Michaelmas Term 1996
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Part 3 Algebraic Topology — Problem Sheet 1

1. Prove that being homotopic is an equivalence relation on the set of maps from one space to another. *Hint*: in proving transitivity - and in very many similar situations - it helps to observe: if a space X is the union of a *finite* number of *closed* subsets X_i then a map $f : X \to Y$ is continuous if its restriction to each X_i is continuous.

2. The circle S^1 can be defined as a subset of $\mathbb{R}^2$ or as a quotient space of the interval [0,1]. Prove that the spaces so defined are homeomorphic. *Hint*: in many questions of this kind one can use the theorem that a continuous bijection from a compact space to a Hausdorff space has a continuous inverse.

3. Prove that $S^n \wedge S^m \cong S^{n+m}$.

4. (a) Show that ~~Do the same for~~ the following three definitions of the real projective space $P^{n-1}_{\mathbb{R}}$ are equivalent:

   (i) $\{x \in \mathbb{R}^n : ||x|| = 1\}/\sim$, where $x \sim x'$ iff $x = \pm x'$

   (ii) $\{x \in \mathbb{R}^n : x \neq 0\}/\sim$, where $x \sim x'$ iff $x = \lambda x'$ with $\lambda \in \mathbb{R}$.

   (iii) $n \times n$ real symmetric matrices A of rank 1 such that $A^2 = A$.

   (b) The Grassmannian $Gr_k\mathbb{R}^n$ is the set of k dimensional vector subspaces of $\mathbb{R}^n$. What are the three corresponding definitions of its topology, and why are they equivalent?

5. If x and y are points of a space X, define $x \sim y$ if there is a path in X from x to y. Prove that this defines an equivalence relation on x.
   If X is a manifold with boundary and corners, prove that X is path-connected if and only if it is connected.

6. If X is a path-connected space, and $x_0, x_1, x_2 \in X$, prove that $\pi_1(X, x_2) = 0$ if and only if any two paths from x_0 to x_1 are homotopic through paths from x_0 to x_1.

7. A space X is the union of two simply connected open subsets U and V. Prove that X is simply connected if $U \cap V$ is path-connected. Deduce that S^n is simply connected if $n > 1$.

8. If X and Y are two spaces, the *compact-open* topology on the set $\mathrm{Map}(X;Y)$ of continuous maps from X to Y is the coarsest topology such that

$$M_{K,U} = \{f \in \mathrm{Map}\,(X;Y) : f(K) \subset U\}$$

   is open for every compact subset K of X and open subset U of Y.
   If X is compact, and Y is a metric space, prove that the metric $\widetilde{d}$ defined by

$$\widetilde{d}(f,g) = \sup_{x \in X}(d(f(x), g(x))$$

   defines the compact-open topology on $\mathrm{Map}(X;Y)$.

9. If X is a compact space, prove that

$$\mathrm{Map}(Z \times X; Y) \cong \mathrm{Map}(Z; \mathrm{Map}(X; Y))$$

as a set. (Here $\mathrm{Map}(X;Y)$ has the compact-open topology).

Prove that a space is simply connected if and only if the space $\mathcal{L}X = \mathrm{Map}(S^1; X)$ is connected.

10. If X is a space with a base-point, prove that the composition map $\Omega X \times \Omega X \to \Omega X$ is associative up to homotopy.

11. The *one-point compactification* of a space X is the space $X^+ = X \cup \{\infty\}$ whose open subsets are the open subsets of X and the sets of the form $V \cup \{\infty\}$, where $X - V$ is a compact closed subset of X. Check that this defines a topology on X^+, and that X^+ is compact.
Prove that the one-point compactification of $\mathbb{R}^n$ is homeomorphic to the standard sphere S^n. *Hint*: use stereographic projection.

Regarding the one-point compactification as a space with base-point ∞, prove that $X^+ \wedge Y^+ \cong (X \times Y)^+$.

Part III Algebraic Topology 1996 Problem Sheet 2

1. (i) Prove that there is a neighbourhood U of the origin in $\mathbb{R}^n$, and a homeomorphism $f_u : D^n \to D^n$ for each $u \in U$, such that (a) $f_u(0) = u$, (b) $f_u(x) = x$ if $\|x\| = 1$, and (c) $(u,x) \mapsto f_u(x)$ is a continuous map $U \times D^n \to D^n$.

(ii) Prove that the forgetful map $\tilde{C}_{k+1}(\mathbb{R}^n) \to \tilde{C}_k(\mathbb{R}^n)$ is a locally trivial fibration.

(iii) For a space X with base-point x_0, let PX denote the space of paths $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$. Define $p : PX \to X$ by $p(\gamma) = \gamma(1)$. Prove that $p : PX \to X$ is a locally trivial fibration if X is a manifold.

(iv) Prove that for <u>any</u> space X the map p has the homotopy lifting property for all spaces.

2. Prove that the map $O_n \to S^{n-1}$ which takes a matrix to its first column is a locally trivial fibration. Deduce that $\pi_k(O_n)$ is independent of n if $n \geq k+2$.

3. Prove that $\pi_i(V_k(\mathbb{R}^n)) = 0$ if $i < n-k$, where V_k is the Stiefel manifold

4. If $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in T}$ are open coverings of X then $\mathcal{U}$ is a <u>refinement</u> of $\mathcal{V}$ if one can choose $\theta : S \to T$ so that $U_\alpha \subset V_{\theta(\alpha)}$ for all α. Prove that θ induces a cochain map $\theta^* : \check{C}(\mathcal{V}) \to \check{C}(\mathcal{U})$. If $\tilde{\theta} : S \to T$ is another choice, prove that θ^* and $\tilde{\theta}^*$ are homotopic. Deduce that one can define <u>Čech cohomology</u> by $\check{H}^k(X) = \varinjlim_{\mathcal{U}} H^k(\mathcal{U})$.

5. Prove that the standard simplex Δ^n can be identified with

$$\{(t_1, \ldots, t_n) \in \mathbb{R}^n : 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n \leq 1\}.$$

What are the $n+1$ faces of Δ^n in this description?

For each p-element subset A of $\{1, \ldots, n\}$ define $\pi_A : \Delta^n \to \Delta^p$ by forgetting t_i when $i \notin A$. If A' is the complement of A, prove that $\pi_A \times \pi_{A'} : \Delta^n \to \Delta^p \times \Delta^{n-p}$ is an embedding, with image $\Delta_{(A)}$, say. Prove that the simplexes $\Delta_{(A)}$ cover $\Delta^p \times \Delta^{n-p}$, and that two of them intersect only along their boundaries.

How is this related to the proof of the homotopy property of H^*?

6. For some $n \in \mathbb{Z}$ let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x+1) = f(x) + n$ for all x. Define $\sim$ on $\mathbb{R}$ by $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$, so that $\mathbb{R}/\sim \,\cong S^1$. Prove that f induces a map $\tilde{f} : S^1 \to S^1$ of degree n.

7. Define $f : \mathbb{C}^n \to \mathbb{C}^n$ by $f(x_1, \ldots, x_n) = (\sigma_1, \ldots, \sigma_n)$, where the σ_i are the elementary symmetric functions of $\{x_1, \ldots, x_n\}$. Prove that f extends to a continuous map $f : S^{2n} \to S^{2n}$, where $S^{2n} = \mathbb{C}^n \cup \{\infty\}$. What is its degree?

What would be answer be if $\mathbb{C}^n$ were replaced by $\mathbb{R}^n$? [Hint: This is easy.]

8. Let

$$\begin{array}{ccccccccc} 0 & \to & A_1 & \to & A_2 & \to & A_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B_1 & \to & B_2 & \to & B_3 & \to & 0 \end{array}$$

be a commutative diagram of abelian groups with exact rows. Prove that if two of the vertical maps are isomorphisms, so is the third.

More generally, if

$$\begin{array}{ccccccccc} A_0 & \to & A_1 & \to & A_2 & \to & A_3 & \to & A_4 \\ {\scriptstyle f_0}\downarrow & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & {\scriptstyle f_4}\downarrow \\ B_0 & \to & B_1 & \to & B_2 & \to & B_3 & \to & B_4 \end{array}$$

is commutative with exact rows prove

f_0 surjective, f_1 and f_3 injective $\Rightarrow$ f_2 injective

f_4 injective, f_1 and f_3 surjective $\Rightarrow$ f_2 surjective.

9. Calculate $H^*(S^1 \times S^1)$ additively by using the Mayer-Vietoris sequence.

10. Calculate $H^*(X_p)$ by the M-V sequence, where X_p is the lens space $S^3/\sim$, with $S^3 \subset \mathbb{C}^2$ and $(z_0, z_1) \sim (z_0', z_1') \Leftrightarrow z_i' = \lambda z_i$ for some λ such that $\lambda^p = 1$. [Consider the regions where $|z_0| \leqslant |z_1|$ and where $|z_0| \geqslant |z_1|$.]

11. Calculate $H^*(P^2_{\mathbb{R}}; A)$ additively, where A is an arbitrary abelian group.

12. Use the M-V sequence to prove that, for any space X,

$$H^*(S^n) \otimes H^*(X) \longrightarrow H^*(S^n \times X)$$

as a ring. Hence determine the ring-structure of $H^*(S^n \times S^m)$.

Algebraic Topology Part III Mich. 1996 Problem Sheet 3

1. Let $\{U_\alpha\}$ be an open covering of a manifold X. Prove that to orient X it is enough to orient each U_α so that for each α, β the induced orientations of $U_\alpha \cap U_\beta$ agree.

2. (i) Establish the Mayer-Vietoris sequence for cohomology with compact supports.
(ii) If an oriented manifold X of dimension n is the union of two open subsets X_1 and X_2 prove that the diagram of Poincaré duality maps

$$\begin{array}{ccc} H^k_{cpt}(X) & \xrightarrow{d_{MV}} & H^{k+1}_{cpt}(X_{12}) \\ \downarrow PD & & \downarrow PD \\ H^{n-k}(X)^* & \xrightarrow{(d_{MV})^*} & H^{n-k-1}(X_{12})^* \end{array}$$

is commutative.

3. (i) Let F be a contravariant additive functor from A-modules to B-modules, where A and B are commutative rings. ("Additive" means that for all A-modules P, Q the map $F^*: \mathrm{Hom}_A(P;Q) \to \mathrm{Hom}_B(F(Q);F(P))$ is a homomorphism of abelian groups.) Prove that F is <u>exact</u>, i.e. takes exact sequences to exact sequences, if it takes short exact sequences (i.e. those of the form $0 \to P \to Q \to R \to 0$) to short exact sequences.
(ii) If $A = \mathbb{Z}$, $B = \mathbb{Z}/n$, and $F(P) = \mathrm{Hom}(P;\mathbb{Z}/n)$, prove that F is not exact. But prove that the same formula does define an exact functor if we take $A = B = \mathbb{Z}/n$.

4. Consider the double covering map $p: S^n \to \mathbb{P}^n_{\mathbb{R}} = \mathbb{P}^n$. If $(x_0,..,x_k)$ are nearby points of $\mathbb{P}^n$ observe that there are exactly two sets $(x_0',\dots,x_k')$ and $(x_0'',\dots,x_k'')$ of nearby points of S^n such that $p(x_i') = p(x_i'') = x_i$. Show that one can define a homomorphism $p_*: H^k(S^n) \to H^k(\mathbb{P}^n)$ by

$$(p_* c)(x_0,..,x_k) = c(x_0',..,x_k') + c(x_0'',\dots,x_k'').$$

If n is odd, prove that $p^* \circ p_*$ is multiplication by 2. Is the same true for $p_* \circ p^*$? Deduce that $\mathbb{P}^n$ is orientable, and also that $2\alpha = 0$ if $\alpha \in H^k(\mathbb{P}^n)$ with $k \neq 0, n$.

5. If X is a connected manifold which is not orientable, prove that $H^n_{cpt}(X;\mathbb{Z}) \cong \mathbb{Z}/2$.

6. Calculate $H^k_{cpt}(X;\mathbb{Z})$ for all k when X is the (open) Möbius band.

7. Let $p: E \to X$ be an n-dimension real vector bundle, and let ε_x be a generator of $H^n(E_x, E_x - \{0\})$ for each $x \in X$. If $\alpha: U \times \mathbb{R}^n \to p^{-1}(U)$ is a local trivialization near x, one says that the family $\{\varepsilon_y\}$ is locally constant near x if $\alpha_y^*(\varepsilon_y) \in H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$ is independent of y for y near x, where $\alpha_y(\xi) = \alpha(y, \xi)$. Prove that this notion does not depend on the choice of α.

8. Let $p: E \to X$ be a real vector bundle. Suppose that an inner product is given on each fibre E_x. Prove that the following are equivalent.

(i) The norm $E \to \mathbb{R}$ is continuous.

(ii) E possesses local trivializations $p^{-1}(U) \cong U \times \mathbb{R}^n$ which take the inner products on the fibres to the standard inner product on $\mathbb{R}^n$.

[Hint First show that (i) $\Rightarrow$ $\langle s_1, s_2 \rangle$ is continuous, when s_1, s_2 are sections. Then use Gram-Schmidt.]

9. ("Invariance of domain") Prove that if U is an open subset of $\mathbb{R}^n$, and $f: U \to \mathbb{R}^n$ is injective and continuous, then $f(U)$ is open. [Use the Alexander duality theorem, assuming that it holds for an arbitrary compact subset of $\mathbb{R}^n$. If D is a small closed disc in U with boundary S, show that the complement of $f(S)$ has two connected components, of which one, say V, is bounded, while the complement of $f(D)$ is connected. Deduce that $f(\mathring{D}) = V$.]

10. Let X be a space such that $H^k(X) = H^k(X; \mathbb{Z})$ is finitely generated and free for each k. Let $\{\xi_i\}_{i \in B_k}$ be a basis for $H^k(X)$, and let $\{\eta_i\}$ be its image in $H^k(X; \mathbb{Z}/p)$, where p is prime. Use the Bockstein sequence to show that $\{\eta_i\}$ is a basis for $H^k(X; \mathbb{Z}/p)$.

Suppose further that X is a compact oriented n-manifold, and let

$$a_{ij} = \int_X \xi_i \xi_j \in \mathbb{Z}, \quad \text{for } i \in B_k \text{ and } j \in B_{n-k}.$$

Use Poincaré duality with coefficients in $\mathbb{Z}/p$ to show that (a_{ij}) is a square matrix with determinant ± 1, and deduce that the duality map $H^k(X) \to \mathrm{Hom}(H^{n-k}(X); \mathbb{Z})$ is an isomorphism.

Part III Algebraic Topology Michaelmas 1996 Problem Sheet 4

1. Let $p : E \to X$ be a smooth real n-dimensional oriented vector bundle on a smooth manifold X. (Thus, in particular, E is a smooth manifold, and the zero-section is a smooth submanifold.) If $s : X \to E$ is a smooth section (i.e. its graph is smooth) whose graph intersects the zero-section $X \subset E$ transversally in Z, prove that Z is co-oriented, and that its cohomology class in X is the Euler class of E.

2. (i) Let U be an open subset of $\mathbb{R}^n$, and $f : U \to \mathbb{R}^n$ a map with an isolated fixed point at $u \in U$. The <u>multiplicity</u> of the fixed point is defined as the degree of the map $x \mapsto (f(x) - x)/\|f(x) - x\|$ from S to S^{n-1}, where S is a small sphere with centre u. If f is a linear map, prove that f has an isolated fixed point if and only if $f - 1$ is invertible, and that its multiplicity is the sign of $\det(f-1)$. Explain this geometrically when $n = 1$.

(ii) Let X be a compact smooth oriented n-manifold. If $f : X \to X$ is a smooth map with transversal fixed points $Z = \{x \in X : f(x) = x\}$, prove that the "number of fixed points", defined as $\int_X \varepsilon_Z$, is $\sum_{x \in Z} \text{sign} \det(Df(x) - 1)$. Derive the Lefschetz formula for this number in terms of the action of f^* on $H^*(X; F)$, where F is a field.

3 By considering the obvious map $\mathbb{R}^n - \mathbb{R}^m \to S^{n-m-1}$ prove that $P^{n-1}_{\mathbb{R}} - P^{m-1}_{\mathbb{R}}$ is homeomorphic to the total space of a vector bundle E on $P^{n-m-1}_{\mathbb{R}}$.

4. The tangent space $T_x X$ to a smooth manifold X at x is characterized by the properties (i) $T_x X = X$ if X is a vector space, and (ii) a smooth open embedding $f : X \to Y$ induces an isomorphism $T_x X \to T_{f(x)} Y$. Prove that the tangent space to the Grassmannian $Gr_k(\mathbb{R}^n)$ at W is <u>canonically</u> isomorphic to $\text{Hom}(W; W^\perp)$.

5. Deduce from the preceding question that $T_x P^n_{\mathbb{C}} \oplus \mathbb{C} \cong \underbrace{L_x^* \oplus \ldots \oplus L_x^*}_{n+1}$, where $\{L_x\}$ is the tautological bundle. Deduce that the Euler class $e_{TP^n_{\mathbb{C}}}$ is $n+1$ times the generator of $H^{2n}(P^n_{\mathbb{C}})$.

Find a map $\mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}$ which is homotopic to the identity and has $n+1$ fixed points.