NOTES ON ALGEBRAIC TOPOLOGY

G. Segal - Michaelmas Term '96

1. Introduction

Algebraic topology is the study of the connectivity properties of topological spaces. A topological space is a set in which there is a notion of "proximity", which enables us to speak, for instance, of a "continuous path" in the space. Formally, we describe proximity by giving a preferred collection of subsets, called the *open subsets*. The intuitive idea is that a subset U of X is open if whenever $x \in U$ then $x' \in U$ for all x' sufficiently close to x. I shall assume the reader understands this concept, together with the related ideas

neighbourhoods
closed sets
continuous maps
homeomorphisms
compact spaces
the product of two spaces
the quotient space of a space by an equivalence relation.

1.1 First examples

I shall begin by describing some simple mathematical situations where ideas of algebraic topology play a crucial role.

A space is *connected* if it is not the union of two disjoint non-empty open subsets. A related concept is *path-connectedness*. A *path* in X from x_0 to x_1 is a continuous map

$$\gamma : [0,1] \to X$$

such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. A space is path-connected if there is a path in it from any point to any other. A path-connected space is connected, and the converse is true too for the sort of spaces we shall be interested in.

The fact that the line $\mathbb{R}$ is connected, but becomes disconnected if a point is removed from it, gives us one of the simplest but most basic results of real analysis.

Proposition 1.1.1 (The Intermediate-Value Theorem) *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $f(x_1) > 0$ for some x_1, and $f(x_2) < 0$ for some x_2, then $f(x) = 0$ for some $x \in \mathbb{R}$.*

For otherwise $\mathbb{R} = f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$ would prove $\mathbb{R}$ was disconnected. ∎

After asking whether a space X is connected, the next simplest topological question is whether it is *simply connected.* For this we consider closed paths γ in X, i.e. continuous maps $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1)$. We say that X is simply connected if any such closed path can be deformed continuously to any other, or, equivalently, if the space $\mathcal{L}X$ of all loops in X is path-connected. (Precise definitions of the words just used will be given presently.) Corresponding to the fact that $\mathbb{R} - \{0\}$ is not connected, we have

Proposition 1.1.2 *(i) The space $\mathbb{R}^2$ is simply connected.*

(ii) The space $\mathbb{R}^2 - \{0\}$ is not simply connected. In fact each closed path γ in $\mathbb{R}^2 - \{0\}$ has a winding number deg $(\gamma) \in \mathbb{Z}$, which does not change if γ is continuously deformed.

(iii) The closed path γ_n given by $\gamma_n(t) = (\cos 2\pi nt,\ \sin 2\pi nt)$ has winding number n.

The winding number — or "degree" — counts, of course, the number of times the path "goes around" the origin in an anticlockwise direction.

Just as the intermediate-value theorem arises from the idea of connectivity, so the fundamental theorem of algebra follows at once from Proposition (1.1.2).

Proposition 1.1.3 (The Fundamental Theorem of Algebra) *Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a monic polynomial with complex coefficients. Then $f(z) = 0$ for some $z \in \mathbb{C}$.*

Proof. Identify the complex numbers $\mathbb{C}$ with $\mathbb{R}^2$. Suppose the theorem is false, and that $f(z) \neq 0$ for all z. Consider the curve γ_R in $\mathbb{C} - \{0\}$ defined by $\gamma_R(t) = f(Re^{2\pi i t})$. When $R = 0$ the curve γ_R becomes a point, and so deg $(\gamma_0) = 0$. By continuity, deg $(\gamma_R) = 0$ for all R. Now choose R so that

$$R > |a_1| + |a_2| + \cdots + |a_n|,$$

and define $\gamma_{R,S}$ for $0 \leqslant S \leqslant 1$ by

$$\gamma_{R,S}(t) = z^n + S(a_1 z^{n+1} + \cdots + a_n),$$

where $z = Re^{2\pi i t}$. This is always a closed path in $\mathbb{C} - \{0\}$. When $S = 1$ we have $\gamma_{R,1} = \gamma_R$, while $\gamma_{R,0}(t) = R^n e^{2\pi i n t}$, so that $\deg(\gamma_{R,0}) = n \neq \deg(\gamma_R) = 0$. This is a contradiction. ∎

Before leaving this theme it is worth mentioning a natural generalization. To pass beyond simple connectivity we can consider the maps $\gamma : S^m \to X$, where

$$S^m = \{\xi \in \mathbb{R}^{m+1} : \|\xi\| = 1\}$$

is the unit sphere in $\mathbb{R}^{m+1}$. We say that X is *m-connected* if any two of these can be deformed into each other. Alongside Proposition (1.1.3) we have

Proposition 1.1.4 *A map $\gamma : S^m \to \mathbb{R}^{m+1} - \{0\}$ has a degree deg $(\gamma) \in \mathbb{Z}$ which does not change under deformation. The natural inclusion has degree 1, and the constant map has degree 0.*

This gives us

Proposition 1.1.5 (The Brouwer fixed-point theorem) *Let $f : D^n \to D^n$ be a continuous map, where D^n is the closed unit ball $D^n = \{\xi \in \mathbb{R}^n : \|\xi\| \leqslant 1\}$. Then f has a fixed point, i.e. $f(x) = x$ for some $x \in D^n$.*

Proof. We use exactly the same steps as in proving (1.1.3). If f has no fixed point, define $\gamma_R : S^{n-1} \to \mathbb{R}^n - 0$ by $\gamma_R(\xi) = R\xi - f(R\xi)$ for $0 \leqslant R \leqslant 1$. Then γ_0 is constant, so deg $(\gamma_R) = 0$ for all R. But γ_1 can be deformed to $\gamma_{1,s}$ for $0 \leqslant s \leqslant 1$, where

$$\gamma_{1,s}(\xi) = \xi - sf(\xi).$$

We have $\gamma_{1,1} = \gamma_1$, while $\gamma_{1,0}$ is the inclusion map $S^{n-1} \to \mathbb{R}^n - \{0\}$, which has degree 1. ∎

1.2 Homotopy

In the preceding discussion, the idea of deforming one map to another is crucial. Such deformations are called *homotopies.*

Definition 1.2.1 *Two maps* $f_0, f_1 : X \to Y$ *are* homotopic *if there is a map* $F : X \times [0,1] \to Y$ *such that*

$$F(x,0) = f_0(x) \quad \textit{and}$$
$$F(x,1) = f_1(x) \quad \textit{for all} \quad x \in X.$$

We shall write $f_0 \simeq f_1$ to indicate this relation.

We also say that two space X and X are *homotopy equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that both composites $g \circ f : X \to X$ and $f \circ g : Y \to Y$ are homotopic to the respective identity maps.

Example The sphere S^{n-1} is homotopy equivalent to $\mathbb{R}^n - \{0\}$, for if $F : S^{n-1} \to \mathbb{R}^n - \{0\}$ is the inclusion and $g : \mathbb{R}^n - \{0\} \to S^{n-1}$ is defined by $g(x) = x / \parallel x \parallel$ then $g \circ f$ is the identity, while

$$(x,t) \longmapsto tx + (1-t)\frac{x}{\parallel x \parallel}$$

is a homotopy from $f \circ g$ to the identity.

1.3 Smoothness

Algebraic topology is at first sight concerned with topological spaces and continuous maps. In life, however, smooth maps are much more important than continuous ones. (I shall use "smooth" to mean "indefinitely often continuously differentiable".) Much of the success of algebraic topology has come from developing techniques for translating questions about smooth maps into questions about the homotopy classes of continuous maps.

To give a very simple example of such a translation, define an *immersed* closed curve in the plane $\mathbb{R}^2$ as a smooth map $\gamma : \mathbb{R} \to \mathbb{R}^2$ such that $\gamma(t+1) = \gamma(t)$ for all $t \in \mathbb{R}$, and such that, in addition, the tangent vector $\dot{\gamma}(t) \in \mathbb{R}^2$ is non-zero for all t. It is obvious that any closed curve in $\mathbb{R}^2$ can be deformed to any other. But an immersed curve γ has an integer invariant $\eta(\gamma) \in \mathbb{Z}$ which is the winding number of the tangent vector, i.e. of the map $\dot{\gamma} : [0,1] \to \mathbb{R}^2 - \{0\}$. It is easy to see that an immersed curve γ_0 can be deformed to another one γ_1 *through immersions* if and only if $\eta(\gamma_0) = \eta(\gamma_1)$.

Although this example is almost trivial, the method it embodies can be applied very widely, and permits one to prove, for example, the surprising theorem that the standard sphere S^2 in $\mathbb{R}^3$ can be deformed through immersions to its mirror image.

1.4 The spaces to be considered

Algebraic topology is mostly concerned with very simple spaces, and certainly with spaces which look very simple *locally*, although they may have interesting global properties. The majority of the spaces we consider will be manifolds : a *manifold* is a space X which is locally homeomorphic to Euclidean space $\mathbb{R}^n$, i.e. each point $x \in X$ has a neighbourhood U which is homeomorphic to an open subset of $\mathbb{R}^n$. A somewhat larger class of spaces, including, for example, the closed disc D^n, consists of manifolds with boundary and corners, defined as spaces locally homeomorphic to $\mathbb{R}^p \times (\mathbb{R}_+)^q$ for some p, q, where

$$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geqslant 0\}.$$

In the mathematical questions where algebraic topology plays a role the relevant spaces most often arise from linear algebra. After the sphere S^{n-1} one of the important examples is the *projective space* $P_{\mathbb{R}}^{n-1}$, consisting of all lines through the origin in $\mathbb{R}^n$. More generally we have the *Grassmannian* $Gr_k(\mathbb{R}^n)$, consisting of all k-dimensional vector subspaces of $\mathbb{R}^n$. There are very many situations where we need to study a varying subspace of $\mathbb{R}^n$, but the most important for us is the map $X \to Gr_k(\mathbb{R}^n)$ defined whenever X is a k-dimensional manifold immersed in $\mathbb{R}^n$, which assigns to a point $x \in X$ the tangent space to X at x.

The other spaces from linear algebra which we shall consider in these lectures are the *orthogonal group* O_n, which is a subspace of $\mathbb{R}^{n^2}$, the *Stiefel manifold* $V_k(\mathbb{R}^n)$, which is the set of ordered k-tuples of orthonormal vectors in $\mathbb{R}^n$, and the complex analogues of the spaces just mentioned, i.e.

$$\mathbb{P}_{\mathbb{C}}^{n-1},\ Gr_k(\mathbb{C}^n),\ U_n, \text{ and } V_k(\mathbb{C}^n).$$

1.5 Electromagnetism and de Rham cohomology

To prove the theorems described in §1.1 we must find a way of defining the "degree" of various maps. The most natural way to do this is revealed to us in the study of electromagnetism. Suppose that we have a wire running along the Z-axis in $\mathbb{R}^3$ and carrying a constant electric current. The current produces a magnetic field which at any point x is perpendicular to the plane containing x and the wire. If we move a magnetic pole around any closed path in $\mathbb{R}^3$ which does not intersect the wire, then the magnetic field does an amount of work on the pole which depends only on the winding number of the path around the wire (as well, of course, as on the strength of the current and the pole). The mathematical fact which is exemplified here is that if we have a vector field v defined in an open subset X of $\mathbb{R}^3$, and curl $v = 0$, then the line-integral

$$\int_\gamma v \cdot ds$$

of v around any closed curve γ in X does not change when the curve γ is continuously deformed. This is Stokes's theorem: if γ_0 and γ_1 are two curves in X which are sufficiently close to each other, then together they form the boundary of a piece of surface Σ, and we have

$$\int_{\gamma_1} v \cdot ds - \int_{\gamma_0} v \cdot ds \;=\; \int_\Sigma (\text{curl } v) \cdot dS \;=\; 0.$$

By means of such vector fields v we can measure the winding numbers of curves, and hence detect that a region is not simply connected.

The failure of connectivity in dimension two is equally easily detected: for this we need vector fields v in X with vanishing divergence, such as the electric field of a charged particle. Consider, for instance, a charge at the origin in $\mathbb{R}^3$, generating an electric field v in $X = \mathbb{R}^3 - \{0\}$. Then the surface integral

$$\int_\Sigma v \cdot dS$$

of v over a closed surface Σ, i.e. the "flux" of the electric field through Σ, counts the number of times Σ wraps around the origin. It does not change when Σ is deformed, for if Σ_0 and Σ_1 are two nearby surfaces we have

$$\int_{\Sigma_1} v \cdot dS - \int_{\Sigma_0} v \cdot dS \;=\; \int_R (\text{div } v) dV \;=\; 0,$$

where R is the 3-dimensional region bounded by Σ_0 and Σ_1.

There is another point to be noted. If we write $I_v(\gamma) = \int_\gamma v \cdot ds$ for the invariant of a closed curve γ defined by a vector field v such that curl $v = 0$, then we see that I_v does not change if we add to v the gradient of any smooth function in X, for the integral of a gradient around a closed curve is zero. Similarly, the invariant $I_v(\Sigma)$ of a closed surface Σ associated to v when div $v = 0$ does not change if we add a curl to v.

With the benefit of hindsight we are led to consider for any open subset X of $\mathbb{R}^3$ the following sequence of vector spaces and linear maps

$$0 \longrightarrow \Omega^0(X) \xrightarrow{grad} \Omega^1(X) \xrightarrow{curl} \Omega^2(X) \xrightarrow{div} \Omega^3(X) \longrightarrow 0,$$

where $\Omega^0(X)$ and $\Omega^3(X)$ both denote the vector space of smooth real-valued functions on X, and $\Omega^1(X)$ and $\Omega^2(X)$ both denote the smooth $\mathbb{R}^3$-valued functions.

First let us consider $\Omega^0(X)$. The gradient of a function f vanishes if and only if f is locally constant, i.e. constant on each connected component of X. Thus the kernel $H^0(X)$ of grad, which is called the 0-dimensional cohomology of X, is a vector space whose dimension is the number of connected components of X.

Now consider $\Omega^1(X)$. We define the 1-dimensional cohomology as the quotient vector space

$$H^1(X) = \frac{\{v \in \Omega^1(X) : \text{curl } v = 0\}}{\{v \in \Omega^1(X) : v \text{ is a gradient}\}}.$$

We have seen that this is the set of invariants which we can define for closed curves in X.

Turning to $\Omega^2(X)$, we have also seen that

$$H^2(X) = \frac{\{v \in \Omega^2(X) : \text{div } v = 0\}}{\{v \in \Omega^2(X) : v \text{ is a curl}\}}.$$

is the set of invariants we can define for closed surfaces in X.

That is as far as we can go, for the operator div is always surjective.

Everything we have said can be generalized quite easily to open subsets X of $\mathbb{R}^n$ for any n. We define $\Omega^k(X)$ as the vector space of smooth functions on X whose values are vectors with $\binom{n}{k}$ components. We think of these functions as "tensor fields" $\alpha_{i_1,\cdots i_k}$ with k indices running from 1 to n and alternating in the indices. It is clear how to define grad : $\Omega^0(X) \to \Omega^1(X)$. Locally, a vector-valued function α_i is a gradient if and only if $\partial\alpha_i/\partial x_i = \partial\alpha_i/\partial x_j$, so we define curl : $\Omega^1(X) \to \Omega^2(X)$ by

$$(\text{curl } \alpha)_{ij} = \partial\alpha_j/\partial x_i - \partial\alpha_i/\partial x_j.$$

In general, we define an operator simply denoted by

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

by

$$(d\alpha)_{i_1} \cdots {}_{i_{k+1}} = \Sigma(-1)^{r-1} \frac{\partial}{\partial x_r} \alpha_{i_1 \cdots \widehat{i_r} \cdots i_{k+1}},$$

where $\wedge$ indicates that the symbol beneath it should be omitted. The operator d has three basic properties, which we shall not prove here.

(i) $d \circ d = 0$

(ii) (the "Poincaré lemma"). If $d\alpha = 0$ then locally $\alpha = d\beta$ for some β.

(iii) ("Stokes's theorem"). If $\alpha \in \Omega^k(X)$, and R is an oriented $(k+1)$-dimensional region in X with k-dimensional boundary ∂R, then

$$\int_R d\alpha = \int_{\partial R} \alpha.$$

These properties make it reasonable to define cohomology groups by

$$H^k(X) = \frac{\{\alpha \in \Omega^k(X) : d\alpha = 0\}}{\{\alpha \in \Omega^k(X) : \alpha = d\beta \text{ for some } \beta\}}.$$

The elements of $H^k(X)$ are invariants of closed oriented k-dimensional regions in X.

We can generalise still further from open subsets of $\mathbb{R}^n$ to arbitrary smooth manifolds, but we shall not pursue that here. The cohomology groups we have been describing are called the *de Rham cohomology groups*, and we shall write them $H^k_{dR}(X)$ when we want to distinguish them from others to be defined presently.

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2. Cohomology

2.1 Cochain complexes

The cohomology of a space X is defined in two steps.

(i) We associate to X a cochain complex $C^{\cdot}$.

(ii) We define $H^k(X) = H^k(C^{\cdot})$.

Definitions A *cochain complex* is a sequence of abelian groups and homomorphisms

$$C^{\cdot} = \{\cdots \xrightarrow{d} C^{k-1} \xrightarrow{d} C^k \xrightarrow{d} C^{k+1} \xrightarrow{d} \cdots\},$$

indexed by $k \in \mathbb{Z}$, such that $d \circ d = 0$.
The *k-cocycles* of $C^{\cdot}$ are $\ker d : C^k \to C^{k+1}$,
the *k-coboundaries* of $C^{\cdot}$ are $\operatorname{im} d : C^{k-1} \to C^k$,
and the k^{th} *cohomology group* is $H^k(C^{\cdot}) = \{k\text{-cocycles}\}/\{k\text{-coboundaries}\}$.

2.2 Alexander cochains

We first fix a "coefficient group" A. It can be any abelian group, but usually will be $\mathbb{Z}$.

A *k-cochain* on X will be an A-valued function $(x_0, \cdots, x_k) \longmapsto c(x_0, \cdots, x_k)$ defined on all $(k+1)$-tuples of points of x which are sufficiently close together. Formally, this means that we consider all pairs (U, c), where U is a neighbourhood of the diagonal in X^{k+1}, and c is a *not necessarily continuous* function $c : U \to A$. On these pairs, we introduce an equivalence relation defined by

$$(U, c) \sim (U', c') \quad \Leftrightarrow \quad c|U'' = c'|U'' \text{ for some neighbourhood } U'' \text{ of the diagonal which is contained in } U \cap U'.$$

The resulting equivalence classes of functions are the k-cochains. They form an abelian group $C^k(X)$ under addition.

Definition 2.2.1 *We define* $d : C^k(X) \to C^{k+1}(X)$ *by*

$$dc(x_0, \cdots, x_{k+1}) = \Sigma(-1)^i c(x_0, \cdots, \hat{x}_i, \cdots, x_{k+1}).$$

Here the notation $\widehat{x}_i$ indicates that x_i is to be omitted.

Proposition 2.2.2 $C^{\cdot}(X)$ *is a cochain complex, i.e.* $d \circ d = 0$.

Proof.

$$ddc(x_0, \cdots, x_{k+2}) = \sum(-1)^i dc(x_0, \cdots, \hat{x}_i, \cdots, x_{k+2})$$

$$= \sum_{j>i}(-1)^{i+j} c(x_0, \cdots, \hat{x}_j, \cdots, \hat{x}_i, \cdots, x_{k+2}) + \sum_{j>i}(-1)^{i+j-1} c(x_0 \cdots \hat{x}_i, \cdots, \hat{x}_j \cdots, x_{k+2})$$

$$= 0.$$

■

We have the following analogue of the Poincaré lemma.

Proposition 2.2.3 *If $c \in C^k(X)$, with $k > 0$, and $dc = 0$, then any point $x \in X$ has a neighbourhood V such that $c|V = db$ for some $b \in C^{k-1}(X)$.*

Here $c|V$ means the element of $C^k(V)$ obtained by restricting c to $(k+1)$-tuples which are contained in V.

Proof Suppose that c is defined in a neighbourhood U of the diagonal in X^{k+1}. Let V be a neighbourhood of x such that $V^{k+1} \subset U$. Then define

$$b(x_0, \cdots, x_{k-1}) = c(x, x_0, \cdots, x_{k-1})$$

for all $x_0, \cdots, x_{k-1}$ in V. We find at once

$$\begin{aligned} db(x_0, \cdots, x_k) &= \Sigma(-1)^i c(x, x_0, \cdots, \hat{x}_i, \cdots, x_k) \\ &= c(x_0, \cdots, x_k) - dc(x, x_0, \cdots, x_k) \\ &= c(x_0, \cdots, x_k). \end{aligned}$$

■

2.3 Other ways of defining cochains

There are many ways of associating cochain complexes to spaces. For the spaces of interest in algebraic topology, they all lead to the same cohomology groups. I shall mention four more, of which the first two — Čech and singular — have the same formal structure as the Alexander cochains, while the other two do not.

Čech cochains

Let $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in S}$ be an open covering of a space X. Define

$$S_k = \{(\alpha_0, \cdots, \alpha_k) \in S^{k+1} : U_{\alpha_0} \cap \cdots \cap U_{\alpha_k} \neq \emptyset\},$$

$$\check{C}^k(\mathcal{U}) = \{\text{all maps } S_k \to A\},$$

and $d : \check{C}^k \to \check{C}^{k+1}$ by the formula of 2.2.1. The proof that $d \circ d = 0$ is unchanged.

Variant Choose a total ordering of S, and omit from S_k all except the $(k+1)$-tuples $(\alpha_0, \cdots, \alpha_k)$ which satisfy $\alpha_0 < \alpha_1 < \cdots < \alpha_k$.

Either way, we shall prove that the cochain complex $C^{\cdot}(\mathcal{U})$ defines the (Alexander) cohomology $H^*(X)$ providing the sets U_α and their non-empty finite intersections are contractible.

Singular cochains

The standard k-simplex is

$$\Delta^k = \{(\lambda_0, \cdots, \lambda_k) \in \mathbb{R}^{k+1} : \lambda_i \geqslant 0, \Sigma\lambda_i = 1\}.$$

It has $k+1$ faces, which are the images of the maps $\theta_i : \Delta^{k-1} \to \Delta^k$ given by

$$\theta_i(\lambda_0, \cdots, \lambda_{k-1}) = (\lambda_0, \cdots .\lambda_{i-1}, 0, \lambda_i, \cdots, \lambda_{k-1}).$$

For any space X, a *singular k-simplex* in X is a continuous map $\phi : \Delta^k \to X$. Let Σ_k be the set of all of these. Define $C^k_{sing}(X) = \{\text{all maps } \Sigma_k \to A\}$, and $d : C^k_{sing} \to C^{k+1}_{sing}$ by

$$dc(\phi) = \Sigma(-1)^i c(\phi \circ \theta_i).$$

The proof that $d^2 = 0$ is as before, noting that $\theta_i\theta_j = \theta_{j-1}\theta_i$ if $i < j$. Singular cohomology coincides with Alexander cohomology for spaces which are well-behaved locally: roughly speaking, for those which are locally contractible.

Cellular cochains

A finite *cell complex* is a compact space X which is the disjoint union of a finite number of subsets B_α – called the *cells* – each homeomorphic to $\mathbb{R}^k$ for some k, and such that $\overline{B}_\alpha - B_\alpha$ is contained in the union of the cells of dimension less than $\dim(B_\alpha)$. Let P_k be the set of cells of dimension k. We define

$$C^k_{\text{cell}}(X) = \{\text{all maps } P_k \to A\}.$$

One can prove that $d : C^k_{\text{cell}} \longrightarrow C^{k+1}_{\text{cell}}$ can be defined, giving a cochain complex whose cohomology is $H^*(X)$.

Examples

(i) $S^n = (\textit{point}) \cup (\mathbb{R}^n)$. This gives us a chain complex with

$$C^k_{\text{cell}}(S^n) = H^k_{\text{cell}}(S^n) = \begin{cases} A & \text{if } k = 0 \text{ or } n \\ 0 & \text{if not.} \end{cases}$$

(ii) $P^n_{\mathbb{C}} = \mathbb{C}^n \cup P^{n-1}_{\mathbb{C}} = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C} \cup (\textit{point})$. This gives us a cochain complex $\mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z}$, and

$$H^k_{\text{cell}}(P^n_{\mathbb{C}}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0, 2, 4, \cdots, 2n \\ 0 & \text{if not} \end{cases}$$

Morse cochains

Let X be a compact smooth manifold. A smooth function $f : X \to \mathbb{R}$ is a *Morse function* if it has only finitely many critical points, all non degenerate. (A *critical point* is a point $x \in X$ where grad(f) vanishes; it is non degenerate if the Hessian matrix $(\partial^2 f/\partial x_i \partial x_j)$ is non singular at x).

The *index* of a non degenerate critical point is the number of negative eigenvalues of the Hessian.

Choose a Morse function f on X, and let Q_k be the set of critical points of index k. We define $C^k(f) = \{\text{all maps } Q_k \to A\}$.

As before, we can define d to get a cochain complex whose cohomology is $H^*(X)$. But really this construction reduces to the cellular method. For each point of X lies on a unique trajectory of the gradient flow of f, and each trajectory has a critical point as its upper limit. The trajectories descending from a critical point of index k sweep out a cell of dimension k.

2.4 The idea of a cohomology class

Intuitively speaking, a k-dimensional cohomology class c of a space X with real coefficients is a rule which associates a number $c(R)$ to each *closed oriented* k-dimensional region R in X. The function $R \mapsto c(R)$ has two properties.

(i) It can be extended in many ways — the extension is *not* given — to an additive function defined for regions R which are not necessarily closed, where "additive" means that $c(R_1 \cup R_2) = c(R_1) + c(R_2)$ if R_1 and R_2 intersect only at their boundaries.

(ii) $c(R) = 0$ if R is the boundary of an oriented $(k+1)$-dimensional region.

The significance of the property (i) is that it expresses the sense in which c is additive for closed regions: if R_1, R_2, and R_3 are regions all with the same boundary $\partial R_1 = \partial R_2 = \partial R_3$, then $R_1 - R_2, R_2 - R_3$, and $R_1 - R_3$ are closed regions, and

$$c(R_1 - R_3) = c(R_1 - R_2) + c(R_2 - R_3).$$

Here $R_1 - R_2$ means $R_1 \cup (-R_2)$, where $-R_2$ denotes R_2 with its orientation reversed. We have $c(-R) = -c(R)$ because $R \cup (-R)$ is the boundary of a collapsed $(k+1)$-dimensional region.

The definition just given is not quite correct if we take the coefficient group to be $\mathbb{Z}$ rather than $\mathbb{R}$, and in any case it is not practical, as it would be laborious to make precise what is meant by a "k-dimensional region". Nevertheless it is the correct idea to keep in mind. If "region" is taken to mean "union of singular simplexes", it reduces precisely to the definition of singular cohomology.

2.5 The basic properties of cohomology

These are (i) functoriality, (ii) homotopy invariance, (iii) the Mayer-Vietoris sequence.

(i) Functoriality

A map $f : X \to Y$ induces a homomorphism $f^* : H^k(Y) \to H^k(X)$ for each k, and one has

1. $(\text{identity})^* = \text{identity}$,
2. $(g \circ f)^* = f^* \circ g^*$.

This is because f induces a cochain map $f^* : C^\cdot(Y) \to C^\cdot(X)$. A *cochain map* $\phi : C^\cdot \to \tilde{C}^\cdot$ is a sequence of maps $\phi : C^k \to \tilde{C}^k$ satisfying $d \circ \phi = \phi \circ d$. Such a map ϕ takes cocycles to cocycles and coboundaries to coboundaries, and hence induces $\phi : H^k(C^\cdot) \to H^k(\tilde{C}^\cdot)$.

(ii) Homotopy invariance

Proposition 2.5.1 *If $f_0 \simeq f_1 : X \to Y$, then $f_0^* = f_1^* : H^k(Y) \to H^k(X)$. In particular, $H^*(X) \cong H^*(Y)$ if X and Y are homotopy-equivalent.*

The topological notion of homotopy is reflected in the algebraic notion of "cochain homotopy". Two cochain maps $\phi_0, \phi_1 : C^\cdot \to \tilde{C}^\cdot$ are *cochain-homotopic* if there is a sequence of maps $h : C^k \to \tilde{C}^{k-1}$ such that

$$\phi_1 - \phi_0 = d \circ h + h \circ d.$$

Clearly this implies that ϕ_0 and ϕ_1 induce the same map $H^k(C^\cdot) \to H^k(\tilde{C}^\cdot)$.

The proof of (2.5.1), when the cohomology is defined using Alexander cochains, is fairly difficult. It is much easier, however, if the space X is compact, so for the moment we shall give the proof only in that case.

If $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ is an open covering of Y, let $\mathcal{U}_k$ denote the neighbourhood of the diagonal in Y^{k+1} consisting of all $(k+1)$-tuples $(y_0, \cdots, y_k)$ which are contained in one of the sets U_α, i.e.

$$\mathcal{U}_k = \bigcup_{\alpha \in S} U_\alpha^{k+1}.$$

Let

$$C^k_{\mathcal{U}}(Y) = \{\text{all maps} \quad \mathcal{U}_k \to A\}.$$

We define $d : C^k_{\mathcal{U}}(Y) \to C^{k+1}_{\mathcal{U}}(Y)$ by the usual formula, and have a cochain complex, and a cochain map $C^{\cdot}_{\mathcal{U}}(Y) \to C^{\cdot}(Y)$. Any element of $H^*(Y)$ obviously comes from a cohomology class of $C^{\cdot}_{\mathcal{U}}(Y)$ for some open covering $\mathcal{U}$ of Y.

Lemma 2.5.2 *If $f_0, f_1, : X \to Y$ are two maps which are $\mathcal{U}$-close, i.e. if for all $x \in X$ there is an $\alpha \in S$ such that both $f_0(x)$ and $f_1(x)$ belong to U_α, then f_0^* and f_1^* are cochain homotopic maps*

$$C^{\cdot}_{\mathcal{U}}(Y) \to C^{\cdot}(X).$$

This lemma implies Proposition 2.5.1 when X is compact, for any element of $H^*(Y)$ comes from some $C^{\cdot}_{\mathcal{U}}(Y)$, and if $\{f_t : X \to Y\}_{t \in [0,1]}$ is a homotopy then we can find a subdivision $0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$ of $[0,1]$ such that f_{t_i} and $f_{t_{i-1}}$ are $\mathcal{U}$-close for $i = 1, \cdots, n$.

Proof. We define

$$h : C^{k+1}_{\mathcal{U}}(Y) \to C^k(X)$$

by the formula

$$(hc)(x_0, \cdots, x_k) = \Sigma(-1)^i c(y_0, \cdots, y_i, z_i, z_{i+1}, \cdots, z_k),$$

where $y_i = f_0(x_i)$ and $z_i = f_1(x_i)$. Notice that the right-hand side is defined providing the points $x_0, \cdots, x_k$ are close enough together. Then

$$\begin{aligned} hdc(x_0, \cdots, x_k) &= \Sigma(-1)^i dc(y_0, \cdots, y_i, z_i, \cdots, z_k) \\ &= \Sigma\{c(y_0, \cdots, y_{i-1}, z_i. \cdots, z_k) - c(y_0, \cdots, y_i, z_{i+1}, \cdots, z_k) \\ &\quad + \Sigma_{j<i}(-1)^{i+j} c(y_0, \cdots, \hat{y}_j, \cdots, y_i, z_i, \cdots, z_k) \\ &\quad - \Sigma_{j>i}(-1)^{i+j} c(y_0. \cdots, y_i, z_i, \cdots, \hat{z}_j, \cdots, z_k)\} \\ &= c(z_0, \cdots, z_k) - c(y_0, \cdots, y_k) - dhc(x_0, \cdots, x_k). \end{aligned}$$

■

(iii) The Mayer-Vietoris sequence

The Mayer-Vietoris sequence tells us about the cohomology of the union of two spaces. If X is the union of two open subsets X_1 and X_2 then a cochain on X is the same thing as a pair of cochains on X_1 and X_2 which agree when restricted to $X_{12} = X_1 \cap X_2$. In other words, we have

Proposition 2.5.3 *There is a short exact sequence of cochain complexes*

$$\begin{array}{ccccccc} 0 \to & C^{\cdot}(X) & \to & C^{\cdot}(X_1) \oplus C^{\cdot}(X_2) & \to & C^{\cdot}(X_{12}) & \to 0 \\ & c & \mapsto & (c|X_1, c|X_2) & & & \\ & & & (c_1, c_2) & \mapsto & (c_1|X_{12}) - (c_2|X_{12}). & \end{array}$$

To say that a sequence of cochain maps

$$0 \to A^{\cdot} \to B^{\cdot} \to C^{\cdot} \to 0 \qquad (2.5.4)$$

is a short exact sequence means simply that for each k the sequence

$$0 \to A^k \to B^k \to C^k \to 0$$

is exact.

Proof of 2.5.3. It is clear that the first map is injective and the second is surjective (in fact the restriction $C^{\cdot}(X_1) \to C^{\cdot}(X_{12})$ is already surjective). What needs to be checked is that if c_1 and c_2 are cochains on X_1 and X_2 which agree on X_{12} then there is a cochain c on X such that $c|X_i = c_i$.

Suppose that c_1 and c_2 are defined in neighbourhoods U_1 and U_2 of the diagonals in X_1^{k+1} and X_2^{k+1}, and that $c_1|U_{12} = c_2|U_{12}$, where U_{12} is a neighbourhood of the diagonal in X_{12}^{k+1}. Let us choose disjoint open subsets V_1 and V_2 of X^{k+1} such that V_1 contains the diagonal of $X - X_2$, and V_2 contains the diagonal of $X - X_1$. This can be done (providing the topological space X is *normal*, which I assume) because $X - X_2$ and $X - X_1$ are disjoint closed subsets of X. Then $V = V_1 \cup U_{12} \cup V_2$ is a neighbourhood of the diagonal in X^{k+1}, and we can define $c : V \to \mathbb{Z}$ by

$$\begin{aligned} c|V_1 &= c_1|V_1 \\ c|V_2 &= c_2|V_2 \\ c|U_{12} &= c_1|U_{12} = c_2|U_{12} \end{aligned}$$

∎

Remark 2.5.5 For this proof, and for the exactness of the Mayer-Vietoris sequence, we do not need X_1 and X_2 to be open subsets of X, but only that their interiors cover X.

One of the most basic and characteristic tools of algebraic topology is the observation that a "long exact sequence" of cohomology groups arises from a short exact sequence like 2.5.4.

Proposition 2.5.6 *To each short exact sequence 2.5.4 of cochain complexes we can associate a map*

$$d : H^k(C^{\cdot}) \to H^{k+1}(A^{\cdot})$$

for each k, and the sequence

$$\cdots \to H^{k-1}(B^{\cdot}) \to H^{k-1}(C^{\cdot}) \xrightarrow{d} H^k(A^{\cdot}) \to H^k(B^{\cdot}) \to H^k(C^{\cdot}) \xrightarrow{d} H^{k+1}(A^{\cdot}) \to \cdots$$

is exact.

Proof. This is by "diagram chasing". We consider the diagram

$$\begin{array}{ccccccc} & & \uparrow & & \uparrow & & \uparrow \\ 0 \to & & A^{k+1} & \to & B^{k+1} & \to & C^{k+1} \to 0 \\ & & \uparrow d & & \uparrow d & & \uparrow d \\ 0 \to & & A^k & \to & B^k & \to & C^k \to 0 \\ & & \uparrow & & \uparrow & & \uparrow \end{array}$$

∎

We first define the map $d : H^k(C^{\cdot}) \to H^{k+1}(A^{\cdot})$. Consider an element of $H^k(C^{\cdot})$ represented by $c \in C^k$ such that $dc = 0$. Choose $b \in B^k$ such that $b \mapsto c$. Then $db \mapsto 0$ in C^{k+1}, so db comes from $a \in A^{k+1}$. And $da = 0$, for $da \mapsto ddb = 0$ in B^{k+2}. So a represents an element of $H^{k+1}(A^{\cdot})$. We check successively that

1. the class of a does not change if the choice of b is changed,
2. the class of a does not change if c is changed by a coboundary, and
3. the map $c \mapsto a$ is a homomorphism from $H^k(C^\cdot)$ to $H^{k+1}(A^\cdot)$.

I shall omit the proof that the long sequence is exact.

Putting together 2.5.3 and 2.5.6 we obtain the Mayer-Vietoris sequence.

Proposition 2.5.7 *There is an exact sequence*

$$\cdots \to H^{k-1}(X_{12}) \to H^k(X) \to H^k(X_1) \oplus H^k(X_2) \to H^k(X_{12}) \to H^{k+1}(X) \to \cdots.$$

I shall sometimes write $d_{MV} : H^{k-1}(X_{12}) \to H^k(X)$ for the "coboundary" map in this sequence. Intuitively it can be imagined as follows. If we are given an element c of $H^{k-1}(X_{12})$ and a closed region R in X, then we can write $R = R_1 \cup R_2$, where $R_i \subset X_i$, and R_1 and R_2 are not closed but intersect in their common boundary R_{12}, which has dimension $k-1$. Then

$$(d_{MV}c)(R) = c(R_{12}).$$

Apart from the Mayer-Vietoris sequence there are many other applications of 2.5.6. I shall mention two.

(i) If Y is a subspace of X we define the *relative cochain complex* $C^\cdot(X,Y)$ as the kernel of the restriction $C^\cdot(X) \to C^\cdot(Y)$, and the *relative cohomology* $H^k(X,Y) = H^k(C^\cdot(X,Y))$. Then we have a long exact sequence

$$\cdots \to H^{k-1}(Y) \to H^k(X,Y) \to H^k(X) \to H^k(Y) \to H^{k+1}(X,Y) \to \cdots.$$

(ii) If $C^\cdot(X)$ denotes the cochains of X with integer coefficients $\mathbb{Z}$, and $C^\cdot(X;\mathbb{Z}/n)$ denotes the cochains with coefficients $\mathbb{Z}/n$, then there is a short exact sequence

$$0 \to C^\cdot(X) \xrightarrow{\times n} C^\cdot(X) \to C^\cdot(X;\mathbb{Z}/n) \to 0,$$

and a long exact sequence, called the *Bockstein sequence*,

$$\cdots \to H^{k-1}(X;\mathbb{Z}/n) \to H^k(X) \xrightarrow{\times n} H^k(X) \to H^k(X;\mathbb{Z}/n) \to H^{k-1}(X) \to \cdots.$$

This enables us to calculate $H^*(X;\mathbb{Z}/n)$ from $H^*(X)$.

2.6 Examples of the use of the Mayer-Vietoris sequence

The sphere

Proposition 2.6.1 *(i) If $n > 0$ then*

$$\begin{aligned} H^k(S^n) &\cong \mathbb{Z} \text{ if } k = 0 \text{ or } n, \\ &= 0 \text{ if not.} \end{aligned}$$

(ii) If $f : S^n \to S^n$ is defined by an orthogonal transformation $f \in O_{n+1}$, then $f^* : H^n(S^n) \to H^n(S^n)$ is multiplication by $det(f) = \pm 1$.

Proof.

(i) Write $S^n = X_1 \cup X_2$, where $X_1 = S^n - \{e_1\}$ and $X_2 = S^n - \{-e_1\}$. Then X_1 and X_2 are each homeomorphic to $\mathbb{R}^n$, and hence contractible, while $X_{12} \cong S^{n-1} \times \mathbb{R}$ is homotopy-equivalent to the equatorial sphere S^{n-1}. From the Mayer-Vietoris sequence we have

$$H^{k-1}(X_1) \oplus H^{k-1}(X_2) \to H^{k-1}(X_{12}) \to H^k(S^n) \to H^k(X_1) \oplus H^k(X_2).$$

We find at once that $H^{k-1}(S^{n-1}) \cong H^k(S^n)$ if $k > 1$, and even when $k = 1$ providing $n > 1$ (so that X_{12} is connected). By induction it is therefore enough to consider the case $n = 1$, and then only $k = 1$ is interesting. As $S^0 = \text{(point)} \amalg \text{(point)}$ we have

$$H^0(\text{point}) \oplus H^0(\text{point}) \to H^0((\text{point}) \amalg (\text{point})) \to H^1(S^1) \to 0,$$

i.e.

$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H^1(S^1) \to 0. \tag{2.6.2}$$

The left-hand map is clearly $(x, y) \mapsto (x - y, x - y)$, so $H^1(S^1) \cong \mathbb{Z}$ in such a way that the right-hand map is $(x, y) \mapsto x - y$.

(ii) We shall use the fact that the Mayer-Vietoris sequence is clearly *natural* in the following sense. If $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ and we have $f : Y \to X$ such that $f(Y_i) \subset X_i$ then the diagram

$$\begin{array}{ccc} H^k(X_{12}) & \xrightarrow{d_{MV}} & H^{k+1}(X) \\ \downarrow f^* & & \downarrow f^* \\ H^k(Y_{12}) & \xrightarrow{d_{MV}} & H^{k+1}(Y) \end{array}$$

commutes.

The group O_{n+1} has just two connected components, distinguished by the sign of $\det(f)$. Elements in the same connected component give homotopic maps $S^n \to S^n$, so it is enough to consider the case where f is reflection in a hyperplane of $\mathbb{R}^{n+1}$. In the inductive argument just used we can assume that $f(X_1) \subset X_1$ and $f(X_2) \subset X_2$, and that f induces a reflection on the equatorial S^{n-1}. By induction we get back to the case $f : S^1 \to S^1$. Then the summands $\mathbb{Z} \oplus \mathbb{Z}$ in the middle of the sequence 2.6.2 are interchanged by f, and so f induces $x \mapsto -x$ on $H^1(S^1)$.

∎

Complex projective space

Proposition 2.6.3

$$\begin{aligned} H^k(P^n_{\mathbb{C}}) &\cong \mathbb{Z} \quad \textit{if} \quad k = 0, 2, 4, \cdots, 2n, \\ &= 0 \quad \textit{if not.} \end{aligned}$$

Proof. Write $X = P^n_{\mathbb{C}}$, and $X = X_1 \cup X_2$, where —using homogeneous coordinates —

$$\begin{array}{lcl} X_1 &=& \{(x_0, \cdot, x_n) : x_0 \neq 0\} \cong \mathbb{C}^n, \\ X_2 &=& X - \{(1, 0, 0, \cdots, 0)\}, \\ X_{12} &\cong& \mathbb{C}^n - \{0\} \simeq S^{2n-1}. \end{array}$$

We think of $P_{\mathbb{C}}^{n-1}$ as the subspace of all points of X with $x_0 = 0$. Thus $P_{\mathbb{C}}^{n-1} \subset X_2$, and the inclusion is a homotopy equivalence in view of the homotopy

$$(x_0, \cdots, x_n) \mapsto (tx_0, x_1, \cdots, x_n)$$

of the identity map of X_2.

The Mayer-Vietoris sequence gives us

$$\cdots \to H^{k-1}(S^{2n-1}) \to H^k(P_{\mathbb{C}}^n) \to H^k(P_{\mathbb{C}}^{n-1}) \to H^k(S^{2n-1}) \to \cdots$$

if $k > 1$, and it is easy to complete the proof by induction on n, beginning with $P_{\mathbb{C}}^0 =$ (point). ∎

2.7 Cohomology with compact supports

For a locally compact space X we can define a sub cochain complex $C^{\cdot}_{cpt}(X)$ of $C^{\cdot}(X)$ by

$$c \in C^{\cdot}_{cpt}(X) \text{ if } c(x_0, \cdots, x_k) = 0$$
unless all x_i belong to some compact subspace K of X.

These are the cochains "with compact supports".

Definition 2.7.1

$$H^k_{cpt}(X) = H^k(C^{\cdot}_{cpt}(X)).$$

Cohomology with compact supports is functorial in two different ways. A *proper* map $f : X \to Y$, i.e. a continuous map such that the inverse-image of every compact subset is compact, induces a cochain homomorphism $f^* : C^{\cdot}_{cpt}(Y) \to C^{\cdot}_{cpt}(X)$, and hence $f^* : H^*_{cpt}(Y) \to H^*_{cpt}(X)$.

But if $i : U \to X$ maps U homeomorphically to an *open* subset of X we also have a cochain map $i_* : C^{\cdot}_{cpt}(U) \to C^{\cdot}_{cpt}(X)$ which "extends by zero", i.e.

$$\begin{aligned} i_*c(x_0, \cdots, x_k) &= c(i^{-1}x_0, \cdots, i^{-1}x_0 \cdots, i^{-1}x_k) \text{ if } \{x_0, \cdots, x_k\} \subset i(U) \\ &= 0 \quad \text{if not.} \end{aligned}$$

This gives $i_* : H^*_{cpt}(U) \to H^*_{cpt}(X)$.

Proposition 2.7.2

$$\begin{aligned} H^k_{cpt}(\mathbb{R}^n) &\cong \mathbb{Z} \quad \textit{if} \quad k = n \\ &= 0 \textit{ if not.} \end{aligned}$$

Furthermore, $g \in GL_n\mathbb{R}$ acts on $H^n_{cpt}(\mathbb{R}^n)$ by multiplication by the sign of det(g).

Proof. Let $i : \mathbb{R}^n \to S^n$ embed $\mathbb{R}^n$ as an open disc $U = i(\mathbb{R}^n)$ in S^n. Let V be another open disc such that $U \cup V = S^n$. I shall show that

$$i_* : H^k_{cpt}(\mathbb{R}^n) \to H^k(S^n)$$

is an isomorphism if $k > 0$. Injectivity : suppose $i_*c = db$ for some $b \in C^{k-1}(S^n)$. Then $b|V = d\beta$ for some $\beta \in C^{k-2}(V)$. Extend β to $\tilde{\beta} \in C^{k-2}(S^n)$. Then $b - d\tilde{\beta}$ has compact support in U, and $i_*c = d(b - d\tilde{\beta})$. The proof of surjectivity is similar. ∎

The calculation of g^* is as in 2.6.1 (ii).

Corollary 2.7.3 *If $f : \mathbb{R}^n \to \mathbb{R}^n$ maps $\mathbb{R}^n$ homeomorphically onto an open subset $f(\mathbb{R}^n) \subset \mathbb{R}^n$ then $f_* : H^n_{cpt}(\mathbb{R}^n) \to H^n_{cpt}(\mathbb{R}^n)$ is multiplication by ± 1.*

Proof. If $f(\mathbb{R}^n)$ is an open disc this follows at once from the proof of 2.7.1. In general it is enough to show that f_* is surjective, which is clear by functoriality, as $f(U)$ always *contains* an open disc of $\mathbb{R}^n$. ∎

One says that such a map f *preserves* or *reverses orientation* according as f_* is $+1$ or -1.

There is a version of the Mayer-Vietoris sequence for cohomology with compact supports. Suppose that $X = U \cup V$ is the union of two open subsets, and let the inclusion maps be

$$\begin{array}{ccc} U \cap V & \xrightarrow{i'} & V \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ U & \xrightarrow{i} & X. \end{array}$$

We have an exact sequence of cochain complexes

$$\begin{array}{ccccccc} 0 \longrightarrow & C^{\cdot}_{cpt}(U \quad V) & \longrightarrow & C^{\cdot}_{cpt}(U) \oplus C^{\cdot}_{cpt}(V) & \longrightarrow & C^{\cdot}_{cpt}(X) & \longrightarrow 0 \\ & c & \longmapsto & (j'_* c, -i'_* c) & & & \\ & & & (c_1, c_2) & \longmapsto & i_* c_1 + j_* c_2, & \end{array}$$

and a corresponding Mayer-Vietoris sequence

$$\cdots \longrightarrow H^q_{cpt}(U \cap V) \longrightarrow H^q_{cpt}(U) \oplus H^q_{cpt}(V) \longrightarrow H^q_{cpt}(X) \longrightarrow H^{q+1}_{cpt}(U \cap V) \longrightarrow \cdots \tag{2.7.4}$$

Proposition 2.7.5 *If X is a locally compact space, and Y is a closed subspace of X, then*

$$H^*_{cpt}(X - Y) \xrightarrow{\cong} H^*_{cpt}(X, Y).$$

In particular, $H^*_{cpt}(X, Y) \cong H^*(X, Y)$ if X is itself compact.

For this we need a lemma.

Lemma 2.7.6 *(i) If $c \in C^k(Y)$ is a cocycle then there is a neighbourhood V of Y in X and a cocycle $\tilde{c} \in C^k(V)$ such that $\tilde{c}|V = c$.*

(ii) If $c \in C^k(X)$ is a cocycle such that $c|Y = 0$ then there is a neighbourhood V of Y in X and a cochain $b \in C^{k-1}(V, Y)$ such that $c|V = db$.

Proof of 2.7.6

(i) Suppose that c comes from a cocycle $c \in C^k_{\mathcal{V}}(Y)$ for some finite open covering $\mathcal{V} = \{V_1, \cdots, V_m\}$ of Y. We can find (e.g. by induction on m) open subsets $\{V'_1, \cdots, V'_m\}$ of X such that $\{V'_i \cap Y\}$ is a shrinkage of $\mathcal{V}$, and

$$V'_{i_0 \cdots i_p} \neq \varnothing \quad \Longrightarrow \quad V_{i_0 \ldots i_p} \neq \varnothing.$$

Define $V = \bigcup V'_i$, and choose a not necessarily continuous map $f : V \to Y$ such that $f|Y$ is the identity and $f(V'_i) \subset V'_i$. Then $\tilde{c} = f^* c$ is a well-defined cocycle of V which restricts to c.

(ii) With the same V, define b by the standard formula

$$b(x_0, \cdots, x_{k-1}) = \Sigma(-1)^i c(x_0, \cdots, x_i, f(x_i), \cdots, f(x_{k-1})).$$

∎

This defines a cochain in $C^{k-1}(V,Y)$ such that $db = c|V$.

Proof of 2.7.5 The map is surjective, for if c is a cocycle in $C^k(X)$ such that $c|Y = 0$ then $c|V = db$ for some b defined in a neighbourhood V of X, and if $\tilde{b}$ is an arbitrary extension of b to X then $c - d\tilde{b}$ has compact support in $X - Y$.

The map is injective, for if c is a cocycle on X with compact support inside $X - Y$, and $c = db$ for some b such that $b|Y = 0$, then b is a cocycle in a neighbourhood of Y, and so $b|V = de|V$ for some e and some neighbourhood V. But then $c = d(b - de)$, and $b - de$ has compact support in $X - Y$. ∎

2.8 The degree of maps $S^n \to S^n$

Definition 2.8.1 *The* degree *of $f : S^n \to S^n$ is the integer q such that $f^*\epsilon_n = q\epsilon_n$, where ϵ_n is a generator of $H^n(S^n) = \mathbb{Z}$.*

Intuitively, the degree is the number of points in $f^{-1}(x)$ for a generic point $x \in S^n$, counted with signs. To make this precise, suppose that there is a small open disc V in S^n such that $f^{-1}(V)$ is the disjoint union of open sets $U_1, \cdots, U_m$ of S^n each mapped homeomorphically onto V of F. (If f is a *smooth* map, the inverse-function theorem says that this is true whenever V is a small neighbourhood of a regular value of f; and Sard's theorem says that almost all points of S^n are regular.) Let e_a (for $a = 1, \cdots, m$) be ± 1 according as $f : U_\alpha \to V$ preserves or reverses orientation, i.e. $f^*\epsilon_V = e_a \epsilon_{U_a}$, where $\epsilon_{U_a} \in H^n_{cpt}(U_a)$ and $\epsilon_V \in H^n_{cpt}(V)$ are the generators which correspond to $\epsilon_n \in H^n(S^n)$ by the map (5.2).

Proposition 2.8.2 *In this situation,* $\deg(f) = \Sigma_{a=1}^m e_a$.

Proof. To calculate $f^*\epsilon_n$ we represent ϵ_n by a cocycle c with compact support in V. Then $f^*\epsilon_n$ is represented by $\Sigma_a (i_a)_*(f|U_\alpha)^*(c)$, where $i_a : U_a \to S^n$ is the inclusion. So

$$f^*\epsilon_n = \Sigma(i_a)_* e_a \epsilon_{U_a} = (\Sigma e_a) \cdot \epsilon_n.$$

∎

2.9 The multiplicative structure of cohomology

If the coefficient group A of our cohomology is a commutative ring, then for any space X we make the cochains $C^{\cdot}(X)$ into an associative graded ring by the bi-additive maps

$$\begin{array}{ccc} C^p(X) \times C^1(X) & \longrightarrow & C^{p+q}(X) \\ (c_1, c_2) & \longmapsto & c_1 \cdot c_2, \end{array}$$

where $(c_1 \cdot c_2)(x_0, \cdots, x_{p+q}) = c_1(x_0, \cdots, x_p) c_2(x_p, \cdots, x_{p+q})$.

The differential $d : C^k \to C^{k+1}$ is an *antiderivation*, in the sense that

$$d(c_1 \cdot c_2) = dc_1 \cdot c_2 + (-1)^p c_1 \cdot dc_2$$

if $c_1 \in C^p$. This implies that the product of cocycles is a cocycle, and that there is an induced multiplication — often called the "cup-product" —

$$H^p(X) \times H^q(X) \to H^{p+q}(X).$$

Theorem 2.9.1 *The multiplication in $H^*(X)$ is anticommutative, i.e.*

$$c_2 \cdot c_1 = (-1)^{pq} c_1 \cdot c_2$$

if $c_1 \in H^p$ and $c_2 \in H^q$.

Proof. Define a cochain map $T : C^{\cdot}(X) \to C^{\cdot}(X)$ by

$$(Tc)(x_0, \cdots, x_p) = (-1)^{\frac{1}{2}p(p+1)} c(x_p, x_{p-1} \cdots, x_0).$$

(Exercise : check that T really is a cochain map!) If $c_1 \in C^p$ and $c_2 \in C^q$ then

$$T(c_1 \cdot c_2) = (-1)^{pq}(Tc_2) \cdot (Tc_1),$$

so the proof will be complete if we show that T induces the identity on $H^*(X)$. The proof of this will be postponed to the next chapter. ∎

The external product

The cup-product gives us a map called the external product

$$\begin{array}{ccc} H^p(X) \times H^q(Y) & \to & H^{p+q}(X \times Y), \\ (c_1, c_2) & \longmapsto & (\pi_1^* c_1) \cdot (\pi_2^* c_2) \end{array}$$

where $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the projections. It is bi-additive, so extends to

$$H^p(X) \otimes H^q(Y) \to H^{p+q}(X \times Y). \tag{2.9.2}$$

NOTES ON ALGEBRAIC TOPOLOGY

G. Segal - Michaelmas Term '96

3. The double complex theorem and its applications

3.1 The theorem

For each $p \in \mathbb{Z}$ let $C^{p\cdot} = (\cdots \longrightarrow C^{p,q-1} \xrightarrow{d} C^{p,q} \xrightarrow{d} \cdots)$ be a cochain complex, and suppose that

$$\cdots \xrightarrow{\varphi} C^{p\cdot} \xrightarrow{\varphi} C^{p+1,\cdot} \xrightarrow{\varphi} \cdots$$

is a sequence of cochain maps such that $\varphi \circ \varphi = 0$. Define

$$\delta : C^{pq} \longrightarrow C^{p+1,q}$$

by $\delta = (-1)^q \varphi$. Then $\delta^2 = 0$, and $d\delta + \delta d = 0$.

The groups $C^{\cdot\cdot}$ with the maps d and δ constitute a *double complex*. I shall always think of $C^{p\cdot}$ as the p^{th} *column*, and $C^{\cdot q}$ as the q^{th} *row*. Furthermore, I shall always assume that, for some p_0, q_0, we have $C^{pq} = 0$ unless $p \geqslant p_0$ and $q \geqslant q_0$.

Definition. The *total complex* $\hat{C}^{\cdot}$ of $C^{\cdot\cdot}$ consists of the groups $\hat{C}^n = \bigoplus_{p+q=n} C^{pq}$ with the differential $\hat{d} = d + \delta$.

The basic theorem about double complexes is:

Proposition 3.1.1 *Let $C^{\cdot\cdot}$ be a double complex whose rows are acyclic (i.e. exact). Then $\hat{C}^{\cdot}$ is acyclic.*

Proof. Without loss of generality, suppose that $C^{pq} = 0$ unless $p, q \geqslant 0$. Let $c \in \hat{C}^n$ satisfy $\hat{d}c = 0$. Write $c = c_0 + c_1 + \cdots + c_n$, with $c_q \in C^{pq}$. Then

$$\begin{aligned} \delta c_0 &= 0 \\ \delta c_1 &= -dc_0 \\ \delta c_2 &= -dc_1, \text{ etc.} \end{aligned}$$

Choose $b_0 \in C^{n-1,0}$ such that $\delta b_0 = c_0$. Then $\delta(c_1 - db_0) = -dc_0 + dc_0 = 0$, so we can choose $b_1 \in C^{n-2,1}$ such that $\delta b_1 = c_1 - db_0$, and similarly $b_2 \in C^{n-3,2}$ such that $\delta b_2 = c_2 - db_1$, etc.

Then $b_k = 0$ if $k > n$, and $\hat{d}(\Sigma b_i) = c$. ∎

The following result is obvious.

Proposition 3.1.2 *Let $C^{\cdot\cdot}$ be a double complex. Let $A^{\cdot\cdot}$ be the double complex got by replacing C^{pq} by 0 when $p \leqslant r$, and $B^{\cdot\cdot}$ be the double complex got by replacing C^{pq} by 0 when $p > r$.*

$$C^{\cdot\cdot} \quad = \quad \boxed{B^{\cdot\cdot} \mid A^{\cdot\cdot}}$$

Then we have an exact sequence of cochain complexes $0 \longrightarrow \hat{A}^{\cdot} \longrightarrow \hat{C}^{\cdot} \longrightarrow \hat{B}^{\cdot} \longrightarrow 0$.

We can put together 3.1.1 and 3.1.2 to obtain

Proposition 3.1.3 *If $0 \longrightarrow B^{\cdot} \longrightarrow A^{0\cdot} \longrightarrow A^{1\cdot} \longrightarrow A^{2\cdot} \longrightarrow \cdots$ is an exact sequence of cochain complexes, then the natural map $B^{\cdot} \longrightarrow \hat{A}^{\cdot}$ is a cohomology equivalence.*

3.2 Some standard exact sequences

The model situation is: let Σ be any set which is expressed as a union $\Sigma = \bigcup_{\alpha \in S} \Sigma_\alpha$. Write

$$\Sigma_{\alpha_0\alpha_1\cdots\alpha_p} = \Sigma_{\alpha_0} \cap \Sigma_{\alpha_1} \cap \cdots \cap \Sigma_{\alpha_p}.$$

Let $F(\Sigma) = \{\text{all maps } f : \Sigma \to A\}$.

Define a cochain complex $\mathcal{F}^{\cdot} = \{0 \longrightarrow F(\Sigma) \xrightarrow{\delta} \Pi_\alpha F(\Sigma_\alpha) \xrightarrow{\delta} \Pi_{\alpha,\beta} F(\Sigma_{\alpha,\beta}) \xrightarrow{\delta} \cdots\}$ by

$$(\delta f)_{\alpha_0} \cdots \alpha_{p+1} = \Sigma(-1)^i (f_{\alpha_0\cdots\hat{\alpha_i}\cdots\alpha_{p+1}} | \Sigma_{\alpha_0\cdots\alpha_{p+1}}).$$

Proposition 3.2.1 *$\mathcal{F}^{\cdot}$ is acyclic.*

Proof. Choose functions $\lambda_\alpha : \Sigma \longrightarrow \{0,1\}$ such that $\lambda_\alpha(x) = 0$ unless $x \in \Sigma_\alpha$, and $\Sigma_\alpha \lambda_\alpha(x) = 1$. Thus the functions λ_α define a partition of Σ into *disjoint* sets supp $(\lambda_\alpha) \subset \Sigma_\alpha$.
Define $h : \Pi F(\Sigma_{\alpha_0\cdots\alpha_p}) \longrightarrow \Pi F(\Sigma_{\alpha_0\cdots\alpha_{p-1}})$ by

$$(hf)_{\alpha_0\cdots\alpha_{p-1}} = \sum_\alpha \lambda_\alpha f_{\alpha\alpha_0\cdots\alpha_{p-1}}. \tag{3.3}$$

Here $\lambda_\alpha f_{\alpha\alpha_0\cdots\alpha_{p-1}}$ is regarded as a function on $\Sigma_{\alpha_0\cdots\alpha_{p-1}}$ which vanishes outside $\Sigma_{\alpha\alpha_0\cdots\alpha_{p-1}}$.
A by now well-known calculation shows that $\delta h + h\delta$ =identity, and hence that $\mathcal{F}^{\cdot}$ is acyclic.
■

Variant The complex $\mathcal{F}^{\cdot}$ contains the *alternating subcomplex* $\mathcal{F}^{\cdot}_{alt}$, consisting of families $\{f_{\alpha_0\cdots\alpha_p}\} \in \Pi F(\Sigma_{\alpha_0\cdots\alpha_0})$ such that $f_{\alpha_0\cdots\alpha_p} = 0$ unless $\alpha_0, \cdots, \alpha_p$ are *distinct*, and

$$f_{\alpha_{\pi(0)}\cdots\alpha_{\pi(p)}} = \text{sign } (\pi) f_{\alpha_0\cdots\alpha_p}$$

if π is a permutation of $\{0, \cdots, p\}$.

The maps δ and h preserve $\mathcal{F}^{\cdot}_{alt}$, and so $\mathcal{F}^{\cdot}_{alt}$ is also acyclic.

If one chooses a total ordering of the index set S then $\mathcal{F}^{\cdot}_{alt}$ is isomorphic to the *ordered* complex $\mathcal{F}^{\cdot}_{ord}$, which has

$$\mathcal{F}^p_{ord} = \Pi_{\alpha_0 < \alpha_1 < \cdots < \alpha_p} F(\Sigma_{\alpha_0\cdots\alpha_p}).$$

The argument of Proposition 3.2.1 can be applied, with minor variations, in very many situations. The most important for us is

Proposition 3.2.3 *Let $\mathcal{U} = \{X_\alpha\}_{\alpha \in S}$ be an open covering of a space X. Then the sequence*

$$0 \longrightarrow C^q(X) \longrightarrow \Pi C^q(X_\alpha) \longrightarrow \Pi C^q(X_{\alpha\beta}) \longrightarrow \cdots$$

is exact for any q.

Remark The proof we shall give works equally well if we consider cochains $c_{\alpha_0\cdots\alpha_p}(x_0,\cdots,x_q)$ which are alternating in $\alpha_0,\cdots,\alpha_p$. Thus for a covering by two open sets we are reproving 2.5.3.

Proof. We first choose a *shrinkage* $\{V_\alpha\}$ of the covering $\{X_\alpha\}$, i.e. another open covering with the same index set such that $\overline{V}_\alpha \subset X_\alpha$ for each α. In a paracompact space any open covering has a shrinkage. Then we choose a partition of unity $\{\lambda_\alpha\}$ consisting of discontinuous functions $\lambda_\alpha : X \to \{0,1\}$ such that supp $(\lambda_\alpha) \subset V_\alpha$. We can now define

$$h : \Pi C^q(X_{\alpha_0\cdots\alpha_p}) \to \Pi C^q(X_{\alpha_0\cdots\alpha_{p-1}})$$

by

$$(hc)_{\alpha_0\cdots\alpha_{p-1}}(x_0,\cdots,x_q) = \Sigma_\alpha\lambda_\alpha(x_0)c_{\alpha_0\cdots\alpha_{p-1}}(x_0,\cdots,x_q).$$

∎

Because of the shrinkage, the formula makes sense for all $x_0,\cdots,x_q$ in a neighbourhood of the diagonal in $X_{\alpha_0\cdots\alpha_{p-1}}$. The rest of the argument is as usual.

We shall also make use of another, easier, variant. For any open covering $\mathcal{U} = \{X_\alpha\}$ of X, let us write $C^{\cdot}_{\mathcal{U}}(X)$ for the cochain complex of cochains which are defined on all $(q+1)$-tuples $(x_0,\cdots,x_q)$ which are completely contained in a set of the covering $\mathcal{U}$. Notice that there is a cochain map $C^{\cdot}_{\mathcal{U}}(X) \to C^{\cdot}(X)$, and that any element of $C^{\cdot}(X)$ comes from $C^{\cdot}_{\mathcal{U}}(X)$ for *some* covering $\mathcal{U}$ of X.

For any set Σ let us write $F^{\cdot}(\Sigma)$ for the cochain complex with

$$F^q(\Sigma) = \{\text{ all maps } \Sigma^{q+1} \to A\}.$$

(Thus $F^{\cdot}(\Sigma) = C^{\cdot}_{\{\Sigma\}}(\Sigma)$.) The argument of 2.2.3 shows that, for any Σ, the complex $F^{\cdot}(\Sigma)$ has the cohomology of a point. On the other hand 3.2.1 gives us

Proposition 3.2.4 *We have an exact sequence of cochain complexes*

$$0 \to C^{\cdot}_{\mathcal{U}}(X) \to \Pi_\alpha F^{\cdot}(X_\alpha) \to \Pi_{\alpha,\beta}F^{\cdot}(X_{\alpha,\beta}) \to \cdots$$

3.3 Comparison theorems

The theorem of the double complex is an extremely powerful method for proving that different cochain complexes have the same cohomology. It was first used by André Weil to prove that de Rham cohomology — the cohomology of the complex $\Omega^{\cdot}(X)$ of differential forms on a smooth manifold X — coincides with Čech cohomology.

The simplest case of Weil's argument comes from Proposition 3.2.4. Combining this with Proposition 3.1.3 we get a cohomology equivalence

$$C^{\cdot}_{\mathcal{U}}(X) \to \hat{F}^{\cdot}(\mathcal{U}), \tag{3.2}$$

where the right-hand side is the total complex of the double complex $F^{\cdot\cdot}(\mathcal{U})$ with $F^{pq}(\mathcal{U}) = \Pi\, F^q\,(X_{\alpha_0\cdots\alpha_p})$. But whenever $X_{\alpha_0\cdots\alpha_p} \neq \emptyset$ the complex $F^{\cdot}(X_{\alpha_0\cdots\alpha_p})$ has the cohomology of a point, so we also have an exact sequence of cochain complexes

$$0 \to \check{C}^{\cdot}(\mathcal{U}) \to F^{\cdot 0}(\mathcal{U}) \to F^{\cdot 1}(\mathcal{U}) \to \cdots,$$

and hence a cohomology equivalence

$$\check{C}^{\cdot}(\mathcal{U}) \to \hat{F}^{\cdot}(\mathcal{U}). \tag{3.3}$$

Putting 3.3.1 and 3.3.2 together gives

Proposition 3.3.3 *For any open covering $\mathcal{U}$ of X the complexes $C^{\cdot}_{\mathcal{U}}(X)$ and $\check{C}^{\cdot}(\mathcal{U})$ have canonically isomorphic cohomology.*

As a first application of this result we have the lemma used in proving the anticommutativity of the cohomology ring.

Proposition 3.3.4 *The cochain map $T : C^{\cdot}(X) \to C^{\cdot}(X)$ defined by*

$$TC(x_0, \cdots, x_k) = (-1)^{\frac{1}{2}k(k+1)}c(x_k, \cdots, x_0)$$

induces the identity on cohomology.

Proof. Any cocyle in $C^{\cdot}(X)$ comes from $C^{\cdot}_{\mathcal{U}}(X)$ for some open covering $\mathcal{U}$. But T acts on the double complex $F^{\cdot\cdot}(\mathcal{U})$ compatibly with the equivalences

$$C^{\cdot}_{\mathcal{U}}(X) \to \hat{F}^{\cdot}(\mathcal{U}) \leftarrow \check{C}^{\cdot}(\mathcal{U}),$$

where T acts trivially on $\check{C}^{\cdot}(\mathcal{U})$. ∎

We shall call a covering $\mathcal{U}$ *contractible* if each X_α, and each finite intersection $X_{\alpha_0\alpha_1\cdots\alpha_p}$, is either empty or contractible.

Proposition 3.3.3 gives us

Proposition 3.3.5 *If $\mathcal{U}$ is a contractible covering of X then $C^{\cdot}(X)$ and $\check{C}^{\cdot}(\mathcal{U})$ — and hence also $C^{\cdot}_{\mathcal{U}}(X)$ — have the same cohomology.*

In other words, the Čech cochains define the "correct" cohomology $H^*(X)$.

Weil considered smooth manifolds X with coverings $\mathcal{U}$ such that each non-empty $X_{\alpha_0\cdots\alpha_p}$ is diffeomorphic to $\mathbb{R}^n$. Then the Poincaré lemma tells us that

$$0 \to \mathbb{R} \to \Omega^0(X_\sigma) \xrightarrow{d} \Omega^1(X_\sigma) \xrightarrow{d} \Omega^2(X_\sigma) \xrightarrow{d} \cdots$$

is exact for each non-empty X_σ, while the argument of 3.2.1. — but taking $\{\lambda_\alpha\}$ to be a smooth partition of unity $\{\lambda_\alpha : X \to \mathbb{R}_+\}$ subordinate to $\mathcal{U}$ — shows that

$$0 \to \Omega^{\cdot}(X) \to \Pi\Omega^{\cdot}(X_\alpha) \to \Pi\Omega^{\cdot}(X_{\alpha\beta}) \to \cdots$$

is exact. If X is a paracompact manifold, and we choose a Riemannian metric on it, then any covering by sufficiently small geodesic balls has the properties required of $\mathcal{U}$, so the standard argument proves

Proposition 3.3.6 *"de Rham's theorem". The de Rham cohomology $H^*(\Omega^{\cdot}(X))$ coincides with $H^*(X;\mathbb{R})$.*

3.4 Homotopy invariance

So far we have proved the homotopy invariance of cohomology only for compact spaces. To prove that homotopic maps $f_0, f_1 : X \to Y$ induce the same homomorphism of cohomology, it is enough to show that

$$p^* : H^*(X) \to H^*(X \times I)$$

is surjective, where $I = [0,1]$. For if $i_t : X \to X \times I$ is $x \mapsto (x,t)$ then $f_t^* = i_t^* F^*$ for some $F : X \times I \to Y$. But if $F^*\xi = p^*\eta$ then $f_t^*\xi = i_t^* p^*\eta = \eta$ is independent of t.

Now any cocycle in $C^{\cdot}(X \times I)$ comes from a cocycle of $C^{\cdot}_{\mathcal{V}}(X \times I)$, where $\mathcal{V}$ is an open covering of $X \times I$ which, using the compactness of I, we can take to be of the form

$\{X_\alpha \times J_\beta\}$, where $\{X_\alpha\}$ is an open covering $\mathcal{U}$ of X, and J_β belongs to an open covering of I that depends on α.

For a given covering $\mathcal{U}$ of X it is helpful to introduce the hybrid cochain complex $\tilde{C}^{\cdot}$ consisting of cochains c on $X \times I$ such that

$$c((x_0, t_0), \cdots, (x_q, t_q))$$

is defined whenever $\{x_0, \cdots, x_q\}$ is contained in some X_α and $(t_0, \cdots, t_q)$ is contained in some neighbourhood V of the diagonal in I^{q+1}. (We identify two such cochains if they agree when V is made smaller.) Thus $\tilde{C}^{\cdot}$ interpolates between $C^{\cdot}_{\mathcal{V}}(X \times I)$ and $C^{\cdot}(X \times I)$

$$C^{\cdot}_{\mathcal{V}}(X \times I) \to \tilde{C}^{\cdot} \to C^{\cdot}(X \times I),$$

and we can assume that the cocycle we are interested in comes from a cocycle of $\tilde{C}^{\cdot}$. The proof will be complete if we show that

$$p^* : C^{\cdot}_{\mathcal{U}}(X) \to \tilde{C}^{\cdot}$$

induces an isomorphism of cohomology. For this, we consider the diagram

$$\begin{array}{ccc} \tilde{C}^{\cdot} & \to & \Gamma^{\cdot\cdot} \\ \uparrow & & \uparrow \\ C^{\cdot}_{\mathcal{U}}(X) & \to & F^{\cdot\cdot}(\mathcal{U}), \end{array}$$

where $\Gamma^{\cdot\cdot}$ is the double complex such that

$$\Gamma^{p,q} = \Pi_{\alpha_0,\cdots,\alpha_p} \tilde{C}^q(X_{\alpha_0,\cdots,\alpha_p} \times I)$$

and $\tilde{C}^q(X_{\alpha_0,\cdots,\alpha_p} \times I)$ denotes the cochains defined for all $(q+1)$-tuples $(x_0, t_0), \cdots, (x_q, t_q)$ with $x_i \in X_{\alpha_0\cdots\alpha_p}$ and $(t_0, \cdots, t_q)$ in a neighbourhood of the diagonal. The argument of (3.2.4) proves that the top horizontal map induces an isomorphism of cohomology, and so it is enough to show that the right-hand vertical map induces a cohomology isomorphism of the total complexes. This reduces, in turn, to showing that $F^{\cdot}(\mathcal{U}_\sigma) \to \tilde{C}^{\cdot}(\mathcal{U}_\sigma \times I)$ is an equivalence for each non-empty $\mathcal{U}_\sigma$. But $F^{\cdot}(\mathcal{U}_\sigma)$ has the cohomology of a point by the argument of (2.2.3), and $\tilde{C}^{\cdot}(\mathcal{U}_\sigma \times I)$ is cochain-homotopy equivalent to $C^{\cdot}(I)$ by the argument of (2.5.2). Finally, $C^{\cdot}(I)$ has the cohomology of a point because we know already that homotopy invariance is true for compact spaces.

3.5 The Künneth theorem

For any two spaces X and Y we have a ring homomorphism

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$$

defined by multiplication of cocycles. Here the left-hand side denotes the graded abelian group whose component in degree k is

$$\bigoplus_{p+q=k} H^p(X) \otimes H^q(Y),$$

and the multiplication on the left is defined by

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^{q_1 p_2}(a_1 a_2) \otimes (b_1 b_2)$$

if $b_1 \in H^{q_1}(Y)$ and $a_2 \in H^{p_2}(X)$. More generally, if X_0 and Y_0 are subspaces of X and Y, we have

$$H^*(X, X_0) \otimes H^*(Y, Y_0) \to H^*(X \times Y, (X \times Y_0) \cup (X_0 \times Y)).$$

The Künneth theorem tells us that in some circumstances these maps are isomorphisms. We shall prove only a fairly easy version.

Proposition 3.5.1 *Let X be a space with a contractible open covering, and suppose each group $H^q(Y, Y_0)$ is finitely generated and free. Then*

$$H^*(X) \otimes H^*(Y, Y_0) \longrightarrow H^*(X \times Y, X \times Y_0)$$

is an isomorphism.

Proof. Let $\mathcal{U} = \{X_\alpha\}$ be a contractible open covering of X. By choosing cocycles representing a basis for each group $H^q(Y, Y_0)$ one can define a cochain map

$$C^{\cdot}(X) \otimes H^*(Y, Y_0) \to C^{\cdot}(X \times Y, X \times Y_0), \tag{3.3}$$

where the left-hand side denotes the cochain complex which in degree k is

$$\bigoplus C^{k-i}(X) \otimes H^i(Y, Y_0),$$

with the differential given by $d(a \otimes b) = da \otimes b$. It is clearly enough to prove that (3.5.2) induces an isomorphism of cohomology.

For this, consider the diagram

$$\begin{array}{ccc} C^{\cdot}(X) \otimes H^* & \to & C^{\cdot\cdot}(\mathcal{U}) \otimes H^* \\ \downarrow & & \downarrow \\ C^{\cdot}(X \times Y,\, X \times Y_0) & \to & C^{\cdot\cdot}(\mathcal{U} \times Y, \mathcal{U} \times Y_0), \end{array}$$

where $H^* = H^*(Y, Y_0)$ and $C^{\cdot\cdot}(\mathcal{U}) \otimes H^*$ denotes the double complex whose p^{th} column is

$$\bigoplus C^{\cdot}(X_{\alpha_0 \cdots \alpha_p}) \otimes H^*.$$

Both horizontal maps induce cohomology isomorphisms (to the total complexes of the double complexes on the right) by (3.2.3). But the right-hand vertical map induces a cohomology isomorphism on each column, because each space $X_{\alpha_0 \cdots \alpha_p}$ is empty or contractible, and so induces a cohomology isomorphism of the double complexes. That completes the proof. ∎

Remark The preceding proof applies equally well if we use a field as our coefficient group, and assume that each vector space $H^q(Y, Y_0)$ is finite dimensional. The assumption that X has a contractible covering is quite unnecessary: it was put in to make the proof simpler.

An important particular case of 3.5.1 is the isomorphism

$$H^i(X) \to H^{i+n}(X \times \mathbb{R}^n, X \times (\mathbb{R}^n - \{0\})) \tag{3.4}$$

given by multiplying by the generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. If X is compact the right-hand side of (3.5.3) is isomorphic to $H^{i+n}_{cpt}(X \times \mathbb{R}^n)$. Another important case is

$$H^m_{cpt}(\mathbb{R}^m) \otimes H^n_{cpt}(\mathbb{R}^n) \xrightarrow{\cong} H^{n+m}_{cpt}(\mathbb{R}^{m+n}), \tag{3.5}$$

which we can derive by including $\mathbb{R}^m$ and $\mathbb{R}^n$ in S^m and S^n and using the Künneth theorem for $S^m \times S^n$.

NOTES ON ALGEBRAIC TOPOLOGY

G. Segal - Michaelmas Term '95

4. Vector bundles and the Thom isomorphism theorem

4.1 Vector bundles

A (real) *vector bundle* on a space X is a family $\{E_x\}_{x\in X}$ of vector spaces indexed by the points of X, together with a topology on the disjoint union $E = \bigcup_{x\in X} E_x$. We require it to be *locally trivial* in the following sense:

> each $x \in X$ has a neighbourhood U in X such that $E|U$ is isomorphic to the trivial family $U \times \mathbb{R}^n$ for some n.

Here $E|U$ denotes the family $\{E_x\}_{x\in U}$, and an isomorphism $E|U \xrightarrow{\cong} U \times \mathbb{R}^n$ means a homeomorphism which maps E_y by a vector-space isomorphism to $\{y\} \times \mathbb{R}^n$ for each $y \in U$.

Terminology.

(i) E is called the *total space* of the bundle.

(ii) The map $\pi : E \to X$ taking E_x to x is the *projection*.

(iii) The vector spaces E_x are the *fibres*.

(iv) A map $s : X \to E$ such that $s(x) \in E_x$ for all $x \in X$ is called a *section* of E.

(v) The *zero-section* is the map $i : X \to E$ such that $i(x) = 0 \in E_x$ for each x.

Vector bundles are examples of a more general concept. A *fibre bundle* on X with fibre a space F is a space Y which looks locally like $X \times F$ in the sense that there is given a map $\pi : Y \to X$, and each $x \in X$ has a neighbourhood U such that $\pi^{-1}U \cong U \times F$ by a homeomorphism taking $Y_y = \pi^{-1}(y)$ homeomorphically to $\{y\} \times F$ for each $y \in U$.

Examples.

(i) A smooth n-dimensional manifold X has a tangent space T_xX at each point x, and $TX = \cup T_xX$ is a vector bundle on X. Thus if $X = S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ then $T_xX = \{\xi \in \mathbb{R}^{n+1} : < x, \xi >= 0\}$, and

$$TX = \{(x,\xi) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} : ||x|| = 1 \text{ and } < x, \xi >= 0\}.$$

(ii) If Y is a smooth m-dimensional submanifold of a smooth n-dimensional manifold X then T_yY is a subspace of T_yX for each $y \in Y$. The quotient space $N_y = T_yX/T_yY$ is the *normal space* to Y at y. If X has a Riemannian structure one can identify N_y with the orthogonal complement of T_yY in T_yX. In any case, $NY = \bigcup_{y\in Y} N_y$ is an $(n-m)$-dimensional vector bundle on Y, with its topology acquired from TX.

(iii) Let X be the Grassmannian $Gr_k(\mathbb{R}^n)$. There is a "tautological" k-dimensional vector bundle E on X whose fibre at x is $E_x = x \subset \mathbb{R}^n$. The total space E is a subspace of $X \times \mathbb{R}^n$.

To see that this bundle is locally trivial, choose an inner product on $\mathbb{R}^n$ and let U be the neighbourhood of x consisting of all y such that $E_y \cap E_x^\perp = 0$. Then define $E|U \to U \times E_x$ by

$$(y, \xi) \mapsto (y, pr_x(\xi)),$$

where $pr_x : \mathbb{R}^n \to E_x$ is orthogonal projection, which maps E_y isomorphically to E_x if $y \in U$.

(iv) The Stiefel manifold $V_k(\mathbb{R}^n)$ is the subspace of $(\mathbb{R}^n)^k$ consisting of orthonormal k-tuples $\{v_1, \cdots, v_k\}$ in $\mathbb{R}^n$. (Thus $V_1(\mathbb{R}^n) = S^{n-1}$, and $V_n(\mathbb{R}^n)$ is the orthogonal group O_n.) If $k \geqslant m$ we can define

$$\pi : V_k(\mathbb{R}^n) \longrightarrow V_m(\mathbb{R}^n)$$

by forgetting the last $k-m$ vectors. This is a fibre bundle with fibre $V_{k-m}(\mathbb{R}^{n-m})$.

There are certain operations we can perform on vector bundles.

(i) If E is a vector bundle on X, and $f : Y \to X$ is a map, we can define the *pull-back* f^*E, which is a vector bundle on Y such that $(f^*E)_y = E_{f(x)}$. The total space of f^*E is a subspace of $Y \times E$. In fact it is the fibre product $Y \times_X E$: if Y_1 and Y_2 are two spaces with given maps $\pi_i : Y_i \to X$, the fibre product $Y_1 \times_X Y_2$ is defined as the subspace

$$\{(y_1, y_2) \in Y_1 \times Y_2 : \pi_1(y_1) = \pi_2(y_2)\}$$

of $Y_1 \times Y_2$.

The pull-back operation is defined for all fibre bundles, not just vector bundles. If f is the inclusion of a subspace Y of X then f^*E is just the restriction $E|Y$.

(ii) If E and F are vector bundles on X the *Whitney sum* $E \oplus F$ is a vector bundle such that $(E \oplus F)_x = E_x \oplus F_x$. Its total space is the fibre product $E \times_X F$.

In proving that f^*E and $E \oplus F$ are vector bundles the only point is to show the local triviality. For this it is helpful to introduce the term *pre-vector-bundle* for a structure which is not necessarily locally trivial. The pull-back and Whitney sum are obviously well-defined operations on pre-vector-bundles. They clearly take trivial bundles to trivial bundles. On the other hand they commute with restriction. So they take vector bundles to vector bundles.

An *inner product* on a vector bundle E is an inner product on each fibre E_x such that $\xi \mapsto \langle \xi, \xi \rangle$ is a continuous map $E \to \mathbb{R}$. This implies that $(\xi, \eta) \mapsto \langle \xi, \eta \rangle$ is a continuous map $E \times_X E \to \mathbb{R}$, for

$$\langle \xi, \eta \rangle = \frac{1}{2}(||\xi + \eta||^2 - ||\xi||^2 - ||\eta||^2).$$

Proposition 4.1.1 *Any vector bundle E on a paracompact base X has an inner product.*

Proof. Let $\{X_\alpha\}$ be an open covering of X such that $E|X_\alpha \cong X_\alpha \times \mathbb{R}^n$. Use a choice of local trivialization for each α to define an inner product $\langle , \rangle_\alpha$ on $E|X_\alpha$. Then define

$$\langle \xi, \eta \rangle = \Sigma \lambda_\alpha(x) \langle \xi, \eta \rangle_\alpha$$

for $\xi, \eta \in E_x$, where $\{\lambda_\alpha\}$ is a partition of unity *subordinate* to $\{X_\alpha\}$, i.e. $\lambda_\alpha : X \to \mathbb{R}_+$ is continuous, and $\text{supp}(\lambda_\alpha) \subset X_\alpha$.

■

Proposition 4.1.2 *If E is a vector bundle with an inner product, one can find local trivializations which are compatible with the inner product.*

Proof. A local trivialization of E over U is the same thing as a sequence of sections $s_1, \cdots, s_n$ of $E|U$ such that $s_1(x), \cdots, s_n(x)$ are a basis of E_x for each $x \in U$. It is compatible with the inner product if each $s_1(x), \cdots, s_n(x)$ is an orthonormal basis. We can get such a sequence from an arbitrary sequence $s_1, \cdots, s_n$ by the Gram-Schmidt orthonormalization process. ■

Proposition 4.1.3 *If E is a vector bundle on X with an inner product, and F is a subbundle of E, let $F_x^\perp$ denote the orthogonal complement of F_x in E_x. Then the subspace $F^\perp = \bigcup F_x^\perp$ of E is a vector bundle.*

Proof. We have only to prove local triviality, so we can assume E and F are trivial. Say $E = X \times \mathbb{R}^n$ and $F = X \times \mathbb{R}^m$, where the inclusion $F \to E$ takes (x, e_i) to $(x, \xi_i(x))$, where $\xi_i : X \to \mathbb{R}^n$ is continuous. (Here $\{e_i\}$ is the standard basis of $\mathbb{R}^m$). Given $x_0 \in X$, choose $\xi_{m+1}, \cdots, \xi_n \in \mathbb{R}^n$ so that

$$\xi_1(x_0), \cdots, \xi_m(x_0), \xi_{m+1}, \cdots, \xi_n$$

is a basis. Then

$$\xi_1(x), \cdots, \xi_m(x), \xi_{m+1}, \cdots, \xi_n \tag{4.1.4}$$

is a basis for all x in a neighbourhood U of x_0. Let $\eta_i(x), \cdots, \eta_n(x)$ be the basis got from (4.1.4) by the Gram-Schmidt process. Then $(x, e_i) \mapsto (x, \eta_i(x))$ defines an isomorphism $E|U \to E|U$ which takes $U \times (0 \oplus \mathbb{R}^{n-m})$ to $F^\perp|U$. ■

A vector bundle E on X is a family of "abstract" vector spaces $\{E_x\}$ parametrized by $x \in X$. For most purposes one can assume, if it is helpful to do so, that all the fibres E_x are subspaces of a fixed large vector space $\mathbb{R}^N$, i.e. that E is a subbundle of the trivial bundle $X \times \mathbb{R}^N$, at least if X is compact. (Actually it is enough for X to be finite dimensional. In general one must replace $\mathbb{R}^N$ by an arbitrary infinite dimensional topological vector space.)

Proposition 4.1.5 *On a compact space X any vector bundle E is a subbundle of a trivial bundle $X \times \mathbb{R}^N$.*

For this we need a lemma.

Lemma 4.1.6 *If E is a vector bundle on a paracompact space X, and $\xi \in E_x$ for some $x \in X$, then there is a section $s : X \to E$ such that $s(x) = \xi$.*

Proof. Let $\alpha : U \times \mathbb{R}^n \to E|U$ be a trivialization in a neighbourhood U of x. Choose $f : X \to \mathbb{R}$ such that $f(x) = 1$ and $\mathrm{supp}(f) \subset U$. Define

$$\begin{aligned} s(y) &= f(y)\alpha(y, \xi) \quad \text{if} \quad y \in U \\ &= 0 \quad \text{if not.} \end{aligned}$$

■

Proof. of 4.1.5 It is enough to find sections $s_1, \cdots, s_N$ of E such that $s_1(x), \cdots, s_N(x)$ span E_x for each $x \in X$. For then we can choose an inner product on E and define $E \to X \times \mathbb{R}^N$ by

$$\xi \mapsto (x\,;\, < s_1(x), \xi >, \cdots, < s_N(x), \xi >)$$

for $\xi \in E_{(x)}$.

To find $s_1, \cdots, s_N$ we choose for each $x \in X$ sections $s_1^x, \cdots, s_n^x$ such that $s_1^x(x), \cdots, s_n^x(x)$ are a basis of E_x. Then $s_1^x(y), \cdots, s_n^x(y)$ span E_y for all y in a neighbourhood U_x of x. Choose a finite number of these neighbourhoods U_x which cover X. The corresponding sections s_i^x are as desired. ■

Corollary 4.1.7 *We have $E = f^*\mathbb{E}$ for some $f : X \to Gr_n(\mathbb{R}^N)$, where $\mathbb{E}$ is the tautological bundle on $Gr_n(\mathbb{R}^N)$.*

Proof. The only point is to show that f is continuous, where $f(x) = [E_x] \in Gr_n(\mathbb{R}^N)$. But in the neighbourhood of any point of X the space E_x is spanned by vectors $\xi_1(x), \cdots, \xi_n(x)$ such that the maps $\xi_i : X \to \mathbb{R}^N$ are continuous. This implies that f is continuous. ∎

Considerably more is true. Let us define Gr_n as the union of the spaces $Gr_{n,N} = Gr_n(\mathbb{R}^N)$ for all N, where we take

$$\cdots \subset \mathbb{R}^N \subset \mathbb{R}^{N+1} \subset \mathbb{R}^{N+2} \subset \cdots$$

in the obvious way. We define the topology of Gr_n by prescribing that a subset is open if its intersection with each $Gr_{n,N}$ is open. I shall leave it as an exercise to show that any compact subset of Gr_n is contained in $Gr_{n,N}$ for some N.

The union of the tautological bundles on $Gr_{n,N}$ is a vector bundle $\mathbb{E}$ on Gr_n.

Proposition 4.1.8 *For any compact space X the map $f \mapsto f^*\mathbb{E}$ defines a $1-1$ correspondence between homotopy classes of maps $X \to Gr_n$ and isomorphism classes of n-dimensional vector bundles on X.*

This is expressed by saying that Gr_n is a *classifying space* for n-dimensional vector bundles.

Proof. We have shown that every bundle is of the form $f^*\mathbb{E}$. We must prove

(i) If $f_0 \simeq f_1 : X \to Gr_n$ then $f_0^*\mathbb{E} \cong f_1^*\mathbb{E}$, and

(ii) If $f_0,\ f_1 : X \to Gr_n$ and $f_0^*\mathbb{E} \cong f_1^*\mathbb{E}$ then $f_0 \simeq f_1$.

For (i) we can assume that $f_0(X)$ and $f_1(X)$ are both contained in $Gr_{n,N}$. If U is a neighbourhood of the diagonal in $Gr_{n,N} \times Gr_{n,N}$ then it is enough to show $f_0^*\mathbb{E} \cong f_1^*\mathbb{E}$ whenever $(f_0(x), f_1(x)) \in U$ for all $x \in X$. Take U to be the set of pairs (V_1, V_2) such that $V_1 \cap V_2^\perp = 0$. For such pairs the orthogonal projection $V_1 \to V_2$ is an isomorphism, and so if $(f_0(X), f_1(x)) \in U$ then we have isomorphism $(f_0^*\mathbb{E})_x \to (f_1^*\mathbb{E})_x$ which fit together to give an isomorphism of bundles.

For (ii) we shall show that if $f_0(X)$ and $f_1(X)$ are contained in $Gr_{n,N}$ then f_0 and f_1 become homotopic in $Gr_{n,2N}$. It is enough to show $f_0 \simeq T \circ f_1$, where $T : Gr_{n,2N} \to Gr_{n,2N}$ is induced by

$$T : \mathbb{R}^N \oplus \mathbb{R}^N \to \mathbb{R}^N \oplus \mathbb{R}^N$$

and $T(\xi, \eta) = (-\eta, \xi)$. (For T can be joined to the identity by a path in SO_{2N}). But if V_0 and V_1 are n-dimensional subspaces of $\mathbb{R}^N$ than an isomorphism $\alpha : V_0 \to V_1$ defines a path γ from $V_0 \oplus 0$ to $0 \oplus V_1$ in $Gr_n(\mathbb{R}^N \oplus R^N)$, where $\gamma(t)$ is the image of

$$ti_0 \oplus (1-t)i_1\alpha : V_0 \to \mathbb{R}^N \oplus \mathbb{R}^N,$$

and $i_0 : V_0 \to \mathbb{R}^N$ and $i_1 : V_1 \to \mathbb{R}^N$ are the inclusions. The paths so defined by isomorphisms $(f_0^*\mathbb{E})_x \to (f_1^*\mathbb{E})_x$ provide a homotopy from f_0 to $T \circ f_1$. ∎

4.2 The Thom isomorphism theorem

When we have a vector bundle $\pi : E \to X$ the interesting cohomology to consider is that of the complex of cochains of E which are supported "near" the zero-section $i(X) \subset E$. In fact, it is simplest to consider the relative cohomology $H^*(E, E^\#)$, where $E^\#$ denotes the complement of the zero-section in E.

If E is the trivial bundle $X \times \mathbb{R}^n$ then the Künneth theorem tells us that $H^{k+n}(E, E^\#)$ is isomorphic to $H^k(X)$ by multiplication by the generator $\varepsilon_n \in H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\})$. We shall now prove a fundamental theorem which shows that $H^k(X) \cong H^{k+n}(E, E^\#)$ for any oriented vector bundle E — in other words, cohomology does not see the twisting of E.

Definition. An *orientation* of a vector bundle E on X is a choice of a generator ε_x of $H^n(E_x, E_x - \{0\}) \cong H^n_{cpt}(E_x)$ for each $x \in X$, the choices being locally constant in the sense that when $E|U \cong U \times E_x$ is a local trivialization in a neighbourhood U of x then for every $y \in U$ the induced isomorphism $E_y \cong E_x$ takes ε_y to ε_x.

Theorem 4.2.1 (The Thom isomorphism theorem.) *If E is an oriented n-dimensional vector bundle on X, then*

(i) there is a unique element $u_E \in H^n(E, E^\#)$ such that $u_E|E_x = \varepsilon_x$ for each $x \in X$, and

(ii) the map $\alpha \mapsto (\pi^*\alpha) \cdot u_E$ is an isomorphism $H^k(X) \to H^{k+n}(E, E^\#)$.

The element u_E is called the *Thom class* of E.

Proof. *(i)* We know from (3.5.3) that the theorem is true if E is trivial. Let $\mathcal{U} = \{X_\alpha\}$ be an open covering of X such that $E|X_\alpha$ is trivial for each α. Then by (3.2.3) the cohomology groups $H^*(X)$ and $H^*(E, E^\#)$ can be calculated from the total complexes of the double complexes $C^{\cdot\cdot}(\mathcal{U})$ and $Q^{\cdot\cdot}$, where

$$\begin{aligned} C^{pq}(\mathcal{U}) &= \bigoplus C^q(X_{\alpha_0\cdots\alpha_p}), \textit{ and} \\ Q^{pq} &= \bigoplus C^q(E|X_{\alpha_0\cdots\alpha_p}, E^\#|X_{\alpha_0\cdots\alpha_p}). \end{aligned}$$

Because the theorem is true for trivial bundles we can find an n-dimensional cocycle u_0 of $Q^{0\cdot}$ representing the Thom class in $E|X_\alpha$ for each α. Furthermore, the image of u_0 in $Q^{1\cdot}$ represents zero in $\bigoplus H^n(E|X_{\alpha\beta}, E^\#|X_{\alpha\beta})$, so it comes from an element $u_1 \in Q^{1,n-1}$. But the columns $Q^{1\cdot}, Q^{2\cdot}$, etc. have zero cohomology below dimension n, so we can find $u_2, u_3, \cdots$ iteratively so that $u_0 + u_1 + u_2 + \cdots$ is a cocycle in the total complex of $Q^{\cdot\cdot}$, and represents the desired Thom class u_E. As the cohomology class of u_0 in $Q^{0\cdot}$ is prescribed by the definition, it is easy to see that the total cohomology class of $u_0 + u_1 + u_2 + \cdots$ is uniquely determined.

(ii) Suppose now that we have a Thom class, represented by a cocycle u_E in $C^n(E, E^\#)$. Then the map $\alpha \mapsto (\pi^*\alpha) \cdot u_E$ defines a map of double complexes

$$C^{\cdot\cdot}(\mathcal{U}) \to Q^{\cdot\cdot},$$

raising degrees in each column by n. It induces an isomorphism of cohomology (raising degrees by n) in each column, and so it induces an isomorphism of the total cohomology. Thus $H^i(X) \to H^{n+i}(E, E^\#)$ is also an isomorphism. ■

The Thom class $u_E \in H^n(E, E^\#)$ defines, of course, an element of $H^n(E)$, which we can identify with $H^n(X)$ by π^* because $\pi : E \to X$ is a homotopy equivalence. The element $e_E \in H^n(X)$ obtained in this way is called the *Euler class* of E. Alternatively, we can say that $e_E = i^*u_E$, where $i : X \to E$ is the zero-section. The characterization of u_E in 4.2.1 (i) shows that e_E is an example of a *characteristic class* for real oriented vector bundles, in the following sense.

Definition. **4.2.2** A rule which associates to a bundle $\pi : E \to X$ of a certain type a cohomology class $c(E) \in H^k(X)$ is a characteristic class if

$$c(f^*E) = f^*c(E)$$

for every map $f : Y \to X$.

The importance of characteristic classes is that they give us a way of describing and distinguishing the possible bundles on a given base-space X.

A first property of the Euler class is

Proposition 4.2.3 *If a real oriented vector bundle E on X has a nowhere-vanishing section s, then $e_E = 0$.*

Proof. We have $e_E = i^*u_E$, where i is the zero-section. But $i \simeq s$, so $e_E = s^*u_E$. This vanishes, for $s(X) \subset E^\#$, and $u_E|E^\# = 0$. ∎

We shall see presently that if n is even the Euler class of the tangent bundle of the sphere S^n is twice the generator of $H^n(S^n)$.

4.3 The Gysin sequence

An immediate application of the Thom isomorphism theorem is to obtain the Gysin exact sequence for a sphere bundle.

Let E be an oriented n-dimensional real vector bundle on X with an inner product, and let S be the fibre bundle formed by the unit spheres S_x in the vector spaces E_x. (The bundle S is locally trivial by 4.1.2). The total space S is clearly homotopy equivalent to $E^\#$ and $\pi : E \to X$ is also a homotopy equivalence, so the cohomology sequence

$$\cdots \to H^{i-1}(E^\#) \to H^i(E, E^\#) \to H^i(E) \to H^i(E^\#) \to \cdots \qquad (4.3.1)$$

becomes (replacing i by $i + n$, and using 4.2.1)

Proposition 4.3.2 *There is a long exact sequence of $H^*(X)$-modules*

$$\cdots \to H^{i+n-1}(S) \to H^i(X) \to H^{i+n}(X) \xrightarrow{\pi^*} H^{i+n}(S) \to \cdots .$$

Here the map $H^i(X) \to H^{i+n}(X)$ is multiplication by the Euler class $e_E \in H^n(X)$.

This sequence is called the *Gysin sequence.* It is a sequence of $H^*(X)$-module homomorphisms because the maps in (4.3.1) are homomorphisms of $H^*(E)$-modules, and we identify $H^*(E)$ as a ring with $H^*(X)$ by π^*. The map $H^{i+n-1}(S) \to H^i(X)$ is called *integration along the fibres.*

As a first application of the Gysin sequence, let us consider the case when the Euler class e_E is zero. Then we can choose $\sigma \in H^{n-1}(S)$ which maps to 1 on integration along the fibres - this element will restrict to the preferred generator of $H^{n-1}(S_x)$ on each fibre. From the Gysin sequence we find at once that the map

$$H^*(X) \oplus H^*(X) \longrightarrow H^*(S)$$

given by $(a, b) \mapsto a + b\sigma$ is an isomorphism of $H^*(X)$-modules. This completely determines $H^*(S)$ as a ring in terms of $H^*(X)$ once one knows $a_0 \in H^{2n-2}(X)$ and $b_0 \in H^{n-1}(X)$ such that $\sigma^2 = a_0 + b_0\sigma$. If, for example, n is even and $H^{2n-2}(X)$ has no elements of order two, then $\sigma^2 = 0$ by anticommutativity.

Proposition 4.3.3 *The cohomology ring of the Stiefel manifold $V_k(\mathbb{C}^n)$ is $\bigwedge(\sigma_{2n-2k+1}, \cdots, \sigma_{2n-3}, \sigma_{2n-1})$, i.e. each group is free abelian, and the ring is generated by the k elements $\sigma_i \in H^i(V_k)$ of the indicated odd dimensions, with no relations other than those of anticommutativity.*

In particular, the rank of $H^i(V_k)$ is the coefficient of t^i in the polynomial

$$\Pi_{i=1,\cdots,k}(1 + t^{2n-2i+1}).$$

Proof. The result is true for $V_1 \cong S^{2n-1}$, and we proceed by induction on k, observing that V_{k+1} is the sphere bundle of a complex vector bundle E on V_k with fibres $\mathbb{C}^{n-k}$. All complex vector bundles are orientable, as the group $GL_n(\mathbb{C})$ is connected. The Euler class e_E belongs to $H^{2n-2k}(V_k)$, which is the zero group by the inductive hypothesis, and the element $\sigma_{2n-2k-1} \in H^{2n-2k-1}(V_{k-1})$ must have square zero because $H^*(V_k)$ is free. ∎

As another example of the use of the Gysin sequence we can calculate the cohomology ring of the complex Grassmannian Gr_k. The sphere bundle S of the tautological bundle $\mathbb{E}$ on Gr_k is homotopy equivalent to Gr_{k-1} by the map taking (V, ξ), where V is a k-dimensional subspace of $\mathbb{C}^\infty$ and ξ is a unit vector in V, to the $(k-1)$-dimensional subspace $V \cap \xi^\perp$. Indeed the map $S \to Gr_{k-1}$ is a sphere bundle whose fibre at W is the infinite dimensional unit sphere in $W^\perp$. I shall omit the proof that this map is a homotopy equivalence: we need only that it induces an isomorphism of cohomology, which I leave as an exercise.

Let us write c_k for the Euler class of $\mathbb{E}$ in $H^{2k}(Gr_k)$. The Gysin sequence gives us

$$\cdots \to H^{i-2k}(Gr_k) \to H^i(Gr_k) \to H^i(Gr_{k-1}) \to H^{i-2k+1}(Gr_k) \to \cdots,$$

which shows that $H^i(Gr_k) \to H^i(Gr_{k-1})$ is an isomorphism if $i < 2k-1$. In particular, the element $c_{k-1} \in H^{2k-2}(Gr_{k-1})$ comes from a unique element, again called c_{k-1}, in $H^{2k-2}(Gr_k)$. In this way we obtain k elements $c_1, \cdots, c_k$ in $H^*(Gr_k)$, with $c_i \in H^{2i}$.

Proposition 4.3.4 *The ring $H^*(Gr_k)$ is the polynomial ring $\mathbb{Z}[c_1, \cdots, c_k]$.*

Proof. This is obviously true when $k = 0$ and Gr_k is a point, so we use induction on k. The inductive hypothesis implies that $H^*(Gr_k) \longrightarrow H^*(Gr_{k-1})$ is surjective, so the Gysin sequence becomes

$$0 \to H^*(Gr_k) \xrightarrow{\times c_k} H^*(Gr_k) \to H^*(Gr_{k-1}) \to 0. \tag{4.3.5}$$

We can identify $H^*(Gr_{k-1})$ with a subring R_{k-1} of $H^*(Gr_k)$, and (4.3.5) implies that $H^*(Gr_k) \cong R_{k-1}[c_k]$. ∎

Remarks The first interesting case of the preceding theorem is when $k = 1$ and $Gr_1 = \mathbb{P}^\infty_{\mathbb{C}}$.

The unit sphere bundle of the tautological bundle over the projective space $\mathbb{P}^{n-1}_{\mathbb{C}}$ is the sphere S^{2n-1}. Applying the Gysin sequence to that gives us

Proposition 4.3.6 *The ring $H^*(\mathbb{P}^{n-1}_{\mathbb{C}})$ is $\mathbb{Z}[c_1]/(c_1^n)$.*

NOTES ON ALGEBRAIC TOPOLOGY

G. Segal - Lent Term '95

5. The cohomology of manifolds

5.1 Orientation

For simplicity we shall confine ourselves to n-dimensional manifolds X which possess a finite convex covering, (i.e. a finite open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ such that each subset $U_{\alpha_0 \cdots \alpha_p}$ is either empty or homeomorphic to $\mathbb{R}^n$).

We shall say that X is of type k if it has a convex covering by k sets. By induction on the type we find

Proposition 5.1.1 *(i) $H^i_{cpt}(X)$ is a finitely generated abelian group for each i.*

(ii) $H^i_{cpt}(X) = 0$ if $i > n$.

(iii) If X is connected, then $H^n_{cpt}(X)$ is cyclic, and $i_* : H^n_{cpt}(\mathbb{R}^n) \to H^n_{cpt}(X)$ is surjective for any open embedding $i : \mathbb{R}^n \hookrightarrow X$ (i.e. a map taking $\mathbb{R}^n$ homeomorphically to an open subset of X).

Proof. Only (iii) needs discussion. If $X = X_0 \cup X_1$, with $X_0 \cong \mathbb{R}^n$ and X_1 and X_{01} of type $k-1$ then the Mayer-Vietoris sequence and the inductive hypothesis give us an exact sequence

$$H^n_{cpt}(X_{01}) \to H^n_{cpt}(X_0) \oplus H^n_{cpt}(X_1) \to H^n_{cpt}(X) \to 0. \tag{5.1.2}$$

Now $H^n_{cpt}(X_0) \cong \mathbb{Z}$, and the map $H^n_{cpt}(X_{01}) \to H^n_{cpt}(X_0)$ is onto, for if X is connected then X_{01} is non-empty, and we can choose an embedding $\mathbb{R}^n \hookrightarrow X_{01}$ and use (2.7.3). So $H^n_{cpt}(X_1) \to H^n_{cpt}(X)$ is surjective, which proves (iii) by induction. ∎

Definition 5.1.3 X *is* oriented *if there is given a generator $\omega_U \in H^n_{cpt}(U)$ for each open subset U of X which is homeomorphic to $\mathbb{R}^n$, and $\omega_U \longmapsto \omega_{U'}$ when $U \subset U'$.*

Remark Using (2.7.3) we see that it is enough to be given ω_U for all *sufficiently small* sets U, e.g. all those which are contained in some set of an open covering $\mathcal{U}$ of X.

Proposition 5.1.4 *(i) If X is oriented there is a unique map*

$$\int_X : H^n_{cpt}(X) \longrightarrow \mathbb{Z}$$

such that $\int_X i_*(\omega_U) = 1$ whenever $i : U \to X$ is the inclusion of an open subset homeomorphic to $\mathbb{R}^n$.

(ii) If in addition X is connected, then $\int_X$ is an isomorphism.

5.2 The Poincaré duality theorem

Let X be an oriented manifold.

Using cohomology with coefficients in any commutative ring A we have an A-bilinear map

$$\begin{aligned} H^k_{cpt}(X) \times H^{n-k}(X) &\to A \\ (\alpha\ ,\ \beta) &\longmapsto \textstyle\int_X \alpha \cdot \beta, \end{aligned} \tag{5.2.1}$$

for the product of any cochain with a cochain of compact support has compact support. Equivalently, we have a map

$$H^k_{cpt}(X) \to H^{n-k}(X)^*, \tag{5.2.2}$$

where M^* denotes the A-dual of an A-module M, i.e. $M^* = \mathrm{Hom}_A(M; A)$.

The contravariant functor $M \mapsto M^*$ from A-modules to A-modules is not well-behaved for most rings A. The best case is the following.

Proposition 5.2.3 *If A is a field, or if $A = \mathbb{Z}/r$ for some integer $r \neq 0$, then $M \mapsto M^*$ is an exact functor, i.e. it takes exact sequences to exact sequences.*

I leave the proof as an exercise.

Definition 5.2.4 *If X is a space with finitely generated cohomology groups, and A is either a field or $\mathbb{Z}/r$, then we define the* homology *groups of X with coefficients in A by $H_i(X) = H^i(X)^*$.*

For the present I shall not define H_i with $\mathbb{Z}$ as coefficients.

Theorem 5.2.5 *(Poincaré duality) If X is an oriented manifold and A is a field or $\mathbb{Z}/r$ then the map*

$$H^k_{cpt}(X) \to H^{n-k}(X)^* = H_{n-k}(X)$$

of (5.2.2) is an isomorphism.

Remark This remains true with integer coefficients when the homology is properly defined.

Proof. Once again we use induction on the type of X, writing $X = X_0 \cup X_1$ as in the proof of (5.1.1). The maps (5.2.2) for X, X_0, X_1, and X_{01} map the Mayer-Vietoris sequence for H^*_{cpt} to the dual of the Mayer-Vietoris sequence for H^*. Commutativity is obvious, except perhaps for

$$\begin{array}{ccc} H^k_{cpt}(X) & \longrightarrow & H^{k+1}_{cpt}(X_{01}) \\ \downarrow & & \downarrow \\ H^{n-k}(X)^* & \longrightarrow & H^{n-k-1}(X_{01})^*. \end{array}$$

This commutes to sign, for if $\alpha \in C^k_{cpt}(X)$ and $\beta \in C^{n-k-1}(X_{01})$ are cocycles then

$$\int_{X_{01}} d_{MV}\alpha \cdot \beta = \pm \int_X \alpha \cdot d_{MV}\beta.$$

■

5.3 Alexander duality

In this section I shall assume that the cohomology has coefficients in a *field.*

Suppose that X is a compact subset of the sphere S^n. Poincaré duality gives us $H^{n-i}_{cpt}(S^n - X) \cong H^i(S^n - X)^*$, for $S^n - X$ is an oriented manifold. By (2.7.4) we have $H^{n-i}_{cpt}(S^n - X) \cong H^{n-i}(S^n, X)$, and, using the sequence for the pair (S^n, X) we obtain

Theorem 5.3.1 *(Alexander duality) If X is a compact subset of S^n then*

$$\begin{aligned} H^{n-i-1}(X) &\cong H^i(S^n - X)^* \qquad \text{if } 0 < i < n-1, \\ \tilde{H}^0(X) &\cong H^{n-1}(S^n - X)^*, \\ H^{n-1}(X) &\cong \tilde{H}^0(S^n - X)^*. \end{aligned}$$

Here the so called *reduced* cohomology group $\tilde{H}^0(Y)$, for any non-empty space Y, means the cokernel of $H^0(point) \to H^0(Y)$. By choosing a point $y \in Y$ we get an isomorphism $H^0(Y) \cong H^0(point) \oplus \tilde{H}^0(Y)$.

Corollary 5.3.2 *If X is a compact subspace of S^n or $\mathbb{R}^n$ then the number of connected components of $S^n - X$ is one more than the dimension of $H^{n-1}(X)$.*

5.4 The cohomology class of a submanifold

It is a basic result of differential topology that if Y is a closed submanifold of a smooth manifold X there is an open neighbourhood U_Y of Y in X which is homeomorphic to the normal bundle NY of Y in X by an embedding $i : NY \to X$ which is the identity on the zero-section and is canonical up to "ambient isotopy". This means that if $i_0, i_1 : NY \to X$ are two choices, there is a continuously varying family of homeomorphisms $\{\varphi_t : X \to X\}$ such that φ_0 is the identity, $\varphi_t|Y$is the identity for all t, and $\varphi_1 \circ i_0 = i_1$. Such a neighbourhood U_Y is called a *tubular neighbourhood* of Y in X. If $y \in Y$ I shall sometimes write $U_{Y,y}$ for the normal disc which is the image of N_yY under $NY \cong U_Y$.

If $m = \dim(X) - \dim(Y)$ is the codimension of Y, and the bundle NY is oriented, the Thom class $u_{NY} \in H^m(NY, NY^\#)$ can be identified with an element of $H^m(U_Y, U_Y - Y) = H^m(X, X - Y)$. Its image ε_Y in $H^m(X)$ is called the *cohomology class of* Y. If Y is compact it is naturally an element of $H^m_{cpt}(X)$. It is always defined if X and Y are oriented.

If X is oriented and connected and Y is a point then ε_Y is a generator of $H^n_{cpt}(X)$.

The cohomology class ε_Y of a submanifold Y is completely characterized by two properties:

(i) it can be represented by a cocycle with support in Y, and

(ii) its restriction to each normal disc $U_{Y,y}$ is the generator of $H^m_{cpt}(U_{Y,y})$.

This follows from the corresponding characterization of the Thom class in (4.2.1). In fact ε_Y can be regarded as a "δ-function along Y", in the following sense.

Proposition 5.4.1 *If $\alpha \in H^{n-m}_{\mathrm{cpt}}(X)$ then*

$$\int_X \varepsilon_Y \cdot \alpha = \int_Y (\alpha|Y).$$

Proof. It is enough to prove this when $X = NY$. In that case, we can assume that $\alpha = \pi^*\beta$, where $\pi : NY \to Y$ is the projection and $\beta \in H^{n-m}_{\mathrm{cpt}}(Y)$. But then we can assume further that β is the cohomology class of a point of Y, and the result follows from (3.5.4). ∎

It is not quite true that every cohomology class of a manifold is the class of some submanifold: to obtain all cohomology classes one would have to include submanifolds with self-intersections and singularities. Nevertheless, the best way to think of the geometric meaning of the multiplication in cohomology is in terms of the following relation with the intersection of submanifolds.

Proposition 5.4.2 *If Y and Z are closed oriented submanifolds of an oriented manifold X, and they intersect transversally, then $\varepsilon_Y \cdot \varepsilon_Z = \varepsilon_{Y \cap Z}$.*

Two submanifolds Y and Z of X of codimensions m and r are said to *intersect transversally* if $Y \cap Z$ is a submanifold W of codimension $m + r$, and $T_xY + T_xZ = T_xX$ for each $x \in Y \cap Z$. In that case

$$NW = (NY)|W \oplus (NZ)|W,$$

and we can assume

$$U_{Y \cap Z} = U_Y \cap Y_Z,$$

and

$$U_{W,w} \cong U_{Y,w} \times U_{Z,w}.$$

Proposition 5.4.2. follows directly from this definition and the characterization of the cohomology classes of submanifolds, together with (3.5.4).

If $f : Z \to X$ is a smooth map of oriented manifolds, and Y is a closed submanifold of X, then f is said to be *transversal to* Y if the derivative $Df(z)$ maps to T_zZ surjectively to $T_yX/T_yY = N_yY$ for all $z \in f^{-1}(Y)$, where $y = f(z)$. Then the implicit function theorem tells us that $f^{-1}(Y)$ is an oriented submanifold of Z, and that $N(f^{-1}Y) = f^*(NY)$. We deduce

Proposition 5.4.3 *In this situation we have $\varepsilon_{f^{-1}Y} = f^*\varepsilon_Y$.*

On a smooth manifold X we can speak of a *smooth* vector bundle E. The total space E is a smooth manifold, and the zero section $i : X \to E$ embeds X as a closed submanifold with normal bundle E. If X is compact and E is oriented, the cohomology class ε_X of the zero section is just the Thom class $u_E \in H^m_{cpt}(E)$.

Proposition 5.4.4 *If a smooth section $s : X \to E$ is transversal to the zero section, the cohomology class of the zero-set $Z = s^{-1}(0) = s^{-1}(i(X))$, is the Euler class of the bundle E, i.e.*

$$\varepsilon_Z = e_E$$

in $H^*(X)$.

Proof. We have $\varepsilon_Z = s^*u_E = i^*u_E = e_E$, because $s \simeq i$. ∎

5.5 The class of the diagonal and the Lefschetz fixed-point theorem

In this section we shall always use cohomology groups with coefficients in a *field*.

If X is a compact oriented n-dimensional manifold we can identify $H^*(X \times X)$ with $H^*(X) \otimes H^*(X)$. We shall find a formula for the class ε_Δ of the diagonal $\Delta \subset X \times X$ in $H^*(X) \otimes H^*(X)$. Let $\{a_i\}$ be a basis of $H^*(X)$, with $a_i \in H^{n-d_i}(X)$, and let $\{a_i^*\}$ be the dual basis, in the sense that $\int_X a_i a_j^* = \delta_{ij}$.

Proposition 5.5.1

$$\varepsilon_\Delta = \Sigma(-1)^{d_i} a_i \otimes a_i^*.$$

Proof. By Poincaré duality for $X \times X$ it is enough to show that

$$\int_{X \times X} \varepsilon_\Delta \cdot (\xi \otimes \eta) = \int_{X \times X} (\Sigma(-1)^{d_i} a_i \otimes a_i^*) \cdot (\xi \otimes \eta)$$

for any $\xi \in H^p(X)$ and $\eta \in H^{n-p}(X)$. But by (5.4.1) the left-hand side is $\int_X \xi\eta$, while

$$\int (a_i \otimes a_i^*)(\xi \otimes \eta) = \begin{cases} (-1)^{p^2} \int_X a_i\xi \int_X a_i^*\eta & \text{if } d_i = p \\ 0 & \text{if } d_i \neq p. \end{cases}$$

So we must prove

$$\int_X \xi\eta = \sum_i \int a_i\xi \int a_i^*\eta.$$

It is even enough to prove this when $\xi = a_j^*$, and then it is obvious. ∎

Now suppose that $f : X \to X$ is a map with non-degenerate fixed points. That means that $F = (id \times f) : X \to X \times X$ is transversal to the diagonal $\Delta \subset X \times X$. The fixed-point set $\{x \in X : f(x) = x\}$ is $F^{-1}\Delta$. It is necessarily finite, and each point $x \in F^{-1}\Delta$ has a sign $\pm$ according as $DF(x) : T_xX \to N_{x,x}\Delta \cong T_xX$ preserves or reverses orientation. The algebraic number of fixed points is $\int_X \varepsilon_{F^{-1}\Delta}$.

Proposition 5.5.2 *(The Lefschetz fixed-point theorem). In the preceding situation, the number of fixed-points of f, counted with signs, is*

$$\Sigma(-1)^k \text{ trace } \{f^* : H^k(X) \to H^k(X)\}.$$

In particular, if f is homotopic to the identity, then the number of fixed points is the Euler number

$$\chi(X) = \Sigma(-1)^k \dim H^k(X).$$

Proof. We have $\varepsilon_{F^{-1}\Delta} = F^*\varepsilon_\Delta = \Sigma(-1)^{d_i} a_i f^*(a_i^*)$. But $\int_X a_i f^*(a_i^*)$ is the (i,i) matrix element of f^* with respect to the basis $\{a_i^*\}$. So

$$\int_X \varepsilon_{F^{-1}\Delta} = \Sigma(-1)^k \text{ tr } \{f^* : H^k(X) \to H^k(X)\}.$$

∎

A closely related result is

Proposition 5.5.3 *If X is a compact oriented manifold, then*

$$\int_X e_{TX} = \chi(X),$$

and hence $\chi(X)$ is the number of zeros of any tangent vector field on X which is transversal to the zero-section.

Proof. Exploiting the fact that the Thom class of TX is the cohomology class of the zero-section in $H^n_{cpt}(TX)$, and also that TX can be identified with the normal bundle to the diagonal in $X \times X$, we have

$$\begin{aligned}\int_X e_{TX} &= \int_{TX} u_{TX} e_{TX} = \int_{U_\Delta} \varepsilon_\Delta^2 = \int_{X\times X} \varepsilon_\Delta^2 \\ &= \int_\Delta \varepsilon_\Delta | \Delta = \Sigma(-1)^{d_i} \int_X a_i a_i^* = \chi(X).\end{aligned}$$

■