Algebraic Topology Notes 1996

Homotopy theory

§1 Spaces of maps

If X and Y are spaces, the "compact-open" topology on the set $\mathrm{Map}(X;Y)$ of continuous maps from X to Y is the coarsest topology such that

$$M_{K,U} = \{ f \in \mathrm{Map}(X;Y) : f(K) \subset U \}$$

is open for every compact subset K of X and every open subset U of Y. Thus the open subsets of $\mathrm{Map}(X;Y)$ are all unions of finite intersections of sets of the form $M_{K,U}$.

The most important case is when X is compact. If X is compact and Y is a metric space then the compact-open topology on $\mathrm{Map}(X;Y)$ can be defined by the metric

$$d(f,g) = \sup_{x \in X} d(f(x), g(x)).$$

Proposition (1.1) If X and Y are both compact then

$$\mathrm{Map}(X; \mathrm{Map}(Y;Z)) \cong \mathrm{Map}(X \times Y; Z)$$

as spaces. //

If X and Y are spaces with base-points x_0 and y_0 I shall write $\mathrm{Map}_0(X;Y)$ for the subspace of base-point-preserving maps, and

$X \wedge Y$ for the quotient space of $X \times Y$ by the equivalence relation which identifies $(X \cup \{y_0\}) \cup (\{x_0\} \cup Y)$ to a single point, which is taken as the base-point in $X \wedge Y$.

Proposition (1.2) If X and Y are both compact, and X, Y, Z have base-points, then

$$\mathrm{Map}_0(X; \mathrm{Map}_0(Y;Z)) \cong \mathrm{Map}_0(X \wedge Y; Z). \qquad //$$

<u>Example</u> For a space X with base-point, the <u>loop space</u> ΩX is $\mathrm{Map}_0(S^1; X)$, with the constant map to x_0 as base-point. The <u>suspension</u> SX is $S^1 \wedge X$. We have

$$\mathrm{Map}_0(SX; Y) \cong \mathrm{Map}_0(X; \Omega Y). \tag{1.3}$$

(In the language of category theory, the functors S and Ω are "adjoint".)

§ 2 Paths and the fundamental group

For a given space X, let $P(x_0; x_1)$ denote the space of paths γ in X from x_0 to x_1, i.e. of maps $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$,

If $\gamma_1 \in P(x_0; x_1)$ and $\gamma_2 \in P(x_1; x_2)$ define $\gamma_2 * \gamma_1 \in P(x_0; x_2)$ by

$$\begin{aligned}(\gamma_2 * \gamma_1)(t) &= \gamma_1(2t) && \text{if } 0 \le t \le \tfrac{1}{2},\\ &= \gamma_2(2t-1) && \text{if } \tfrac{1}{2} \le t \le 1.\end{aligned}$$

Then $* : P(x_0; x_1) \times P(x_1; x_2) \longrightarrow P(x_0; x_2)$ is continuous.

If $\alpha, \beta \in P(x_0; x_1)$ I shall write $\alpha \sim \beta$ if α and β are connected by a path in $P(x_0; x_1)$, i.e. if there is a map $F : [0,1] \times [0,1] \to X$ such that

$$\begin{aligned} F(t,0) &= \alpha(t),\\ F(t,1) &= \beta(t),\\ F(0,s) &= x_0,\\ F(1,s) &= x_1. \end{aligned}$$

<u>Proposition (2.1)</u>

(i) If $\alpha \sim \tilde{\alpha}$ in $P(x_0; x_1)$ and $\beta \sim \tilde{\beta}$ in $P(x_1; x_2)$ then $\beta * \alpha \sim \tilde{\beta} * \tilde{\alpha}$.

(ii) $\gamma * (\beta * \alpha) \sim (\gamma * \beta) * \alpha$ when both are defined.

(iii) If $1_x \in P(x; x)$ denotes the constant path at x then

$$1_{x_1} * \gamma \sim \gamma * 1_{x_0} \sim \gamma \quad \text{for any } \gamma \in P(x_0; x_1).$$

(iv) If $\gamma \in P(x_0; x_1)$ define $\gamma^{-1} \in P(x_1; x_0)$ by $\gamma^{-1}(t) = \gamma(1-t)$. Then

$$\gamma^{-1} * \gamma \sim 1_{x_0} \quad \text{and} \quad \gamma * \gamma^{-1} \sim 1_{x_1}. \qquad //$$

These statements can be summed up by saying that

(a) the points of X are the objects of a <u>category</u> in which the set of morphisms from x_0 to x_1 is the set $\pi_0(P(x_0, x_1))$ of path-components of $P(x_0; x_1)$, and

(b) the category is a <u>groupoid</u>, i.e. every morphism is invertable.

The groupoid is called the <u>fundamental groupoid</u> of X.

<u>Definition (2.2)</u> The <u>fundamental group</u> of X at x_0 is the group

$$\pi_1(X, x_0) = \pi_0(P(x_0; x_0)) = \pi_0(\Omega X).$$

<u>Remarks</u> (a) The proofs of (2.1) (ii), (iii), (iv), properly considered, actually prove rather more, namely that (ii) two <u>maps</u> $P(x_0; x_1) \times P(x_1; x_2) \times P(x_2; x_3) \longrightarrow P(x_0; x_3)$ are <u>homotopic</u>, (iii) three maps $P(x_0; x_1) \longrightarrow P(x_0; x_1)$ ... etc.

In particular, ΩX is a "group in the category of spaces and homotopy-classes of maps".

(b) A path α from x_0 to x_1 induces an isomorphism $\alpha_* : \pi_1(X, x_0) \to \pi_1(X,$ and if β is another such path then β_* differs from α_* by conjugation by $\alpha^{-1} \circ \beta \in \pi_1(X, x_1)$.

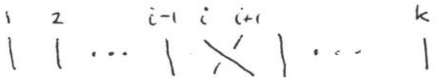
<u>Examples</u>

(a) $\pi_1(S^n) = 0$ if $n > 1$. (See Question 7 of Problem Sheet 1.)

(b) $\pi_1(S^1) \cong \mathbb{Z}$. For any path $\gamma : [0,1] \longrightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$ such that $\gamma(0) = \gamma(1) = 1$ can be written uniquely $\gamma(t) = e^{2\pi i \varphi(t)}$, where $\varphi : [0,1] \longrightarrow \mathbb{R}$ is continuous and such that $\varphi(0) = 0$ and $\varphi(1) \in \mathbb{Z}$. The map $\gamma \longmapsto \varphi(1)$ defines $\pi_1(S^1) \xrightarrow{\cong} \mathbb{Z}$. In fact ΩS^1, which is a group under pointwise multiplication, is the disjoint union of connected components $\{\Omega_n\}_{n \in \mathbb{Z}}$, where Ω_n is the paths from 0 to n in $\mathbb{R}$. Notice that $\Omega_0 \cong \Omega\mathbb{R}$ is a real vector space, and each coset Ω_n is an affine space of Ω_0.

(c) Let $C_k(\mathbb{R}^2)$ be the space of unordered k-tuples of distinct points of $\mathbb{R}^2$. The <u>braid group on k strings</u> Br_k is, by definition, $\pi_1(C_k(\mathbb{R}^2))$. ("Braid" is American for "plait".) It can be generated by $k-1$ elements $g_1, \ldots, g_{k-1}$ subject to the so-called "braid relations"

$$\left.\begin{array}{llll} g_i g_{i+1} g_i & = & g_{i+1} g_i g_{i+1} & \text{for } 1 \leq i \leq k-2 \\ g_i g_j & = & g_j g_i & \text{for } |i-j| > 1. \end{array}\right\} \qquad (2.3)$$

The element g_i can be depicted ,

and the first relation in (2.3) by

(For comparison, the symmetric group S_k has generators $g_1, \ldots, g_{k-1}$ and relations (2.3) together with $g_i^2 = 1$. And $\pi_1(C_k(\mathbb{R}^n)) \cong S_k$ if $n > 2$.)

§ 3 <u>Homotopy groups</u>

For a space X with base-point x_0, define

$$\pi_k(X, x_0) = \pi_0(\Omega^k X) = \pi_1(\Omega^{k-1} X) =$$
$$= \{\text{homotopy classes of base-point preserving maps } S^k \to X\}.$$

The third expression shows that $\pi_k(X, x_0)$ is a group. The last equality follows from (1.2), together with $S^1 \wedge S^p \cong S^{p+1}$.

<u>Proposition (3.1)</u> $\pi_k(X, x_0)$ is abelian if $k > 1$.

<u>Proof</u> The map $* : \Omega^{k-1} X \times \Omega^{k-1} X \longrightarrow \Omega^{k-1} X$ (got by regarding $\Omega^{k-1} X$ as $\Omega(\Omega^{k-2} X)$) induces a group homomorphism

$$\pi_1(\Omega^{k-1} X) \times \pi_1(\Omega^{k-1} X) \longrightarrow \pi_1(\Omega^{k-1} X)$$

taking $(\alpha, 1)$ to α and $(1, \beta)$ to β. But $(\alpha, 1)$ commutes with $(1, \beta)$. //

The homotopy groups of spaces are hard to calculate. The basic case is $\pi_k(S^n)$, which is still unknown in general. The main known facts are

Theorem (3.2)
(i) $\pi_k(S^n) = 0$ if $k < n$.
(ii) $\pi_n(S^n) \xrightarrow{\cong} \mathbb{Z}$ by the "degree" or "winding number".
(iii) $\pi_k(S^n)$ is finite if $k > n$, except that
(iv) $\pi_{2n-1}(S^n) \cong \mathbb{Z} \oplus$ (finite group) when n is even.
(v) $\pi_{n+m}(S^n)$ is independent of n when $n > m+1$.
(vi) $\pi_{n+m}(S^n)$ has a canonical decomposition as $J_m \oplus C_m$ when $n > m+1$, where J_m is a cyclic group of known order. If p is an odd prime then

$$p^a \text{ divides } |J_m| \iff (2p-2)p^{a-1} \text{ divides } m+1.$$

Remarks (a) The map $\pi_{2n-1}(S^n) \to \mathbb{Z}$ for even n is called the Hopf invariant.
(b) There is a homomorphism $\pi_k(S^n) \to \pi_{k+1}(S^{n+1})$ given by suspension. It is an isomorphism when $k < 2n-1$.
(c) The power of 2 which divides $|J_m|$ is given by a slightly more complicated rule. Experimentally, J_m is the biggest part of $\pi_{n+m}(S^n)$. It is the image of $\pi_m(GL_n\mathbb{R}) \to \pi_m(\Omega^n S^n) = \pi_{n+m}(S^n)$, where $GL_n\mathbb{R} \hookrightarrow \Omega^n S^n$ by one-point compactification.

Proof that $\pi_k(S^n) = 0$ if $k < n$

It is enough to show that any map $f: S^k \to S^n$ is homotopic to a map which is not surjective, for $S^n - \{\text{point}\} \cong \mathbb{R}^n$ is contractible. That can be done by approximating f by a smooth map $g: S^k \to S^n$. For $f \simeq g$ if $d(f,g) < \pi$, and $g(S^k)$ has measure zero in S^n if g is smooth. Alternatively, we can use a piecewise-linear approximation. For this we need some definitions.

Definition (3.3)

(i) If S is a finite set, the standard simplex Δ_S is

$$\Delta_S = \{(\lambda_\alpha)_{\alpha \in S} \in \mathbb{R}^S : \text{all } \lambda_\alpha \geq 0 \text{ and } \Sigma \lambda_\alpha = 1\}$$

(ii) A <u>simplicial scheme</u> (S, Σ) is a finite set S (of "vertices") together with a set Σ of subsets of S such that $\sigma \in \Sigma, \tau \subset \sigma \Rightarrow \tau \in \Sigma$.

(iii) The <u>realization</u> $|(S, \Sigma)|$ of a simplicial scheme is the subspace of Δ_S consisting of all $(\lambda_\alpha)_{\alpha \in S}$ such that $\{\alpha : \lambda_\alpha > 0\}$ belongs to Σ.

(iv) A <u>polyhedral subdivision</u> of a space X is a simplicial scheme (S, Σ) together with a homeomorphism $X \cong |(S, \Sigma)|$.

<u>Example</u> If $S = \{0, 1, \ldots, k\}$ and $\Sigma = \{$all subsets except S itself$\}$ then $|(S, \Sigma)| \cong S^{k-1}$.

I shall <u>assume</u> that for any open covering of S^k there is a subdivision $S^k \cong |(S, \Sigma)|$ such that each simplex Δ_σ (for $\sigma \in \Sigma$) is contained in a set of the covering.

Returning to our proof, regard S^n as $\mathbb{R}^n \cup \infty$, with ∞ as base-point. Given $f : S^k \to S^n$ choose a decomposition of S^k such that, for each simplex Δ_σ, either $\text{diameter}(f(\Delta_\sigma)) < 1$ or else $f(\Delta_\sigma) \subset \{\xi \in \mathbb{R}^n \cup \infty : \|\xi\| > 10\}$. Let X be the union of the simplexes Δ_σ of S^k such that $f(\Delta_\sigma)$ meets $2D^n = \{\xi \in \mathbb{R}^n : \|\xi\| \leq 2\}$. Define $\tilde{f} : X \to \mathbb{R}^n$ by

$$\tilde{f}(\alpha) = f(\alpha) \quad \text{if } \alpha \text{ is a vertex}$$

$$\tilde{f}|\Delta_\sigma \text{ is linear, i.e. } \tilde{f}((\lambda_\alpha)_{\alpha \in \sigma}) = \sum \lambda_\alpha f(\alpha).$$

Choose a continuous function $\rho : S^k \to \mathbb{R}_+$ such that $\rho = 1$ on $f^{-1}(D^n)$ and $\rho = 0$ outside of X.

Define $f_t : S^k \to \mathbb{R}^n \cup \infty$ for $0 \leq t \leq 1$ by $f_t(x) = (1 - t\rho(x)) f(x) + t\rho(x)\tilde{f}(x)$.

Then $f_1 \simeq f_0 = f$. But $f_1(S^k) \cap D^n = \tilde{f}(X) \cap D^n$ is a finite union of simplexes of dimension $\leq k$. So $f_1(S^k) \not\supset D^n$, and f_1 is not surjective. //

<u>Determination of $\pi_n(S^n)$</u>

Still regarding S^n as $\mathbb{R}^n \cup \infty$, we begin by constructing a special family of maps $S^n \to S^n$.

First choose a fixed map $\varphi : S^n \to S^n$ which is homotopic to the identity, and is such that $\varphi(\xi) = \infty$ if $\|\xi\| \geq 1$.
Then define $\tilde{\varphi} : S^n \to S^n$ by $\tilde{\varphi}(x_1, \dots, x_n) = \varphi(-x_1, x_2, \dots, x_n)$.

Now let A and B be two disjoint finite subsets of $\mathbb{R}^n$ such that $\|x-y\| \geq 2$ if $x, y \in A \cup B$. Define $f_{A,B} : S^n \to S^n$ by

$$
\begin{aligned}
f_{A,B}(x) &= \varphi(x-\xi) \quad \text{if } \|x-\xi\|<1 \text{ for some } \xi \in A \\
&= \tilde{\varphi}(x-\xi) \quad \text{if } \|x-\xi\|<1 \text{ for some } \xi \in B \\
&= \infty \quad \text{otherwise.}
\end{aligned}
$$

It is clear that the homotopy class of $f_{A,B}$ depends only on the number of points in A and B. In fact it depends only on the difference $|A|-|B|$. For if $e_1 = (1\ 0 \dots 0) \in A$ and $-e_1 \in B$, and all the other points of $A \cup B$ are distant ≥ 2 from 0, then the formula

$$
\begin{aligned}
f_t(x) &= f_{A,B}(x) \quad \text{if} \\
&= \varphi(x-(1-2t)e_1) \quad \text{if } \|x-(1-2t)e_1\| \leq 1 \text{ and } x_1 \geq 0 \\
&= \tilde{\varphi}(x+(1-2t)e_1) \quad \text{if } \|x+(1-2t)e_1\| \leq 1 \text{ and } x_1 \leq 0 \\
&= f_{A,B}(x) \quad \text{otherwise}
\end{aligned}
$$

defines a homotopy between $f_{A,B}$ and $f_{A-\{e_1\}, B-\{-e_1\}}$.

I shall now prove that any map $f : S^n \to S^n$ is homotopic to a map of the form $f_{A,B}$. That will complete the more difficult half of the calculation of $\pi_n(S^n)$. The other half of the proof is the fact that maps of different degrees are not homotopic, which I shall postpone for the present.

We first follow precisely the argument used above for $\pi_k(S^n)$ to replace f by a map such that $f^{-1}(D^n)$ is contained in a finite union of simplexes each mapped linearly by f. By choosing $y \in D^n$ which is not in the image of any simplex of dimension $< n$, and then translating f by $-y$, we deform f to a map with the property

(†) $f^{-1}(0)$ is finite, and for each $x \in f^{-1}(0)$ there is $g_x \in GL_n\mathbb{R}$ such that $f(x') = g_x \cdot (x'-x)$ for all x' in a neighbourhood of x.

Now f is homotopic to $\varphi_\varepsilon \circ f$, where $\varphi_\varepsilon(y) = \varphi(\varepsilon^{-1} y)$.

Because $\varphi_\varepsilon(y) = \infty$ when $\|y\| \geqslant \varepsilon$, the map $\varphi_\varepsilon \circ f$, for sufficiently small ε, depends only on the finite set $f^{-1}(0)$, the matrices g_x for $x \in f^{-1}(0)$, and the number ε. From the following lemma we easily see that $\varphi_\varepsilon \circ f$ is homotopic to one of the maps $f_{A,B}$.

Lemma (3.5) Any $g \in GL_n\mathbb{R}$ can be joined by a path to either 1 or $\begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$.

Proof It is enough to show that if $n > 1$ any g can be joined to a matrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$ with $h \in GL_{n-1}\mathbb{R}$. To do this, choose an orthonormal basis $\{e_1, v_2, v_3, \ldots, v_n\}$ of $\mathbb{R}^n$ such that

$$g e_1 = \lambda \cos\theta \; e_1 + \lambda \sin\theta \; v_2$$

with $\lambda = \|g e_1\|$. Define $g_t \in GL_n\mathbb{R}$ by
$$\begin{cases} g_t e_1 = \cos t\theta \; e_1 + \sin t\theta \; v_2 \\ g_t v_2 = -\sin t\theta \; e_1 + \cos t\theta \; v_2 \\ g_t v_i = v_i \quad \text{for } i > 2 . \end{cases}$$

Then $g_t^{-1} g$ is a path in $GL_n\mathbb{R}$ ~~to~~ from g to a matrix of the form $\begin{pmatrix} \lambda & * \\ 0 & h \end{pmatrix}$. This can be joined linearly to $\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}$ in $GL_n\mathbb{R}$. //

§4 Fibrations

Definition (4.1) A *locally trivial fibration* is a map $p: Y \to X$ such that each point $x \in X$ has a neighbourhood U over which p is *trivial*, i.e. such that $p^{-1}(U)$ is homeomorphic to $U \times p^{-1}(x)$ by a fibre-preserving map, i.e. by a map taking $p^{-1}(x')$ to $\{x'\} \times p^{-1}(x)$ for all $x' \in U$.

Vocabulary

X is called the *base* of the fibration
Y . . . . *total space* . . . .
p . . . . *projection*
$p^{-1}(x)$ . . . . *fibre* at x.

A map $s: X \to Y$ such that $p \circ s = \text{identity}$, i.e. such that $s(x) \in p^{-1}(x)$ for all x, is called a *section*.

Remark If the base is connected, then all fibres are homeomorphic.

Examples

(i) The archetypal example is the *Möbius band*, which has base S^1 and fibres homeomorphic to $\mathbb{R}$. It is the case $n = 2$, $k = 1$ of the fibration $p: Aff_k(\mathbb{R}^n) \longrightarrow Gr_k(\mathbb{R}^n)$, where $Aff_k(\mathbb{R}^n)$ is the space of k-dimensional affine subspaces of $\mathbb{R}^n$, and p takes a subspace to the parallel subspace through 0. The fibres of p are copies of $\mathbb{R}^{n-k}$.

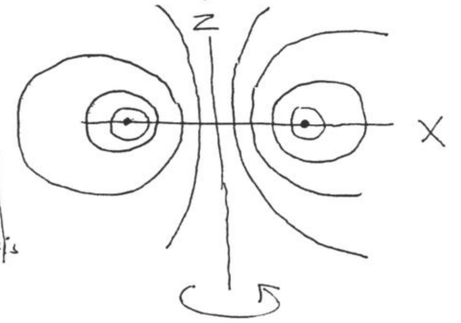
(ii) The *Hopf fibration* $p: S^3 \to S^2$ is given by $p(z_0, z_1) = z_1/z_0 \in \mathbb{C} \cup \{\infty\}$ where $S^3 = \{(z_0, z_1) \in \mathbb{C}^2 : |z_0|^2 + |z_1|^2 = 1\}$. Its fibres are circles, any two of which are linked. (Another description is as $p: \mathbb{R}^3 \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ with

$p^{-1}(0) = (Z\text{-axis}) \cup \{\infty\}$

$p^{-1}(\infty) = $ unit circle in XY-plane

$p^{-1}(\{z \in \mathbb{C} : |z| = \lambda\}) = \left\{\begin{array}{l}\text{torus } T_\lambda \text{ obtained} \\ \text{by rotating a circle in} \\ \text{the XZ-plane about the Z-axis}\end{array}\right.$

$p^{-1}(z) = \left\{\begin{array}{l}\text{curve on } T_{|z|} \\ \text{which links once} \\ \text{both } p^{-1}(0) \text{ and } p^{-1}(\infty)\end{array}\right.$



There are three similar Hopf fibrations $p: S^1 \to S^1$ with fibre S^0, $p: S^7 \to S^4$ with fibre S^3, and $p: S^{15} \to S^8$ with fibre S^7, got by the same formula but with $\mathbb{C}$

replaced by $\mathbb{R}$, $\mathbb{H}$ = (quaternions), and $\mathbb{O}$ = (Cayley numbers).

(iii) Let $\tilde{C}_k(\mathbb{R}^n)$ = {ordered distinct k-tuples in $\mathbb{R}^n$} $\subset \mathbb{R}^{kn}$.
Define $p : \tilde{C}_k(\mathbb{R}^n) \to \tilde{C}_{k-1}(\mathbb{R}^n)$ by forgetting the k^{th} point.
Then the fibre at $(\xi_1, \ldots, \xi_{k-1})$ is $\mathbb{R}^n - \{\xi_1, \ldots, \xi_{k-1}\}$.

(iv) The <u>Stiefel manifold</u> $V_k(\mathbb{R}^n)$ = {ordered orthonormal k-tuples in $\mathbb{R}^n$} $\subset \mathbb{R}^{kn}$.
(E.g. $V_1(\mathbb{R}^n) = S^{n-1}$ and $V_n(\mathbb{R}^n) = O_n$.)
Define $p : V_k(\mathbb{R}^n) \to V_{k-1}(\mathbb{R}^n)$ by forgetting the k^{th} vector.
The fibres now are $\cong S^{n-k}$.

(v) Let $p : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$ take a k-tuple to the subspace it spans.
The fibres are $\cong O_k$.

(vi) If G is a Lie group — e.g. a closed subgroup of $GL_n(\mathbb{R})$ — and H is a closed subgroup then $p : G \to G/H$ is a locally trivial fibration with fibres $\cong H$. (The Hopf fibration is the case G = unit quaternions = SU_2, $H = \mathbb{T}$ = unit complex numbers = $\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : |\lambda| = 1 \right\} \subset SU_2$.

(vii) For a smooth manifold X let T_xX be the vector space of tangent vectors to X at x, and $TX = \bigcup_{x \in X} T_xX$. Let $p : TX \to X$ take T_xX to x.
There is a topology on TX such that p is a locally trivial fibration.
E.g. if $X = S^{n-1} \subset \mathbb{R}^n$ then $T_xX \cong \{ y \in \mathbb{R}^n : \langle x, y \rangle = 0 \}$, and
$TX = \{ (x, y) \in \mathbb{R}^n \oplus \mathbb{R}^n : \|x\| = 1 \text{ and } \langle x, y \rangle = 0 \}$.
Notice that $V_2(\mathbb{R}^n)$ = {unit tangent vectors to S^{n-1}} $\subset TS^{n-1}$.

<u>Sample proofs of local triviality</u>

(ii) For $p : S^3 \to S^2$, let $\mathbb{T} = \{ u \in \mathbb{C} : |u| = 1 \}$. Define

$$p^{-1}(S^2 - \{\infty\}) \cong (S^2 - \{\infty\}) \times \mathbb{T} \quad \text{by} \quad (z_0, z_1) \mapsto (z_1/z_0,\ z_0/|z_0|),$$
$$p^{-1}(S^2 - \{0\}) \cong (S^2 - \{0\}) \times \mathbb{T} \quad \text{by} \quad (z_0, z_1) \longmapsto (z_1/z_0,\ z_1/|z_1|).$$

(v) For $p : V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$, given $P \in Gr_k(\mathbb{R}^n)$ let $U = \{ Q \in Gr_k(\mathbb{R}^n) : Q \cap P^{\perp} =$
Then U is a neighbourhood of P, and $p^{-1}(U) \to U \times p^{-1}(P)$ by
$(v_1, \ldots, v_k) \mapsto$ (Gram-Schmidt orthonormalization of $(w_1, \ldots, w_k)$, where w_i is the orthogonal projection of v_i on to P.)

The exact sequence for a fibration

A sequence of groups and homomorphisms

$$\cdots \to G_{k-1} \xrightarrow{\varphi_{k-1}} G_k \xrightarrow{\varphi_k} G_{k+1} \to \cdots$$

is exact if $\mathrm{image}(\varphi_{k-1}) = \ker(\varphi_k)$ for all k. (The notion makes sense even for sets with base-points.)

Let $p : Y \to X$ be a locally trivial fibration.
Choose base-points $y_0 \in Y$ and $x_0 \in X$ such that $p(y_0) = x_0$.
Let $F = p^{-1}(x_0)$. The inclusion $i : F \to Y$ and projection $p : Y \to X$ induce homomorphisms $i_* : \pi_k(F, y_0) \to \pi_k(Y, y_0)$ and $p_* : \pi_k(Y, y_0) \to \pi_k(X, x_0)$.

Theorem (4.1) There is a homomorphism $\partial : \pi_k(X, x_0) \to \pi_k(F, y_0)$, for $k \geq 1$, such that the sequence

$$\cdots \to \pi_{k+1}(X) \xrightarrow{\partial} \pi_k(F) \xrightarrow{i_*} \pi_k(Y) \xrightarrow{p_*} \pi_k(X) \xrightarrow{\partial} \pi_{k-1}(F) \to \cdots$$
$$\cdots \to \pi_1(X) \xrightarrow{\partial} \pi_0(F) \to \pi_0(Y) \to \pi_0(X)$$

is exact.

Example $\pi_k(S^1) = 0$ for $k > 1$, as ΩS^1 is a sequence of copies of the vector space $\Omega \mathbb{R}$, which is contractible. So applying (4.1) to the Hopf fibration $p : S^3 \to S^2$ we find that

$$p_* : \pi_k(S^3) \xrightarrow{\cong} \pi_k(S^2) \quad \text{for } k \geq 3$$

In particular, $\pi_3(S^2) \cong \mathbb{Z}$, with generator p.

Remark In (4.1) the terms $\pi_0(\ )$ are only sets, but the proof of exactness below gives us a little more information than the exactness indicates, as follows.

Proposition (4.2) The group $\pi_1(X, x_0)$ acts on the set $\pi_0(F)$. The set of orbits is the subset of $\pi_0(Y)$ which maps to the base-point in $\pi_0(X)$, and the stabilizer of the component of $y \in F$ is the image of $\pi_1(Y, y)$ in $\pi_1(X, x_0)$.

The construction of the map ∂ of (4.1) depends on the "lifting of homotopies". For a given map $p : Y \to X$, if $f : Z \to X$ then a <u>lift</u> of f means a map $g : Z \to Y$ such that $p \circ g = f$.

<u>Definition (4.3)</u> (a) A map $p : Y \to X$ has the <u>homotopy lifting property</u> (HLP) for a space Z if for every homotopy(*) $\{f_t : Z \to X\}$ and every lift $g_0 : Z \to Y$ of $f_0 : Z \to X$ there is a homotopy $\{g_t\}$ of g_0 which lifts $\{f_t\}$.

(b) $p : Y \to X$ has the <u>relative HLP</u> for (Z, Z_0), where Z_0 is a closed subspace of Z if for every homotopy $\{f_t : Z \to X\}$, every lift g_0 of f_0, and every <u>partial</u> lift, i.e. every homotopy $\{h_t : Z_0 \to Y\}$ which lifts $\{f_t | Z_0\}$ and has $h_0 = g_0 | Z_0$, there is a homotopy $\{g_t\}$ of g_0 which both lifts $\{f_t\}$ and extends $\{h_t\}$.

$$\begin{tikzcd}
(Z \times \{0\}) \cup (Z_0 \times [0,1]) \arrow[r, "g_0 \cup \{h_t\}"] \arrow[d, hook] & Y \arrow[d, "p"] \\
Z \times [0,1] \arrow[r, "\{f_t\}"'] \arrow[ur, dashed, "\{g_t\}"] & X
\end{tikzcd}$$

<u>Theorem (4.4)</u> A locally trivial fibration has the relative HLP for any polyhedron Z and subpolyhedron Z_0

In fact much more is true. Providing X is paracompact (e.g. metrizable) a locally trivial fibration has the HLP for <u>any</u> space Z.

A map p with the property of (4.4) is called a <u>Serre fibration</u>, while one with the HLP for all Z is called a <u>Hurewicz fibration</u>. There is a still weaker concept called a <u>quasifibration</u>, which we shall meet later.

(locally trivial) $\Rightarrow$ (Hurewicz) $\Rightarrow$ (Serre) $\Rightarrow$ (quasifibration).

Our proof of the homotopy sequence is valid for Serre fibrations, but the exactness holds even for quasifibrations.

Assuming (4.4) for the moment, let us give the definition of $\partial : \pi_k(X) \to \pi_{k-1}(F)$. We represent an element f of $\pi_k(X)$ by a

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(*) The notation $\{f_t\}$ means that $(t, z) \mapsto f_t(z)$ is a continuous map $[0,1] \times Z \to X$.

homotopy $\{f_t : S^{k-1} \to X\}$ beginning and ending at the base-point (e.g. $f_0(S^{k-1}) = f_1(S^{k-1}) = x_0$). We lift $\{f_t\}$ to $\{g_t\}: S^{k-1} \to Y\}$, beginning with $g_0(S^{k-1}) = y_0$. Then $g_1 : S^{k-1} \to Y$ maps S^{k-1} into $p^{-1}(x_0) = F$, and represents ∂f.

To see that ∂ is well-defined we must check two things. First, if we replace the lift $\{g_t\}$ by another lift $\{\tilde{g}_t\}$ we can apply the relative HLP to the homotopy $\{\hat{f}_t : S^{k-1} \times [0,1] \to X\}$ given by $\hat{f}_t(x,s) = f_t(x)$. Regard g_t and $\tilde{g}_t$ together as a lift of $\hat{f}_t | (S^{k-1} \times \{0,1\})$, and extend this partial lift of $\hat{f}_t$ to a lift $\hat{g}_t$, beginning with $\hat{g}_0(S^{k-1} \times [0,1]) = y_0$. Then $\hat{\ }_1 : S^{k-1} \times [0,1] \to Y$ is a homotopy between g_1 and $\tilde{g}_1$ in F.

Secondly, we must check that changing the homotopy $\{f_t\}$ to another one $\{\tilde{f}_t\}$ which represents the same element of $\pi_k(X)$ does not change the class of g_1. For this, think of $\{f_t\}$ and $\{\tilde{f}_t\}$ as maps $[0,1] \times S^{k-1} \to X$. If we lift a homotopy between them, starting with $\{g_{t,0}\} = \{g_t\}$, we get $\{g_{t,s}\}$, and can assume that $g_{0,s}(S^{k-1}) = y_0$ for all $s \in [0,1]$. Then $\{\tilde{g}_t = g_{t,1}\}$ is a lift of $\{\tilde{f}_t\}$, while $\{g_{1,s}\}$ is a homotopy between g_1 and $\tilde{g}_1$ through maps $S^{k-1} \to F$.

The map $\partial : \pi_k(X) \to \pi_{k-1}(F)$ is now well-defined, and we must show that it is a homomorphism. Let us consider the case $k = 2$. Represent two elements of $\pi_2(X)$ by maps $f, \tilde{f} : [0,1] \times [0,1] \to X$, both taking the boundary of the square to x_0. Then $[f] + [\tilde{f}]$ in $\pi_2(X)$ is represented by a map which can be depicted $\begin{array}{|c:c|}\hline f & \tilde{f} \\ \hline\end{array}$. This is homotopic to $\begin{array}{|c:c|}\hline x_0 & \tilde{f} \\ \hdashline f & x_0 \\ \hline\end{array} = F$.
Regard F as a homotopy of its left-hand edge, and lift it to a map $[0,1] \times [0,1] \to Y$ taking the left-hand, top, and bottom edges all to y_0. If g and $\tilde{g}$ are lifts of $f, \tilde{f}$ then we can clearly choose the lift of F to be $\begin{array}{|c:c|}\hline y_0 & \tilde{g} \\ \hdashline g & h \\ \hline\end{array}$, where h is constant along horizontal lines. The end of this lift, i.e. its right-hand edge, is $\tilde{g}_1 * g_1$, where g_1 and $\tilde{g}_1$ are

the ends of g and $\tilde{g}$. So $\partial F = (\partial\tilde{f}) * (\partial f)$.

The case $k > 2$ can be treated by essentially the same argument.

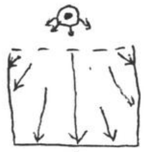
The proof of the exactness of the sequence (4.1) can now be performed by repeated homotopy liftings; there seems no point in spelling it out.

Proof of (4.4)

I shall assume that for any open covering $\{U_\alpha\}$ of a polyhedron W there is some polyhedral subdivision of W for which each simplex Δ_σ is contained in some U_α.

Choose an open covering $\{U_\alpha\}$ of X such that $p: Y \to X$ is trivial over each U_α. Then, given $f: Z \times [0,1] \to X$, choose a subdivision of Z and a subdivision $0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0,1]$ such that $f(\Delta_\sigma \times I_i) \subset$ some U_α, for each i and each simplex Δ_σ of Z, where $I_i = [t_{i-1}, t_i]$.

We are given $g: W \to Y$, where $W = (Z \times \{0\}) \cup (Z_0 \times [0,1]) \subset Z \times [0,1]$, and must extend it to $g: Z \times [0,1] \to Y$ lifting f. We make the extension inductively over the subsets $\Delta_\sigma \times I_i$, ordering them lexicographically first by i and then by the dimension of σ. For the inductive step one has a map $h_\sigma: W_\sigma \to F$, and must extend it to $g_\sigma: \Delta_\sigma \times I_i \to F$, where $W_\sigma = (\Delta_\sigma \times \{t_{i-1}\}) \cup (\partial\Delta_\sigma \times I_i)$. We can define $g_\sigma = h_\sigma \circ r_\sigma$, where $r_\sigma: \Delta_\sigma \times I_i \to W_\sigma$ is the identity on W_σ. A picture of r_σ is



I_i

Δ_σ

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Remark A better, if slightly more sophisticated, way of treating the homotopy theory of a Hurewicz fibration is to lift the homotopy $\Omega X \times F \times [0,1] \to X$ given by $(\gamma, y, t) \mapsto \gamma(t)$ to a map $\Omega X \times F \times [0,1] \to Y$ beginning with $(\gamma, y, 0) \mapsto y$. The end of the lift is a map $m: \Omega X \times F \to F$ which is an "action of ΩX on F up to homotopy", in the sense that (a) $1.y = y$ and (b) the two obvious maps $\Omega X \times \Omega X \times F \to F$ are homotopic. (The map m is unique up to homotopy.)

The map $\partial: \pi_k(X) \to \pi_{k-1}(F)$ is then induced by the map $\Omega X \to F$ expressing the "action of ΩX on $y_0 \in F$".