set Δ of subsets of V called simplices such that $\{v\} \in \Delta$ and every subset of a simplex is a simplex ($B \subseteq A \in \Delta \Rightarrow B \in \Delta$).

$J \subseteq \Pi$. Let $W_J = \langle w_\alpha \mid \alpha \in J \rangle$.

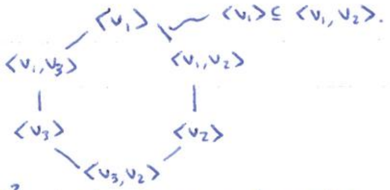
More groups: $GL_n(k) \geqslant \{$monomial matrices - permutations, eg $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)\} \cong S_n$.

The Coxeter complex is an example of a simplicial complex. $\mathcal{C} = \{w C_I : w \in W, I \subset \Pi\}$,
span has dimension $n - |I|$.

where vertices are rays.

Example: $V = \langle v_1, .., v_n \rangle$. Define vertices as subspaces with basis given by some subset of $\{v_1, .., v_n\}$.
Eg: $\langle v_1 \rangle$, $\langle v_1, v_2 \rangle$, .., $\langle v_1, .., v_n \rangle$. We'll say two vertices V_1, V_2 lie on the same line if either $V_1 \subseteq V_2$, or $V_2 \subseteq V_1$. Edges are chains of subspaces of length 2.
The faces of this geometry are chains of subspaces $V_1 \subseteq \cdots \subseteq V_i$. These are called flags.
$\langle v_1 \rangle \subseteq \langle v_1, v_2 \rangle \subseteq \langle v_1, v_2, v_3 \rangle \subseteq \cdots \subseteq \langle v_1, .., v_n \rangle$. This is called a flag complex; it is a simplicial complex.

$V = \langle v_1, v_2, v_3 \rangle$. Then $\mathcal{C}(V) \cong \mathcal{C}(A_2)$



Exercise: Can you give a set of vectors in $\mathbb{R}^3$ which is a root system, whose Coxeter complex is isomorphic to the flag complex defined by $V = \langle v_1, .., v_4 \rangle$

$GL_3(k)$ acts on $V = \langle v_1, v_2, v_3 \rangle$

Parabolic subgroups: $\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$, $\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \rightarrow$ stabilises $\langle v_3 \rangle$ $(\subseteq \langle v_1, v_2, v_3 \rangle)$

$\hookrightarrow$ stabiliser of $\langle v_3 \rangle \subseteq \langle v_3, v_2 \rangle$ $(\subseteq \langle v_1, v_2, v_3 \rangle)$

$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \rightarrow$ stabilises $\langle v_2, v_3 \rangle$.

Exercise: $Sp_{2n}(k)$ - What is the finite group inside here?

Definition: Parabolic subgroups of Weyl Groups:

Let $J \subseteq \Pi$. Set V_J = span of $J \subseteq V$ and $\Phi_J = \Phi \cap V_J$, $W_J = \langle w_\alpha : \alpha \in J \rangle$.

Proposition 1.15: Φ_J is a system of roots for V_J. J is a set of simple roots and the Weyl Group is W_J.

Remember that the other sets of simple roots are just $w\Pi \supset J'$. $W_{J'} = \langle w_{w(\alpha)} : \alpha \in J \rangle = \langle w w_\alpha w^{-1} : \alpha \in J \rangle$
$= W_J^w$.

Definition 1.16: The subgroups W_J and their conjugates are called parabolic subgroups.

Proposition 1.17: (i) The stabiliser of wC_J in W is W_J^w.

(ii) Each element of $\mathcal{C}$ can be transformed into an element C_J by action of W.

(iii) The parabolic subgroups of W are the stabilisers in W of elements of $\mathcal{C}$.

Proof: Let $D_J = \{ w \in W : w(\alpha) \in \Phi^+ \text{ for all } \alpha \in J \}$ - not a subgroup, but a set of coset representatives for W_J in W - use induction on $l(w)$.

Suppose $w \in W \setminus W_J$, $w(C_J) = C_J$, $\exists\, d_J \neq 1$, $w(C_J) = d_J(C_J)$. $\exists$ some $\alpha \in \Pi \setminus J$ such that $d_J(\alpha) \in \Phi$

Let $v \in C_J \Rightarrow (v, \alpha) > 0 \Rightarrow (d_J v, -d_J \alpha) < 0$ and $-d_J \alpha \in \Phi^+$, $d_J v \in \bar{C}$, but then should have $(d_J v, -d_J \alpha) \geq 0$ ※

"$\{ v \in V : (v, \alpha) = 0 \ \forall \alpha \in J ; (v, \alpha) > 0, \alpha \in \Pi \setminus J \}$.

Theorem 1.18: $\mathcal{C}_{\leq} \cong \{ W_J^w : J \subset \Pi, w \in W \}_{\leq \text{opp}}$

Ie, we have correspondance: $W \longrightarrow \mathcal{C}$

parabolic subgroups $\longleftrightarrow$ stabilisers of facets.

Suppose we have Coxeter group W, but know nothing about any underlying root system. Can still look at $\{ W_J^w : J \subset \Pi, w \in W \}_{\leq \text{op}}$.

Flag complexes: $GL_n(k)$. Fix basis $v_1, \ldots, v_n$.

$\mathcal{C}(V) = \{ V_1 \subset V_2 \subset \cdots \subset V_i : i = 1, \ldots, n \}$. $V_i = \langle v_{i_1}, \ldots, v_{i_j} \rangle$ $j \geq i$.

Elements of $\mathcal{C}(V)$ correspond to elements of $GL_n(k)$. If we have a different basis we get new parabolic subgroups of $GL_n(k)$ which are conjugates of the previous one. The collection of all Coxeter complexes defined by different bases is an example of a building.

2. Lie Algebras.

Definition: (i) L is a semisimple Lie algebra if the largest soluble ideal, rod $L = \{0\}$.
[no proper ideals in which $[\,,\,]$ is trivial).
(ii) L is simple if it no ideals except L, $\{0\}$.
(iii) A Cartan subalgebra H of L is a nilpotent subalgebra which equals its normaliser. ie: (a) $[[\cdots[[H,H],H],\dots,H] = \{0\}$, some r.
(b) If $[x,h] \in H$ $\forall h$, then $x \in H$.

Example 2.2: $A = M_{l+1}(\mathbb{C})$, $[x,y] = xy - yx$. $L = \{x \in A : \operatorname{tr} x = 0\}$ is a simple Lie algebra of dimension $l(l+1)$. Example of a Cartan subalgebra $= \{h = \operatorname{diag}(\lambda_0, \dots, \lambda_l) \in L\}$.
$[h, \Sigma a_{ij} e_{ij}] = \Sigma (\lambda_i - \lambda_j) a_{ij} e_{ij}$. ($\uparrow$ abelian. $\Rightarrow$ nilpotent.)
In this example, L decomposes as a direct sum subspaces $\mathbb{C} e_{ij}$, $i \neq j$, invariant under H. $L = H \bigoplus_{i \neq j} \mathbb{C} e_{ij}$. $[h, e_{ij}] = (\lambda_i - \lambda_j) e_{ij}$.

Proposition 2.3: (i) L has a Cartan subalgebra
(ii) If L is simple then $[H,H] = 0$.
(iii) $L = H \oplus L_1 \oplus \cdot \oplus L_k$ where L_i is 1-dimensional subspace invariant under action of H. (Cartan decomposition).

For (iii), $\{\operatorname{ad} h : h \in H\}$ is a commuting set of semi-simple endomorphisms, so simultaneously diagonalisable.

Definition: The rank of $L = \dim_{\mathbb{C}} H = l$.

For each L_i we have a linear functional $\alpha_i \in H^*$ on H defining the action of H on L_i.
For all $h \in H$, $e_i \in L_i$, $[h, e_i] = \alpha_i(h) e_i$.

Definition: $\Phi = \{\alpha_1, \dots, \alpha_k\}$ are the roots of L and the subspaces are called root-subspaces (relative to H). Then α_i are non-zero (since H is self-normalising) and distinct.

Definition 2.6: The Killing Form $K(x,y)$ is defined on $L \times L$ by $K(x,y) = \operatorname{tr}(\operatorname{ad} x . \operatorname{ad} y)$
- non-singular, symmetric, positive definite form on L.

Lemma 2.7: The Killing Form is non-degenerate when restricted to H.
Proof: Claim H is orthogonal. L_α, $\alpha \in \Phi$, $(\operatorname{Tr}([xy]z) = \operatorname{Tr}(x[yz]))$.
$\alpha \neq 0 \Rightarrow \exists\, h \in H$ such that $\alpha(h) \neq 0$.
If $x \in H$, $y \in L_\alpha$: $0 = K([h,x], y) = -K([x,h], y) = -K(x, [h,y]) = -K(x, \alpha(h) y)$
$= -\alpha(h) K(x,y)$.

If $z \in H$ and $K(z, H) = 0$ then $K(z, L) = 0 \Rightarrow z = 0$.

So we can use $K(,)$ to identify H with its dual H^* in the usual way. ie, for each $\alpha^* \in H^*$ $\exists\ \alpha \in H$ such that $\alpha^* h = K(\alpha, h)$.

Lemma 2.8: Φ spans H.

Proof: Suppose not. Then $\exists\ h \in H$ such that $\alpha(h) = 0\ \forall\ \alpha \in \Phi$. $\Rightarrow [h, L_\alpha] = 0$. This, and $[h, H] = 0$ imply $h \in Z(L) = 0$.

Let $V_{\mathbb{R}} = \mathbb{R}$-span of Φ. $\dim V_{\mathbb{R}} = \dim_{\mathbb{C}} H$, because each root $\alpha \in \Phi$ is a rational combination of a basis of roots. (If $V_{\mathbb{Q}} = \mathbb{Q}$-span of Φ, then $\dim V_{\mathbb{Q}} = \dim V_{\mathbb{R}}$).

Proposition 2.9: (a) Φ is a set of non-zero vectors spanning $H_{\mathbb{R}} = V_{\mathbb{R}}$

(b) If $\alpha \in \Phi$ and $\lambda\alpha \in \Phi$ then $\lambda = \pm 1$.

(c) $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

-ie, Φ is a root system in $(H_{\mathbb{R}}, K(,))$

Proof: See Lie Algebras course.

We are looking for a basis of $L = H \oplus \sum_{\alpha \in \Phi} L_\alpha$, with integral structure constants. Ie, $[x_i, x_j] = \sum \lambda_{ijk} x_k$, $\lambda_{ijk} \in \mathbb{Z}$.

A basis $\{\alpha_1, \ldots, \alpha_l\} = \Pi$ for Φ is a basis for H. We define $h_{\alpha_i} = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$

Lemma 2.10: $\Phi^* = \{h_\alpha : \alpha \in \Phi\}$ is a root system and $\Pi^* = \{h_{\alpha_i} : \alpha_i \in \Pi\}$ is a basis for Φ^*.

Corollary: $h_\alpha = \sum \lambda_i h_{\alpha_i}$, $\lambda_i \in \mathbb{Z}$.

Choose $e_\alpha \in L_\alpha$. $[h_{\alpha_i}, e_\alpha] = \frac{(\alpha, 2\alpha_i)}{(\alpha_i, \alpha_i)} e_\alpha$. $\frac{2(\alpha,\alpha_i)}{(\alpha_i,\alpha_i)} \in \mathbb{Z}$ as Φ is a root system.

$[h_{\alpha_i}, h_{\alpha_j}] = 0$.

If $e_\alpha \in L_\alpha$, $e_\beta \in L_\beta$, then $[e_\alpha, e_\beta] \in L_{\alpha+\beta}$. $[e_\alpha, e_{-\alpha}] \in L_0 = H$.

Lemma 2.11: (i) For all $\alpha, \beta \in \Phi$, $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

(ii) We can choose $e_\alpha \in L_\alpha$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$.

Proof: (i) Take $x \in L_\alpha$, $y \in L_\beta$, $h \in H$. $\mathrm{ad}\,h([x,y]) = [h,[x,y]] = [[h,x],y] + [x,[h,y]]$ - Jacobi identity.

$= \alpha(h)[x,y] + \beta(h)[x,y] = (\alpha+\beta)(h)[x,y]$.

(ii) We just have to prove that if $x \in L_\alpha$, $y \in L_{-\alpha}$, then $[x,y] \in \mathbb{C}h_\alpha \setminus \{0\}$.

Take $h \in H$. $(h, [x,y]) = ([h,x], y) = \alpha(h)(x,y) = (h_\alpha, h) . \frac{(\alpha,\alpha)}{2}(x,y) = (h, (x,y)\frac{(\alpha,\alpha)}{2} h_\alpha)$

$\Rightarrow (h, [x,y] - (x,y) . \frac{(\alpha,\alpha)}{2} h_\alpha) = 0\ \ \forall\ h \in H$.

Since $(,)$ is non-degenerate on $H \Rightarrow [x,y] = (x,y)\frac{(\alpha,\alpha)}{2} h_\alpha$

(**Exercise:** Prove that $(x,y) \neq 0$ - See Lemma 2.7)

1. $[h_{\alpha_i}, e_\alpha] = (\alpha, \frac{2\alpha_i}{(\alpha_i,\alpha_i)}) e_\alpha \in \mathbb{Z}e_\alpha$
2. $[h_{\alpha_i}, h_{\alpha_j}] = 0$
3. $[e_\alpha, e_{-\alpha}] = h_\alpha = \sum \lambda_i h_{\alpha_i}$, $\lambda_i \in \mathbb{Z}$.
4. $[e_\alpha, e_\beta] = 0$ if $\alpha + \beta \notin \Phi$.

5. $[e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}$ if $\alpha+\beta \in \Phi$. (Prove later...)

Example: Matrices: L_α , L_β , $L_{\alpha+\beta}$

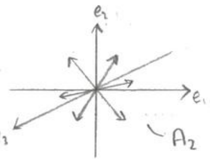
$\begin{matrix} \circ & \times & \circ \\ & \circ & \circ \\ & & \circ \end{matrix}$, $\begin{matrix} \circ & \circ & \circ \\ & \circ & * \\ & & \circ \end{matrix}$, $\begin{matrix} \circ & \circ & * \\ & \circ & \circ \\ & & \circ \end{matrix}$

Cartan subalgebra - H, diagonal. $h = (\lambda_0, .., \lambda_l)$. $[h, e_{ij}] = (\lambda_i - \lambda_j)\, e_{ij}$.

$\Phi = \{ \alpha_{ij} : H \to \lambda_i - \lambda_j : i \neq j \}$. $(\lambda_0 + .. + \lambda_l = 0)$.

$\Pi = \{ \alpha_{01}, \alpha_{12}, .., \alpha_{l-1,l} \}$.

Consider $l=2$.



$\{ \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 : \Sigma \lambda_i = 0 \}$.

The root system of sl_{l+1} is called A_l.

A root system Φ is irreducible if $\Phi = \Phi_1 \cup \Phi_2$ and $(\Phi_1, \Phi_2) \Rightarrow \Phi_1 = 0$ or $\Phi_2 = 0$.
We define length $|x| = \sqrt{(x,x)}$ and angle by $(x,y) = |x|.|y| \cos\theta$.
Knowing $\Pi \Rightarrow$ know W, as $W = \langle s_\alpha : \alpha \in W \rangle$ and $\alpha \in \Phi \Leftrightarrow \alpha = w\alpha_i$, some $w \in W$, $\alpha_i \in \Pi$.
if $\alpha, \beta \in \Pi$, $(\alpha,\beta) < 0$. Assume that Φ is continuous.

θ:	$\pi/2$	$2\pi/3$	$4\pi/5$	$5\pi/6$
	$\lvert\alpha\rvert, \lvert\beta\rvert$ independent	$\lvert\alpha\rvert = \lvert\beta\rvert$	$\sqrt{2}\lvert\alpha\rvert = \lvert\beta\rvert$ $\lvert\alpha\rvert = \sqrt{2}\lvert\beta\rvert$	$\sqrt{3}\lvert\alpha\rvert = \lvert\beta\rvert$ $\lvert\alpha\rvert = \sqrt{3}\lvert\beta\rvert$.

The Dynkin diagram has a vertex for every simple root.

θ	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
length	$\lvert\alpha\rvert, \lvert\beta\rvert$ independent	$\lvert\alpha\rvert = \lvert\beta\rvert$	$\sqrt{2}\lvert\alpha\rvert = \lvert\beta\rvert$, $\lvert\alpha\rvert = \sqrt{2}\lvert\beta\rvert$ ↓	$\sqrt{3}\lvert\alpha\rvert = \lvert\beta\rvert$, $\lvert\alpha\rvert = \sqrt{3}\lvert\beta\rvert$ ↓
notation	• •	•—•	α •⇐• β	α •⇚• β

$L = H \oplus \sum_{\alpha \in \Phi} L_\alpha$

For $\alpha \in \Phi$ define $h_\alpha = \frac{2\alpha}{(\alpha,\alpha)}$. Then,

1. $[h_{\alpha_i}, e_\alpha] = (\frac{2\alpha_i}{(\alpha_i,\alpha_i)}, \alpha)\, e_\alpha = A_{\alpha\alpha_i} e_\alpha$, where $A_{\alpha_i\alpha}$ = Cartan Integer. $(A_{\alpha_i\alpha_j})$ = Cartan matrix
2. $[h_{\alpha_i}, h_{\alpha_j}] = 0$
3. $[h_\alpha, h_{-\alpha}] = e_\alpha$ - we still have free choice of $e_\alpha : \alpha \in \Phi^+$, but then $e_{-\alpha}$ is determined.
4. $[e_\alpha, e_\beta] = 0$ if $\alpha+\beta \notin \Phi$
5. $[e_\alpha, e_\beta] = N_{\alpha,\beta}\, e_{\alpha+\beta}$ if $\alpha+\beta \in \Phi$ (Convention: $N_{\alpha,\beta} = 0$ if $\alpha+\beta \notin \Phi$).

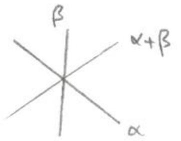
Theorem: (i) Let Φ be an irreducible root system. Then there exists a simple Lie algebra over $\mathbb{C}$ with root system Φ.

(ii) Let L and L' be simple Lie algebras over $\mathbb{C}$ with Cartan subalgebras H and H', of the same dimension. Let $\alpha_1, .., \alpha_l$ and $\alpha'_1, .., \alpha'_l$ be simple roots for L and L'. A_{ij}, A'_{ij} Cartan integers, $[e_{\alpha_i}, e_{\alpha_i}] = h_{\alpha_i}$. If $(A_{ij}) = (A'_{ij})$ then $\exists$ a unique isomorphism $\theta : L \to L'$ such that $\theta(h_{\alpha_i}) = h_{\alpha'_i}$, $\theta(e_{\alpha_i}) = e_{\alpha'_i}$, $\theta(e_{-\alpha_i}) = e_{-\alpha'_i}$.

Definition 2.15: Let $\alpha, \beta \in \Phi$. The α-string through β is the maximal sequence of roots:

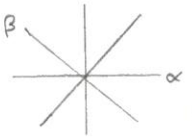
$-p\alpha+\beta, .., -\alpha+\beta, \beta, \alpha+\beta, .., q\alpha+\beta$.

A_2:



$\beta, \alpha+\beta$

B_2:



$\beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha.$

G_2: $\beta, \beta+\alpha, \beta+2\alpha, \beta+3\alpha$

Lemma 2.16: (i) Let $\alpha, \beta \in \Phi$ be linearly independent. Then the set $\{i\alpha + j\beta \in \Phi : i, j \in \mathbb{Z}\}$ forms a root system of type $A_1 \times A_1$, A_2, B_2, or G_2.

(ii) If $\beta + r\alpha \in \Phi$, $r>0$, then $\beta + i\alpha \in \Phi$ for $0 \leq i \leq r$.

(iii) $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} = A_{\alpha,\beta} = p-q$ (see definition 2.15).

(iv) α-strings have length 0, 1, 2, 3 or 4.

Proof: (i) This is easy.

(ii) $\beta + i\alpha$ is a root, and $\beta + (1+i)\alpha \notin \Phi$, and $\beta + (j-1)\alpha \notin \Phi$ (and all in between), $\beta + j\alpha \in \Phi$, $i<j$. Show that if $(\alpha,\beta)>0$ then $\alpha-\beta \in \Phi$, and if $(\alpha,\beta)<0$, then $\alpha+\beta \in \Phi$. Then $(\beta+i\alpha, \alpha) > 0$, $(\beta+j\alpha,\alpha) < 0$ – contradiction.

(iii) $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} = p-q$, where $-p\alpha+\beta, \dots, q\alpha+\beta$ is the α-string through β. Look at the image of the α-string under w_α: $\beta - p\alpha - 2\left(\frac{(\alpha, \beta-p\alpha)}{(\alpha,\alpha)}\right)\alpha, \dots, \beta+q\alpha - 2\left(\frac{(\alpha,\beta+q\alpha)}{(\alpha,\alpha)}\right)\alpha$.
Since the α-string is unique, we get $\beta+q\alpha = \beta - p\alpha - 2\frac{(\alpha,\beta-p\alpha)}{(\alpha,\alpha)}\alpha$ and $\beta+p\alpha = \beta - q\alpha - 2\frac{(\alpha,\beta+q\alpha)}{(\alpha,\alpha)}\alpha$
Thus $\beta+q\alpha = \beta - p\alpha - A_{\alpha\beta}\alpha + 2p\frac{(\alpha,\alpha)}{(\alpha,\alpha)}\alpha$. So $\beta+q\alpha = \beta+p\alpha - A_{\alpha\beta}\alpha \Rightarrow p-q = A_{\alpha\beta}$.

(iv) Have bounded $A_{\alpha\beta} \leq 4$ so, 'choose "β" at beginning of string so $p=0$'.

$[e_\alpha, e_\beta] = N_{\alpha\beta}\, e_{\alpha+\beta}$. $e_\alpha : \alpha \in \Phi^+$, $[e_\alpha, e_{-\alpha}] = h_\alpha$.

$[e_{-\alpha}, e_{-\beta}] = N_{-\alpha,-\beta}\, e_{-(\alpha+\beta)}$

Lemma 2.17: $N_{\alpha,\beta} \cdot N_{-\alpha,-\beta} = -(p+1)^2$ if $\alpha, \beta, \alpha+\beta \in \Phi^+$.

$[[e_\alpha, e_\beta], e_{-\alpha}] + [[e_\beta, e_{-\alpha}], e_\alpha] + [[e_{-\alpha}, e_\alpha], e_\beta] = 0$ (Jacobi).

$N_{\alpha\beta}[e_{\alpha+\beta}, e_{-\alpha}] + N_{\beta,-\alpha}[e_{\beta-\alpha}, e_\alpha] + [-h_\alpha, e_\beta] = 0$

So, $N_{\alpha\beta} \cdot N_{\alpha+\beta,-\alpha}\, e_\beta + N_{\beta,-\alpha} \cdot N_{\beta-\alpha,\alpha}\, e_\beta - A_{\alpha\beta}\, e_\beta = 0$.

$(N_{\alpha,\beta} N_{\alpha+\beta,-\alpha} + N_{\beta,-\alpha} N_{\beta-\alpha,\alpha} - A_{\alpha\beta})\, e_\beta = 0$ – ⊕

↳ $(\alpha+\beta) + (-\alpha) + (-\beta) = 0$.

$r+s+t=0$, $r,s,t \in \Phi$. $[[e_r, e_s], e_t] + [[e_s, e_t], e_r] + [[e_t, e_r], e_s] = 0$.

$\Rightarrow N_{r,s}[e_{-t}, e_t] + N_{s,t}[e_{-r}, e_r] + N_{t,r}[e_{-s}, e_s] = 0$

$\Rightarrow 2N_{r,s}\, t/(t,t) + 2N_{s,t}\, r/(r,r) + 2N_{t,r}\, s/(s,s) = 0$

$t+r+s=0$, $t = -r-s$

$\Rightarrow \left(\frac{N_{s,t}}{(r,r)} - \frac{N_{r,s}}{(t,t)}\right) r + \left(\frac{N_{t,r}}{(s,s)} - \frac{N_{r,s}}{(t,t)}\right) s = 0$

Since r and s are linearly independent roots (as otherwise $r = \pm s$, so $t = 0$ or $2r$ - not allowed).

Lemma 2.18: $\frac{N_{s,t}}{(r,r)} = \frac{N_{r,s}}{(t,t)} = \frac{N_{t,r}}{(s,s)}$, provided that $s+t+r=0$, $s,t,r \in \Phi$.

Return to ⊕: $N_{\alpha,\beta}\frac{N_{-\alpha,-\beta}}{(\alpha+\beta,\alpha+\beta)} \cdot (-\beta,-\beta) + \frac{N_{-\alpha,\alpha-\beta}}{(\beta,\beta)} \cdot (\alpha-\beta,\alpha-\beta) \cdot \underset{-N_{\alpha,\beta-\alpha}}{\underset{\shortparallel}{N_{\beta-\alpha,\alpha}}} - A_{\alpha\beta} = 0$

$M_{\alpha\beta} - M_{\alpha,\beta-\alpha} - A_{\alpha\beta} = 0$

Repeat with $\alpha, \beta-\alpha$ instead of α,β:

Now get $M_{\alpha,\beta-\alpha} - M_{\alpha,\beta-2\alpha} = A_{\alpha,\beta-\alpha} = A_{\alpha,\beta} - 2$ } go along the α-string.

$\vdots \qquad \hookrightarrow = \frac{2(\alpha,\beta-\alpha)}{(\alpha,\alpha)} = A_{\alpha,\beta} - 2$

$M_{\alpha,\beta-p\alpha} = A_{\alpha\beta} - 2p$

All all these up: $M_{\alpha,\beta} = (p+1)A_{\alpha\beta} - p(p+1)$

Lemma: If $\alpha, \beta, \alpha+\beta$ are roots, then $\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)} = \frac{p+1}{q}$.

Proof: $(p+1) - q\left(\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)}\right) = (p+1) - q\left(1 + A_{\beta\alpha} + \frac{(\alpha,\alpha)}{(\beta,\beta)}\right)$ $\qquad A_{\beta\alpha} = \frac{(\alpha,\alpha)}{(\beta,\beta)} A_{\alpha\beta}$.

$= A_{\alpha\beta} + 1 - q\left(\frac{(\alpha,\alpha)}{(\beta,\beta)}\right) - q\frac{(\alpha,\alpha)}{(\beta,\beta)} A_{\alpha\beta}$

$= \underset{A}{\underset{\shortparallel}{(A_{\alpha\beta}+1)}}\underset{B}{\underset{\shortparallel}{(1 - q^{(\alpha,\alpha)}/(\beta,\beta))}}$

Task: show A or B = 0.

Case (i): If $(\alpha,\alpha) \geqslant (\beta,\beta)$, then $\left|\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\right| \leq \left|\frac{2(\alpha,\beta)}{(\beta,\beta)}\right|$. Now, $\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} \lneqq 4$ (as equality occurs if $\alpha = \pm\beta$, which is not the case when $\alpha+\beta$ is also a root).

Then $\left|\frac{2(\alpha,\beta)}{(\alpha,\alpha)}\right| = 0$ or 1; so $A_{\alpha\beta} = -1$ ($\Rightarrow A = 0$), or 0 or 1.

If $A_{\alpha\beta} = 0, 1$: $(\beta,\beta) < (\beta+\alpha, \beta+\alpha) < (\beta+2\alpha, \beta+2\alpha)$ which isn't possible in 2-d root system (can't have three different root lengths).

If $\beta + 2\alpha \notin \Phi$, $q = 1$. $(\beta+\alpha,\beta+\alpha) > (\beta,\beta)$ or (α,α), so $(\beta,\beta) = (\alpha,\alpha)$, hence $B = 0$.

Case (ii): $(\alpha,\alpha) < (\beta,\beta) \Rightarrow (\alpha+\beta,\alpha+\beta) \leq (\beta,\beta)$, other three root sizes.

$\Rightarrow (\alpha,\beta) < 0 \Rightarrow (\alpha-\beta,\alpha-\beta) > (\beta,\beta) > (\alpha,\alpha)$, so $\alpha - \beta \notin \Phi$, thus $p = 0$.

(Thus $p - q = A_{\alpha\beta} \Rightarrow A_{\alpha\beta} = -q$). $\left|\frac{2(\beta,\alpha)}{(\beta,\beta)}\right| < \left|\frac{2(\beta,\alpha)}{(\alpha,\alpha)}\right|$, so $\frac{2(\beta,\alpha)}{(\beta,\beta)} = -1, 0, 1$, so must be -1.

$A_{\alpha\beta} = \frac{-2(\alpha,\beta)}{(\alpha,\alpha)} \cdot \frac{(\beta,\beta)}{2(\beta,\alpha)}$, thus $q = (\beta,\beta)/(\alpha,\alpha)$ so $B = 0$.

Now we have $M_{\alpha,\beta} = (p+1)A_{\alpha\beta} - (p+1)p$. $\qquad M_{\alpha\beta} = N_{\alpha,\beta} \cdot N_{-\alpha,-\beta} \cdot \frac{(\beta,\beta)}{(\alpha+\beta,\alpha+\beta)}$

$\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)} = \frac{p+1}{q}$, hence $N_{\alpha,\beta} N_{-\alpha,-\beta} = (p+1)^2/q \cdot (p-q) - p\frac{(p+1)^2}{q} = -(p+1)^2$

This finishes the proof of lemma 2.17.

$[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$, $[e_{-\alpha}, e_{-\beta}] = N_{-\alpha,-\beta} e_{-\alpha-\beta}$.

Want θ, a Lie algebra automorphism, such that $\theta(e_\alpha) = -e_{-\alpha}$. Then $\theta([e_\alpha, e_\beta]) = \theta(N_{\alpha\beta} e_{\alpha+\beta})$

$\overset{\shortparallel}{[e_{-\alpha}, e_{-\beta}]} = N_{-\alpha,-\beta} e_{-\alpha-\beta}$.

Hence this forces $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. Combining this with lemma 2.17 we have that $N_{\alpha,\beta} = \pm(p+1)$ and $N_{-\alpha,-\beta} = \mp(p+1)$ - integers.

Isomorphism Theorem: $\quad L \qquad L'$

$H \qquad H'$

$\Pi \subset \Phi \qquad \Phi' \supset -\Pi$

If $(A_{\alpha\beta}) = (A'_{\alpha\beta})$ and we choose generators for the $L_\alpha : \alpha \in \Pi$, (e_α) and (e_α'), and $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $[e'_\alpha, e'_{-\alpha}] = h'_\alpha$, then there exists a unique isomorphism $\theta : L \xrightarrow{\sim} L'$, $h_\alpha \mapsto h_{\alpha'}$, $e_\alpha \mapsto e_\alpha'$, $e_{-\alpha} \mapsto e'_{-\alpha}$.

In particular, if $L = L'$, and we choose a different set of simple roots, get an automorphism.

$w \in W$, $\Pi' = w\Pi$. Choose e_α' such that $[e_{\alpha'}, e_{-\alpha'}] = h_{\alpha'}$, $\alpha' \in \Pi'$.

Then we are guaranteed a unique Lie algebra automorphism: $h_\alpha \mapsto h_{\alpha'}$, $e_\alpha \mapsto e'_\alpha$, $e_{-\alpha} \mapsto e'_{-\alpha}$, $\alpha \in \Pi$.

Now choose $\Pi' = -\Pi$. If we choose $e_{\alpha'} = e_{-\alpha}$, $e_{-\alpha'}$ is forced, and $[e_{\alpha'}, e_{-\alpha'}] = h_{\alpha'} = -h_\alpha$, so $e_{-\alpha'} = -e_\alpha$.

Note that $\theta^2 = \mathrm{Id.}$, by the uniqueness part of the isomorphism theorem.

Need to check $\alpha \in \Phi \setminus \Pi$. Take $e_\alpha \in L_\alpha \setminus \{0\}$. $[e_\alpha, e_\beta] = \lambda e_{\alpha+\beta}$, so can determine $\theta(e_{\alpha+\beta})$ from $\theta(e_\alpha), \theta(e_\beta)$ as θ is a Lie automophism.

Want to show $e_\alpha = [\cdot\cdot[[e_{\alpha_{i_1}}, e_{\alpha_{i_2}}], e_{\alpha_{i_3}}], \cdots, e_{\alpha_{i_k}}]$ where $\alpha_{i_j} \in \Pi$, ie, we can write $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_k}$ where $\alpha_{i_1} + \cdots + \alpha_{i_j} \in \Phi$ $\forall_j \leq k$.

Lemma: $\alpha \in \Phi^+$ can be expressed as $\alpha = \alpha_{i_1} + \cdots + \alpha_{i_k}$, $\alpha_{i_j} \in \Pi$, such that $\alpha_{i_1} + \cdots + \alpha_{i_j} \in \Phi$, $\forall_j \leq k$.

Proof: induction on $ht(\alpha) = k$, $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$, $(\alpha,\alpha) = \sum n_i(\alpha, \alpha_i) > 0$, $n_i \geq 0$. $\Rightarrow \exists i$, $(\alpha, \alpha_i) > 0$.

If $\alpha \notin \Pi$ then α, α_i are linearly independent. $2(\alpha,\alpha_i)/(\alpha_i,\alpha_i) = p - q > 0$.

So $p > 0$ and hence $\alpha - \alpha_i \in \Phi$. $ht(\alpha - \alpha_i) = k-1$. Use induction.

$\theta(e_\alpha) = [\cdot\cdot[-e_{-\alpha_{i_1}}, -e_{-\alpha_{i_2}}], -e_{-\alpha_{i_3}}]\cdots, -e_{-\alpha_{i_k}}] = d e_{-\alpha}$ for some $d \neq 0$.

$\theta(e_{-\alpha}) = d^{-1}\theta^2(e_\alpha) = d^{-1} e_\alpha$.

$\theta(\lambda e_\alpha) = \lambda d e_{-\alpha}$, $\lambda^2 d(\lambda^{-1} e_\alpha)$.

$\exists \lambda \in \mathbb{C}$ such that $\lambda^2 = -1/d$.

Now choose $e_\alpha' = \lambda e_\alpha$ and $e_{-\alpha}' = \lambda^{-1} e_\alpha$ and then $\theta(e_\alpha) = -e_{-\alpha}$.

And this is a Chevalley basis.

Theorem: Chevalley Basis Theorem. - for simple Lie algebras.

Let L be a simple Lie algebra over $\mathbb{C}$ and $L = H \oplus \sum_{\alpha \in \Phi} L_\alpha$ be the Cartan decomposition. Let $h_\alpha \in H$ be the co-roots corresponding to α. Then for each $\alpha \in \Phi$ an element e_α can be chosen in L_α such that $\{h_{\alpha_i} : \alpha_i \in \Pi, e_\alpha : \alpha \in \Phi\}$ is a basis for L, and:

1. $[h_{\alpha_i}, e_\alpha] = A_{\alpha_i \alpha} e_\alpha$
2. $[h_{\alpha_i}, h_{\alpha_j}] = 0$
3. $[e_\alpha, e_{-\alpha}] = h_\alpha$.
4. $[e_\alpha, e_\beta] = 0$ if $\alpha + \beta \notin \Phi$.
5. $[e_\alpha, e_\beta] = \pm(p+1) e_{\alpha+\beta}$ if $\alpha+\beta \in \Phi$, where p is the greatest positive integer such that $\beta - p\alpha \in \Phi$.

Such a basis is called a Chevalley basis.

Proof: $N_{\alpha,\beta} N_{-\alpha,-\beta} = -(p+1)^2$

Then we produced a Lie algebra automorphism θ such that if $e_{\alpha_i} \in L_{\alpha_i}$ and $\alpha_i \in \Pi$ then $\theta(e_{\alpha_i}) = -e_{-\alpha_i}$. $[e_{\alpha_i}, e_{-\alpha_i}] = h_{\alpha_i}$

Then we made a guess: $\underset{L_\alpha}{\underset{\Cap}{\tilde{e}_\alpha}} = [\cdots[e_{\alpha_{i_1}}, e_{\alpha_{i_2}}]\cdots, e_{\alpha_{i_j}}]$

$[\tilde{e}_\alpha, \tilde{e}_{-\alpha}] = h_\alpha$. $\tilde{e}_{-\alpha} = \lambda[\cdot\cdot[e_{-\alpha_{i_1}}, e_{-\alpha_{i_2}}]\cdots, e_{-\alpha_{i_j}}] \in L_{-\alpha}$.

But $\theta(e_\alpha) = e_{-\alpha}$, so choose $\mu \in \mathbb{C}$ such that $\mu^2 = -\lambda^{-1}$ and set $e_\alpha = \mu\tilde{e}_\alpha$.

$\theta(e_\alpha) = \mu\theta(\tilde{e}_\alpha) = \mu\lambda^{-1}\tilde{e}_{-\alpha} = -\mu^{-1}\tilde{e}_{-\alpha} = -e_{-\alpha}$., $(e_{-\alpha} = \mu^{-1}\tilde{e}_{-\alpha})$.

Question: does λ have to be -1 anyway?

$$\left.\begin{array}{l} \theta[e_\alpha, e_\beta] = \theta(N_{\alpha,\beta} e_{\alpha+\beta}) = -N_{\alpha,\beta} e_{-\alpha-\beta} \\ \overset{\shortparallel}{[-e_{-\alpha}, -e_\beta]} = N_{-\alpha,-\beta}, e_{-\alpha-\beta} \end{array}\right\} \Rightarrow N_{-\alpha,-\beta} = -N_{\alpha,\beta} \Rightarrow N_{\alpha,\beta} = \pm(p+1).$$

Define $L_{\mathbb{Z}} = \sum_{\alpha_i \in \Pi} \mathbb{Z} h_{\alpha_i} \oplus \sum_{\alpha \in \Phi} \mathbb{Z} e_\alpha$, then $L_{\mathbb{Z}}$ is a Lie algebra over $\mathbb{Z}$.

For any field k, we can define $L_k := L_{\mathbb{Z}} \otimes k$.

Exercise: Show that $sl(l+1, k)$, whenever char $k \mid l+1$, then this has a one-dimensional centre.

3. Chevalley Groups.

Lemma 3.1: Let L be a finite dimensional Lie algebra over a field of characteristic 0. Let δ be a derivation (ie, δ is a linear map and $\delta[x,y] = [\delta x, y] + [x, \delta y]$) which is nilpotent (ie, $\delta^n = 0$).

Then $exp(\delta) = 1 + \delta + \frac{\delta^2}{2!} + \dots + \frac{\delta^{n-1}}{(n-1)!}$ is an automorphism of L.

Proof: $exp(\delta)$ is linear. It has inverse $exp(-\delta)$, so the map is non-singular. What is $exp(\delta)[x,y]$?

Leibnitz rule: $\delta^r[x,y] = \sum_{i=0}^{r} \binom{r}{i} [\delta^i(x), \delta^{r-i}(y)]$, and so $\frac{1}{r!}\delta^r[x,y] = \sum_{i+j=r} [\frac{\delta^i(x)}{i!}, \frac{\delta^j(y)}{j!}]$.

Then, $exp\,\delta[x,y] = \sum_{r\geq 0} \frac{\delta^r}{r!}[x,y] = \sum_{i\geq 0}\sum_{j\geq 0} [\frac{\delta^i(x)}{i!}, \frac{\delta^j(y)}{j!}] = [exp\,\delta(x), exp\,\delta(y)]$.

If $x \in L$, then $adx(y) = [xy]$ is a derivation (via Jacobi identity). Define the following:

Definition 3.2: The inner automorphisms are defined by $Inn(L) = \{ exp(adx) : adx \text{ nilpotent} \}$.

These automorphisms are actually conjugation by elements in the universal enveloping algebra.

Claim: ade_α is nilpotent.

Proof: $ade_\alpha(e_\alpha) = 0$, so $ade_\alpha(L_\alpha) = 0$. $ade_\alpha H \subset L_\alpha \Rightarrow (ade_\alpha)^2 H = 0$.

$ade_\alpha L_{-\alpha} \subset H$; so $(ade_\alpha)^3 L_{-\alpha} = 0$. α, β linearly independent : $(ade_\alpha)^{q+1} L_\beta = 0$.

Hence, $t \in \mathbb{C}$ then $ad\,te_\alpha$ is a nilpotent derivation.

Definition 3.3: Let L be a semisimple Lie algebra with Cartan decomposition $L = H \oplus \sum_{\alpha\in\Phi} L_\alpha$, and Chevalley basis $\{h_{\alpha_i} : \alpha_i \in \Pi, e_\alpha : e_\alpha \in \Phi\}$ We define, for $t \in \mathbb{C}$ and $\alpha \in \Phi$, $x_\alpha(t) = exp\,tade_\alpha$.

The Chevalley Group over $\mathbb{C}$ (corresponding to L) is $\langle x_\alpha(t) : \alpha \in \Phi, t \in \mathbb{C} \rangle$.

Recall the following relations:
1. $[h_{\alpha_i}, e_\alpha] = A_{\alpha_i\alpha} e_\alpha$
2. $[h_{\alpha_i}, h_{\alpha_j}] = 0$
3. $[e_\alpha, e_{-\alpha}]\, h_\alpha$
4. $[e_\alpha, e_\beta] = 0$ if $\alpha + \beta \notin \Phi$
5. $[e_\alpha, e_\beta] = \pm(p+1) e_{\alpha+\beta}$ if $\alpha+\beta \in \Phi$.

$L = H \oplus \sum_{\alpha\in\Phi} L_\alpha$.

(i) $x_\alpha(t) e_\alpha = \sum_0^\infty \frac{ad(te_\alpha)^n}{n!} e_\alpha = 1 . e_\alpha + 0 \cdots$

(ii) $x_\alpha(t) h_{\alpha_i} = h_{\alpha_i} + (-A_{\alpha_i\alpha} t e_\alpha)$

(iii) $x_\alpha(t) e_{-\alpha} = (1 + tode_\alpha + \frac{t^2}{2} ade_\alpha^2) e_{-\alpha} = e_{-\alpha} + th_\alpha - \frac{t^2}{2} . 2e_\alpha = e_{-\alpha} + th_\alpha - t^2 e_\alpha$.

(iv) α, β linearly independent $\Rightarrow$

$$x_\alpha(t)(e_\beta) = e_\beta + N_{\alpha\beta} t e_{\alpha+\beta} + \frac{t^2}{2!} N_{\alpha,\alpha+\beta} N_{\alpha,\beta} e_{2\alpha+\beta} + \dots + \frac{t^q}{q!} N_{\alpha,\beta} N_{\alpha,\alpha+\beta} \cdots N_{\alpha,(q-1)\alpha+\beta} . e_{q\alpha+\beta}$$
$$= \sum_{i=0}^{q} M_{\alpha,\beta,i} t^i e_{i\alpha+\beta}$$

Why is $M_{\alpha,\beta,i}$ an integer? Up to sign, it is $\frac{(p+1)\cdots(p+i)}{i!} = \binom{p+i}{i} \in \mathbb{Z}$. Hence $x_\alpha(t)$ transform the Chevalley basis into a linear combination of the basis where the coefficients are non-negative integral powers of t with rational integer coefficients

Let us define a map $x_\alpha(t)$ of L_K for each $t \in K$, $\alpha \in \Phi$ (where K is now an arbitrary field) by the conditions (i) - (iv).

Proposition 3.4: $x_\alpha(t)$ is an automorphism of L_K.

Proof: $x_\alpha(t)$ is a linear homomorphism by definition, with inverse $x_\alpha(-t)$.

If we have an identity $w[x_{\alpha_{i_1}}(t_1) \cdots x_{\alpha_{i_j}}(t_j)] = 1$ for all $t_i \in \mathbb{C}$, eg $x_\alpha(t)x_\alpha(-t) = 1$. Since the coefficients are given by the way this acts on a basis of polynomials in $\mathbb{Z}[t_1, \dots, t_n]$. These polynomials are zero $\forall\, t_i \in \mathbb{C}$, hence they must be identically zero $\Rightarrow$ the identity can be transfered to any field k.

$x_\alpha(t)[u, u_2] = [x_\alpha(t)u_1, x_\alpha(t)u_2]$ - compare coefficient having written out u_i as an expansion in terms of the basis.

Definition 3.5: Let K be a field and L a Lie algebra of type $X \in \{A_l, B_l, C_l, D_l, E_{6,7,8}, F_4, G_2\}$. Then we define the adjoint Chevalley group of type X over K to be $X(K) = \langle x_\alpha(t) : t \in K, \alpha \in \Phi \rangle \leq \mathrm{Aut}(L_K)$. (This group is independent of the choice of Chevalley basis).

If $\varphi: L \to M_n(\mathbb{C})$ is a representation of L, there is a way to construct a Chevalley group corresponding to φ. What we've done is to take the representation given by the adjoint map. $\mathbb{C}^n$ - the idea is to construct a lattice analogous to $L_\mathbb{Z}$ invariant under $\exp(\varphi(e_\alpha))$ or $\frac{1}{m!}\varphi(x_\alpha)^m$.

Definition 3.6: Let v be a finite dimensional L-module. A lattice in V ($\mathbb{Z}$-span of some basis) is called admissable if $\varphi(e_\alpha)^m/m! \cdot M \subset M$ $\forall\, m$ and $\alpha \in \Phi$.

Proposition 3.7: Every finite dimensional L-module has an admissable lattice. The corresponding Chevalley group $\langle \exp t\varphi(e_\alpha) : t \in K, \alpha \in \Phi \rangle \leq \mathrm{Aut}(V_K)$, $V_K = M \otimes K$.

$A_1(K) \cong PSL_2(K)$:

Lemma: Let L be a simple Lie algebra over $\mathbb{C}$ with a representation $\varphi: L \to (M_n(\mathbb{C}), xy - yx)$. If $\varphi(y)$ is a nilpotent matrix, then $\mathrm{ad}\,y$ is a nilpotent derivation of L, and $\varphi(\exp(y).x) = \exp(\varphi(y))\,\varphi(x)\,\exp(\varphi(y))^{-1}$.

Proof: $\varphi\left(\frac{(\mathrm{ad}\,y)^k . x}{k!}\right) = \sum_{i+j=k} \frac{\varphi(y)^i}{i!} . x . \frac{(-\varphi(y))^j}{j!}$. $\varphi(y)$ nilpotent, so $\frac{\varphi(\mathrm{ad}\,y)^k}{k!} = 0$ for large enough k.

$\Rightarrow$ is a nilpotent derivation.

$\varphi(\exp(\mathrm{ad}\,y)x) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\varphi(y)^i}{i!} . \varphi(x) . \frac{\varphi(-y)^j}{j!} = \exp\varphi(y) . \varphi(x)(\exp\varphi(y))^{-1}$.

Lemma: If K is an arbitrary field then $SL_2(K)$ is generated by $\begin{pmatrix}1&t\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\t&1\end{pmatrix}$ for $t \in K$.

Proof: $\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix} = \begin{pmatrix}1&(\alpha-1)\gamma^{-1}\\0&1\end{pmatrix}\begin{pmatrix}1&0\\\gamma&1\end{pmatrix}\begin{pmatrix}1&(\delta-1)\gamma^{-1}\\0&1\end{pmatrix}$, $\gamma \neq 0$.

$= \begin{pmatrix}1&0\\(\delta-1)\beta^{-1}&1\end{pmatrix}\begin{pmatrix}1&\beta\\0&1\end{pmatrix}\begin{pmatrix}1&0\\(\alpha-1)\beta^{-1}&1\end{pmatrix}$ $\beta \neq 0$.

$= \begin{pmatrix}1&0\\\alpha^{-1}-1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\\alpha-1&1\end{pmatrix}\begin{pmatrix}1&-\alpha^{-1}\\0&1\end{pmatrix}$, $\beta = \gamma = 0$, $\delta = \alpha^{-1}$.

$L(A_1) = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} : \text{trace} = 0\right\}$ - $sl_2(K)$.

Chevalley basis is given by $h_\alpha = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $e_\alpha = \begin{pmatrix}0&1\\0&0\end{pmatrix}$, $e_{-\alpha} = \begin{pmatrix}0&0\\1&0\end{pmatrix}$

$A_1(K) \cong PSL_2(K)$. Take $a \in L$.

$x_\alpha(t).a = \exp(te_\alpha).a.\exp(te_\alpha)^{-1} = \begin{pmatrix}1&t\\0&1\end{pmatrix}.a.\begin{pmatrix}1&-t\\0&1\end{pmatrix}$

So define $\Psi: SL_2(K) \to A_1(K)$ by sending $m \mapsto (a \mapsto mam^{-1})$ = product of $x_\alpha(t)$ and $x_{-\alpha}(t)$.

$\|$ product of $\begin{pmatrix}1&t\\0&1\end{pmatrix}$ and $\begin{pmatrix}1&0\\t&1\end{pmatrix}$.

The map is surjective because $A_1(K) := \langle \underset{\psi\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)}{\underset{\shortparallel}{x_\alpha(t)}}, \underset{\psi\left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right)}{\underset{\shortparallel}{x_{-\alpha}(t)}} \rangle$

Kernel $=$ $m \in SL_2(K)$ such that $ma = am$ for all $a \in SL_2(K)$, $= \{\pm I\}$. Hence $A_1(K) \cong PSL_2(K)$

In general, to identify what $X(K)$ is, find a matrix representation of $L(X)$ and then look at the group $\bar{G}$ generated by $\exp(te_\alpha)$. $1 \to \underset{\text{centre of } G.}{\underset{\shortparallel}{Z}} \to \bar{G} \to X(K) \to 1$

Proposition 3.9: (i) $A_l(K) \cong PSL_{l+1}(K)$

(ii) $B_l(K)$. $O_n(K,f) = \{T \in GL_n(K) : (x,y)_f = (Tx,Ty)_f$, where $(x,y) = \frac{1}{2}(f(x+y) - f(x) - f(y))\}$.

$\Omega_n(K,f) = O_n(K,f)' = [O_n, O_n]$

$P\Omega_n(K,f) = \Omega / Z \cap \Omega$

$B_l(K) : P\Omega_{2l+1}(K, f_B)$, $f_B = x_0^2 + x_1 x_{-1} + \cdots + x_l x_{-l}$.

(iii) $C_l(K) = PSL_{2l}(K)$

(iv) $D_l(K) = P\Omega_{2l}(K, f_D)$, $f_D = x_1 x_{-1} + \cdots + x_l x_{-l}$.

Proof: See Carter - "Simple groups of Lie type" for details.

Definition 3.10: Set $X_\alpha = \{x_\alpha(t) : t \in K\} \leq X(K)$. Eg. $A_1(K)$: $X_\alpha = \begin{pmatrix}1&t\\0&1\end{pmatrix}$, $X_{-\alpha} = \begin{pmatrix}1&0\\t&1\end{pmatrix}$.

$x_\alpha(t_1) x_\alpha(t_2) = \exp(t_1 \operatorname{ad} e_\alpha) . \exp(t_2 \operatorname{ad} e_\alpha) = \exp((t_1+t_2) \operatorname{ad} e_\alpha) = x_\alpha(t_1+t_2)$, because $\operatorname{ad} e_\alpha$ "commutes with itself". Thus we have an isomorphism of additive groups, $(K,+) \to X_\alpha(t)$

$t \longmapsto x_\alpha(t)$.

Understand a presentation for $X(K)$: $X(K) = \langle x_\alpha(t) : \alpha \in \Phi, t \in K \rangle$.

Set $U = \langle X_\alpha : \alpha \in \Phi^+ \rangle$.

Lemma: U operates unipotently on L_K

Proof: For $\alpha = n_1\alpha_1 + \cdots + n_l\alpha_l$, have $h(\alpha) = \sum_{i=1}^{l} n_i$. Let $L_i = \sum_{h(\alpha)=i} L_\alpha$. Then $L_K = \bigoplus_i L_i$.

If $x \in L_i$, then $x_\alpha(t)$ for $\alpha \in \Phi^+$ satisfies $x_\alpha(t)x - x \in \sum_{j>i} L_j$ - check the way that $x_\alpha(t)$ acts on Chevalley basis.

Take $u \in U$, u is a product of $x_\alpha(t)$ $\Rightarrow$ u is unipotent.

L a simple L-algebra of type $X \in \{A_l, B_l, C_l, D_l, E_{6,7,8}, F_4, G_2\}$. $X(K) = \{x_\alpha(t) : \alpha \in \Phi, t \in K\}$, where $x_\alpha(t) = \exp t(\operatorname{ad} e_\alpha)$ for $t \in \mathbb{C}$.

$x_\alpha(t) \cdot e_\alpha = e_\alpha$.

$x_\alpha(t) \cdot h_\beta = h_\beta - A_{\beta\alpha} t e_\alpha$

$x_\alpha(t) \cdot e_{-\alpha} = e_{-\alpha} + t h_\alpha - t^2 e_\alpha$.

$x_\alpha(t) \cdot e_\beta = \sum_{i=0}^{q} M_{\alpha,\beta,i} t^i e_{i\alpha+\beta}$.

$A_1(K) = PSL_2(K)$. $A_l \sim PSL_{l+1}$. $B_l \sim$ orthogonal, $2l+1$

$C_l \sim$ symplectic group $2l$. $D_l \sim$ orthogonal group, $2l$.

$C_l \longleftrightarrow \{T \in M_{2l}(K) : T'A + AT = 0\} = L(C_l)$, $A = \begin{pmatrix}0 & I_l\\ -I_l & 0\end{pmatrix}$.

$x_\alpha(t)$ acts on $x \in L(C_l)$ by conjugation. $\exp(t \operatorname{ad} e_\alpha) = \exp(te_\alpha).x.\exp(te_\alpha)^{-1}$.

$M_{2l}(K) \geq \bar{G} = \langle \exp(te_\alpha) : t \in K, \alpha \in \Phi \rangle \to C_l(K)$

$\exp(te_\alpha) \longmapsto x_\alpha(t)$.

Exercise: (i) The kernel of this homomorphism σ is the centre of $\bar{G}$.

(ii) If T is nilpotent and $T \in L(C_l)$ then $(\exp T)' A \exp(T) = A$ (ie, $\bar{G}$ is a subgroup of $Sp_{2l}(K)$).

$X_\alpha(t) = \{x_\alpha(t) : t \in K\}$. $U = \langle X_\alpha(t) : \alpha \in \Phi^+ \rangle$ - unipotent subgroup.

What is the lower central series of this?

$U_i = \langle X_\alpha(t) : \alpha \in \Phi^+, h(\alpha) = i \rangle$.

sl_{l+1}: what is U_i in this case? $i \to \begin{pmatrix} 1 & 0 & \cdots & \overset{j}{\ast} \\ & \ddots & & 1 \; 0 \\ & & & 0 \\ 0 & & & 1 \end{pmatrix}$. $U = \begin{pmatrix} 1 & \ast \\ 0 & 1 \end{pmatrix}$, $U_i = \begin{pmatrix} 1 & 0 \cdots 0 & \ast \\ & \ddots & \vdots \\ 0 & & 1 \end{pmatrix}$.

$x_\alpha(t) : \alpha = e_i - e_j = (e_i - e_{i+1}) + \dots + (e_{j-1} - e_j)$.

$h_t(\alpha) = j - i$

$\alpha, \beta \in \Phi$. What is $x_\alpha(t) x_\beta(t) x_\alpha(t)^{-1}$

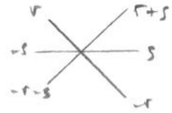
$x_\alpha(t) x_\beta(u) x_\alpha(t)^{-1} = x_\alpha(t) \exp(\mathrm{ad}\, u e_\beta) x_\alpha(t)^{-1}$.

Lemma: $\mathrm{ad}\, y$ nilpotent and ϑ an automorphism of L $\Rightarrow$ $\vartheta \exp \mathrm{ad}\, y \, \vartheta^{-1} = \exp(\mathrm{ad}\, \vartheta y)$.

Proof: $LHS(x) = \theta \sum \frac{1}{i!}[y, \dots, y, \vartheta^{-1} x] = \sum \frac{1}{i!}[\vartheta y, \dots, \vartheta y, x] = RHS(x)$.

$$\exp[\mathrm{ad}(x_\alpha(t) u e_\beta)] = \exp \mathrm{ad}\left(\sum_{i=0}^{q} M_{\alpha,\beta,i} t^i u e_{i\alpha+\beta}\right) \quad (*)$$
$$= \exp\left(\sum_{i=0}^{q} M_{\alpha,\beta,i} t^i u \, \mathrm{ad}\, e_{i\alpha+\beta}\right)$$

If $\alpha + \beta \notin \Phi$ then $x_\alpha(t) x_\beta(u) x_\alpha(t)^{-1} = x_\beta(u)$



If $\alpha + \beta \in \Phi$ then we know $\langle i\alpha + j\beta \in \Phi, i, j \in \mathbb{Z} \rangle \cong A_2, B_2, G_2$.

Suppose $\alpha + 2\beta$, $(3\alpha + 2\beta) \notin \Phi$. This implies if $i\alpha + j\beta \in \Phi$ and $i, j > 0$ then $j = 1$.

$\Rightarrow \{\mathrm{ad}\, e_{i\alpha+\beta} : i > 0\}$ commute.

If $\alpha + \beta \notin \Phi$, $\mathrm{ad}\, e_\alpha \, \mathrm{ad}\, e_\beta - \mathrm{ad}\, e_\beta \, \mathrm{ad}\, e_\alpha = \mathrm{ad}[e_\alpha, e_\beta] = 0$.

$\Rightarrow x_\alpha(t) x_\beta(u) x_\alpha(t)^{-1} = \prod_{i=0} x_{i\alpha+\beta}(M_{\alpha,\beta,i} t^i u)$

(ii) Suppose $\alpha + 2\beta \in \Phi$, and $\{i\alpha + j\beta \in \Phi\} \cong B_2$. (check this means $2\alpha + \beta$, $\alpha + 3\beta \notin \Phi$).

$$x_\alpha(t) x_\beta(u) x_\alpha(t)^{-1} = \exp(\underset{\xi}{\underset{\shortparallel}{u\,\mathrm{ad}\, e_\beta}} + \underset{\eta.}{\underset{\shortparallel}{N_{\alpha,\beta}}} \, tu \, \mathrm{ad}\, e_{\alpha+\beta})$$

$$[\xi, \eta] = \xi\eta - \eta\xi = N_{\alpha,\beta} N_{\beta,\alpha+\beta} tu^2 \, \mathrm{ad}\, e_{\alpha+2\beta} \quad \text{commutes with } \xi, \eta.$$
$$= \exp \xi . \exp \eta . \exp(-\tfrac{1}{2}[\xi, \eta]) = x_\beta(u) x_{\alpha+\beta}(N_{\alpha,\beta} tu) x_{\alpha+2\beta}(\overbrace{N_{\alpha,\beta} N_{\beta,\alpha+\beta}}^{M_{\alpha\beta,2}} tu^2)$$

Unipotent subgroup: $\langle X_\alpha(t) : \alpha \in \Phi^+ \rangle$.

$H \oplus L_1 \oplus \dots \oplus L_n$ where $L_i = \langle e_\alpha : ht(\alpha) = i \rangle$.

$V_i = \{x \in U : x.e_\alpha = e_\alpha \mod L_i \oplus \dots \oplus L_n\}$

$\cup\!\!|$ $\qquad e_\alpha \in L_+$

$U_i = \{x_\alpha(t) : ht(\alpha) = i\}$

$x_\alpha(t) \underset{\displaystyle \exp(\sum M_{\alpha,\beta,i} t^i u \, \mathrm{ad}\, e_{i\alpha+\beta})}{\underset{\shortparallel}{x_\beta(\alpha)}} x_\alpha(t)^{-1} = x_\beta(u) \mod U_{i+1}$. $ht(\beta) = i$.

$= \prod \exp(M_{\alpha,\beta,i} t^i u \, \mathrm{ad}\, e_{i\alpha+\beta})$ if they commute.

Now if $\alpha + 2\beta \notin \Phi$ (eg A_2), or $\alpha + 2\beta \in \Phi$ (eg. B_2)

then $x_\alpha(t) x_\beta(u) x_\alpha(t)^{-1} = \exp(u\,\mathrm{ad}\, e_\beta + N_{\alpha,\beta} tu e_{\alpha+\beta})$. $[\varphi, \psi]$ commutes with φ, ψ.

$[\varphi, \psi] = M_{\alpha,\beta,2} tu^2 \, \mathrm{ad}\, e_{2\alpha+\beta}$. $\hookrightarrow \underbrace{x_\beta(u) x_{\alpha+\beta}}_{\varphi}(N_{\alpha,\beta} tu) \underbrace{x_{\alpha+2\beta}}_{\psi}(M_{\alpha,\beta,2} tu^2)$.

Lemma: $\varphi, \psi : V \to V$ and φ, ψ, $[\varphi,\psi] = \varphi\psi - \psi\varphi$ are nilpotent. $[\varphi,\psi]$ commutes with φ, ψ.
Then $\varphi + \psi$ is nilpotent, and $\exp(\varphi+\psi) = \exp(\varphi)\exp(\psi)\exp(-\frac{1}{2}[\varphi,\psi])$.

Proof: $\frac{(\varphi+\psi)^n}{n!} = \sum_{i,j,k} \frac{\varphi^i}{i!} \cdot \frac{\psi^j}{j!} \cdot \frac{[\varphi,\psi]^k}{2^k \cdot k!} (-1)^k$

$\frac{1}{2}(\varphi+\psi)^2 = \frac{1}{2}\varphi^2 + \frac{1}{2}\psi^2 + \varphi\psi - \frac{1}{2}[\varphi,\psi]$

Induction: $\psi\varphi^i = \varphi^i\psi - i\varphi^{i-1}[\varphi,\psi]$. It then all works.

Baker-Campbell-Hausdorff formula: $G \underset{\exp}{\overset{\log}{\rightleftarrows}} L$.

$x, y \in L$: $\exp(x)\exp(y) = ?$

$\exp X \exp Y = \sum_{\alpha = (i_1, j_1, \dots, j_k)} i_\alpha X^{i_1} Y^{j_1} \dots X^{i_k} Y^{j_k} = \sum d_\alpha [X, \dots, X, Y, \dots, Y, X, \dots, Y]$.

$\exp(x)\exp(y) = \exp(x + y + \frac{1}{2}[x,y] + \frac{1}{12}[x,y,y] - \frac{1}{12}[x,x,y])$

The cases for G_2: $2\alpha + 3\beta \in \Phi$, $3\alpha + 2\beta \in \Phi$. Case analysis is more tedious.

Theorem 3.12 (Chevalley's Commutator Formula): Let α, β be linearly independent, and $t, u \in k$.

$[x_\beta(u), x_\alpha(t)] = x_\beta(u)^{-1} x_\alpha(t)^{-1} x_\beta(u) x_\alpha(t)$
$= \prod_{i,j>0} x_{i\alpha + j\beta}(c_{i,j,\alpha,\beta}(-t^i u^j))$ arranged in order of increasing $i+j$.

We could order this wrt $\alpha < \beta$ which defined Φ^+ (ie, $\beta - \alpha \in \Phi^+$)

$C_{i,1,\alpha,\beta} = M_{\alpha,\beta,i}$
$C_{1,j,\alpha,\beta} = (-1)^j M_{\beta,\alpha,j}$
$C_{3,2,\alpha,\beta} = \frac{1}{3} M_{\alpha+\beta,\alpha,2}$
$C_{2,3,\alpha,\beta} = -\frac{2}{3} M_{\alpha+\beta,\beta,2}$. ($C_{i,j,\alpha,\beta}$ takes values $\pm 1, \pm 2, \pm 3$)

Theorem 3.13: (i) U is nilpotent and $U = U_1 \geq U_2 \geq \dots \geq U_n = 1$ is a central series.
(is this the lower central series?)
(ii) If $u \in U$ then $u = \prod_{\alpha \in \Phi^+} x_\alpha(t_\alpha)$, where the expression is unique (ordered wrt height).
Similar for $V = \langle x_\alpha(t) : \alpha \in \Phi^-, t \in k \rangle$

In fact, linearly independent roots generate a nilpotent group.

Uniqueness: descending induction on n.

$u = \prod_{ht(\alpha) \geq m} x_\alpha(t_\alpha) = \prod_{ht(\alpha) \geq m} x_\alpha(t_\alpha')$

$u \cdot e_\beta$. $ht(\beta) = m$. if $ht(\alpha) > m$ and $-\beta + \alpha \in \Phi$ then $-\beta + \alpha \in \Phi^+$, as $ht > 0$.
If $ht(\alpha) = m$, then $-\beta + \alpha \notin \Phi$ because $ht \neq 0$.
$u \cdot e_{-\beta} = e_{-\beta} + t_\alpha h_\beta + x$, where $x \in \sum_{\alpha \in \Phi^+} L_\alpha > 0$.
$= e_{-\beta} + t_{\alpha'} h_\beta + x'$
$\Rightarrow t_\alpha = t_{\alpha'}$, identifying coefficients. Is this okay?

$h_\beta = n_1 h_{\alpha_1} + \dots + n_l h_{\alpha_l}$, $n_i \in \mathbb{Z}$. We might have $h_\beta = 0$ in char p, ie $p \mid n_i$ $\forall i$.
Now, $\beta = w(\alpha_i)$, $w \in$ Weyl group.
So $w(h_{\alpha_i}) = h_\beta$. Represent w wrt basis given by $h_{\alpha_1}, \dots, h_{\alpha_l}$, by a matrix, with coefficients in $\mathbb{Z}$.

i^{th} row $\rightarrow \begin{pmatrix} & * & \\ n_1 & \cdots & n_l \\ & * & \end{pmatrix}$. $p \mid n_i$ $i = 1, \dots, l$ $\Rightarrow p \mid \det$. But $w^2 = 1$, so $\det = \pm 1$ #

$\langle x_\alpha(t), x_{-\alpha}(t) \rangle \cong PSL_2(k)$.

We know $X(A_1) \cong PSL_2(k)$.

$\langle e_\alpha, h_\alpha, e_{-\alpha} \rangle = sl_2$ (see Lie Algebras)

$x_\alpha(t) \cdot e_\alpha = e_\alpha$

$\cdot\, e_{-\alpha} \mapsto e_{-\alpha} + th_\alpha + t^2 e_\alpha$

$\cdot\, h_\beta \mapsto h_\beta - A_{\beta,\alpha} t e_\alpha$

$\cdot\, e_\beta \mapsto \Sigma \cdots e_{i\alpha+\beta}$.

Want to show that within each Chevalley group is a copy of the Weyl group.

$\varphi_\alpha : SL_2 \to \langle x_\alpha(t), x_{-\alpha}(t) \rangle \leq X(k)$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts, sending $\alpha \mapsto -\alpha$; so guess $\varphi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ will correspond to reflection w_α.

$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-\alpha}(t)$, $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_\alpha(t)$.

What is $\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}$ in terms of $x_\alpha(t), x_{-\alpha}(t)$? $\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ if $n_\alpha(t) = \varphi\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}$, $n_\alpha(t) := x_\alpha(t)\, x_{-\alpha}(-t^{-1})\, x_\alpha(t)$

$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$: $h_\alpha(t) = n_\alpha(t)\, n_\alpha(1)^{-1}$. : $\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$

$N = \langle n_\alpha(t) : \alpha \in \Phi, t \in k \rangle \geqslant H = \langle h_\alpha(t) : \alpha \in \Phi, t \in k \rangle$.

Lemma 3.15: (i) $n_\alpha(t) . h_\beta = h_{w_\alpha(\beta)} = w_\alpha(h_\beta)$

(ii) $n_\alpha(t) \cdot e_\beta = \eta_{\alpha,\beta}\, t^{-A_{\alpha,\beta}} . e_{w_\alpha(\beta)}$. $\eta_{\alpha,\beta} = \pm 1$.

(iii) $h_\alpha(t) h_\beta = h_\beta$

(iv) $h_\alpha(t) . e_\beta = t^{A_{\alpha,\beta}} e_\beta$.

Proof: $n_\alpha(t) h_\alpha = h_{-\alpha}$, if $h \in H$ (and here, H is the Cartan subalgebra), $(h_\alpha, h) = 0$. $n_\alpha(t) . h = 0$.
(check definition of action)

Certainly, $n_\alpha(t) . h_\alpha \in \langle L_\alpha, H_\alpha, L_{-\alpha} \rangle = sl_2$. Now, $n_\alpha(t)$ acts on $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h_\alpha$.

$\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = h_{-\alpha}$.

Now let $x = n_\alpha(t) . e_\beta = \sum_{i \in \mathbb{Z}} t^i x_i$, where $x_i \in L_{\beta+i\alpha}$ (+) (check this: $x_\alpha(t) e_\beta = \Sigma M_{\alpha,\beta,i}\, t^i e_{i\alpha+\beta}$)

Let $h \in H$. $[h,x] = [h, n_\alpha(t) . e_\beta] = n_\alpha(t) . [n_\alpha(t)^{-1} h, e_\beta]$

$= n_\alpha(t) . (\beta(n_\alpha(t)^{-1} h) e_\beta) = \beta(n_\alpha(t)^{-1} h)\, n_\alpha(t) e_\beta = \beta(n_\alpha(t)^{-1} h) x$.

So we just need to show $n_\alpha(t)^{-1} h = n_\alpha(t^{-1}) h = w_\alpha(h)$ – follows from (i), and so

$\beta(n_\alpha(t)^{-1} h) = \beta(w_\alpha(h))$ which is $(\beta, w_\alpha(h)) = (w_\alpha \beta, h)$

$= w_\alpha \beta(h)$

$w_\alpha \beta = \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \alpha$, so power of t is $-A_{\alpha\beta}$ by (+) as $x \in L_{\beta - A_{\alpha,\beta}\alpha}$.

Chevalley's Commutator Formula: $[x_\alpha(t), x_\beta(t)] = \prod_{i,j>0} x_{i\alpha+j\beta}(c_{ij\alpha\beta}(-t)^i u^j)$

$\varphi_\alpha : SL_2(K) \to \langle X_\alpha, X_{-\alpha} \rangle$

$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \longmapsto x_\alpha(t)$

$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto x_{-\alpha}(t)$

$\varphi_\alpha \begin{pmatrix} 0 & t \\ t^{-1} & 0 \end{pmatrix} = x_\alpha(t)\, x_{-\alpha}(t^{-1})\, x_\alpha(t) =: n_\alpha(t)$.

$\varphi_\alpha \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = n_\alpha(t)\, n_\alpha(1) =: h_\alpha(t)$

Return to above lemma: Recall $x_\alpha(t) . e_\beta = \sum_{i=0}^{q} M_{\alpha,\beta,i}\, t^i e_{i\alpha+\beta}$. So $n_\alpha(t) . e_\beta = \sum_{i \in \mathbb{Z}} t^i x_i = x$, $x_i \in L_{i\alpha+\beta}$
$h \in H(L)$. $[h,x] = w_\alpha \cdot \beta(h) . x \Rightarrow x \in w_\alpha(\beta) = L_{\beta - A_{\alpha,\beta}\alpha} \Rightarrow n_\alpha(t) e_\beta = t^{-A_{\alpha,\beta}} . x_{w_\alpha(\beta)}$. $x_{w_\alpha(\beta)} = \pm e_{w_\alpha(\beta)}$
$n_\alpha(1)$ is an automorphism of $L_{\mathbb{Z}}$ as $x_\alpha(1)$ is an automorphism of $L_{\mathbb{Z}}$, because $x_\alpha(\frac{-1}{t})$ is its inverse. Hence $\eta_{\alpha,\beta} = \pm 1$. – sign will depend on $N_{\alpha,\beta} = \pm(p+1)$

Exercise: (i) $\eta_{\alpha,\alpha} = -1 = \eta_{\alpha,-\alpha}$

(ii) $\eta_{\alpha,\beta} = \eta_{\alpha,-\beta}$

(iii) $\eta_{\alpha,\beta} \cdot \eta_{\alpha, w_\alpha(\beta)} = (-1)^{A_{\alpha,\beta}}$.

Lemma 3.16: Define $n_\alpha = n_\alpha(1)$.

(i) $n_\alpha \cdot x_\beta(t) . n_\alpha^{-1} = x_{w_\alpha(\beta)} . (\eta_{\alpha,\beta} t)$

(ii) $h_\alpha(t) . x_\beta(u) . h_\alpha(t)^{-1} = x_\beta(t^{A_{\alpha,\beta}} u)$

(iii) $n_\alpha . h_\beta . n_\alpha^{-1} = h_{w_\alpha(\beta)}(t)$.

Proof: (i) $n_\alpha \cdot x_\beta(t) \cdot n_\alpha^{-1} = n_\alpha \cdot \exp t \operatorname{ad} e_\beta \cdot n_\alpha^{-1}$

$= \exp[t. \operatorname{ad}(n_\alpha \cdot e_\beta)] = \exp . t \operatorname{ad}(\eta_{\alpha\beta} . e_{w_\alpha(\beta)}) = x_{w_\alpha(\beta)}(t. \eta_{\alpha\beta})$

(ii) Similar method.

(iii) If $v \in L_\gamma$ then $n_\alpha^{-1} v \in L_{w_\alpha(\gamma)}$, so $n_\alpha . h_\beta(t) . n_\alpha^{-1}(v) = n_\alpha t^{A_{\beta, w_\alpha(\gamma)}} (n_\alpha^{-1} . v) = t^{A_{w_\alpha(\beta), \gamma}} v = h_{w_\alpha(\beta)} . v$.

Theorem: Let L be a simple Lie algebra of type X $(L \neq A_1)$ and k a field. For each root α of L, $t \in k$, define a symbol $x_\alpha(t)$. Let $\bar{G}$ = abstract group generated by $x_\alpha(t)$ subject to:

(i) $x_\alpha(t_1) . x_\alpha(t_2) = x_\alpha(t_1 + t_2)$

(ii) $[x_\alpha(u), x_\beta(t)] = \prod_{i,j>0} x_{i\alpha+j\beta}(c_{ij\alpha\beta}(-t)^i u^j)$

(iii) $h_\alpha(t_1) h_\alpha(t_2) = h_\alpha(t_1 t_2)$

Let $Z = Z(\bar{G})$. Then $X(k) \cong \bar{G}/Z$. $\bar{G}$ is called the universal Chevalley group.

If $N \leq Z$, then $\bar{G}/N$ are the Chevalley groups corresponding to different representations of L.

Proof: omitted.

$L_0 = \mathbb{Z}$-span of root Φ

$\rotatebox{90}{$\subseteq$}$

$L_1 =$ lattice of weights.

A_l: $L_1/L_0 \cong \mathbb{Z}/(l+1)\mathbb{Z}$. $X(k) = PSL_{l+1}(k)$, $\bar{G} = SL_{l+1}(k)$.

B_l: $L_1/L_0 \cong \mathbb{Z}/2\mathbb{Z}$. $X(k) = PSO_{2l+1}(k) = SO_{2l+1}$, $\bar{G} = Spin_{2l+1}$

C_l: $L_1/L_0 \cong \mathbb{Z}/2\mathbb{Z}$ $X(k) = PSp_{2l}$, $\bar{G} = Sp_{2l}$

D_{2n+1}: $L_1/L_0 \cong \mathbb{Z}/4\mathbb{Z}$. $X(k) = PSO_{4n+2}$, $\bar{G} = Spin_{4n+2}$

D_{2n}: $L_1/L_0 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ $X(k) = PSO_{4n}$. $\bar{G} = Spin_{4n}$.

Definition: B, Borel subgroup, $= \langle U, H \rangle$

Theorem 3.18: (a) $U \lhd B$, and $B = UH$

(b) $U \cap H = 1$

(c) $H \lhd N$

(d) $\exists$ a homomorphism $\varphi : W \to N/H$ which is onto, and $\varphi(w_\alpha) = H n_\alpha(t)$ $\forall \alpha \in \Phi$.

(e) φ is an isomorphism

Proof: (a) H normalises U because $h_\alpha(t) X_\beta h_\alpha(t)^{-1} = X_\beta$, so clear.

(b) If $x \in U \cap H$, x is unipotent, but also diagonal (wrt Chevalley basis), so $x = 1$.

(c) part (iii) of lemma 3.16: $n_\alpha h_\beta(t) n_\alpha^{-1} = h_{w_\alpha(\beta)}(t)$. $n_\alpha(t) = h_\alpha(t) . n_\alpha \Rightarrow H \lhd N$.

(d),(e): $W = \langle w_\alpha : w_\alpha^2 = 1, \alpha \in \Pi, w_\alpha w_\beta w_\alpha^{-1} = w_{w_\alpha(\beta)} \rangle$.

$H n_\alpha(t) n_\alpha(1)^{-1} . n_\alpha(1) = H n_\alpha(1)$. Write this as $\bar{w}_\alpha = H n_\alpha(1)$.

$\bar{w}_\alpha^2 = 1$: $n_\alpha(1)^{-1} = n_\alpha(-1)$: $n_\alpha(1) n_\alpha(-1) = 1 \in \bar{w}_\alpha^2$, hence $\bar{w}_\alpha^2 = 1$.

$n_\alpha n_\beta n_\alpha^{-1} = n_\alpha x_\beta(1) x_{-\beta}(-1) x_\beta(1) n_\alpha^{-1} = x_{w_\alpha(\beta)}(c)\, x_{-w_\alpha(\beta)}(c)\, x_{w_\alpha(\beta)}(c). = n_{w_\alpha(\beta)}(c) \in \overline{w}_{w_\alpha(\beta)}.$

Hence $\overline{w}_\alpha \overline{w}_\beta \overline{w}_\alpha^{-1} = \overline{w}_{w_\alpha(\beta)}$.

Hence $\exists\, \varphi : W \twoheadrightarrow N/H$. So it remains to show $\ker\varphi$ is trivial.

$w \in \ker\varphi$, $w = w_{\alpha_1} \cdots w_{\alpha_r}$, hence $n_{\alpha_1} \cdots n_{\alpha_r} \in H$. Conjugate X_α by $\underset{\overset{\|}{n \in H}}{n_{\alpha_1} \cdots n_{\alpha_r}}$, we get $X_{w(\alpha)}$, but $n \in H \Rightarrow X_{w(\alpha)} = X_\alpha$.

But $X_\alpha \neq X_\beta$ if $\alpha \neq \beta \Rightarrow w(\alpha) = \alpha \ \forall \alpha \in \Phi$. Thus $w = 1$.

Definition: A pair of groups B, N of G is called a **(B,N)-pair** if

(BN1): G is generated by B and N. ($n_\alpha X_\beta n_\alpha^{-1} = X_{w_\alpha(\beta)}$ - W acts transitively on Φ)

(BN2): $B \cap N$ is a normal subgroup of N ($= H$).

(BN3): The group $W = N/B\cap N$ is generated by a set of elements w_i ($i \in I$) such that $w_i^2 = 1$.

(BN4): $\underset{n_i \mapsto w_i}{N \to W}$ and $n \in N$, then $Bn_iB.\, BnB \subseteq Bn_inB \cup BnB$.

(BN5): $n_i B n_i \neq B$.

BN2: $B = U.H$, $N \cap U = 1 \Rightarrow B \cap N = H$.

$W = \langle w_\alpha : \alpha \in \Pi \rangle$. $J \subseteq \Pi$. $W_J = \langle w_\alpha : \alpha \in J \rangle$, and its conjugates.

$$\begin{pmatrix} * & * & & * \\ * & * & & \\ & 0 & \ddots & \\ & & & * \end{pmatrix} = \underset{\overset{\|}{B \cup B n_\alpha B}}{\langle B, n_\alpha \rangle} \longleftrightarrow \langle w_\alpha \rangle. \qquad P_J = \langle B, n_\alpha : \alpha \in J \rangle \longleftrightarrow \begin{pmatrix} \boxed{*} & & * \\ & \boxed{*} & \\ & & \boxed{*} \end{pmatrix} \text{-parabolic.}$$

$B \leq P_J \leq G$.

Lemma: Let $\alpha \in \Pi$. Then $B \cup Bn_\alpha B$ is a subgroup.

Proof: $n_\alpha B n_\alpha \subseteq B \cup B n_\alpha B$. $B = U.H$, define $U_\alpha = \prod_{\beta \in \Phi^+ \setminus \alpha} X_\beta$. $U = X_\alpha . U_\alpha$.

(using α simple, and Chevalley commutator formula).

$n_\alpha B n_\alpha = n_\alpha X_\alpha U_\alpha H n_\alpha^{-1} = X_{-\alpha} . n_\alpha . U_\alpha n_\alpha^{-1} . H$

Need to show (i) $n_\alpha U_\alpha n_\alpha^{-1} = U_\alpha$

(ii) $X_{-\alpha} \subseteq B \cup B n_\alpha B$.

Claim: X_α and $X_{-\alpha}$ normalise U_α.

$[X_{-\alpha}, X_\alpha]$ - Chevalley commutator formula. We get X_γ, $\gamma = -i\alpha + j\beta$, $i, j > 0$, $\beta \in \Phi^+$, $\beta \neq \alpha$, so $\gamma \in \Phi^+$ (as $\beta \neq \pm\alpha$ so other term > 0 in its representation hence $-i\alpha + j\beta \in \Phi^+$).

↳ ie its representation as a linear combination of simple roots.

and $\gamma \neq \alpha$. So $[X_{-\alpha}, U_\alpha] \subseteq U_\alpha$.

Since $n_\alpha = x_\alpha(1) x_{-\alpha}(-1) x_\alpha(1) \Rightarrow n_\alpha$ normalises U_α.

$x_{-\alpha}(t) = x_\alpha(t^{-1}) n_\alpha(-t^{-1}) x_\alpha(t^{-1})$, $[n_\alpha(t) = n_\alpha(1) h_\alpha(-t^{-1}).]$

$= x_\alpha(t^{-1}) h_\alpha(-t^{-1}) n_\alpha x_\alpha(t^{-1}) \in B n_\alpha B$ (If $t = 0$ then $x_{-\alpha}(0) = 1 \in B$).

Note: BN5: $n_i B n_i \neq B$ as $n_\alpha X_\alpha n_\alpha^{-1}$, $\alpha \in \Pi$, $\subseteq X_{-\alpha}$.

Proposition: If $\alpha \in \Pi$ and $n \in N$. Then $BnB.\, Bn_\alpha B \subseteq Bnn_\alpha B \cup BnB$.

If $w = Hn$, then if $w(\alpha) \in \Phi^+$ then $BnB.\, Bn_\alpha B \subseteq Bnn_\alpha B$.

Exercise: Show if $w(\alpha) \in \Phi^-$ then $BnB.\, Bn_\alpha B$ intersects non-trivially with $Bnn_\alpha B$ and BnB.

Proof: $w(\alpha) \in \Phi^+$. $(BnB)(Bnn_\alpha B) = Bn X_\alpha U_\alpha H n_\alpha B. = B(n X_\alpha n^{-1})(n n_\alpha)(n_\alpha^{-1} U_\alpha H n_\alpha) B$

↳ $= Bnn_\alpha B$.

$n X_\alpha n^{-1} = X_{w(\alpha)} \subseteq B$

$n_\alpha^{-1} U_\alpha n_\alpha \subseteq U_\alpha$.

If $w(\alpha) \in \Phi^-$, $w' = ww_\alpha(\alpha) \in \Phi^+$. Choose $n' \in N$ such that $w' = Hn'$. $Bn'B.Bn_\alpha B = Bn'n_\alpha B$

$BnB.Bn_\alpha B = Bn'n_\alpha B.Bn_\alpha B. = (Bn'B)(Bn_\alpha B)(Bn_\alpha B) = Bn'B(B \cup Bn_\alpha B) = Bn'B \cup Bn'n_\alpha B$

$= Bnn_\alpha B \cup BnB \quad (n' = nn_\alpha)$.

Corollary: $Bn_\alpha B.BnB \subseteq Bn_\alpha nB \cup BnB$

Corollary: $X(K)$ has a (B,N)-pair.

Theorem (Bruhat Decomposition): (a) $G = \bigcup_{w \in W} Bn(w)B$, $W = n(w)H$.

(b) $Bn(w)B = Bn(w')B \iff w = w'$.

Proof: (a) $\bigcup_{w \in W}$ contains generator for G and is closed under multiplication by these generators – BN4

(b) By induction on $l(w)$. If $l(w) = 0$, $n(w') \in B \cap N \subset H \Rightarrow w = 1$

Assume $l(w) > 0$. Choose $\alpha \in \Pi$, $l(ww_\alpha) < l(w)$.

$n(w)n(w_\alpha) \in Bn(w')B.Bn(w_\alpha)B \subseteq Bn(w')B \cup Bn(w')n(w_\alpha)B = Bn(w)B \cup Bn(w')n(w_\alpha)B$.

Induction $\Rightarrow ww_\alpha = w$ or $= w'w_\alpha$, $w_\alpha \neq 1 \Rightarrow w = w'$.

Theorem (Parabolic Subgroups): If $J \subset \Pi$, $W_J = \langle w_\alpha : \alpha \in J \rangle$, $P_J = \bigcup_{w \in W_J} BwB$, then.

(a) P_J is a group and is called a parabolic subgroup, and any conjugate of P_J is also called parabolic.

(b) $\{P_J : J \subset \Pi\}$ are all distinct $(W_J = W_{J'} \Rightarrow J = J')$

(c) If $G \geqslant H \geqslant B$ then $H = P_J$ for some J.

(d) P_J and P_I are not conjugate.

(e) $N_G(P_J) = P_J$.

(f) $P_J \cap P_I = P_{I \cap J}$.

(g) $B \cup BwB$ is a group $\iff w = 1$ or $w = w_\alpha$, $\alpha \in \Pi$.

Exercise: Prove (d) – (g).

Proof of (c):

Lemma: $l(w_iw) \geqslant l(w) \Rightarrow Bn_iB.BnB \subseteq Bn_inB$.

$l(w) \geqslant l(w_iw) \Rightarrow Bn_iB.Bn_inB \subseteq BnB. \qquad w_i \leftrightarrow n_i, \quad w \leftrightarrow n.$

Proof: Induction on $l(w)$. $w = w'w_j$ such that $l(w') = l(w) - 1$, $w' \leftrightarrow n'$. Suppose result is false.

Thus $Bn_iB.BnB \cap BnB \neq \phi$. $n_iBn' \cap BnBn_j \neq \phi$.

Now, $l(w_iw') \geqslant l(w') \Rightarrow$ by induction $n_iBn' \subseteq Bn_in'B$.

Hence $Bn_in'B \cap BnBn_j \neq \phi$

$B\overset{n'}{n}n_jB \cup BnB$ by BN4.

Hence $n_in' = nn_j$ or $n_in' = n$. But $n_in' = nn_j \Rightarrow n_i = 1$ #.

$n_in' = n \Rightarrow n_in = n' \Rightarrow l(w_iw) = l(w') < l(w)$ – #.

Exercise: Try to find a more geometric proof: $w(\alpha_i)$ is a positive root if $l(w_iw) \geqslant l(w)$.

Lemma: Suppose $w = w_{i_1} \cdots w_{i_k}$, $l(w) = k$, $J = \{i_1, \dots i_k\}$. Then:

(i) $\langle B, n \rangle$, (ii) $\langle B, nBn^{-1} \rangle$, (iii) $P_J = BW_JB$

are all the same.

Lemma: If $B \leq K \leq G$ then $K = P_J$ for some $J \subseteq M$.

Lemma: If $n \in K$, $n \leftrightarrow w \in W$, $w = w_{\alpha_{i_1}} \cdots w_{\alpha_{i_k}}$, $l(w) = k$. Then $\langle B, n \rangle = \langle B, nBn^{-1} \rangle = P_J$, $J = \{\alpha_{i_1}, \ldots, \alpha_{i_k}\}$.

Proof: Clearly $\langle B, n \rangle \supset \langle B, nBn^{-1} \rangle$, $P_J \supseteq \langle B, n \rangle$. To get the other inclusions we use induction.

We want to prove that $w_{\alpha_{i_1}} \in \langle B, nBn^{-1} \rangle$. Sufficient to prove $X_{-\alpha_{i_1}} \subseteq \langle B, nBn^{-1} \rangle$, because $X_{\alpha_{i_1}} \leq B$.

Now, $l(w) = 1 + l(w_{\alpha_{i_2}} \cdots w_{\alpha_{i_k}})$. $l(w) = \# \{ \alpha \in \Phi^+ : w(\alpha) \in \Phi^- \}$. The only roots which change sign under $w_{\alpha_{i_1}}$ are $\pm \alpha_{i_1}$. Thus $\exists \beta \in \Phi^+$, $w(\beta) = -\alpha_{i_1}$.

Then $n X_\beta n^{-1} = X_{w(\beta)} = X_{-\alpha_{i_1}}$. $\beta \in \Phi^+ \Rightarrow X_\beta \leq B$.

Proof of previous lemma: $K = \bigcup_{\substack{n \in N_0 \subset N \\ W_0 \subset W}} BnB$ if $n \in N_0 \Rightarrow P_{J_n} \leq K$, $J_\alpha = \{\alpha_{i_1} \cdots \alpha_{i_k}\}$,

where $w = w_{\alpha_{i_1}} \cdots w_{\alpha_{i_k}}$. So K is generated by P_{J_n}. So K is P_J, where $J = \bigcup_{n \in N_0} J_n$

Theorem: Let G be a group with a (B,N)-pair. $W = \{ w_i : i \in I \}$.

If (i) $G = G'$

(ii) B is soluble

(iii) $\bigcap_{g \in G} gBg^{-1} = 1$

(iv) $I = J \cup K$ such that $[w_j, w_k] = 1$, $j \in J$, $k \in K \Rightarrow J = \phi$ or $K = \phi$.

Then G is a simple group.

Lemma: If $l(w_i w) > l(w)$ then $Bn_iB . BnB = Bn_inB$ - proved earlier.

Proof of Theorem: $G_1 \triangleleft G$, $G_1 B = P_J$ for some $J \subseteq I$. Take K complementary to J, $K = I \setminus J$.

$l(w_j w_k) > l(w_k)$, $w_j \in W_J$, $w_k \in W_K$. $W_J \cap W_K = W_{J \cap K} = 1$

$Bn_jB . Bn_k . B = Bn_jn_kB$. Now, $Bn_jB \cap G_1 \neq \phi$. $n_k B n_j B n_k \cap G_1 \neq \phi$

$$\begin{array}{c} n_k B n_j B n_k \\ \rotatebox{90}{$\subseteq$} \\ n_k B n_j n_k B \subseteq B n_j n_k B \cup B n_k n_j n_k B \end{array}$$

Suppose $Bn_jn_kB \cap G_1 \neq \phi$. $n_jn_k \in W_J \Rightarrow n_k \in W_J \cap W_K = 1 \Rightarrow \#$

So $B n_k n_j n_k B \cap G_1 \neq \phi \Rightarrow w_k w_j w_k = w_j$ (reason to come later)

$\forall j \in J, k \in K$. Then (iv) $\Rightarrow J = \phi$ or $K = \phi$.

Then if $K = \phi$, $G_1 B = G$. If we look at G/G_1, $G/G_1 = G_1B/G_1 \cong B/B \cap G_1$, and B is soluble, by (ii). If $G_1 < G$, then $[G, G] = G' \subsetneqq G$ $- \#$. So $G_1 = G$.

If $J = \phi$, then $G_1 \leq B$. Hence by (iii), $G_1 = \{1\}$.

Note: $w_k w_j w_k \in W_J \cap W_{\{j,k\}} = W_{J \cap \{j,k\}} = W_{\{j\}} \Rightarrow w_k w_j w_k = w_j$

Proposition: The Chevalley Groups $X(K)$ are simple groups for all fields K, and simple Lie Algebras L of type X, except for $A_1(\mathbb{F}_2)$, $A_1(\mathbb{F}_3)$, $B_2(\mathbb{F}_2)$, $G_2(\mathbb{F}_2)$

Exercise: Why are the exceptions not simple?

Proof: (b) $B = U \rtimes H$ soluble.

(d) Dynkin diagram = Coxeter diagram of the Weyl group W, is connected : W cannot be decomposed into subsets W_J, W_K, $[W_J, W_K] = 1$ because L is simple and hence Dynkin diagram is connected.

(c) $G_1 \trianglelefteq G$, $G_1 \leq B$. $\exists\ \underset{\underset{n_0 \in N.}{\downarrow}}{w_0}: \Phi^+ \to \Phi^-$. $n_0 U n_0^{-1} = V = \{ X_\alpha(t) : \alpha \in \Phi^- \}$

But $B = U.H$. $G_1 \leq U.H \cap V.H = H$. H normalises $U \Rightarrow [U, H] \leq U$. Thus $[U, G_1] \subseteq U \cap G_1$ (G_1 normal) and $U \cap G_1 \subseteq U \cap H = 1$. Similarly for V. So $G_1 \leq Z(G(K))$.

Our Chevalley Groups have trivial centre: $H = \langle h_\alpha(t) : \alpha \in \Phi, t \in K^* \rangle$, $h_\alpha(t) e_\beta = t^{A_{\alpha,\beta}} e_\beta$.

Let $h \in H$. Claim $h \in Z(G) \Rightarrow h = 1$. If $a \in \mathbb{Z}\Phi$, $\chi_{\alpha,t}(a) = t^{2(\alpha,a)/(\alpha,\alpha)}$, $\chi_{\alpha,t} \in \mathrm{Hom}(\mathbb{Z}\Phi, K^*)$.

Thus $h_\alpha(t) e_\beta = \chi_{\alpha,t}(\beta) e_\beta$. Any $h \in H$ has some $\chi \in \mathrm{Hom}(\mathbb{Z}\Phi, K^*)$ such that $h = h(\chi)$, i.e. $h(\chi) e_\beta = \chi(\beta) e_\beta$.

$\Rightarrow h(\chi) . x_\beta(1) h(\chi)^{-1} = x_\beta(\chi(\beta)) = x_\beta(1)$ as $h(\chi) \in Z(G)$. Hence by uniqueness, $\chi(\beta) = 1\ \forall \beta \in \Phi$

$\Rightarrow h = 1$ as $\chi = 1$.

$G_1 = 1 \Rightarrow \bigcap_{g \in G} g B g^{-1} = 1$.

Exercise: Show $Z(G) = 1$.

Proof: if $ZB = G$ then ~~#~~ : $N_G(B) = B$.

Finally check which of these are perfect groups:

Lemma: $G = G'$ except for $G = A_1(\mathbb{F}_2), A_1(\mathbb{F}_3), B_2(\mathbb{F}_2), G_2(\mathbb{F}_2)$

Let $s \in K$, $x_\alpha(s) = h_\alpha(t) x_\alpha(u) h_\alpha(t)^{-1} = x_\alpha(t^2 u)$. Thus $x_\alpha(u)^{-1} h_\alpha(t) x_\alpha(u) h_\alpha(t)^{-1} = x_\alpha((t^2-1)u)$.

Choose $u = \frac{s}{t^2-1}$, then $x_\alpha(s) \in G'$. We have now done all fields, except $\mathbb{F}_2, \mathbb{F}_3$.

$[x_\beta(t), x_\gamma(u)] = - x_{\beta+\gamma}(N_{\beta,\gamma} tu) \prod'_{\substack{i\beta + j\gamma,\ i>1, j\geq 1 \\ j>1, i \geq 1}}$

We can now deal with any root $\alpha = \beta + \gamma$ such that $b\beta + c\gamma \in \Phi \Rightarrow b = c = 1$ or $b = c = 0$ and $N_{\beta,\gamma} \neq 0$. So "if we can get an A_2 in the system" - can find necessary β, γ spanning an A_2, $N_{\beta\gamma} = \pm(p+1)$, $\beta - p\gamma \in \Phi$, $\beta - (p+1)\gamma \notin \Phi$. In A_2, $p = 0$, so $N_{\beta\gamma} = \pm 1$.

So case analysis shows that in all cases A_l, D_l, E_l where we have the same root lengths. We can choose β, γ to generate A_2 and $p = 0$. Also can do this for the following: B_l: α long, C_l: α short, G_2: α long. A few cases left.

Definition: A building is a simplicial complex Δ which can be expressed as a union of subcomplexes Σ (called apartments) such that:

(B0) Each apartment is Coxeter complex

(B1) For any two simplices $A, B \in \Delta$ there is an apartment Σ containing both A, B.

(B2) If Σ, Σ' are two apartments containing A and B then there is an isomorphism (of the building) between Σ and Σ' fixing A and B pointwise.

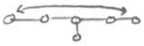
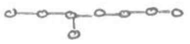
If G is a group with a B,N-pair then

$\Delta(G, B) = \{$ parabolic subgroups P_J^g, $g \in G$, $J \subseteq I$, opposite inclusion $\}$.

Twisted Chevalley Groups.

Field automorphism: If $f: K \to K$ is an automorphism of K, the underlying field, then $f: X(K) \to X(K)$
$x_\alpha(t) \mapsto x_\alpha(f(t))$ is an automorphism.

Symmetries of Dynkin diagram: Graph automorphism: Let ρ be a permutation of the Dynkin diagram preserving the symmetry of it.

A_l :		
B_l :		NONE
C_l :		NONE
D_l :		
D_4 :		S_3
E_6 :		
E_7 :		NONE
E_8 :		NONE
F_4 :		NONE
G_2 :		NONE

Let τ be the linear transformation of $V = \Phi \otimes \mathbb{R}$ such that $\tau(\alpha_i) = \rho(\alpha_i)$.

Exercise: $\tau(\Phi) = \Phi$, τ an isometry.

$x_\alpha(t) \mapsto x_{\tau(\alpha)}(t)$ for simple roots α.

$x_\alpha(t) \mapsto x_{\tau(\alpha)}(\gamma_\alpha(t))$ where $\gamma_\alpha = \pm 1$ for general $\alpha \in \Phi$

ie, $\exists\ \gamma_\alpha$ such that $\gamma_\alpha = \pm 1$ and $\tau : X(K) \to X(K)$ is an automorphism.

$$x_\alpha(t) \mapsto x_{\tau(\alpha)}(\gamma_\alpha t)$$

Definition: $V' = \{v \in V : \tau(v) = v\}$ V' = projection of v onto v'. $\frac{1}{n}(v + \tau(v) + \cdots + \tau^{n-1}(v))$, where τ has order n. $W' = \{w \in W : w\tau = \tau w\}$.

W' is a reflection group. Let $\Phi' = \{\alpha' \in V' : \alpha \in \Phi\}$, $\Pi' = \{\alpha' \in V' : \alpha \in \Pi\}$.
Then Φ' is like a root system for W' except that α and 2α can be in Φ'.

Definition: If $G = X(K)$ admitting an automorphism σ, we define $U' = \{x \in U : \sigma(x) = x\}$.
$V' = \{x \in V : \sigma(x) = x\}$. $G' = \langle U', V' \rangle$. $H' = G' \cap H$, $N' = G' \cap N$, $B' = B \cap G'$.
Then (B', N') is a B-N pair for G'.

G' is called a twisted Chevalley group if $\sigma = \tau f$ with τ a graph automorphism, f a field automorphism such that $(\tau f)^n = 1$ where n = order of τ.

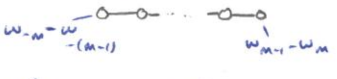
(Can define automorphisms for B_2, G_2, F_4 in characteristic 2 or 3, $x_\alpha(t) \mapsto x_{\tau(\alpha)}(t^{\chi(\alpha)})$, where $\chi(\alpha) = 1$ if α is short, $\chi(\alpha) = 2$, if α is long. If $k = \mathbb{F}_2$ or $\mathbb{F}_3$, this can define an automorphism)

$G = SL_{l+1} \Longleftrightarrow A_l$.

$$\left.\begin{array}{l} l+1 = 2m+1 \\ l+1 = 2m \end{array}\right\} \text{two cases.}$$

Let's index the rows and columns using $-m$ to m and omitting 0 when $l+1$ is even $(=2m)$.

$\omega_i : \mathrm{diag}(\lambda_1, \ldots, \lambda_m) = \lambda_i$. Roots $\omega_i - \omega_j$.



So $\tau : \omega_i \mapsto -\omega_{-i}$ (easy check).

V' = span of $\{\omega_{-i} - \omega_i : i > 0\}$. Set $\omega_i' = \omega_{-i} - \omega_i$.

Now if $i, j \neq 0$, $(\omega_i - \omega_j)' = \frac{1}{2}(\omega_i - \omega_j - \omega_{-i} + \omega_{-j}) = \frac{1}{2}\{-(\omega_i - \omega_{-i}) + (\omega_j - \omega_{-j})\}$

$$= \begin{cases} \frac{1}{2}(\pm\omega_k' \pm \omega_l') & k, l > 0 \\ \omega_k'. \end{cases}$$

If $i=0$ or $j=0$, $w_i - w_j \to \frac{1}{2}w_k'$, $k>0$.
So note $\frac{1}{2}w_k'$ and w_k' will be 'roots', in the odd case.
So if $l+1 = 2m$ then Φ' is of type C_m.
If $l+1 = 2m+1$ then Φ' is of type BC_m (as $\frac{1}{2}$ a root can be a root).
Check $W' = \{w \in W : w\tau = \tau w\} \cong W(\Phi')$.
The effect of $\tau : x_\alpha(t) \mapsto x_{\tau(\alpha)}(\gamma_\alpha t)$ is the same as $a x^{-t} a^{-1}$, where $a = \begin{pmatrix} 0 & \epsilon_m \\ \epsilon_{-m} & 0 \end{pmatrix}$,
$x \in SL_{l+1}(K)$. So $\tau : x \mapsto a x^{-t} a^{-1}$ where $(x^t)^{-1} = x^{-t}$.
So if $\tau(x) = x$, then $x a x^t = a$ – an orthogonal or symplectic group.

$$\Rightarrow \begin{cases} G^\tau = Sp_n & l = 2m \\ G^\tau = SO_n & l = 2m+1 \end{cases}$$

Suppose we had an involution $f : K \to K$, eg. complex conjugation. The effect of τf is
$x_\alpha(t) \to x_{\tau(\alpha)}(\gamma_\alpha f(t))$. If f is denoted by a bar, $x_\alpha(t) = x_{\tau(\alpha)}(\gamma_\alpha \bar{t})$.
So the above becomes $x \mapsto a\overline{(x^t)^{-1}}a^{-1}$. Thus $x a \bar{x}^t = a$. Then in this case $G^\sigma = SU_n$.