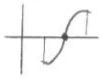
Algebraic Topology

Many theorems in maths are essentially topological. For example:

Intermediate Value Theorem:  - considers connected intervals.

Fundamental Theorem of Algebra: $f(z) = z^n + a_1 z^{n-1} + \dots + a_n$, $f: \mathbb{C} \to \mathbb{C}$. Consider the winding number:  Any closed path in $\mathbb{C} - \{0\}$ "winds" a number of times about 0. Any continuous deformation of the path has the same winding number.

Take $R > |a_1| + \dots + |a_n|$, and $\gamma: [0,1] \to \mathbb{C} - \{0\}$, $\gamma(0) = \gamma(1)$; $\gamma(\theta) = R^n e^{2\pi i n\theta}$.

This has winding number n. Let $\gamma_R(\theta) = R^n e^{2\pi i n\theta} + a_1 R^{n-1} e^{2\pi i (n-1)\theta} + \dots$,

and $\gamma_{R,S}(\theta) = R^n e^{2\pi i n\theta} + S(a_1 R^{n-1} e^{2\pi i (n-1)\theta} + \dots)$. Varying S gives a deformation from γ_R (winding number $=1$) to γ (winding number $= n$) - ※.

Brouwer Fixed Point Theorem: $f: D^n \to D^n$, D^n = closed unit ball in $\mathbb{R}^n = \{\xi \in \mathbb{R}^n : \|\xi\| \leq 1\}$.

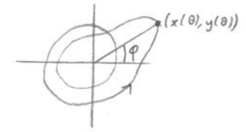
Then $f(\xi) = \xi$ for some ξ. Suppose not, and consider $g: S^{n-1} \to \mathbb{R}^n - \{0\}$, $g(\xi) = f(\xi) - \xi$. Let $g_R(\xi) = f(R\xi) - R\xi$, $0 \leq R \leq 1$.

Take map $\xi \mapsto Sf(\xi) - \xi$, $\|\xi\| = 1$. Obtain a similar contradiction.

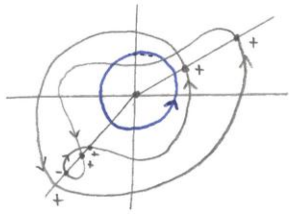
Maps $f_0, f_1 : X \to Y$ are homotopic iff $\exists$ continuous map $F: X \times [0,1] \to Y$ such that $F(x,0) = f_0(x)$, $F(x,1) = f_1(x)$. Often write $F(x,t) = f_t(x)$.

Winding number.

Suppose we have a closed path in $\mathbb{C} - \{0\}$, defined by $(x(\theta), y(\theta))$, $0 \leq \theta \leq 1$



Define the winding number as: $\frac{1}{2\pi} \int_0^1 \frac{y(\theta)\dot{x}(\theta) - x(\theta)\dot{y}(\theta)}{x(\theta)^2 + y(\theta)^2} \, d\theta$ = "$\frac{1}{2\pi} d\varphi$" " $\frac{1}{2\pi i} \frac{dz}{z}$



Draw a half-line from 0 to ∞, and count the crossings, taking the signs into account. (Avoid tangent points). Here the line crosses twice, giving winding number 2. By projecting onto S^1, we get $f: S^1 \to S^1$, and can count the number of points in $f^{-1}(y_0)$, $f^{-1}(y_1)$ - a continuous, integer-valued map.

There is a difference between algebraic topology and point-set topology. For example, consider k electrons moving in the plane. We could represent them in $\mathbb{R}^2 \times \dots \times \mathbb{R}^2 \cong \mathbb{R}^{2k}$. But electrons cannot occupy the same point, so we use $\tilde{C}_{k}(\mathbb{R}^2) = \{$distinct ordered k-tuples in $\mathbb{R}^2\} = \mathbb{R}^{2k} - \{$fat diagonal$\}$. $\subset \mathbb{R}^{2k}$

As they are indistinguishable, could have $C_k(\mathbb{R}^2) = \{$unordered distinct k-tuples$\} = \tilde{C}_k(\mathbb{R}^2)/\sim$

For $k=2$, this is essentially a circle, by identifying antipodal points of S^1.

For $k=3$, it is more complicated. In fact, $C_3(\mathbb{R}^2) \cong \mathbb{R}^3 \times$ (complement of trefoil knot).

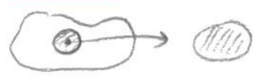
Consider the space of {Lines in $\mathbb{R}^2$}. Identify each line with the line through the origin parallel to it. We get {Lines} $\to S^1$. In fact, {Lines} $\cong$ Mobius band

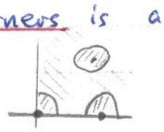
$\mathbb{P}^{n-1}_{\mathbb{R}}$ = 1-dimensional vector subspaces of $\mathbb{R}^n$. $= \mathbb{R}^{n-1} \cup$ "∞" $= (\mathbb{R}^n - \{0\})/v \sim \lambda v, \lambda \neq 0$.
$\cong$ real $n \times n$ symmetric matrix of rank 1 with trace 1. $\subset \mathbb{R}^{n^2}$.

$\mathbb{P}^2 \cong D^2$, disc $\subset \mathbb{R}^2$, with opposite points of $S^1 \subset D^2$ identified.

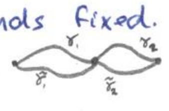
Define the Grassmannian, $Gr_k(\mathbb{R}^n)$ of k-dimensional vector subspaces of $\mathbb{R}^n$.
So, $Gr_1(\mathbb{R}^n) = \mathbb{P}^{n-1}$.

$SO_3 = \{3 \times 3$ real matrices A with $\det A = +1$, $A^T A = 1\}$ = positions of pivoted rigid body
$\cong \mathbb{P}^3$.

All these spaces are manifolds. A manifold is a topological space locally homeomorphic to $\mathbb{R}^n$ for some n.  $\subset \mathbb{R}^n$

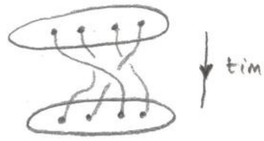
A manifold with boundary and corners is a space locally homeomorphic to $\mathbb{R}^n_+ = \{(x_1, .., x_n) \in \mathbb{R}^n : x_i \geqslant 0\}$. : 

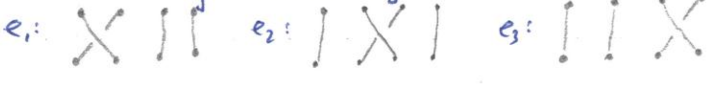
A path in a space X is a continuous map $\gamma : [0,1] \to X$. Its "ends" are $\gamma(0), \gamma(1)$.
If γ_1 is a path from x_0 to x_1, and γ_2 a path from x_1 to x_2, define $\gamma_2 * \gamma_1$ to be the path from x_0 to x_2 by $(\gamma_2 * \gamma_1)(t) = \begin{cases} \gamma_1(2t), & \text{if } t \in [0, 1/2] \\ \gamma_2(2t-1), & \text{if } t \in [1/2, 1] \end{cases}$.

Put an equivalence relation on paths, $\gamma_1 \sim \gamma_2$, if they have the same ends and are homotopic leaving the ends fixed.
$\sim$ is compatible with concatenation 

So, $*$ is defined on equivalence classes, and it is associative. That is, $\gamma_3 * (\gamma_2 * \gamma_1) \sim (\gamma_3 * \gamma_2) * \gamma_1$.
For any $x \in X$, there is a constant path at x, $1_x : [0,1] \to X$, with $\gamma * 1_x = 1_x * \gamma = \gamma$
Given γ, a path from x_0 to x_1, define γ^{-1}, a path from x_1 to x_0, by $\gamma^{-1}(t) = \gamma(1-t)$
Clear: $\gamma^{-1} * \gamma = 1_{x_0}$; $\gamma * \gamma^{-1} = 1_{x_1}$

If we fix $x_0 \in X$, a base-point, then paths beginning and ending at x_0 form a group, $\pi_1(X, x_0)$, the fundamental group of X at x_0.

Example: $\pi_1(C_k(\mathbb{R}^2), x_0)$ = "Braid group on k-strings" 

This can be generated by:


$e_1 e_2 e_1 = e_2 e_1 e_2$, $e_1 e_3 = e_3 e_1$, etc.

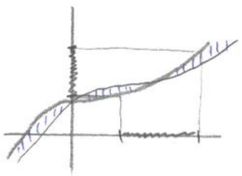
We have a space X, with some base-point $x_0 \in X$. $\pi_1(X, x_0)$ – fundamental group.
We may define $\pi_n(X, x_0)$ = homotopy classes of maps $\varphi: S^n \to X$ such that $\varphi(s_0) = x_0$, where s_0 is a chosen base-point in S^n.

Eg: 

S^2 is swept out by a series of circles, formed by the intersection of S^2 with a rotating plane through base-point, perpendicular to the page. A "path is the space of loops".

For a space X with base-point x_0, define the based loop space $\Omega X = Map_0(S^1; X)$, the space of base-point preserving maps $S^1 \to X$

$Map(X; Y)$, X compact, Y metric. If we have $f, g: X \to Y$, have metric $\tilde{d}(f,g) = \sup_{x \in X} d(f(x), g(x))$



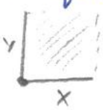
Since $\sup d$ is not necessarily $< \infty$, consider a compact subset of the source space.
K compact, $\subset X$, U open, $\subset Y$. Consider $\{f: X \to Y : f(K) \subset U\}$. This is open.

We can consider $Z \to Map(X; Y)$, and this is $Z \times X \to Y$. We want $Map(Z; Map(X;Y)) \cong Map(Z \times X; Y)$, if X is compact, as sets. It is true as spaces if X and Z are compact.

Definition: $\pi_2(X; x_0) = \pi_1(\Omega X; \omega_0)$, where $\omega_0 \in \Omega X$ is the constant loop at x_0.
$\pi_k(X; x_0) = \pi_{k-1}(\Omega X; \omega_0)$. $= \pi_1(\overbrace{\Omega \ldots \Omega}^{k-1} X; \omega_0)$ = set of connected path-components of $\Omega^k X =: \pi_0(\Omega^k X)$.
With this notation, $\pi_1(X; x_0) = \pi_0(\Omega X)$.

Theorem: $\pi_k(X; x_0)$ is abelian if $k > 1$.
Proof: A map $f: X \to Y$ such that $f(x_0) = y_0$ induces a homomorphism $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$.
$\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$, as groups.
Consider $f: \Omega X \times \Omega X \to \Omega X$, $(\gamma_1, \gamma_2) \mapsto \gamma_2 * \gamma_1$ – this is continuous.
So we have a homomorphism $f_* \pi_1(\Omega X \times \Omega X) \to \pi_1(\Omega X)$, ie, $\pi_1(\Omega X) \times \pi_1(\Omega X) \to \pi_1(\Omega X)$.
Finally, $f_*(\gamma, \omega_0) = \gamma = f_*(\omega_0, \gamma)$, because the composite $\Omega X \to \Omega X \times \Omega X \to \Omega X$; $\gamma \mapsto (\gamma, \omega_0) \mapsto \gamma * \omega_0$, is homotopic to the identity, $\Omega X \to \Omega X$.
Now, if π is a group and $\exists$ homomorphism $\pi \times \pi \to X$ where $(\gamma, 1) \mapsto \gamma$ and $(1, \gamma) \mapsto \gamma$, then the map is multiplication and the image is abelian.

$\pi_k(X, x_0) = \pi_0(\Omega^k X) \cong$ homotopy classes of maps $S^k \to X$. Recall that $Map(X \times Y; Z) \cong Map(X; Map(Y; Z))$ as spaces, if X, Y are compact.
Now, $Map_0(X; Map_0(Y; Z)) \cong Map_0(X \wedge Y; Z)$, where for two spaces X, Y with base-points x_0, y_0, $X \wedge Y$ = quotient space of $X \times Y$ which identifies the subspace $(x_0 \times Y) \cup (X \times y_0)$ to one point.



'axes' shrunk to one base-point.

$\wedge$ is often called "smash product".

So, $\mathrm{Map}_0(S^1 \wedge S^1; X) = \Omega\Omega X = \mathrm{Map}_0(S^1; \mathrm{Map}_0(S^1; X))$.

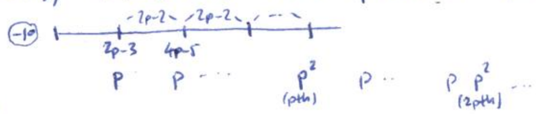
Lemma: $S^p \wedge S^q = S^{p+q}$.

What is $\pi_k(S^n)$?

If $k<n$, then $\pi_k(S^n)$ is trivial. If $k=n$, then $\pi_n(S^n) \cong \mathbb{Z}$ - winding number.
If $k>n$, then $\pi_k(S^n)$ finite, except for $\pi_{2n-1}(S^n)$ when n is even.
So, for n even, $\pi_{2n-1}(S^n) \to \mathbb{Z}$ via the Hopf invariant, a homeomorphism, and $\pi_{2n-1}(S^n) \cong \mathbb{Z} \oplus (\text{finite})$.

We can map $\pi_k(S^n) \to \pi_{k+1}(S^{n+1})$, via suspension, and this is an isomorphism if $k < 2n-1$.

$\pi_{n+m}(S^n) = (\text{cyclic group } J_m) \oplus (\text{unknown})$, for $n>m$. The power of the prime p in the order of J_m is described by:



If $m+1 = p^a(2p-2)$, then order has p^{a-1}.

Return to $k \leq n$.
Notice that $S^n - (\text{point}) \cong \mathbb{R}^n$ is contractible, ie, the identity map is homotopic to a constant map.

$S^2 \simeq$ boundary of a tetrahedron:  Can subdivide each triangle into smaller triangles, and do so successively until the sphere is ~~eff~~ covered by arbitrarily small triangular pieces.

We will consider $S^2 \to S^n$, $n>2$.

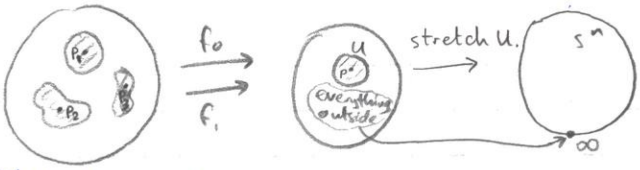
Lemma: If $f_0, f_1 : X \to S^n$ and $d(f_0, f_1) < \pi$, then $f_0 \simeq f_1$.

Lemma: Given $f: S^2 \to S^n$, any continuous map, we can find a triangulation of S^2 such that $f(\text{any triangle})$ has diameter $< \pi/2$.

Proof: Uses the Lebesgue covering lemma: If $\{U_\alpha\}$ is any open covering of a compact metric space X then $\exists\ \delta>0$ such that any subset of X of diam $< \varepsilon$ is $\subset$ some U_α.

Given such a map and triangulation, define new map $\tilde{f}: S^2 \to S^n$ such that $\tilde{f} = f$ on the vertices and $\tilde{f}$ is "linear" on each edge, and each triangle. But $\tilde{f}$ is not surjective, because its image is contained in a finite union of geodesic triangles. So $\tilde{f}$ constant. So $\pi_2(S^n)$ is trivial. $(n>2)$.

Now consider $f: S^n \to S^n$. Take p, not a base-point. Suppose $f_0, f_1: S^n \to S^n$, such that $f_0|_{f_0^{-1}(U)} = f_1|_{f_1^{-1}(U)}$ for some neighbourhood U of p. Then $f_0 \simeq f_1$.



$f^{-1}(p) = \{p_1, \dots, p_m\}$.

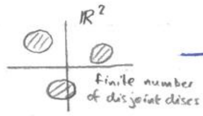
From now on, it is usually more convenient to consider $S^n \cong \mathbb{R}^n \cup \{\infty\}$, with ∞ as a base-point.

<u>Step 1</u>: Deform f to a map which is linear in the neighbourhood of a finite number of points $\{p_1, \dots, p_m\} = f^{-1}(p)$.

<u>Step 2</u>: $A, B \in GL_n(\mathbb{R})$ can be joined by a path in $GL_n(\mathbb{R})$ if sign det$A$ = sign det$B$. So A can be deformed to I_n or $\begin{pmatrix} -1 & 0 \\ 0 & \ddots \end{pmatrix}$.

Once more (from the top...): $\pi_n(S^n)$.

Take $f: S^n \to S^n$. Consider $n=2$. $S^2 = \mathbb{R}^2 \cup \{\infty\}$.



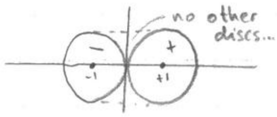
$\longrightarrow D^2 \xrightarrow{\text{onto}} S^2 \cong D^2/S^1$.

(D^2 = closed unit disc in $\mathbb{R}^2$)

Each disc is classified + or −. If the disc is +, map it to D^2 by translation. If −, map by translation followed by a reflection.

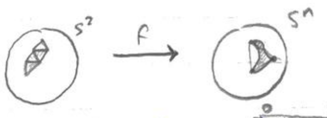
We now have: (union of discs) $\to D^2 \to S^2$. Extend to $\mathbb{R}^2 \cup \{\infty\} \to S^2$, by mapping everything else to ∞.

Suppose we have:

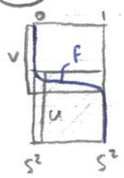


Deform this little region, by pushing the two discs together.

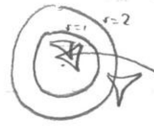
$\pi_k(S^n)$, $k<n$.



Define a homotopy $S^2 \to S^2$:



Eg:

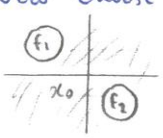


Say, $U = r<2$, $V = r>1$

disc (inside triangles) $\sim D^2 \to S^2$.

Composition law in $\pi_n(X, x_0)$. We have two maps $f_0, f_1: S^n \to X$, $(\infty \mapsto x_0)$. Think of them as maps $f_i: D^n \to X$ such that $f_i(S^{n-1}) = x_0$.

Now choose two disjoint balls $D_1^n, D_2^n \subset \mathbb{R}^n$. Define $(f_1 * f_2)(x) = \begin{cases} f_1(x) & \text{if } x \in D_1 \\ f_2(x) & \text{if } x \in D_2 \\ x_0 & \text{otherwise} \end{cases}$



So $f_1 * f_2: \mathbb{R}^n \cup \{\infty\} \to X$, $(\infty \mapsto x_0)$.

Swapping the discs corresponds to abelian nature of homotopy groups π_n, $n \geq 2$

For π_1, cannot swap in: 

If we map a Möbius band on to a circle: The pre-image of a small interval looks like that interval product the unit interval.

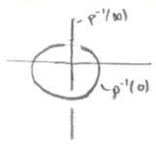
Consider $\tilde{C}_3(\mathbb{R}^2) \xrightarrow{p} \tilde{C}_2(\mathbb{R}^2)$. $p^{-1}(\{x_1, x_2\}) = \mathbb{R}^2 - \{x_1, x_2\}$.
$\mathbb{R}^2 - \{\text{any two points}\} \cong \mathbb{R}^2 - \{\text{any other 2 points}\}$, although  and  are clearly distinct maps.

Definition: A <u>locally trivial fibration</u> is a map $p: Y \to X$ such that for each $x \in X$ $\exists$ a neighbourhood U of x in X such that $p^{-1}(U) \overset{h}{\cong} U \times p^{-1}(x)$, by a homeomorphism h such that $h(p^{-1}(y)) \subset \{y\} \times p^{-1}(x)$. - (*)
The spaces $p^{-1}(x)$ are called the <u>fibres</u>, X is the <u>base</u>, Y is the <u>total space</u>.
(*) $\Leftrightarrow$ h is "fibre-preserving" $\Leftrightarrow$ $h(p_1^{-1}(x)) \subset p_2^{-1}(x)$ $\forall x$.

$$\begin{array}{ccc} Y_1 & \xrightarrow{f} & Y_2 \\ & p_1 \searrow \quad \swarrow p_2 & \\ & X & \end{array}$$

Examples: (a) $\tilde{C}_k(\mathbb{R}^n) \xrightarrow{p} \tilde{C}_m(\mathbb{R}^n)$ if $m<k$, where p is 'forgetting the last $k-m$ points.
(b) $p: Y = S^3 \to X = S^2$ - "Hopf map". Think of S^3 as the unit sphere in $\mathbb{C}^2$ as $\mathbb{C} \cup \{\infty\}$, the Riemann sphere. Define $p(z_1, z_2) = z_1/z_2 \in \mathbb{C} \cup \{\infty\}$.
So, $p^{-1}(\infty) = $ circle. $p^{-1}(\lambda) = (z_1, z_2)$ where $z_1 = \lambda z_2$, so $|z_1|^2 + |z_2|^2 = 1 \Rightarrow (1+|\lambda|^2)|z_2|^2 = 1$
$\Rightarrow |z_2| = \frac{1}{\sqrt{1+|\lambda|^2}}$.

Alternatively, for (b), consider: $S^3 = \mathbb{R}^3 \cup \{\infty\} \longrightarrow S^2 = \mathbb{R}^2 \cup \{\infty\}$.



This can be generalised to $S^7 \to S^4$ be using quaternions. Or, to $S^{15} \to S^8$, via Cayley numbers, or octernions.

Examples: (c): $S^{n-1} \to \mathbb{P}^{n-1}_{\mathbb{R}}$, (unit vectors in $\mathbb{R}^n$) $\mapsto$ (rays in $\mathbb{R}^n$). $p^{-1}(\text{point}) = $ (two points).

Definition: If the fibres are discrete, a locally trivial fibration is called a <u>covering space</u>.

If we have $S^{n-1} \to \mathbb{P}^{n-1}_{\mathbb{R}}$, each fibre $\cong$ (2 points).
$S^{2n-1} \subset \mathbb{C}^n \to \mathbb{P}^{n-1}_{\mathbb{C}}$, each fibre $\cong$ (circle).
$S^3 \to \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

Consider the <u>Stiefel manifold</u> $V_k(\mathbb{R}^n) = \{(v_1, \dots, v_k) \in (\mathbb{R}^n)^k : \langle v_i, v_j \rangle = \delta_{ij}\}$.
Thus, $V_1(\mathbb{R}^n) = S^{n-1}$, $V_n(\mathbb{R}^n) = O_n$.
Define $p: V_k(\mathbb{R}^n) \to V_{k-1}(\mathbb{R}^n)$ by 'forgetting' v_k. Each fibre $p^{-1}(v_1, \dots, v_{k-1})$ is a unit sphere S^{n-k} in $\mathbb{R}^n \ominus (\mathbb{R}v_1 + \dots + \mathbb{R}v_{k-1})$.
There is also a map $p: V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n)$ where $p(v_1, \dots, v_k) = $ subspace spanned by $v_1, \dots, v_k$

Take G, a closed subgroup of $GL_n(\mathbb{R})$, some n, and take H a closed subgroup of G. G/H = space of left cosets gH = quotient space of G. Then, the map $p: G \to G/H$, $g \mapsto gH$, is a locally trivial fibration with fibres $\cong H$.

Example: $G = S^3$ = unit quaternions $= SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : |a|^2 + |b|^2 = 1 \right\}$.
$H = S^1$ = unit complex numbers $= \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : |a| = 1 \right\}$.

X a smooth manifold, $Y = TX$ = all tangent vectors to X. $p: Y \to X$.
Eg. $X = S^{n-1}$, $TX = \{ (\xi, \eta) : \xi \in S^{n-1}, \eta \in \mathbb{R}^n, \langle \xi, \eta \rangle = 0 \}$.
(Unit tangent vectors to S^{n-1}) $\cong V_2(\mathbb{R}^n)$.

Consider map $p: V_k(\mathbb{R}^n) \to V_{k-1}(\mathbb{R}^n)$. Prove this is locally trivial.
Take $(v_1, \dots, v_{k-1}) \in V_{k-1}$. Write $\underline{v} = (v_1, \dots, v_{k-1})$. Let F = fibre at $\underline{v} = S^{n-k}$. Define U = all $(w_1, \dots, w_{k-1})$ such that $(\mathbb{R}w_1 + \dots + \mathbb{R}w_{k-1})^\perp$ doesn't meet F.
Define $F \times U \to V_k(\mathbb{R}^n)$, $(\xi, (w_1, \dots, w_{k-1})) \mapsto$ Gram-Schmidt orthonormalisation of $(w_1, \dots, w_{k-1}, \xi)$.

Consider Hopf map, $p: S^3 (\subset \mathbb{C}^2) \to S^2 (= \mathbb{C} \cup \{\infty\})$, $(z_1, z_2) \mapsto z_1/z_2$.
$p^{-1}(\mathbb{C}) \cong \mathbb{C} \times S^1$, $\left(\frac{z e^{i\theta}}{r}, \frac{e^{i\theta}}{r} \right) \leftrightarrow (z, e^{i\theta})$, where $r = \sqrt{1+|z|^2}$.
$p^{-1}(S^2 - \{0\}) \cong (S^2 - \{0\}) \times S^1$, $\left(\frac{e^{i\theta}}{\rho}, \frac{z^{-1} e^{i\theta}}{\rho} \right) \leftrightarrow (z, e^{i\theta})$, where $\rho = \sqrt{1 + |z|^{-2}}$.

A section of the bundle is a map $s: X \to Y$ (continuous) such that $s(x) \in$ fibre of x. Ie, $p \circ s = id_X$, where $p: Y \to X$.

$p: S^3 \rightleftarrows S^2$. If we have a section, we have a map $S^2 \to S^3 \to S^2$, but $S^2 \to S^3$ is homotopic to a constant $\Rightarrow id_{S^2} \simeq$ const $- \divideontimes$.

Long exact sequence of homotopy groups for a fibration

$\pi_k(X \times F) = \pi_k(X) \times \pi_k(F)$. Have $p: Y \to X$, assume Y, X path-connected.
p induces a homomorphism $p_*: \pi_k(Y) \to \pi_k(X)$.
Take $x_0 \in X$, $y \in p^{-1}(x_0) = F$. $i: F \hookrightarrow Y$ induces a homomorphism $i_*: \pi_k(F) \to \pi_k(Y)$.
Clearly $p_* \circ i_* = 0$. In fact, the sequence (for $k \geq 1$) $\pi_k(F) \xrightarrow{i_*} \pi_k(Y) \xrightarrow{p_*} \pi_k(X)$ is exact, ie image (i_*) = kernel (p_*).

If the bundle $Y \cong X \times F$, then $0 \to \pi_k(F) \underset{\text{injective}}{\to} \pi_k(Y) \underset{\text{surjective}}{\to} \pi_k(X) \to 0$ is exact.

Theorem: $\exists$ a homomorphism $\partial: \pi_k(X) \to \pi_{k-1}(F)$ for all $k \geq 1$ such that the sequence $\cdots \xrightarrow{\partial} \pi_n(F) \to \pi_n(Y) \to \pi_n(X) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots \to \pi_1(X) \xrightarrow{\partial} \pi_0(F) \to \pi_0(Y)$, is exact.

Example: $p: S^3 \to S^2$, $F \cong S^1$. We have: $\underset{0}{\pi_3(S^1)} \to \underset{\cong \mathbb{Z}}{\pi_3(S^3)} \to \underset{\cong \mathbb{Z}}{\pi_3(S^2)} \xrightarrow{\partial} \underset{0}{\pi_2(S^1)} \to \cdots$
And, $\pi_k(S^1) = 0$ if $k > 1$.

so this is an isomorphism.

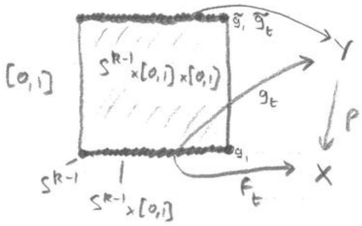
Homotopy Lifting Property: Suppose we have
$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "g_0"] \arrow[r, "f_0"] & X \end{tikzcd}$$
, g_0 is such that

$p \circ g_0 = f_0$ is a lift of f_0. ("H.L. Property").

H.L. Theorem: If $\{f_t\}_{0\le t\le 1}$ is a homotopy $f_t: Z \to X$, and we have a lift g_0 of f_0. Then $\exists$ homotopy $\{g_t\}: Z \to Y$ such that $p \circ g_t = f_t$.

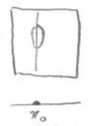
Definition of $\partial: \pi_k(X) \to \pi_{k-1}(F)$: Represent an element of $\pi_k(X)$ by a map $S^k \to X$. Think of it as a path in the space of maps $S^{k-1} \to X$, ie, as a homotopy $f_t: S^{k-1} \to X$. f_0 is constant at x_0; take g_0 constant at y_0. Choose g_t such that $p \circ g_t = f_t$. Consider $g_1: S^{k-1} \to p^{-1}(f(S^{k-1})) = p^{-1}(x_0) = F$. So g_1 represents an element $\partial(f)$ in $\pi_{k-1}(F)$.

To prove that the class of g_1 in $\pi_{k-1}(Y)$ is well-defined, use "relative homotopy lifting theorem". Given a homotopy $\{f_t\}: Z \to X$, and a lift of f_0 to $g_0: Z \to Y$, and a lift of $(f_t | Z_0): Z_0 \to X$ for some closed $Z_0 \subset Z$ to $h_t: Z_0 \to Y$ such that $h_0 = g_0 | Z_0$, we can lift f_t to $g_t: Z \to Y$ such that $h_t = g_t | Z_0$.



Suppose we have two lifts g_t, $\tilde g_t$ of $f_t: S^{k-1} \to X$. Consider the homotopy $\hat f_t: S^{k-1} \times [0,1] \to X$, where $\hat f_t(x,s) = f_t(x)$. Then g_t and $\tilde g_t$ together are a lift of $\hat f_t | S^{k-1} \times \{0,1\} \subset S^{k-1} \times [0,1]$.
Let $\hat g_t: S^{k-1} \times [0,1] \to Y$ be a lift extending g_t and $\tilde g_t$.
Then $\hat g_1$ is a map $S^{k-1} \times [0,1] \to Y$ and is a homotopy between g_1 and $\tilde g_1$.
Also, $p \circ \hat g_1 = \hat f_1 =$ constant map to x_0. So $\hat g_1: S^{k-1} \times [0,1] \to F = p^{-1}(x_0)$.

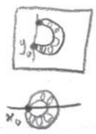
We shall prove that ∂ is a homomorphism of groups. But first consider the exactness of $\pi_k(F) \to \pi_k(Y) \to \pi_k(X)$.



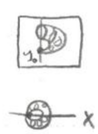
sphere $\to$ fibre is mapped to x_0, so im $\subset$ ker

lift homotopy.

So have exactness at $\pi_k(Y)$.
Now consider $\pi_{k+1}(X) \xrightarrow{\partial} \pi_k(F) \to \pi_k(Y)$

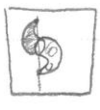


lift homotopy from X up to Y.



can extend image.

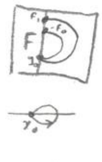
And at $\pi_k(Y) \to \pi_k(X) \xrightarrow{\partial} \pi_{k-1}(F)$.



extend homotopy in Y so end-points are on the fibre

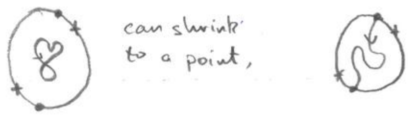
so stand still in X.

At the end of the sequence, have: $\pi_1(Y,y_0) \to \pi_1(X,x_0) \to \pi_0(F = p^{-1}(x_0)) \to \pi_0(Y) \to \pi_0(X) =$ point if X is path-connected.
The isotropy group of the action of $\pi_1(X,x_0)$ on $\pi_0(F)$ at $y_0 \in \pi_0(F)$ is the image of $\pi_1(Y,y_0)$ in $\pi_1(X,x_0)$.
The group $\pi_1(X,x_0)$ acts on the set $\pi_0(F)$ and the orbit space is the subset of $\pi_0(Y)$ which is $(p_*)^{-1}$(component containing x_0).

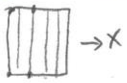


A path at x_0 in X induces a path in Y, starting and ending in the fibre F. But, by above, the end points lie in the same component of the fibre.

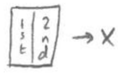
Examples: Consider $p: S^n \to \mathbb{P}^n_{\mathbb{R}}$. Fibres have two points. $\pi_1(S^n) = 0$ if $n > 1$.
So, $\underset{0}{\pi_1(S^n)} \to \pi_1(\mathbb{P}^{n-1}) \to \pi_0(F) \to$ point. So $\pi_1(\mathbb{P}^{n-1})$ is a group with two elements:



Why is $\partial: \pi_k(X) \to \pi_{k-1}(F)$ a homomorphism? Take $k=2$. Start with two maps $[0,1] \times [0,1] \to X$ taking boundary to x_0.



"Add" maps:



Can deform this to:

∂(2nd map) — (RH edge) = ∂(composite).
$\to Y$, after lifting.
all else $\to x_0$.
represents ∂(1st map).

Given $S = \{0, \dots, n\}$, can define a simplex on S, $\Delta_S \subset \mathbb{R}^{n+1}$ as $\{(\lambda_0, \dots, \lambda_n) : \lambda_i \geq 0, \sum \lambda_i = 1\}$.
Take $\sigma \subset S$, get $\Delta_\sigma \subset \Delta_S$.

For a given simplex, we may subdivide it into a union of smaller simplexes. So suppose we have $X \cong$ (polyhedron), with $\{U_\alpha\}$ an open covering of X. Then $X \cong$ (new polyhedron) such that each simplex $\subset$ some U_α.

Cohomology.- Introduction.

Take U, an open subset of $\mathbb{R}^3$. Consider the following:

$$\left\{ \begin{matrix} \text{smooth maps} \\ U \to \mathbb{R} \end{matrix} \right\} \xrightarrow[\text{grad.}]{} \left\{ \begin{matrix} \text{smooth maps} \\ U \to \mathbb{R}^3 \end{matrix} \right\} \xrightarrow[\text{curl.}]{} \left\{ \begin{matrix} \text{smooth maps} \\ U \to \mathbb{R}^3 \end{matrix} \right\} \xrightarrow[\text{div.}]{} \left\{ \begin{matrix} \text{smooth maps} \\ U \to \mathbb{R} \end{matrix} \right\}.$$

For example, let $U = \mathbb{R}^3 - (z\text{-axis})$.

Suppose we have $v : U \to \mathbb{R}^3$ with $\operatorname{curl} v = 0$.
Say $v = \frac{1}{2\pi}\left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 0\right)$.

$\operatorname{curl} v = 0 \iff$ locally, $v = \operatorname{grad} f$, some smooth f.
Clearly, if $v_i = \frac{\partial f}{\partial x_i}$, then $\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i}$, so $\operatorname{curl} v = 0$.

For v above, what is f? $f = \tan^{-1}(y/x)$, or $-\tan^{-1}(x/y)$.
If $v = \operatorname{grad} f$ globally, then $\int \langle v, ds \rangle = \int \langle \operatorname{grad} f, ds \rangle = f(\text{end}) - f(\text{start}) = 0$.

So if we split our loop into segments, then deforming a segment continuously does not alter the integral, if f is defined locally, ie, on each segment.

Such v defines an invariant I_v for closed paths in U which does not change if the path is deformed. Consider $v + \operatorname{grad} f$. $\operatorname{curl}(v + \operatorname{grad} f) = 0$. $I_{v+\operatorname{grad} f} = I_v$.
$I_{v+\operatorname{grad} f}(\gamma) = \int_\gamma \langle v + \operatorname{grad} f, ds \rangle = \int_\gamma \langle v, ds \rangle = I_v(\gamma)$.

$\{\text{Invariants for closed curves in } U\} \leftrightarrow \frac{\operatorname{Ker}(\text{curl})}{\operatorname{image}(\text{grad})}$

Eg: If $U = \mathbb{R}^3 - (z\text{-axis})$, have $\frac{\ker(\text{curl})}{\text{image}(\text{grad})} \cong \mathbb{R}$, spanned by the given element.

Now consider $U = \mathbb{R}^3 - \{0\}$. This is simply-connected, but we must consider a surface - either enclosing or not enclosing the origin.
So, consider $w: U \to \mathbb{R}^3$ such that $\text{div}\, w = 0$. Then, locally, $w = \text{curl}\, v$ for some $v: U \to \mathbb{R}^3$.
Suppose we have w such that $\text{div}\, w = 0$. Then we get an invariant for any closed surface $\Sigma \subset U$, $I_w(\Sigma) =$ "flux of w through Σ" $= \int_\Sigma \langle w, dS \rangle$.

Suppose we deform a small part of Σ:

So $\int_{\Sigma'} - \int_\Sigma = \int_{\partial R} \langle w, dS \rangle = \int_R (\text{div}\, w)\, d(\text{vol.}) = 0$.

Example: Electric field of point charge at 0. $w = \frac{1}{4\pi}\left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$.
$\int_\Sigma \langle w, ds \rangle =$ number of times Σ wraps around 0.

We get such an invariant for surfaces from each element of $\frac{\ker(\text{div})}{\text{image}(\text{curl})}$.

Eg: If $U = \mathbb{R}^3 - \{0\}$, this is $\cong \mathbb{R}$, spanned by the given element.

Now, $\frac{\ker(\text{grad})}{0} =$ vector space whose basis $\leftrightarrow$ connected components of U.
We may begin our sequence from before with $0 \to \{\text{smooth } U \to \mathbb{R}\} \to \cdots$, giving exactness.

We shall assign to each space X a cochain complex C, ie, a sequence of abelian groups and homomorphisms: $\cdots \to C^{-1} \xrightarrow{d} C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots$ with $d.d = 0$.
(May write d as $d_k: C^k \to C^{k+1}$)
Usually $C^k = 0$ for $k < 0$.

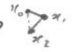
Then we define $H^k(X) = H^k(C) = \frac{\ker d_k}{\text{image}\, d_{k-1}}$, [Makes sense: $d.d = 0 \Rightarrow \text{im}\, d \subset \ker d$], the kth cohomology group of X.
(May write as $H^k(X; A)$, A our chosen coefficient group).

de Rham's Theorem: (cohomology defined for a smooth manifold by differential forms) $\cong H^*(X; \mathbb{R})$.

Alexander-Spanier Cochains.

Given a space X, and a coefficient group A, C^k = functions, not necessarily continuous, sending $(x_0, .., x_k)$, (points of X which are "sufficiently close together") to $c(x_0, .., x_k) \in A$.

Suppose we have $c(x_0, x_1) \in A$ whenever x_0 and x_1 are close, and suppose whenever

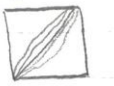
x_0, x_1, x_2 are close we have $c(x_1,x_2) - c(x_0,x_2) + c(x_0,x_1) = 0$. Or, $c(x_0,x_2) = c(x_0,x_1) + c(x_1,x_2)$.
(So, deforming a path by adding a point does not change things: 
Then it would make sense to define $\int_\gamma c$ for any closed path γ in X, and we would get a homotopy invariant of the path.

Define $d: C^k \to C^{k+1}$ by $(dc)(x_0,..,x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i c(x_0,..,\hat{x}_i,..,x_{k+1})$.
Notice that d is a homomorphism and $dd=0$.

"Sufficiently close together" means $\exists$ neighbourhood U_k of the diagonal in X^{k+1} such that $c(x_0,..,x_k)$ is defined whenever $(x_0,..,x_k) \in U_k$.

We have $H^k = \dfrac{\ker d: C^k \to C^{k+1}}{\text{image } d: C^{k-1} \to C}$. $dc = 0 \iff c$ is closed $\iff c$ is a cocycle.
$c = db \iff c$ is a coboundary.

We identify two cochains if they agree when restricted to some smaller common neighbourhood of the diagonal.



If $\mathcal{U} = \{U_\alpha\}$ is an open covering of X, then we get a neighbourhood U_k of the diagonal in X^{k+1}, by $(x_0,..,x_k) \in U_k \iff$ all x_i are in some U_α.
Or, $U_k = \bigcup_\alpha \underbrace{U_\alpha \times \cdots \times U_\alpha}_{k+1} \subset X^{k+1}$.

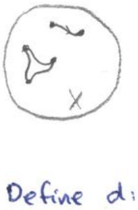
Define $C^k_{\mathcal{U}}(X) = \text{Maps}(U_k; A)$, cochains defined when $(x_0,..,x_k) \in U_k$.
Then $d: C^k_{\mathcal{U}}(X) \to C^{k+1}_{\mathcal{U}}(X)$. So $C_{\mathcal{U}}(X)$ is a cochain complex.

If S is a directed set, ie partially ordered, and for any $\alpha, \beta \in S$ $\exists$ $\gamma \in S$ with $\begin{cases}\alpha \le \gamma \\ \beta \le \gamma\end{cases}$.
Suppose $\{A_\alpha\}_{\alpha \in S}$ are abelian groups, and we have homomorphisms $\theta_{\alpha\beta}: A_\alpha \to A_\beta$ when $\alpha \le \beta$, such that $\theta_{\beta\gamma} \circ \theta_{\alpha\beta} = \theta_{\alpha\gamma}$ if $\alpha \le \beta \le \gamma$. Then $\varinjlim_\alpha A_\alpha$ is the group of all pairs $\{(a,\alpha) : \alpha \in S, a \in A_\alpha\}$ subject to the obvious equivalence relation.

Example: S = integers >0, ordered by divisibility. For $n \in S$, let $A_n = \frac{1}{n}\mathbb{Z} \cong \mathbb{Z}$.
Define $\theta_{nm}: \frac{1}{n}\mathbb{Z} \to \frac{1}{m}\mathbb{Z}$, $\frac{a}{n} \mapsto \frac{(am)/n}{m}$, when $n|m$. Then $\varinjlim_n \frac{1}{n}\mathbb{Z} = \mathbb{Q}$.

Singular cochains of X.

Let $S_k(X)$ = continuous maps : $\Delta^k \to X$, Δ^k = standard simplex with vertices $\{0,..,k\}$.



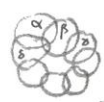
Define $C^k_{sing}(X)$ = all maps $S_k(X) \to A$.
To define $d: C^k_{sing} \to C^{k+1}_{sing}$, notice that there are obvious linear maps $\partial_i : \Delta^k \to \Delta^{k+1}$ such that $\partial_i(j) = \begin{cases} j & \text{if } j<i \\ j+1 & \text{if } j \ge i\end{cases}$, $0 \le i \le k$. 

Define $d: C^k_{sing}(X) \to C^{k+1}_{sing}(X)$, by $(dc)(\sigma) = \sum (-1)^i c(\sigma \circ \partial_i)$.
Check that $d \circ d = 0$. So we can define $H^k_{sing}(X, A)$, as we have a cochain complex.

Consider $H^0_{AS}(X;\mathbb{Z})$ and $H^0_{sing}(X;\mathbb{Z})$.
Singular: If σ is a 1-simplex, $dc(\sigma) = c(\sigma_1) - c(\sigma_0) = 0 \Rightarrow$ functions constant on path components.
$C^0_{AS}(X)$ = all maps: $X \to \mathbb{Z}$. Now we want $c(x_1) - c(x_0) = 0$ whenever x_0, x_1 are sufficiently close.
$H^0_{AS}(X;\mathbb{Z})$ = continuous maps $X \to \mathbb{Z}$, and $\mathbb{Z}$ has the discrete topology.
Difference between H^0_{AS} and H^0_{sing} ≡ difference between "connected" and "path-connected".

Čech Cochains of X.

Choose an open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in S}$ of X. $\check{C}^k(\mathcal{U})$ = all maps $S_k \to A$ where $S_k = \{(\alpha_0, \dots, \alpha_k) \in S \times \dots \times S,\ U_{\alpha_0} \cap \dots \cap U_{\alpha_k} \neq \phi\}$



Here, define $c(\alpha\beta)$, but not $c(\alpha\gamma)$.

– simplexes its "nerve"

$dc = 0 \iff c(\alpha\beta) - c(\beta\delta) + c(\alpha\delta) = 0$
" $dc(\alpha\beta\delta)$, defined, as $\alpha \cap \beta \cap \delta \neq \phi$.

We shall prove that in all cases, $\check{H}^k(\mathcal{U}) \cong H^k(C_{\mathcal{U}}(X))$.

Suppose X is a finite cell complex.

X is compact, and partitioned into regions $X_\alpha \cong \mathbb{R}^k$, some k.

Example:

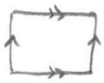
4 pieces: $A, B \cong \mathbb{R}^2$
The line $\cong \mathbb{R}$, the point $\cong \mathbb{R}^0$.

Let Σ_k = set of k-cells, ie cells $\cong \mathbb{R}^k$.
Euler number: $\chi(X) = \sum (-1)^k |\Sigma_k|$, doesn't depend on the subdivision.

Define $C^k_{cell}(X)$ = maps $\Sigma_k \to A$. Can define $d: C^k_{cell} \to C^k_{cell}$ such that $d \cdot d = 0$.

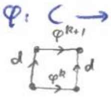
Example: $H^k_{cell}(S^n) = H^k(\underset{1\ 0\text{-cell}}{A} \to 0 \to 0 \to \cdots \to \underset{1\ n\text{-cell}}{A} \to 0 \dots) = \begin{cases} A & \text{if } k = 0, n. \\ 0 & \text{if not.} \end{cases}$

$H^k_{cell}(S^1 \times S^1)$



1 0-cell, 2 1-cells, 1 2-cells.
Get $A \xrightarrow{d=0} A \oplus A \xrightarrow{d=0} A$

Suppose we have $f: X \to Y$, a continuous map. We will define $f^*: C(Y) \to C(X)$.
Consider a cochain map $\varphi: C \to \tilde{C}$, = a sequence of homomorphisms $\varphi^k: C^k \to \tilde{C}^k$, such that $d^k \varphi^k = \varphi^{k+1} d^k$



Clearly a continuous map $f: X \to Y$ gives a cochain map $f^*: C(Y) \to C(X)$
"contravariant functor".

$*$ satisfies: (i) $id^* = id$.

(ii) If $X \xrightarrow{f} Y \xrightarrow{g} Z$, get $C(X) \xleftarrow{f^*} C(Y) \xleftarrow{g^*} C(Z)$. $(g \circ f)^* = f^* \circ g^*$.

And, $(f^* c)(x_0, -, x_{k+1}) = c(f(x_0), -, f(x_{k+1}))$.

A cochain maps $\varphi: C \to \tilde{C}$ takes cocycles to cocycles, coboundaries to coboundaries, and hence induces a homomorphism $H^k(C) \to H^k(\tilde{C})$.
(If $dc = 0$ then $d(\varphi c) = \varphi(dc) = 0$; if $c = db$, then $\varphi c = \varphi db = d(\varphi b)$).

So we have: $\begin{pmatrix}\text{spaces,}\\ \text{continuous}\\ \text{maps}\end{pmatrix} \xrightarrow[\text{contravariant functor}]{} \begin{pmatrix}\text{cochain complexes,}\\ \text{cochain maps}\end{pmatrix} \xrightarrow[\text{covariant functor}]{} \begin{pmatrix}\text{graded abelian groups,}\\ \text{homomorphisms}\end{pmatrix}$

Homotopy invariance: If f_0 and f_1 are homotopic maps $X \to Y$, then $f_0^* = f_1^*: H^k(Y) \to H^k(X)$.

Mayer-Vietoris: If $X = X_1 \cup X_2$ such that the interiors of X_1 and X_2 cover X, then we have a sequence of homomorphisms $d_{MV}: H^k(X_1 \cap X_2) \to H^{k+1}(X_1 \cup X_2)$, such that the sequence: $0 \to H^0(X) \xrightarrow{\alpha} H^0(X_1) \oplus H^0(X_2) \xrightarrow{\beta} H^0(X_1 \cap X_2)$ ---- (cont.)

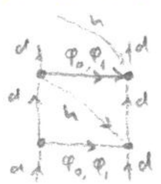
$c \mapsto (c|_{X_1}, c|_{X_2})$

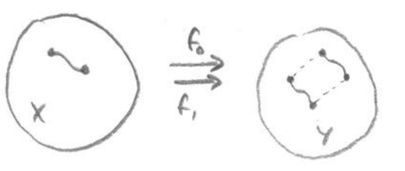
$(c_1, c_2) \mapsto (c_1|\cap) - (c_2|\cap)$, $\cap = X_1 \cap X_2$.

$\xrightarrow{d_{MV}} H^1(X) \to H^1(X_1) \oplus H^1(X_2) \to H^1(\cap) \xrightarrow{d_{MV}} H^2(X) \to H^2(X_1) \oplus H^2(X_2) \to \cdots$

is exact.

Deal with homotopy invariance first.
If $C, \tilde{C}$ are cochain complexes, then cochain maps $\varphi_0, \varphi_1: C \to \tilde{C}$ are homotopic if $\exists$ a sequence of homomorphisms $h^k: C^k \to \tilde{C}^{k-1}$, such that
$d^{k-1} h^k + h^{k+1} d^k = \varphi_1^k - \varphi_0^k$



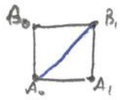


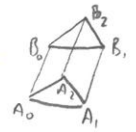
& homotopy $h: X \times I \to Y$.

Define $h(c)(\sigma)$, where c is a $(k+1)$-cochain on Y, σ is a k-simplex on X, to be $c(h(\sigma \times I))$, where $h(\sigma \times I)$ is a prism in Y of dimension $k+1$.
$= \Sigma\, c(\text{simplexes making up } h(\sigma \times [0,1]))$.

Lemma: if φ_0, φ_1 are homotopic cochain maps $C \to \tilde{C}$ then the induced maps of cohomology are the same.

Proof: Take an element c of C^k representing an element of $H^k(C)$. Then $dc = 0$.
But $\varphi_1(c) - \varphi_0(c) = d(hc) + hdc = d(hc)$.
So $\varphi_0(c)$, $\varphi_1(c)$ represent the same element of $H^k(\tilde{C})$.

We wish to subdivide the prism into simplexes. Consider 
Can subdivide into $A_0B_0B_1$ and $A_0A_1B_1$.

For  , split as: $A_0B_0B_1B_2$, $A_0A_1B_1B_2$, $A_0A_1A_2B_2$.

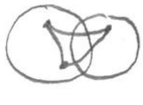
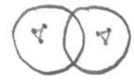
Suppose $f_0, f_1 : X \to Y$ are two maps such that $f_0(x)$ and $f_1(x)$ are "very close" $\forall x$.
Then define $h: C^k(Y) \to C^{k-1}(X)$ by $(hc)(x_0, .., x_{k-1}) = \sum_{i=0}^{k-1} (-1)^i c(y_0 \dots y_i z_i \dots z_{k-1})$,
where $f_0(x_i) = y_i$, $f_1(x_i) = z_i$.

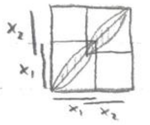
Now calculate $dhc + hdc$ for $c \in C^k(Y)$. See printed notes.

"Very close": we assume X compact, and a finite covering. Then, for a set of points 'close together' in X, with f_0, f_1 of them also close, then c (points) is defined.

Recall: $X = X_1 \cup X_2$, with $\mathring{X}_1 \cup \mathring{X}_2 = X$. Let $X_{12} = X_1 \cap X_2$. We have a sequence

$$0 \to F(X) \xrightarrow{\text{restrictions}} F(X_1) \oplus F(X_2) \xrightarrow{\text{difference of restrictions}} F(X_{12}) \to 0$$

We wish to do the same for AS-cochains. $0 \to C^k(X) \to C^k(X_1) \oplus C^k(X_2) \to C^k(X_{12}) \to 0$.
(AS is good. cf: singular cochains:  But for AS, must have: )

Consider $X = I$:  We may pick a smaller neighbourhood such that they agree on the intersection.

Using normality here: two closed disjoint subsets are contained in two open disjoint subsets. (Stronger than Hausdorff).

We get:

$$\begin{array}{ccccccccc} 0 & \to & C^{k+1}(X) & \to & C^{k+1}(X_1) \oplus C^{k+1}(X_2) & \to & C^{k+1}(X_{12}) & \to & 0 \\ & & \uparrow d & & \uparrow d \oplus d & & \uparrow d & & \\ 0 & \to & C^k(X) & \to & C^k(X_1) \oplus C^k(X_2) & \to & C^k(X_{12}) & \to & 0 \end{array}$$

Suppose we have:

$$\begin{array}{ccccccccc} 0 & \to & A^{k+1} & \xrightarrow{h} & B^{k+1} & \xrightarrow{g} & C^{k+1} & \to & 0 \\ & & & & \uparrow d & & \uparrow d & & \\ 0 & \to & A^k & \to & B^k & \xrightarrow{f} & C^k & \to & 0 \end{array}$$

Take $x \in C^k$ s.t. $dx = 0$
Choose $y \in B^k$ s.t. $f: y \mapsto x$
Then, (commutes) $g \cdot dy = 0$.
Take $z \in A^{n+1}$ s.t. $h(z) = dy$
Then $h(dz) = ddy = 0$
So z is a cocycle.

So we get (cocycles in C^k) $\dashrightarrow$ (cocycles in A^{k+1})
$$\downarrow$$
well-defined $\longrightarrow H^{k+1}(A)$

We get a long exact sequence of homology groups. (Exactness: see IIB notes).

Relative cohomology for $Y \subset X$. $C^k(X,Y) = \ker : C^k(X) \xrightarrow{\text{restriction}} C^k(Y)$.
Get $0 \to C^k(X,Y) \to C^k(X) \to C^k(Y) \to 0$. Define $H^k(X,Y) = H^k(C(X,Y))$.
Get a long exact sequence:
$$\cdots \to H^{k-1}(Y) \xrightarrow{d} H^k(X,Y) \to H^k(X) \to H^k(Y) \xrightarrow{d} \cdots$$

We prefer to work with the coefficient group as a field, as we can then deal with vector spaces. To get enough information, have
$$H^k(X;\mathbb{Z}) \longleftrightarrow \begin{cases} H^k(X;\mathbb{Q}) \\ H^k(X;\mathbb{Z}/n)\ \forall n. \end{cases}$$

We have an obvious exact sequence: $0 \to C^k(X;\mathbb{Z}) \xrightarrow{\times n} C^k(X;\mathbb{Z}) \xrightarrow{\text{mod } n} C^k(X;\mathbb{Z}/n) \to 0$.
This yields the Bockstein sequence:
$$\cdots \to H^k(X;\mathbb{Z}) \xrightarrow{\times n} H^k(X;\mathbb{Z}) \xrightarrow[\text{mod } n]{} H^k(X,\mathbb{Z}/n) \xrightarrow{d} H^{k-1}(X;\mathbb{Z}) \to \cdots$$

$$H^k(S^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0, n \\ 0 & \text{if not.} \end{cases}$$

Now, X is connected $\iff H^0(X;\mathbb{Z}) \cong \mathbb{Z}$.
$f: S^n \to S^n$ gives $f^*: \begin{array}{ccc} H^n(S^n) & \to & H^n(S^n) \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ 1 & \longmapsto & \deg f \end{array}$

Consider S^n as two hemispheres:  . $S^n = D^n_+ \cup D^n_-$. $D^n_+ \cap D^n_- = S^{n-1}$.
But the interiors do not fully cover S^n, so let the discs overlap: 
Then, the intersection is like a 'thickened' sphere.
We have: $S^n = X_1 \cup X_2$, $X_1 \cap X_2 \cong S^{n-1} \times [0,1] \simeq S^{n-1}$.

In many cases, if X is compact, $X = X_1 \cup X_2$, X_i compact, then we can write $X = X_1' \cup X_2'$, $X_{12}' = X_1' \cap X_2'$, such that $X_1 \hookrightarrow X_1'$, $X_2 \hookrightarrow X_2'$, $X_{12} \hookrightarrow X_{12}'$ are all homotopy equivalence and $\mathring{X}_1' \cup \mathring{X}_2' = X$.
Then we have a Mayer-Vietoris sequence for $X = X_1 \cup X_2$, whenever X_1 and X_2 are compact.

Return to $S^n = D^n_+ \cup D^n_-$, $S^{n-1} = D^n_+ \cap D^n_-$. We get:
$$\cdots \to H^{k-1}(D_+) \oplus H^{k-1}(D_-) \to H^{k-1}(S^{n-1}) \to H^k(S^n) \to H^k(D_+) \oplus H^k(D_-) \to \cdots$$
Now, $H^k(D^n) \cong H^k(\text{point})$, by homotopy property, $= 0$ if $k > 0$.
If $k > 1$ then $H^{k-1}(S^{n-1}) \cong H^k(S^n)$. So $H^1(S^1) \cong H^2(S^2) \cong H^3(S^3) \cong \cdots$
If $k < n$, get down to $H^1(S^m)$, $m > 1$.

So we have reduced this to 2 cases: $\begin{cases} H^1(S^m),\ m > 1 \\ H^k(S^1). \end{cases}$

$H^k(S^1)$:  Have $H^{k-1}(pt \amalg pt) \to H^k(S^1) \to H^k(pt) \oplus H^k(pt) \to \cdots$

if $k>1$: $H^k(S^1)=0$. If $k=0$: $H^0(S^1)=\mathbb{Z}$, as S^1 connected.
If $k=1$, sequence becomes: ~~$H^0 H^0(pt) \to H^0(pt \amalg pt) \to H^0(S^1) \to$~~

$$0 \to H^0(S^1) \to H^0(pt)\oplus H^0(pt) \to H^0(pt \amalg pt.) \to H^1(S^1) \to 0$$

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z} & \to & \mathbb{Z}\oplus\mathbb{Z} & \to & \mathbb{Z}\oplus\mathbb{Z} & \to & (?) & \to 0. \\ & & & (a,b) & \longmapsto & (a-b,a-b) & & \wr\wr ? & \\ & & & & & & & \mathbb{Z}. & \\ & & & & & (p,q) & \longmapsto & (p-q) & \end{array}$$

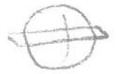
If $g: S^n \to S^n$ comes from an orthogonal matrix $g \in O_{n+1}$. Then $\deg(g) = \det(g) = \pm 1$.
So $g_* = \det(g)\times : H^n(S^n) \to H^n(S^n)$.
By homotopy property, it is enough to show that O_{n+1} has two path components, and that $g_* = -id.$ for one g with $\det(g) = -1$.

Eg: $g = \begin{pmatrix} -1, & 0 \\ 0 & \ddots_{1} \end{pmatrix} \in O_{n+1}$

We have an isomorphism:

$$\begin{tikzcd} H^{n-1}(S^{n-1}) \arrow[r, "d_{mv}", "\cong"'] \arrow[d, "g_*"] & H^n(S^n) \arrow[d, "g_*"] \\ H^{n-1}(S^{n-1}) \arrow[r, "\cong"] & H^n(S^n) \end{tikzcd}$$

Choose coordinates, so that



g reflects $\updownarrow$

Reflection induces

$$\begin{array}{ccc} H^1(S^1) & \xleftarrow{\times(-1)} & H^1(S^1) \\ \shortparallel & & \shortparallel \\ \mathbb{Z} & \longleftarrow & \mathbb{Z} \\ -n & \longmapsfrom & n \end{array}$$

Suppose we have $f: X \to Y$. $H^k(X) \xleftarrow{f^*} H^k(Y)$.

$$\begin{array}{ccccc} X & = & X_1 & \cup & X_2 \\ f\downarrow & & \downarrow f & & \downarrow f \\ Y & = & Y_1 & \cup & Y_2 \end{array}$$

$$\begin{tikzcd} \arrow[r] & H^{k-1}(Y_{12}) \arrow[r, "d_{mv}"] \arrow[d, "f^*"] & H^k(Y) \arrow[r] \arrow[d, "f^*"] & {} \\ \arrow[r] & H^{k-1}(X_{12}) \arrow[r, "d_{mv}"] & H^k(X) \arrow[r] & {} \end{tikzcd}$$

commutes.

Recall the statement that O_n has two path-connected components, as has $GL_n(\mathbb{R})$.
$GL_n(\mathbb{R}) \cong O_n \times \mathbb{R}^{\frac{1}{2}n(n+1)}$. $\underset{GL_n(\mathbb{R})}{\underset{\cap}{A}} = \underset{O_n}{\underset{\cap}{g}} \; P$ ↖ positive definite symmetric

Get: $P = (A^tA)^{1/2}$
$g = AP^{-1}$.

$P = e^Q$ gives (positive definite symmetrics) $\cong$ (all symmetrics). $\cong \mathbb{R}^{\frac{1}{2}n(n+1)}$.

Also, $GL_n(\mathbb{R}) \cong O_n \times$ (upper triangular matrices with positive diagonal elements), via the Gram-Schmidt process.

In O_n, elements have determinant ± 1. So can write $O_n = SO_n \amalg$ (other coset).
Enough to show that SO_n is path connected. $g = (v_1, \dots, v_n)$.
Given v, choose an orthonormal basis $e_1, \tilde{e}_2, \tilde{e}_3, \dots$ such that $v = (\cos\theta)e_1 + (\sin\theta)\tilde{e}_2$.
So get g_t with matrix $\begin{pmatrix} \cos t\theta & -\sin t\theta & \\ \sin t\theta & \cos t\theta & \\ & & 1 \end{pmatrix}$. Consider $g_t^{-1}g \dots$

Have $\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \ddots & \\ 0 & & & \end{pmatrix}$, so done by induction.

Definition: The support of a cochain $c \in C^k(X)$ is the smallest closed set F such that $c|_{(X-F)} = 0$.

$x \notin F \iff c|(\text{some neighbourhood of } x) = 0$.

Define a cocycle $c \in C^1(\mathbb{R})$ by $c(x,y) = \begin{cases} 1 & \text{if } x<0 \text{ and } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$.

Or $c(x,y) = \varphi(y) - \varphi(x)$.

This is a cocycle. Its support is $\{0\}$.

Cohomology with compact supports.

$C^k_{cpt}(X) = \{ c \in C^k(X) : \text{supp}(x) \text{ is compact} \}$

Want $d(C^k_{cpt}(X)) \subset C^{k+1}_{cpt}(X)$.

Need: X to be locally compact and Hausdorff.

Locally compact $\iff$ every compact subset $\subset$ interior of a larger compact subset.

So, we can define $H^k_{cpt}(X) = H^k(C_{cpt}(X))$.

Clearly we have a homomorphism $H^k_{cpt}(X) \to H^k(X)$.

$H^1_{cpt}(\mathbb{R}) = \mathbb{Z}$, generated by above $c \in C^1_{cpt}(\mathbb{R})$. But $H^1(\mathbb{R}) = 0$, so the map is not injective. Had $c(x,y) = \varphi(y) - \varphi(x) = (d\varphi)(x,y)$. $\varphi \in C^0(X)$. φ does not have compact support.

Claim: $H^k_{cpt}(\mathbb{R}^n) \cong \mathbb{Z}$ if $k = n$, and 0 if not.

Notice that if X is an open subset of Y, then $C^\bullet_{cpt}(X) \subset C^\bullet_{cpt}(Y)$, by "extension by zero"

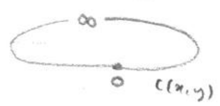
So $H^k_{cpt}(X) \xrightarrow{i_*} H^k_{cpt}(Y)$

$i: X \to Y$, inclusion.

Check: if $f: X \to Y$ is proper (ie, continuous and $f^{-1}(\text{compact}) = \text{compact}$) then we have $f^*: H^*_{cpt}(Y) \to H^*_{cpt}(X)$.

If Y is compact then $H^*_{cpt}(Y) = H^*(Y)$.

So $\underbrace{H^k_{cpt}(\mathbb{R}^n)}_{H^k(S^n) \cong \text{ if } k \neq 0} \to H^k_{cpt}(\underbrace{\mathbb{R}^n \cup \{\infty\}}_{= S^n}) = H^k(\mathbb{R}^n \cup \{\infty\})$.



So can see $H^1_{cpt}(\mathbb{R}) \cong H^1(S^1)$.

$k=0$. $H^0_{cpt}(\mathbb{R}^n) = 0$.

Claim: Map $H^k_{cpt}(\mathbb{R}^n) \to H^k(S^n)$ is an isomorphism.

Onto: Suppose c represents an element of $H^n(S^n)$. Consider S^n as $\mathbb{R}^n \cup \{\infty\}$.

Let D be a compact disc around ∞, say $D = \{x : \|x\| \geq 1\}$. $H^n(D^n) = 0$, so $c|_D = db$ for some $b \in C^{n-1}(D)$. Extend b to $\tilde{b} \in C^{n-1}(S^n)$.

Then $c - d\tilde{b}$ has support not meeting D. So $c - d\tilde{b} = \tilde{c}$ has compact support, and so represents an element of $H^n_{cpt}(\mathbb{R}^n)$.

But $c, \tilde{c}$ represent the same element of $H^n(S^n)$.

Injective: Suppose $c \in C^k_{cpt}(\mathbb{R}^n)$. Extend it to $c \in C^k(S^n)$.
Suppose $c = db$ for some $b \in C^{k-1}(S^n)$. Now $db|(\text{some } D \text{ around } \infty \text{ as before}) = 0$.
So $b|D = de$ for some $e \in C^{k-2}(D)$. Extend e to $\tilde{e} \in C^{k-2}(S^n)$.
$b - d\tilde{e}$ has compact support, so represents same element of $H^k_{cpt}(\mathbb{R}^n)$ as
$c - d(b - d\tilde{e}) = c - db = 0$.

$$\underset{\mathbb{Z}}{\overset{\wr\wr ?}{H^n_{cpt}(\mathbb{R}^n)}} \xrightarrow{\cong} H^n(S^n).$$ If U is any open disc in S^n (so $U \cong \mathbb{R}^n$), then
$H^n_{cpt}(U) \underset{i_*}{\xrightarrow{\cong}} H^n(S^n)$.

$f: S^n \to S^n$, $\deg f \in \mathbb{Z}$, $= |f^{-1}(y)|$ for generic $y \in S^n$.
Suppose that there is an open disc $V \subset S^n$ such that $f^{-1}(V) = U_1 \amalg \dots \amalg U_m$,
where each U_i is mapped homeomorphically to V by f.
Then $\deg(f) = \sum_{i=1}^{m} \varepsilon_i$, where $\varepsilon_i = \pm 1$ according as $U_i \to V$ preserves or reverses orientation.

$$\begin{array}{ccc} H^n_{cpt}(U_i) & \overset{f^*}{\underset{f_*}{\rightleftarrows}} & H^n_{cpt}(V) \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$ $f^* = (f_*)^{-1}$.

$H^n_{cpt}(U) \xrightarrow{\cong} H^n(S^n)$; choose a preferred generator.

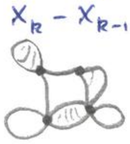
$H^n_{cpt}(\mathbb{R}^n) \xrightarrow{A_*} H^n_{cpt}(\mathbb{R}^n)$ with $A: \mathbb{R}^n \to \mathbb{R}^n$ in $GL_n\mathbb{R}$, then $A_* = \mathrm{sign}(\det(A))$

$$\begin{array}{ccc} H^n(S^n) & \xleftarrow{f^*} & H^n(S^n) \cong \mathbb{Z} \\ & & \uparrow \cong \\ & & H^n_{cpt}(V) \cong \mathbb{Z} \end{array}$$

Return to proof of: $H^n_{cpt}(\mathbb{R}^n) \xrightarrow{\cong} H^n(S^n)$
What we showed was: if X compact, Y a closed subspace,
$H^n_{cpt}(X - Y) \underset{i_*}{\xrightarrow{\cong}} H^n(X, Y)$

For, suppose $Y \subset$ compact neighbourhood $\tilde{Y}$ such that $Y \hookrightarrow \tilde{Y}$ is a homotopy equivalence.

X, cell complex.
X compact space. We have: $X_0 \subset X_1 \subset \dots \subset X_m = X$, closed subspaces,
$X_k - X_{k-1}$ = finite disjoint union of open subsets of X_k, each homeomorphic to $\mathbb{R}^k$.
(called open k-cells).



C^k_{cell} = free abelian group on the set of k-cells.
$\cong H^k_{cpt}(X_k - X_{k-1}) \cong H^k(X_k, X_{k-1}) \to H^{k+1}(X_{k+1}, X_k) = C^{k+1}_{cell}$
$(X_{k-1} \hookrightarrow X_k \hookrightarrow X_{k+1})$

$C(X_k, X_{k-1}) = \ker C(X_k) \longrightarrow C(X_{k-1})$.

If $X \supset Y \supset Z$, can write $H^k(X,Y) \longrightarrow H^k(X,Z) \to H^k(Y,Z) \to H^{k+1}(X,Y) \to \cdots$

$0 \to C(X,Y) \to C(X,Z) \to C(Y,Z) \to 0.$ ($\{f$ vanishing on $Y\} \subset \{f$ vanishing on $Z\}$).

Suppose we have:
$$\begin{array}{ccccccc} Z' & & Y' & & X' & & W' \\ \cap & & \cap & & \cap & & \cap \\ Z & \subset & Y & \subset & X & \subset & W \end{array}$$

$$\begin{array}{ccccc} H^k(Y,Z) & \xrightarrow{d} & H^{k+1}(X,Y) & \xrightarrow{d} & H^{k+2}(W,X) \\ \downarrow\cong & & \downarrow & & \downarrow\cong \\ H^k(Y',Z') & \to & H^{k+1}(X',Y') & \longrightarrow & H^{k+2}(W',X') \end{array}$$ – ? Hmm..

<u>Example:</u> $Gr_k(\mathbb{C}^n)$ Have matrix: $\left(\overbrace{\qquad\qquad}^{n}\right)k$.

We can reduce to echelon form by row operations: eg $\begin{pmatrix} 0 & 1 & * & - & - & * \\ 0 & 0 & 0 & 1 & * & - & * \\ 0 & - & 0 & 1 & * \end{pmatrix} k$

So $Gr_k(\mathbb{C}^n)$ = set of reduced echelon matrices of size $n \times k$, rank k.

Suppose the leading 1's appear in columns $p_1, \ldots, p_k$.

For each $1 \le p_1 < \ldots < p_k \le n$, the matrices with echelon form of type $(p_1, \ldots, p_k)$ are $\cong \mathbb{C}^{\delta(p_1,\ldots,p_k)}$

$\delta(p_1, \ldots, p_k) = \#$ 'stars' in matrix not in a column p_i, any i.

Consider the $\binom{n}{k}$ k-tuples $(p_1 < \ldots < p_k)$. Suppose ν_m have 'dimension' m,

ie $m = 2\delta(p_1, \ldots, p_k)$.

Then $C^m_{cell} \cong \mathbb{Z}^{\nu_m} = 0$ unless m is even.

So $C^0_{cell} = \mathbb{Z} \to 0 \to \mathbb{Z}^{\nu_2} \to 0 \to \mathbb{Z}^{\nu_4} \to \cdots$

So $H^m(Gr_k(\mathbb{C}^n)) = \mathbb{Z}^{\nu_m}$

$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$. If $yx = txy$, $(tx = xt,\ ty = yt)$,

$(x+y)^2 = x^2 + (1+t)xy + y^2$. $(x+y)^n = \sum \binom{n}{k}_t x^{n-k} y^k$.

$$\binom{n}{k}_t = \frac{(t^n-1)(t^{n-1}-1)\cdots(t^{n-k+1}-1)}{(t^k-1)(t^{k-1}-1)\cdots(t-1)} = \sum_m \nu_{2m} t^m.$$

If $t = q$, then $\binom{n}{k}_q = |Gr_k(\mathbb{F}_q^n)|$ (for q a prime power).

Return to earlier situation:

$$\begin{array}{ccccccc} Z & \subset & Y & \subset & W & \subset & W \\ \parallel & & \parallel & & \cup & & \parallel \\ Z & \subset & Y & \subset & X & \subset & W \end{array}$$

$$\begin{array}{ccccc} H^k(Y,Z) & \xrightarrow{d} & H^{k+1}(X,Y) & \xrightarrow{d} & H^{k+2}(W,X) \\ \parallel & & \uparrow & & \\ H^k(Y,Z) & \xrightarrow{d} & H^{k+1}(W,Y) & \xrightarrow{d} & H^{k+2}(W,W) = 0. \end{array}$$

X, cell complex, $X_0 \subset X_1 \subset \cdots$

$H^i_?(X_k, X_{k-1}) = 0$ if $i \ne k$, $= C^k_{cell}$ if $i = k$.

<u>Lemma:</u> $H^k(X_m, X_{k-1})$ is independent of m if $m \ge k+1$.

$$H^{k+1}(X_{m+1}, X_m) \swarrow \quad \nwarrow\cong\ H^k(X_{m+1}, X_m) \leftarrow H^k(X_{m+1}, X_m) = 0.$$

$0 \swarrow$

Corollary: $Z^k_{cell} = \text{im } H^k(X, X_{k-1}) \hookrightarrow H^k(X_k, X_{k-1}) = C^k_{cell}$

$$C^k_{cell} \xrightarrow{d} H^{k+1}(X_{k+1}, X_k)$$

$0 = H^k(X_{k+1}, X_k) \to H^k(X_m, X_{k-1}) \to H^k(X_k, X_{k-1})$
if $m \geq k+1$

$H^k_{cell} = Z^k_{cell} /$ image of $H^{k-1}(X_{k-1}, X_{k-2}) \to H^k(X, X_{k-1})$

So H^k_{cell} = image of $H^k(X, X_{k-1}) \xrightarrow{\text{onto}} H^k(X, X_{k-2})$

$= H^k(X, X_{k-2})$

Finally, $H^k(X, X_m)$ is independent of m if $m \leq k-2$.

$H^0(X_m, X_{m-1}) \to H^k(X, X_m) \xrightarrow{\cong} H^k(X, X_{m-1})$ and hence $\cong H^k(X)$.

$Gr_1(\mathbb{C}^n) = \mathbb{P}^{n-1}_{\mathbb{C}}$. $\mathbb{P}^0_{\mathbb{C}} \subset \mathbb{P}^1_{\mathbb{C}} \subset \cdots \subset \mathbb{P}^n_{\mathbb{C}}$. $\mathbb{P}^k_{\mathbb{C}} - \mathbb{P}^{k-1}_{\mathbb{C}} \cong \mathbb{C}^k$

$\underset{0}{\mathbb{Z}} \to \underset{1}{0} \to \underset{2}{\mathbb{Z}} \to 0 \to \mathbb{Z} \to \cdots \to \underset{2n}{\mathbb{Z}}$ $\quad H^i(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Z}$ if i is even and $\leq 2n$

$= 0$ if not.

Now suppose we have $\mathbb{P}^0_{\mathbb{R}} \subset \cdots \subset \mathbb{P}^n_{\mathbb{R}}$. $\mathbb{P}^k_{\mathbb{R}} - \mathbb{P}^{k-1}_{\mathbb{R}} \cong \mathbb{R}^k$.

Get: $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \cdots$

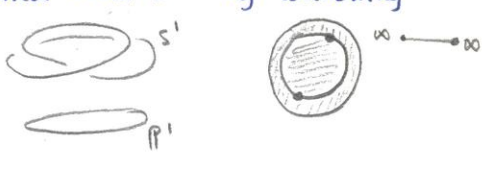
So $H^i(\mathbb{P}^n_{\mathbb{R}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i \text{ is odd and} \neq n \\ \mathbb{Z} & \text{if } i = n \text{ is odd.} \\ \mathbb{Z}/2 & \text{if } i \text{ is even and} \leq n. \\ 0 & \text{if } i > n \end{cases}$

Suppose X_k is obtained from X_{k-1} by attaching a finite number of k-cells, by attaching maps $f_i : S^{k-1} \to X_{k-1}$

$X_k - X_{k-1} = \amalg$ copies of $\mathbb{R}^k$. $X_k = (X_{k-1} \amalg D^k_{(1)} \amalg \cdots \amalg D^k_{(r)}) / \sim$,

where $\sim$ identifies $S^{k-1}_{(i)} \subset D^k_{(i)}$ with part of X_{k-1} by $x \sim f_i(x)$

Proof of $H^i(\mathbb{P}^n_{\mathbb{R}})$ claims: $\mathbb{P}^k - \mathbb{P}^{k-1} \cong \mathbb{R}^k$. $\mathbb{P}^k$ is obtained from $\mathbb{P}^{k-1}$ by attaching a k-cell by the obvious maps. $f : S^{k-1} \to \mathbb{P}^{k-1}$



$C^{k-1}_{cell} = H^{k-1}(X_{k-1}, X_{k-2}) \xrightarrow{d} H^k(X_k, X_{k-1}) = C^k_{cell}$

restriction: $H^{k-1}(X_{k-1}, X_{k-2}) \to H^{k-1}(X^{k-1})$; $\quad H^k(X_k, X_{k-1}) \xrightarrow{\hat{f}_i^*} H^k(D^k, S^{k-1}) = \mathbb{Z}$; $\quad C^k_{cell} \to \mathbb{Z}$ projection onto ith $\mathbb{Z}$

$H^{k-1}(X^{k-1}) \xrightarrow{f_i^*} H^{k-1}(S^{k-1}) \xrightarrow{d} H^k(D^k, S^{k-1})$

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{f_i} & X_{k-1} \\ \cap & & \cap \\ D^k & \xrightarrow{\hat{f}_i} & X_k \end{array}$$

In other words, the ith component of $d : C^{k-1}_{cell} \to C^k_{cell}$ is

$H^{k-1}(X_{k-1}, X_{k-2}) \to H^{k-1}(X_{k-1}) \xrightarrow{f^*} H^{k-1}(S^{k-1}) = \mathbb{Z}$.

We get
$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & H^{k-1}(\mathbb{P}^{k-1}, \mathbb{P}^{k-2}) \arrow[r] \arrow[d, "f^*"] & H^{k-1}(\mathbb{P}^{k-1}) \arrow[r, "f^*"] & H^{k-1}(S^{k-1}) = \mathbb{Z}. \\ \mathbb{Z} \oplus \mathbb{Z} \cong & H^{k-1}(S^{k-1}, S^{k-2}) \arrow[urr, "\text{restriction}"'] & & \end{tikzcd}$$

The S^k's are double covers of the $\mathbb{P}^k$'s :

(as ½ spheres)

$c = c_1 + c_2$, $c_2 = \theta^* c_1$ where $\theta : S^{k-1} \to S^{k-1}$ is $x \mapsto -x$

$c_2 = (-1)^k c_1$

We wish to define products of cocycles.
$C^p(X) \times C^q(X) \longrightarrow C^{p+q}(X)$.
Let $(ab)(x_0, \dots, x_{p+q}) = a(x_0, \dots, x_p) b(x_p, \dots, x_{p+q})$. Assume that the coefficients form a ring.
This is obviously associative and bi-additive.
It is not commutative, because of the way we choose a specific order on the vertices.
In fact, d is an antiderivation for multiplication of cochains.
Ie, $d(a.b) = da.b + (-1)^p a.db$ $(a \in C^p)$.
Hence, cocycle . cocycle = cocycle.
cocycle . coboundary = coboundary.

So we have a well-defined bi-additive associative map from $H^p(X) \times H^q(X) \to H^{p+q}(X)$, ie $H^*(X)$ is a graded ring.

Theorem: It is an anticommutative graded ring, (ie, $ba = (-1)^{pq} ab$, $a \in H^p$, $b \in H^q$) providing the coefficient ring is commutative.

Lemma: We have a cochain map $T : C^p(X) \to C^p(X)$, $(Tc)(x_0, \dots, x_p) = (-1)^{\frac{1}{2}p(p+1)} c(x_p, \dots, x_0)$, such that $Ta . Tb = (-1)^{pq} T(ab)$

It is enough to prove that T^* is the identity on $H^*(X)$.

[Hint for lemma: $\frac{1}{2}(p+q)(p+q+1) - \frac{1}{2}p(p+1) - \frac{1}{2}q(q+1) = pq$].

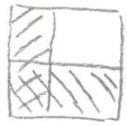
$T : C^p \to C^p$ induces the identity on $H^* \Rightarrow H^*$ is an anticommutative graded ring.
(Proof later).

Suppose $X \supset Y_1, Y_2$. $C^p(X, Y_1) \times C^q(X, Y_2) \to C^{p+q}(X, Y_1 \cup Y_2)$ induces
$H^p(X, Y_1) \times H^q(X, Y_2) \to H^{p+q}(X, Y_1 \cup Y_2)$.
X locally compact: $C^p_{cpt}(X) \times C^q(X) \to C^{p+q}_{cpt}(X)$. Similarly, $H^p_{cpt}(X) \times H^q(X) \to H^{p+q}_{cpt}(X)$.
H^*_{cpt} is a graded module over H^*.

X, Y two spaces. $f^*(a.b) = (f^*a).(f^*b)$. $X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$ give ring homomorphisms $H^*(X) \xrightarrow{\pi_1^*} H^*(X \times Y) \xleftarrow{\pi_2^*} H^*(Y)$.

We get the "external product" $H^p(X) \times H^q(Y) \to H^{p+q}(X \times Y)$, $(a,b) \mapsto (\pi_1^* a)(\pi_2^* b)$

$H^p(X, X_1) \times H^q(Y, Y_1) \to H^{p+q}(X \times Y, (X_1 \times Y) \cup (X \times Y_1))$.



Thus, $H^p_{cpt}(\mathbb{R}^p) \times H^q_{cpt}(\mathbb{R}^q) \to H^{p+q}_{cpt}(\mathbb{R}^{p+q})$,

takes $(\varepsilon_p, \varepsilon_q)$ to $\varepsilon_p \cdot \varepsilon_q$, where ε_p is a generator.

$H^1_{cpt}(\mathbb{R})$. Take $\varepsilon_1(x,y) = \vartheta(x) - \vartheta(y)$, where $\vartheta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

So we have an explicit cocycle representing the generator of $H^p_{cpt}(\mathbb{R}^p)$.

Its support is $\{0\} \subset \mathbb{R}^p$.

Recall: we found $H^k(S^n)$ by considering S^n as $D^n_+ \cup D^n_-$, with $D^n_+ \cap D^n_- = S^{n-1}$.

Can assume $\infty \in S^{n-1}$. So $H^k(S^n, \infty) \cong H^k(\mathbb{R}^n) = \begin{cases} \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$.

$H^k(D^k, \text{point}) = 0$ for all k.

We get $0 \to H^{k-1}(S^{n-1}, \infty) \xrightarrow{d_{mv}} H^k(S^n, \infty) \to 0$.

$(D^n_+ \times X) \cup (D^n_- \times X) = S^n \times X$ $\qquad$ $H^k(D^n \times X, \text{point} \times X) = 0$ for all k.

$(D^n_+ \times X) \cap (D^n_- \times X) = S^{n-1} \times X$.

Lemma: $H^{k-1}(S^{n-1} \times X, \{\infty\} \times X) \underset{\cong}{\xrightarrow{d_{mv}}} H^k(S^n \times X, \{\infty\} \times X)$, for all k, n.

Lemma: $H^k(S^n \times X) \cong H^k(X) \oplus H^{k-n}(X)$ for all k, n, by the map

$(a, b) \mapsto (\pi_2^* a) + (\pi_1^* b).(\pi_2^* \varepsilon_n)$. $[\pi_1 : S^n \times X \to S^n,\ \pi_2 : S^n \times X \to X]$.

Get $H^{k-n}(X) \xrightarrow{d_{mv}, \cong} H^k(S^n \times X, \{\infty\} \times X)$

$b \longmapsto \pi_1^*(\varepsilon_n) . \pi_2^*(b)$.

$$\begin{tikzcd}
H^k(S^n, \infty) \arrow[r, "(d_{mv})^n"] \arrow[d, "\times \pi_2^*(a)"'] & H^{k+n}(S^n, \infty) \arrow[d, "\times \pi_2^*(a)"] \\
H^k(S^n \times X, \infty \times X) \arrow[r, "(d_{mv})^n"] & H^{k+n}(S^n \times X, \infty \times X)
\end{tikzcd}$$

$$\begin{tikzcd}
\cdots \arrow[r, "0"] & H^k(S^n \times X, \infty \times X) \arrow[r, hook] \arrow[d, phantom, "\parallel"] & H^k(S^n \times X) \arrow[r, "(\text{restriction})^*", "\text{onto}"'] & H^k(\infty \times X) \arrow[r, "0"] \arrow[l, bend left, "\pi_2^*"] \arrow[loop below, "\text{id.}"] & \cdots \\
& H^{k-n}(X) & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\infty \times X \arrow[r, hook] \arrow[rr, bend right, "\text{id.}"'] & S^n \times X \arrow[r, "\pi_2"] & X
\end{tikzcd}$$

So $H^k(S^n \times X) \cong H^k(X) \oplus H^{k-n}(X)$.

Künneth Theorem: Recall we had X, Y. $H^p(X) \times H^q(Y) \longrightarrow H^{p+q}(X \times Y)$, bi-additive.

$A \times B \longmapsto A \otimes B$.

$\mathbb{Z}\otimes\mathbb{Z}\to\mathbb{Z}$. $\quad A\otimes(\mathbb{Z}/n)\xrightarrow{\cong} A/nA \qquad (\mathbb{Z}/n)\otimes(\mathbb{Z}/m)\to\mathbb{Z}/(m,n)$

$(n,m)\longmapsto n\otimes m \qquad a\otimes k\longmapsto ka$

$(\mathbb{Z}/m)\otimes\mathbb{Q}=0$

$$k\otimes q=\underbrace{(k\otimes\tfrac{q}{n})+\dots+(k\otimes\tfrac{q}{n})}_{n}=kn\otimes\tfrac{q}{n}=0.$$

$X,Y.\ \bigoplus_{p+q=n}\left(H^p(X)\otimes H^q(Y)\right)\xrightarrow[\text{bi-additive}]{} H^n(X,Y)$ — true if, eg, $H^p(X)$ is free and f.g. $\forall p$.

$\sum a_i\otimes b_i\longmapsto\sum a_ib_i$

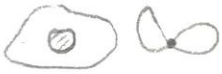
$$\left(H^0(S^n)\otimes H^k(Y)\right)\oplus\left(H^n(S^n)\otimes H^{k-n}(Y)\right)\xrightarrow{\cong}H^k(S^n\times Y).$$

Corollary: $H^p(S^p)\times H^q(S^q)\to H^{p+q}(S^{p+q})$

takes $(\varepsilon_p,\varepsilon_q)\longmapsto\varepsilon_{p+q}$.

Corollary: $H^p_{cpt}(\mathbb{R}^p)\times H^q_{cpt}(\mathbb{R}^q)\xrightarrow{\cong}H^{p+q}_{cpt}(\mathbb{R}^{p+q})$.

Manifold = Hausdorff space locally homeomorphic to $\mathbb{R}^n$, some n.



"convex covering" = open covering $\{U_\alpha\}$ of X such that each U and each non-empty finite intersection $U_{\alpha_0}\cap\dots\cap U_{\alpha_k}\cong$ some $\mathbb{R}^n$

$\cong\mathbb{R}^n$

$\cap$ = annulus, not $\cong\mathbb{R}^n$

There is always a convex covering on a Riemannian manifold.
Put a metric on the manifold (smooth), and obtain geodesics. A small neighbourhood of a point has the property that there is a unique geodesic between points. These give a convex covering.

Orientability.

Suppose U is an open subset of $\mathbb{R}^n$ which is homeomorphic to $\mathbb{R}^n$.
$i: U\to\mathbb{R}^n$, inclusion. Then $i_*: H^n_{cpt}(U)\to H^n_{cpt}(\mathbb{R}^n)$

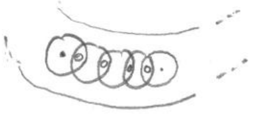
$\mathbb{Z}\longrightarrow\mathbb{Z}$

For we have proved that if U_0 is an open disc in $\mathbb{R}^n$, then $H^n_{cpt}(U_0)\xrightarrow{\cong}H^n_{cpt}(\mathbb{R}^n)\xrightarrow{\cong}H^n(S^n)$, and we can suppose that $U_0\subset U\subset\mathbb{R}^n$.

$$\underbrace{H^n_{cpt}(U_0)\to H^n_{cpt}(U)\xrightarrow{\therefore\ \text{onto}}H^n_{cpt}(\mathbb{R}^n)}_{\text{onto.}}$$

Definition: An orientation of an n-manifold X is a choice of a generator ω_U of $H^n_{cpt}(U)$ for every open $U\subset X$ which is $\cong\mathbb{R}^n$, such that if $j: U\hookrightarrow U'$ then $j_*\omega_U=\omega_{U'}$.

Exercise: (i) If X is connected then it has at most two orientations.

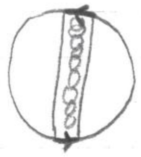


Fixing an orientation on one chart induces the same orientation on any compatible chart intersecting it. This covers the whole manifold.

Similarly: (ii) It is enough to give ω_U compatibly for all sufficiently small U, say all U of diam $< \varepsilon$

(iii) If $\{U_\alpha\}$ is an open convex covering of X it is enough to give ω_{U_α} for each α, compatibly wrt each non-empty $U_\alpha \cap U_\beta$.

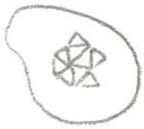
$\mathbb{P}^{2m}_{\mathbb{R}}$ is not orientable:



Definition of orientation $\Rightarrow$ if we give an orientation on a manifold, then we have the orientation on any submanifold.

So we would get an orientation on the Möbius band #.

Poincaré duality theorem. (we will prove for manifolds with finite convex covering)
X oriented, and we use $H^*(\,; A)$ where A is a field, or $\mathbb{Z}/m$ (any $m \neq 0$).
Then there is a canonical isomorphism $\int_X : H^n_{cpt}(X; A) \xrightarrow{\cong} A$,



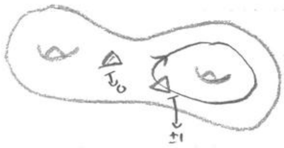
and the bilinear map $H^p_{cpt}(X) \times H^q(X) \to A$, $(q = n-p)$

$$(\alpha, \beta) \mapsto \int_X \alpha.\beta \in A,$$

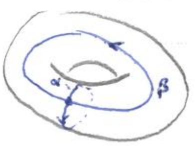
puts H^p_{cpt} and H^q in duality, ie induces

$$H^p_{cpt}(X) \xrightarrow{\cong} \mathrm{Hom}(H^q(X), A)$$
$$H^q(X) \xrightarrow{\cong} \mathrm{Hom}(H^p_{cpt}(X), A).$$

In particular, if X is compact, $H^p(X)^* \cong H^{n-p}(X)$.



One-dimensional closed submanifold induces a map on two-dimensional cochains, by mapping a simplex to # intersections with the curve.



$H^1(X) = A \oplus A$. Think of α, β as 1-forms. Then their product is a 2-form on the torus, so $H^1 \times H^1 \to A$

$$(\alpha, \beta) \mapsto 1 = \int_X \alpha.\beta.$$

Shrink a meridian of the torus:

α is now a boundary, so this does not satisfy the duality theorem.

Lemma: X an n-manifold with finite convex covering, then

(i) $H^i(X)$ is finite generated over A for all i.

(ii) $H^i(X) = 0$ if $i > n$.

(iii) If X is connected then $A \cong H^n_{cpt}(U) \longrightarrow H^n_{cpt}(U)$ is onto for any open $U \cong \mathbb{R}^n$.

Prove by induction on the number k of sets in a convex covering of X.
Say X is of type k if $\exists$ a convex covering with k sets.
Result is true if X is of type 1. Suppose $X = U_1 \cup \dots \cup U_k$ is a convex covering.
Let $X' = U_2 \cup \dots \cup U_k$ - of type $k-1$.
$U_1 \cap X'$ is also of type $k-1$ (Consider:  $\cap = (1 \cap 3) \cup (1 \cap 2)$).
There is a Mayer-Vietoris sequence:

$$\dots \xrightarrow{d_{MV}} H^i_{cpt}(U_1 \cap X') \longrightarrow H^i_{cpt}(U_1) \oplus H^i_{cpt}(X') \longrightarrow H^i_{cpt}(X) \xrightarrow{d_{MV}} H^{i+1}_{cpt}(U_1 \cap X') \to \dots$$

(We have the reverse ordering, since instead of going down via restriction, we go up via compactification.)
Coming from the exact sequence:

$$0 \longrightarrow C^k_{cpt}(X' \cap U_1) \longrightarrow C^k_{cpt}(U_1) \oplus C^k_{cpt}(X') \longrightarrow C^k_{cpt}(X) \longrightarrow 0$$

$$\begin{tikzcd} & U_1 \cap X' \arrow[dl, "j'"'] \arrow[dr, "i'"] & \\ U_1 \arrow[dr, "i"'] & & X' \arrow[dl, "j"] \\ & X & \end{tikzcd}$$

$$\alpha \longmapsto (j'_* \alpha, -i'_* \alpha)$$
$$(\alpha, \beta) \longmapsto i_* \alpha + j_* \beta.$$

So we have $H^i_{cpt}(U_1) \oplus H^i_{cpt}(X')$ and $H^{i+1}_{cpt}(U_1 \cap X')$ finitely generated.
These have f.g. image in $H^i_{cpt}(X)$, is f.g. quotient, respectively.
Hence $H^i_{cpt}(X)$ is f.g. (in the cases we are considering i.e field or $\mathbb{Z}/m$).
This does (i), and (ii) follows.
For (iii),

$$\begin{tikzcd} H^n_{cpt}(U_1 \cap X') \arrow[r] \arrow[rr, bend right, "\text{onto, by induction}"'] & H^n_{cpt}(U_1) \oplus H^n_{cpt}(X') \arrow[r] \arrow[r, bend left, "\text{onto}"] & H^n_{cpt}(X) \arrow[r] & H^{n+1}_{cpt}(U_1 \cap X') \\ & & & \overset{\shortparallel}{0}, \text{ by (ii)} \end{tikzcd}$$

Now we assume X is oriented and connected. To construct $\int_X : H^n_{cpt}(X) \to A$:
It is characterised by $\int_X i_* w_U = 1$ for $i : \underset{\mathbb{R}^n}{\underset{\wr\wr}{U}} \hookrightarrow X$

$$\begin{tikzcd} \arrow[r, "d_{MV}"] & H^n_{cpt}(U_1 \cap X') \arrow[r] \arrow[d, "\int_{X' \cap U_1}"] & H^n_{cpt}(U_1) \oplus H^n_{cpt}(X') \arrow[r, "\text{onto}"] \arrow[d, "\cong \,\oplus\, \int_{X'}"] & H^n_{cpt}(X) \arrow[r] \arrow[d, dashed] & 0 \\ & A \arrow[r, "\text{subtraction}"'] & A \oplus A \arrow[r, "\text{add}"] & A & \end{tikzcd}$$

X an oriented n-manifold. $\exists$ unique homomorphism $\int_X : H^n_{cpt}(X) \to \mathbb{Z}$, such that $\int_X i_*(w_U) = 1$ for each $U \subset X$ with $U \cong \mathbb{R}^n$. If $X = X_1 \amalg X_2$, this is okay, as $H^n_{cpt}(X) = H^n_{cpt}(X_1) \oplus H^n_{cpt}(X_2)$. We proved this by induction.
If X is connected, $X = U_1 \cup \dots \cup U_k$, then we can order the U_i such that $X = U_1 \cup X'$, $U_1 \cap X' \neq \phi$. $(X' = U_2 \cup \dots \cup U_k)$  ← order, for if this were U_1, X' not connected

Get:

$$\begin{tikzcd}
H^n_{cpt}(U_1 \cap X') \arrow[r, "\int_{X' \cap U_1}"] \arrow[d] & \mathbb{Z} \arrow[d] & a \arrow[d, mapsto] \\
H^n_{cpt}(U_1) \oplus H^n_{cpt}(X') \arrow[r, "\cong"] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[d, "\text{additive}"] & (a,-a) \quad (*). \\
H^n_{cpt}(X) \arrow[r, "\text{“ ”}"] \arrow[d] & \mathbb{Z} & \\
0. & &
\end{tikzcd}$$

If X is not orientable, this breaks down at some point. Consider $(*)$, above. We get $(\pm 1, \pm 1)$. If both are the same then the map down gives $\mathbb{Z}$ as before. If different, we get $\mathbb{Z}/2$.

In fact, for X connected, $H^n_{cpt}(X) = \begin{cases} \mathbb{Z} & \text{if } X \text{ orientable} \\ \mathbb{Z}/2 & \text{if not.} \end{cases}$

$\mathbb{P}^n_{\mathbb{R}}$ is not orientable if n is even, but is if n is odd.
Know from before that $H^n(\mathbb{P}^n) = \mathbb{Z}/2$ if n is even.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, leftrightarrow, "="] & H^2_{cpt}(\mathbb{R}^2) \arrow[r] & H^2(\mathbb{P}^2) \arrow[r] \arrow[d, "f^*"] & \mathbb{Z} \arrow[d, equal] \\
\mathbb{Z} \arrow[u, "\times(-1)"] \arrow[r, "="] & H^2_{cpt}(\mathbb{R}^2) \arrow[r] & H^2(\mathbb{P}^2) \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

$(x,y) \longmapsto (1,x,y)$, $\quad (x,y) \mapsto (x,-y)$

$f^*(w,x,y) = (w,x,-y) = (-w,-x,y)$
Get $f^* \simeq \mathrm{id}$. #.

X an oriented n-manifold. Take A = field or $\mathbb{Z}/n$.

$H^p_{cpt}(X) \times H^{n-p}(X) \longrightarrow A$
$(a, b) \longmapsto \int_X a \cdot b$
$H^p_{cpt}(X) \longrightarrow H^{n-p}(X)^* = \mathrm{Hom}_A(\ ,A)$.

$$\begin{tikzcd}[column sep=small]
H^p_{cpt}(X'') \arrow[r] \arrow[d, "\cong"] & H^p_{cpt}(U_1) \oplus H^p_{cpt}(X') \arrow[r] \arrow[d, "\cong"] & H^p_{cpt}(X) \arrow[r] \arrow[d] & H^{p+1}_{cpt}(X'') \arrow[r] \arrow[d, "\cong"] & H^{p+1}_{cpt}(U_1) \oplus H^{p+1}_{cpt}(X') \arrow[d, "\cong"] \\
H^{n-p}(X'')^* & H^{n-p}(U_1)^* \oplus H^{n-p}(X')^* & H^{n-p}(X)^* & H^{n-p-1}(X'')^* & H^{n-p-1}(U_1)^* \oplus H^{n-p-1}(X')^* \\
H^{n-p}(X'') & H^{n-p}(U_1) \oplus H^{n-p}(X') \arrow[l] & H^{n-p}(X) \arrow[l] & H^{n-p-1}(X'') \arrow[l] & H^{n-p-1}(U_1) \oplus H^{n-p-1}(X') \arrow[l]
\end{tikzcd} \quad (⊛)$$

Use induction. Base of induction is when $X \cong \mathbb{R}^n$.
Have $A = H^n_{cpt}(\mathbb{R}^n) \to H^0(\mathbb{R}^n)^* = A^*$. $\quad A \cong \mathrm{Hom}(A, A) = A^*$.

Lemma: If $P \to Q \to R$ is exact sequence of A-modules, then $P^* \leftarrow Q^* \leftarrow R^*$ is also exact.

Note: need A a field or $\mathbb{Z}/n$. Eg: $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$ is exact. $0 \leftarrow \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z}$ is not.

So we get an exact sequence in ⊛, and the diagram commutes, so the middle map is an isomorphism. This proves the Poincaré duality theorem.

Let X be a compact subspace of $\mathbb{R}^n$. Then $\mathbb{R}^n - X$ is an oriented n-manifold.
So $H^p_{cpt}(\mathbb{R}^n - X) \cong H^{n-p}(\mathbb{R}^n - X)^*$. Consider: $H^p_{cpt}(S^n - X) = H^{n-p}(S^n - X)^*$.
But $H^p_{cpt}(S^n - X) = H^p(S^n, X)$.

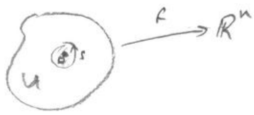
Have: $H^{p-1}(S^n) \to H^{p-1}(X) \to H^p(S^n, X) \to H^p(S^n) \to H^p(X)$.
If $1 < p < n$: $\quad \hookrightarrow = 0 \quad \therefore \ \iota = \cong. \quad \hookrightarrow = 0$
So $H^{p-1}(X) \cong H^{n-p}(S^n - X)^* \cong H^{n-p}(\mathbb{R}^n - X)^*$ (as $1 < p < n$).
This is the Alexander duality theorem.

To include dimension 0, we replace $H^0(Y)$ by $\tilde{H}^0(Y)$ - "reduced cohomology", which is the cokernel of $H^0(pt) \xrightarrow{} H^0(Y)$. $H^0(Y) = A \oplus \tilde{H}^0(Y)$.
$\overset{\shortparallel}{A} \qquad pt \leftrightarrows Y$

Example: If $X \subset \mathbb{R}^2$ and X is homeomorphic to S^1 - "X is a Jordan Curve."
$A = H^1(X)^* \cong \tilde{H}^0(\mathbb{R}^2 - X) \cong A$. So $H^0(\mathbb{R}^2 - X) \cong A \oplus A$, ie $\mathbb{R}^2 - X$ has 2 connected components.

"Invariance of domain". U open, $\subset \mathbb{R}^n$. $f: U \to \mathbb{R}^n$ is 1-1 and continuous, then $f(U)$ is open.

Why does this follow from Alexander Duality?



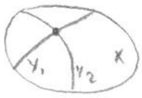
$f(S) \subset f(D) \subset \mathbb{R}^n$
$\ \ \wr\wr$ compact, connected complement
$\ S^n$ complement has two connected components.


Image of D must be exactly one of the two components of $\mathbb{R}^n - f(S)$.

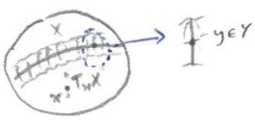
Thom Isomorphism Theorem.


Y closed submanifold $\subset X$.


$H^n(S^n) = H^n_{cpt}(\mathbb{R}^n)$


"codimension of the intersection = sum of codimensions of Y_1, Y_2".

Vector bundles.


$N_y Y \cong T_yX / T_yY$. = normal space at y.

A vector bundle on X is a space E with a map $p: E \to X$ and a vector space structure on each fibre $E_x = p^{-1}(x)$, such that $p: E \to X$ is locally trivial by maps which respect the vector space structure, ie each $x \in X$ has a neighbourhood U in X such that $\exists$ homeomorphism $h: p^{-1}(U) \to E_x \times U$, taking E_y isomorphically to $E_x \times \{y\}$.
(In this course, the vector spaces are finite dimensional).

Examples: (i) A smooth n-dimensional manifold X has a tangent bundle TX which is a vector bundle on X with fibre T_xX ($\cong \mathbb{R}^n$) at $x \in X$.

(ii) If Y is a closed submanifold of X, then Y has a normal bundle NY whose fibre at $y \in Y$ is $N_y Y = T_yX/T_yY \cong (T_yY)^\perp$.

(iii) "Tautological" bundle on $Gr_{\mathbb{R}}(\mathbb{R}^n)$, $E = \{(\xi, W) \in \mathbb{R}^n \times Gr_{\mathbb{R}}(\mathbb{R}^n) : \xi \in W\}$.
This has fibre W at $W \in Gr_{\mathbb{R}}(\mathbb{R}^n)$. $E_W \cong W$ as vector spaces.
Local trivialisation near $W \in Gr_{\mathbb{R}}(\mathbb{R}^n)$: Let U be all W' with $W' \cap W^\perp = 0$.
Then define $p^{-1}(U) \to W \times U$ by $W' \to W \times \{w'\}$, $pr: W' \to W$ is orthogonal projection.
$\xi \mapsto (pr(\xi), \{w'\})$

(iv) Similarly, have complex tautological bundle on $Gr_{\mathbb{R}}(\mathbb{C}^n)$.
In particular, have a bundle with fibres $\mathbb{C}$ on $\mathbb{P}^{n-1}_{\mathbb{C}}$.

Consider $\mathbb{P}^{n-1}_{\mathbb{C}}$. Have $(z_0, .. z_{n-1}) \in \mathbb{C}^n$. If $\varphi(z_0, .., z_{n-1})$ is homogeneous of degree d, then $\varphi(\lambda \underset{\sim}{z}) = \lambda^d \varphi(\underset{\sim}{z})$. Although not quite a function on $\mathbb{P}^{n-1}_{\mathbb{C}}$, it does make sense to consider, say $\varphi(\underset{\sim}{z}) = 0$.
Consider $\varphi|_L$, where L is a line in $\mathbb{C}^n$, ie a point in $\mathbb{P}^{n-1}_{\mathbb{C}}$. It belongs to the 1-dimensional vector space of homogeneous functions of degree d on L.
Ie, $\varphi|_L \in (L^*)^{\otimes d}$.
We have a complex line bundle on $\mathbb{P}^{n-1}_{\mathbb{C}}$ whose fibre at L is $(L^*)^{\otimes d}$.
(This is $\mathcal{O}(d)$ is algebraic geometry). A homogeneous polynomial of degreed on $\mathbb{C}^n$ is a section of this bundle with fibre $(L^*)^{\otimes d}$.

Oriented n-dimensional real vector bundle $p: E \to X$ is one with a given choice of a generator $w_x \in H^n_{cpt}(E_x)$ for each $x \in X$ such that $\{w_x\}$ is locally constant, in the sense that each x has a neighbourhood U such that for some local trivialisation $h: p^{-1}(U) \to E_x \times U$. h takes w_y to w_x for all $y \in U$.

V real vector space of dimension n. $H^n_{cpt}(V) \cong \mathbb{Z}$.
(i) orientation of V = choice of a generator of $H^n_{cpt}(V)$.
(ii) Let $\mathcal{B}$ = {ordered basis for V}. Then $\mathcal{B}$ consists of two equivalence classes, $\mathcal{B}_L$ and $\mathcal{B}_R$ such that $(v_1, .., v_n) \sim (w_1, .., w_n) \Leftrightarrow \det(\text{transition}) > 0$.
'Orientation of V' = choice of one equivalence class.
Choosing a basis $\leftrightarrow$ map $\mathbb{R}^n \to V$ giving an isomorphism $H^n_{cpt}(\mathbb{R}^n) \cong H^n_{cpt}(V)$.
Assume we have picked some generator of $H^n_{cpt}(\mathbb{R}^n)$. An equivalent basis gives an equivalent generator.

Note that any complex vector bundle is oriented, because there is a preferred class of bases over $\mathbb{R}$, namely those of the form $\{v_1, iv_1, .., v_n, iv_n\}$ where $\{v_1, .., v_n\}$ is a basis over $\mathbb{C}$.
$A_1 + iA_2 = A \in GL_n(\mathbb{C})$. $\det\begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix} = |\det A|^2$.

Theorem: If E is an oriented real n-dimensional vector bundle on X (paracompact), then

(i) there is a unique element $w_E \in H^n(E, E^\#)$, $\quad [w_x \in H^n(E_x, E_x - \{0\})]$
where $E^\#$ = all non-zero vectors in E, which restricts to w_x for each $x \in X$.
w_E is called the Thom class of the bundle.

(ii) multiplication by w_E is an isomorphism, $H^k(X) \to H^{k+n}(E, E^\#)$ for all k.

$$\begin{array}{ccc} & (p^*)\wr\!\wr & \\ & H^k(E) & \\ \alpha & \longmapsto & p^*(\alpha).w_E. \end{array}$$

$$p: E \rightleftarrows X$$

"zero section"

is homotopy equivalence.

Apply this to a tubular region about a submanifold: 

$w_E \in H^n(E) = H^n(X)$. In $H^n(X)$, w_E is called the Euler class of E.

Let $p: E \to X$ be an oriented real vector bundle of dimension n. Assume that X has a finite open covering $\{U_\alpha\}_{\alpha=1,\dots,m}$ such that $E|U_\alpha$ is trivial.
Say "E is of type $\leq m$" in this case.
We will prove the Thom isomorphism theorem by induction on m.
When $m=1$, $E = X \times \mathbb{R}^n$. $\quad H^{k-n}(X) \xrightarrow{\cong} H^k(X \times \mathbb{R}^n, X \times (\mathbb{R}^n - \{0\}))$
$$c \longmapsto p^*(c).\varepsilon_n, \quad \varepsilon_n = \text{generator of } H^n(\mathbb{R}^n, \mathbb{R}^n - \{0\}).$$

Write $X = U_1 \cup X'$
$E = E_1 \cup E'$, where $E_1 = p^{-1}(U_1)$, $E' = p^{-1}(X')$.

$$\left.\begin{array}{l} E|_{U_1 \cap X'} = E'' \to X'' = U_1 \cap X' \\ E' \to X' \end{array}\right\} \text{ bundles of type } m-1.$$

First prove that $H^k(E, E^\#) = 0$ for $k < n$
$\cong H^0(X)$ for $k = n$.

$E = E_1 \cup E'$. Get:
$$\begin{array}{ccccccc} H^{k-1}(E'', E''^\#) & \to & H^k(E, E^\#) & \to & H^k(E_1, E_1^\#) \oplus H^k(E', E'^\#) & \to & H^k(E'', E''^\#) \to \cdots \\ \text{For } k<n: \quad \| & & \therefore = 0 & & \| & & \| \\ 0 & & & & 0 & & 0 \end{array}$$

For $k=n$,
$$\begin{tikzcd} H^{n-1}(E'', E''^\#) \arrow[r] & H^n(E, E^\#) \arrow[r, hook] & H^n(E_1, E_1^\#) \oplus H^k(E', E'^\#) \arrow[r] & H^k(E'', E''^\#) \arrow[r] & \cdots \\ 0 \arrow[u, equal] \arrow[r] & H^0(X) \arrow[u, "w_E\ \therefore\ \cong"] \arrow[r] & H^0(U_1) \oplus H^0(X') \arrow[u, "\cong w_{E_1}\ \uparrow\cong\ \times w_{E'}\ \uparrow\cong"] \arrow[r] & H^0(X'') \arrow[u, "w_{E''}", "\cong"'] \arrow[r] & \cdots \end{tikzcd}$$

Now, return to:
$$\begin{tikzcd} H^{k-1}(E'', E''^\#) \arrow[r] & H^k(E, E^\#) \arrow[r] & H^k(E_1, E_1^\#) \oplus H^k(E', E'^\#) \arrow[r] & H^k(E'', E''^\#) \arrow[r] & \cdots \\ H^{k-n-1}(X'') \arrow[u, "\times w_{E''}"'] \arrow[r] & H^{k-n}(X) \arrow[u, "\therefore \cong"] \arrow[r] & H^{k-n}(U_1) \oplus H^{k-n}(X_1) \arrow[u, "\cong\ \uparrow\cong"] \arrow[r] & H^{k-n}(X'') \arrow[u, "w_{E''}", "\cong"'] & \end{tikzcd}$$

So done.

X a smooth manifold of dimension n, Y a closed submanifold of dimension m.
NY = vector bundle on Y of dimension $n-m$. $\quad N_y Y = T_y X / T_y Y$.

Theorem: $\exists$ open neighbourhoods U_Y of Y in X such that $NY \cong U_Y$, diffeomorphism, and zero section: $Y \to NY \to U_Y$ is the inclusion, and they are canonical up to "ambient isotopy", ie if U_Y' is another, then $\exists$ diffeomorphism $f: X \to X$

such that $f|_Y = \mathrm{id.}$, $f(U_Y) = U'_Y$, and
$$\begin{tikzcd} & NY \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ U_Y \arrow[rr, "f"] & & U'_Y \end{tikzcd}$$
commutes,

and f is isotopic to the identity, ie f is homotopic to the identity through diffeomorphisms.

If U_Y is a tubular neighbourhood of Y, write $U_{Y,y}$ for the part corresponding to N_yY.

$U_{Y,y} \cong \mathbb{R}^{n-m} \cong N_yY$.

Suppose Y is co-oriented, ie NY is oriented.

$$\left(\begin{array}{l} T_yX \cong T_yY \oplus N_yY. \\ H^n_{cpt}(T_yX) \xleftarrow{\cong} H^m_{cpt}(T_yY) \otimes H^{n-m}_{cpt}(N_yY). \end{array} \right)$$

Then, define $\varepsilon_Y \in H^{n-m}(X)$, the "cohomology class of Y", to be the image of ω_{NY} under: $\omega_{NY} \in H^{n-m}(NY, (NY)^*) \xrightarrow{\cong} H^{n-m}(U_Y, U_Y - Y) \xrightarrow{\text{extend by zero}} H^{n-m}(X)$.

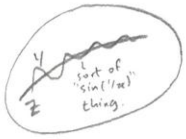
ε_Y has two properties:

(i) $\mathrm{supp}(\varepsilon_Y) = Y$.

(ii) comes from an element $\tilde{\varepsilon}_Y \in H^{n-m}(X, X-Y)$ such that $\tilde{\varepsilon}_Y|_{U_{Y,y}}$ is the preferred generator of $H^{n-m}(U_{Y,y}, U_{Y,y} - \{y\})$

By the Thom isomorphism theorem, this characterises ε_Y completely.

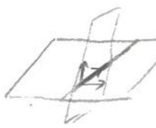
Theorem: $\varepsilon_Y \cdot \varepsilon_Z = \varepsilon_{Y \cap Z}$, provided Y and Z are cooriented closed submanifolds which intersect transversally.



This is bad. We get "points of accumulation". Things don't look like $\mathbb{R}^n$ anymore. Eek!

Y, Z intersect transversally $\iff N_xY \cap N_xZ = 0$, for all $x \in Y \cap Z$.

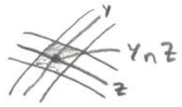
$\iff (T_xY) + (T_xZ) = T_xX$.



sum is not necessarily direct

$\iff N_x(Y \cap Z) = (N_xY) \oplus (N_xZ)$.

In this situation, $U_Y \cap U_Z = U_{Y \cap Z}$, in such a way that $U_{Y \cap Z, x} \cong U_{Y,x} \times U_{Z,x}$, under $N_x(Y \cap Z) \cong (N_xY) \oplus (N_xZ)$.



$\mathrm{supp}(\varepsilon_Y) = Y$, $\mathrm{supp}(\varepsilon_Z) = Z$, so $\mathrm{supp}(\varepsilon_Y \cdot \varepsilon_Z) \subset Y \cap Z$.

Comes from $\tilde{\varepsilon}_Y \cdot \tilde{\varepsilon}_Z$. Must check that $(\tilde{\varepsilon}_Y \cdot \tilde{\varepsilon}_Z)|_{U_{Y \cap Z, x}}$ represents the preferred generator of $H^k(U_{Y \cap Z, x}, U_{Y \cap Z, x} - \{x\})$.

This follows from $H^p(\mathbb{R}^p, \mathbb{R}^p - \{0\}) \times H^q(\mathbb{R}^q, \mathbb{R}^q - \{0\})$
$(\varepsilon_p \quad , \quad \varepsilon_q)$

$\varepsilon_p \cdot \varepsilon_q$ generates $H^{p+q}(\mathbb{R}^{p+q}, \mathbb{R}^{p+q} - \{0\})$.

X, smooth n-manifold, Y a closed submanifold, dimension n, cooriented. $\varepsilon_Y \in H^{n-m}(X)$.

Proposition: If $\alpha \in H^m_{cpt}(X)$ then $\int_X \varepsilon_Y . \alpha = \int_Y \alpha|_Y$ $\left(= \int_Y i^*\alpha,\ i: Y \to X\right)$. (with X oriented).

Proof: May as well assume that $X = U_Y$. So we may assume that $X = NY$, in which case $\alpha = p^*\alpha_0$ for some $\alpha \in H^m_{cpt}(Y)$, $p: NY \to Y$.

We want to prove that $\int_{NY} \omega_{NY} \cdot p^*(\alpha_0) = \int_Y (p^*\alpha_0)|_Y = \int_Y \alpha_0$.

Enough to prove this when $\alpha = \omega_U$ for some open $U \subset Y$, with $U \cong \mathbb{R}^n$, U "small".

We can assume therefore that $NY = U \times \mathbb{R}^{n-m}$, and $\omega_{NY}|(U \times \mathbb{R}^n)$ is $q^*\varepsilon_{n-m}$, $q: U \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$.

We are therefore reduced to $\varepsilon_m . \varepsilon_{n-m} = \varepsilon_n$.

$p: E \to X$, real vector bundle, dimension n. $E_x = p^{-1}(x)$. Choose an inner product $\langle , \rangle$ on E_x for each $x \in X$. Then (a) $\|\cdot\|: E \to \mathbb{R}$ is continuous. $\Leftrightarrow$ (b) $\exists$ local trivialisations $p^{-1}(U) \cong U \times \mathbb{R}^n$ which respect the inner product.

Proposition: There always exist continuous inner products on E, providing X is paracompact.

Proof: Choose an open covering $\{U_\alpha\}$ of X such that $\underset{p^{-1}(U_\alpha)}{\overset{\parallel}{E|_{U_\alpha}}} \cong U \times \mathbb{R}^n$.

We can define $\langle , \rangle_\alpha$ on E_x for $x \in E_\alpha$ by using the local trivialisation.

Choose partition of unity $\{\lambda_\alpha\}$ subordinate to $\{U_\alpha\}$, ie (i) $\lambda_\alpha : X \to \mathbb{R}_+$ is continuous,

(ii) support$(\lambda_\alpha) = \overline{\{x : \lambda_\alpha(x) > 0\}} \subset U_\alpha$, and (optionally, but nice to have)

(iii) $\{\lambda_\alpha\}$ is locally finite, ie each $x \in X$ has a neighbourhood U such that $\lambda_\alpha | U = 0$ except for finitely many α's,

(iv) $\Sigma \lambda_\alpha = 1$.

Finally, if $\xi, \eta \in E_x$, define $\langle \xi, \eta \rangle = \sum_\alpha \lambda_\alpha(x) \langle \xi, \eta \rangle_\alpha$

$\hookrightarrow = 0$ if $x \notin U_\alpha$.

Suppose E has an inner product. Write $S_E = \{\xi \in E : \|\xi\| = 1\}$. This is a locally trivial fibre bundle on X with fibre S^{n-1}. Notice that $S_E \hookrightarrow E^\#$ is a homotopy equivalence.

[Use $\xi/\|\xi\| \leftarrow\!\!\text{-}\!\text{-}\!\text{-}\ \xi$. Have homotopy $t\,\xi/\|\xi\| + (1-t)\xi$]

Gysin Sequence.

E oriented, dimension n. $\exists$ long exact sequence obtained by considering

$$\cdots \to H^k(X) \xrightarrow{\cong} H^{k+n}(E, E^\#) \to H^k(E) \xrightarrow{\text{restriction}} H^k(E^\#) \to \cdots$$

forget where it vanishes

$\alpha \longmapsto (p^*\alpha)\,\omega_E$

$= p^*(\alpha)\, e_E$ in $H^k(E)$

$= \alpha . e_E$, restriction to zero-section

$p: E \rightleftarrows X$, homotopy equivalence

$H^k(E^\#)$: S_E

which is:

$$\cdots \to H^{k+n-1}(S_E) \longrightarrow H^k(X) \longrightarrow H^{k+n}(X) \longrightarrow H^{k+n}(S_E) \to H^{k+1}(X) \to \cdots$$

ring homomorphism

multiplication by the Euler class $e_E \in H^n(X)$.

map of graded modules over $H^*(X)$.

$p: S_E \to X$. $p^*: H^k(X) \to H^k(S_E)$, homomorphism of graded rings, makes $H^*(S_E)$ into an $H^*(X)$-module.

$$\cdots \to \underset{\alpha}{\overset{\in}{H^{k-1}(Y)}} \to H^k(X,Y) \to H^k(X) \to \cdots$$

homomorphism of $H^*(X)$-modules.

$$\alpha \longleftrightarrow \tilde{\alpha} \in C^{k-1}(X)$$
$$\downarrow$$
$$d\tilde{\alpha} \in C^k(X,Y)$$

Example: $X = \mathbb{P}^{n-1}_{\mathbb{C}}$. $H^k(X) = \begin{cases} \mathbb{Z} & \text{if } k \text{ is even and } \leq 2(n-1) \\ 0 & \text{otherwise.} \end{cases}$

Could use Poincaré duality. $\varepsilon_Y \in H^{2p}(X)$, $Y = \mathbb{P}(\mathbb{C}^{n-p}) \subset \mathbb{P}(\mathbb{C}^n)$.
If we take $\varepsilon_Z \in H^{2q}(X)$, $\varepsilon_Y \cdot \varepsilon_Z = \varepsilon_{Y \cap Z}$. If we took generators a_p of $H^{2p}(X)$, a_q of $H^{2q}(X)$, then $a_p \cdot a_q = a_{p+q}$.
If c = generator of $H^2(\mathbb{P}^{n-1})$, then as a ring, $H^*(\mathbb{P}^{n-1}) \cong \mathbb{Z}[c]/(c^n)$.
as graded rings.

$\exists$ complex line bundle E on $\mathbb{P}^{n-1}_{\mathbb{C}}$. $E \subset \mathbb{P}^{n-1} \times \mathbb{C}^n$
$E_L = L \subset \mathbb{C}^n$.
S_E = unit sphere in $\mathbb{C}^n$, $= S^{2n-1}$. $\{(L, \xi) : \xi \in L, \|\xi\| = 1\}$ can disregard, since $\exists$ only one ray for each unit vector
$p: S^{2n-1} \to \mathbb{P}^{n-1}_{\mathbb{C}}$. Use the Gysin sequence.

$$\underset{0}{\overset{\parallel}{H^{k+1}(S^{2n-1})}} \to H^k(\mathbb{P}) \xrightarrow[\times e_E \in H^2(\mathbb{P})]{\therefore \cong} H^{k+2}(\mathbb{P}) \to \underset{0}{\overset{\parallel}{H^{k+2}(S^{2n-1})}}, \quad \text{for suitable } k.$$

At end of sequence:

$k = -1$: $\cdots \to \underset{0}{H^{-1}(\mathbb{P})} \to \underset{\therefore = 0}{H^1(\mathbb{P})} \to \underset{0}{H^1(S^{2n-1})} \to \cdots$

$k = 0$: $\cdots \underset{0}{H^1(S^{2n-1})} \to \underset{= \mathbb{Z}}{H^0(\mathbb{P})} \to \underset{\therefore = \mathbb{Z}}{H^2(\mathbb{P})} \to \underset{0}{H^2(S^{2n-1})}$ and so on.

Example: $\mathbb{P}^{2n-1}_{\mathbb{R}}$ is a circle bundle over $\mathbb{P}^{n-1}_{\mathbb{C}}$.

$$\mathbb{P}_{\mathbb{R}}(\mathbb{R}^{2n}) \longrightarrow \mathbb{P}_{\mathbb{C}}(\mathbb{C}^n)$$
$$L \longmapsto L + iL.$$

We get a Gysin sequence: $H^*(\mathbb{P}_{\mathbb{R}}) \to H^*(\mathbb{P}_{\mathbb{C}}) \xrightarrow{\times(\text{euler class}) = 2c} H^*(\mathbb{P}_{\mathbb{C}}) \to H^*(\mathbb{P}_{\mathbb{R}}) \to \cdots$
$\mathbb{Z}[c]/(c^n) \longrightarrow \mathbb{Z}[c]/(c^n)$, $c \in H^2(\mathbb{P}^{n-1}_{\mathbb{C}})$

$H^*(\mathbb{P}^{2n-1}_{\mathbb{R}}) = \mathbb{Z}[c]/(c^n, 2c)$.

We get: $\dfrac{\mathbb{Z}[c]}{(c^n)} \xrightarrow{\times 2c} \dfrac{\mathbb{Z}[c]}{(c^n)} \to (\,?\,)$

$\mathbb{Z} \cong \mathbb{Z}c^{n-1}$

Yields: $0 \to \dfrac{\mathbb{Z}[c]}{(c^n, 2c)} \xrightarrow{\text{ring hom.}} H^*(\mathbb{P}^{2n-1}_{\mathbb{R}}) \xrightarrow{\text{degree } -1} \mathbb{Z} \to 0$
in degree $2n-1$.

Generators of $\frac{\mathbb{Z}[c]}{(c^n, 2c)}$ are : $1, c, c^2, \dots, c^{n-1}$, and $2c = 2c^2 = \dots = 0$.

So it is $\cong \mathbb{Z} \oplus (\mathbb{Z}/2)c \oplus (\mathbb{Z}/2)c^2 \oplus \dots \oplus (\mathbb{Z}/2)c^{n-1}$ degree $2n-2$.

Get homology sequence: $\mathbb{Z}\ 0\ \mathbb{Z}/2\ 0\ \mathbb{Z}/2\ 0 \dots 0\ \underbrace{\mathbb{Z}/2}_{2n-2}\ \underbrace{\mathbb{Z}}_{2n-1}\ 0 \dots$

$H^{2n-1}(\mathbb{P}^{2n-1}_{\mathbb{R}}) \xrightarrow{\cong} \mathbb{Z}$
$\omega \longmapsto 1$
Get $\mathbb{Z}[c,\omega]/(c^n, 2c, c\omega, \omega^2)$.

Why is $\mathbb{P}^{2n-1}_{\mathbb{R}} = S(E)$, where E = complex 1-dimensional bundle on $\mathbb{P}^{n-1}_{\mathbb{C}}$?
Take $E_L = (L^*)^{\otimes 2}$ = homogeneous quadratic complex-valued functions on $L \subset \mathbb{C}$.
$\lambda \in L$, $x\lambda \mapsto kx^2$ $(k \in \mathbb{C})$.
Define an inner product on E_L, $\langle \varphi_1, \varphi_2 \rangle = \mathrm{Re}\left(\frac{\overline{\varphi_1(\lambda)}\cdot\varphi_2(\lambda)}{\|\lambda\|^4}\right)$, for any $\lambda \neq 0$ on L.

Unit vectors: $|\varphi(\lambda)|^2 = |\lambda|^4$. Check that $S_E = \mathbb{P}^{2n-1}_{\mathbb{R}}$

Why is the Euler class $2c$?
We know e (line bundle) $\in H^2(\mathbb{P}^{n-1}_{\mathbb{C}}) = \mathbb{Z}c$.
so $\hookrightarrow = mc$.
We can get $m = \pm 2$ if we can show that, say, H^2 is $\mathbb{Z}/2$. The $\pm$ is irrelevant. We will come back to this later.

Now, $e(L \otimes M) = e(L) + e(M)$, $e(L^*) = -e(L)$.
$e((L^*)^{\otimes 2}) = -2e(L) = -2c$

$\mathbb{P}^{2n-2}_{\mathbb{R}} \hookrightarrow \mathbb{P}^{2n-1}_{\mathbb{R}}$. Complement: $\mathbb{P}^{2n-1}_{\mathbb{R}} - \mathbb{P}^{2n-2}_{\mathbb{R}} \cong \mathbb{R}^{2n-1}$.

$$H^k(\mathbb{P}^{2n-1}, \mathbb{P}^{2n-2}) \to H^k(\mathbb{P}^{2n-1}) \to H^k(\mathbb{P}^{2n-2}) \to H^{k+1}_{cpt}(\mathbb{R}^{2n-1})$$
$\| ?$
$$H^k_{cpt}(\mathbb{R}^{2n-1}) = \begin{cases} 0 & \text{unless } k = 2n-1 \\ \mathbb{Z} & \text{if } k = 2n-1. \end{cases}$$

At top, $k = 2n-1$, : $0 \to \mathbb{Z} \to H^{2n-1}(\mathbb{P}^{2n-1}) \to H^{2n-1}(\mathbb{P}^{2n-2}) = 0$
$\therefore \cong \mathbb{Z}$

Theorem: $H^*(\mathbb{P}^{2n-2}_{\mathbb{R}}) \cong \mathbb{Z}[c]/(c^n, 2c)$.

$H^{2n-2}(\mathbb{P}^{2n-2}) = (\mathbb{Z}/2)c^{n-1}$. $H^{2n-1}(\mathbb{P}^{2n-1}) \cong \mathbb{Z}$.

Suppose E is a real oriented n-dimensional vector bundle on X.
$e_E \in H^n(X;\mathbb{Z})$. - "Euler class of E".
We want it to be 'natural', ie for $f: Y \to X$, can form a vector bundle f^*E on Y whose fibre at $y \in Y$ is $E_{f(y)}$.

$$\begin{array}{ccc} (f^*E)_y = E_{f(y)} & & \{E_x\}_{x\in X} \\ \downarrow & & \downarrow \\ f: Y & \to & X \end{array}$$

"Natural" $\Leftrightarrow$ $e_{f^*E} = f^* e_E$
Why is f^*E a vector bundle on Y?
$f^*E \subset Y \times E$ - gives f^*E a topology.
$(y, \xi \in E_{f(y)})$
Why locally trivial? Notice that $(f^*E)|_U$, for open $U \subset Y$,
$= (f|U)^*(E|f(U))$

The triviality of this argument cannot be over-estimated. "I don't want to be a dentist" - G Segal 26/11/96.

But for any $y \in Y$, $\exists$ neighbourhood V of $f(y)$ such that $E|V \cong \mathbb{R}^n \times V$.
Choose U such that $f(U) \subset V$. Then $(f^*E)|U = (f|U)^*(f(U) \times \mathbb{R}^n) = U \times \mathbb{R}^n$.

The Euler class is a characteristic class, because $w_{f^*E} \in H^n(f^*E, (f^*E)^*)$.
$= f^* w_E$.

Chern Classes.

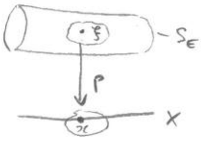
Suppose E is a complex n-dimensional vector bundle on X. We shall define $c_k(E) \in H^{2k}(X;\mathbb{Z})$ for $k = 1, .., n$, which are characteristic classes.
↳ the kth Chern class of E.

$c_n(E) \in H^{2n}(X)$.
$\overset{\parallel}{e_E}$, regarding E as an oriented $2n$-dimensional real vector bundle.

Consider S_E = sphere bundle on E. Consider the Gysin sequence.

$$H^{k-2n}(X) \xrightarrow{\times e_E} H^k(X) \xrightarrow{p^*} H^k(S_E) \longrightarrow H^{k-2n+1}(X) ..$$

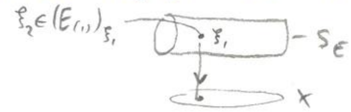
So if $k \le 2n-2$, then $H^k(X) \xrightarrow{p^*, \cong} H^k(S_E)$.
But there is a complex vector bundle of dimension $n-1$ on S_E, whose fibre at $\xi \in E_x$ is $\xi^\perp \subset E_x$.
$\|\xi\| = 1$, $\xi^\perp \cong \mathbb{C}^{n-1}$



Call this bundle $E_{(1)}$ on S_E
It has an Euler class $e_{E_{(1)}} \in H^{2n-2}(S_E) \underset{p^*}{\cong} H^{2n-2}(X)$
Write $c_{n-1}(E) = (p^*)^{-1}(e_{E_{(1)}})$

orthonormal pair in E_x.

Consider now the bundle $E_{(2)}$ on $S_{E_{(1)}}$, whose fibre at $(\xi_1, \xi_2) \in E_x$ is $(\xi_1, \xi_2)^\perp \cong \mathbb{C}^{n-2}$.



Repeating this: $H^k(X) \xrightarrow{p^*,\cong} H^k(S_E) \xrightarrow{p_1^*,\cong} H^k(S_{E_{(1)}})$, if $k \le 2n-4$.
$e_{E_{(2)}} \in H^{2n-4}(X)$.

We have constructed a sequence of bundles $V_k(E)$ on X whose fibres at x are $V_k(E_x)$ for $k = 1, 2, ..$ See: $V_1(E_x) = S(E_x)$, $V_0(E) = X$.
$V_k(E)$ is a fibre bundle over $V_{k-1}(E)$, with fibre $S^{2n-2k+1}$.

E complex n-dimensional bundle on X, with inner product.

$$\begin{tikzcd}
E \arrow[d, "\text{fibre } \mathbb{C}^n"] & E^{(1)} \arrow[d, "\text{fibre } \mathbb{C}^{n-1}"] & E^{(2)} \arrow[d, "\text{fibre } \mathbb{C}^{n-2}"] & E^{(3)} \arrow[d, "\text{fibre } \mathbb{C}^{n-3}"] & \\
X & S_E = V_1(E) \arrow[l, "\text{fibre } S^{2n-1}"'] & V_2(E) \arrow[l, "\text{fibre } S^{2n-3}"'] & V_3(E) \arrow[l, "\text{fibre } S^{2n-5}"'] & \cdots \arrow[l]
\end{tikzcd}$$

$$c_{n-k}(E) = e_{E(k)} \in H^{2n-2k}(V_k(E)) \cong H^{2n-2k}(X)$$

Real oriented bundle E, fibre $\mathbb{R}^n$.

$$\xrightarrow{\times e_E} H^k(X) \xrightarrow[\text{ring hom.}]{} H^k(S_E) \xrightarrow[\substack{\text{module hom} \\ \text{over } H^*(X)}]{} H^{k-n+1}(X) \xrightarrow{\times e_E} \cdots$$

If E has a nowhere-vanishing section $s: X \to E$, then $S_E \to X$ has a section $s: X \to S_E$.
Gysin sequence $\Rightarrow$ $e_E = 0$.

Theorem: If X is a smooth manifold and $s: X \to E$ is a smooth section transversal to the zero-section $X \to X$, then $e_E = \varepsilon_Z$, where $Z = s^{-1}(0) \subset X$

Return to above complex bundle. If $E \to X$ has k sections $s_1, \dots, s_k$ such that $\{s_1(x), \dots s_k(x)\}$ are linearly independent for all $x \in X$, then $V_k(E) \to X$ has a section, and so $c_{n-k+1}(E) = 0$. In general, $c_{n-k+1}(E) = \varepsilon_{Z_k}$, where Z_k is the 'submanifold' where k generic smooth sections become linearly dependent.

Return to real oriented $E \to X$, fibre $\mathbb{R}^n$. Suppose $e_E = 0$. Then we have

$$0 \to H^k(X) \to H^k(S_E) \to H^{k-n+1}(X) \to 0$$
$$\sigma \longmapsto 1$$

$\sigma \in H^{n-1}(S_E)$, automatically true that $\sigma(\text{fibre}) =$ generator of $H^{n-1}(S^{n-1})$

We have an isomorphism of $H^*(X)$-modules:

$$H^*(X) \oplus H^*(X) \longrightarrow H^*(S_E)$$
$$(a, b) \longmapsto a + \sigma b = p^*(a) + \sigma p^*(b), \quad p: H^*(X) \hookrightarrow H^*(S_E).$$

If we now determine $\sigma^2 = a_0 + \sigma b_0$, then we know $H^*(S_E)$ as a ring in terms of $H^*(X)$.

Example: $H^*(V_k(\mathbb{C}^n)) = \Lambda_{\mathbb{Z}}(\sigma_{2n-1}, \sigma_{2n-3}, \dots, \sigma_{2n-2k+1})$ = "exterior algebra on classes σ_i of dimension i".

(So $\sigma_i^2 = 0$, $\sigma_i \sigma_j = -\sigma_j \sigma_i$.) $\cong H^*(S^{2n-1} \times S^{2n-3} \times \dots \times S^{2n-2k+1})$

but $V_k(\mathbb{C}^n) \not\cong S^{2n-1} \times \dots \times S^{2n-2k+1}$.

$V_n(\mathbb{C}^n) = U_n$, so $H^*(U_n) = \Lambda_{\mathbb{Z}}(\sigma_1, \sigma_3, \dots, \sigma_{2n-1})$

Proof: Induction on k. $V_k(\mathbb{C}^n) \to V_{k-1}(\mathbb{C}^n)$ is S_E for a bundle E with fibre $\mathbb{C}^{n-k+1}$.
$e_E \in H^{2n-2k+2}(V_{k-1}) = 0$, by induction.
So $H^*(V_k) = H^*(V_{k-1}) \oplus \sigma_{2n-2k+1} H^*(V_{k-1})$. But $2\sigma^2 = 0$, by anticommutativity, and by induction $H^*(V_k)$ is a free abelian group, so $\sigma^2 = 0$.

$f : X' \to X$, smooth map of smooth manifolds. $Y \subset X$, closed, smooth, cooriented.
$\varepsilon_Y \in H^*(X)$. $f^*(\varepsilon_Y) = \varepsilon_{f^{-1}(Y)} \in H(X')$, if f is transversal to Y.

Implicit Function Theorem: If $x \in X$ then $f^{-1}(x)$ is smooth if $Df(x') : T_{x'}X' \to T_x X$ is surjective for all $x' \in f^{-1}(x)$.
Similarly, $f^{-1}(Y)$ is smooth if $Df(x') : T_{x'}X' \to T_x X \to T_xX / T_xY = N_x Y$ is surjective for each $x' \in f^{-1}(Y)$. "f is transversal to Y".

Then $Df(x') : N_{x'}(f^{-1}(Y)) \to N_x(Y)$

X compact, oriented, smooth, n-dimensional manifold. To calculate $\varepsilon_\Delta \in H^n(X \times X; F)$, with F a field, where Δ = diagonal $\subset X \times X$, $\Delta \cong X$.

Künneth Theorem: $\bigoplus_{p+q=n} H^p(X) \otimes H^q(X) \xrightarrow{\cong} H^{p+q}(X \times X)$.

Choose a basis $\{a_i\}$ for $H^*(X)$ with $a_i \in H^{d_i}(X)$. Let $\{a_i^*\}$ be the dual basis, in the sense that $\int_X a_i^* a_j = \delta_{ij}$, $a_i^* \in H^{n-d_i}(X)$.

Theorem: $\varepsilon_\Delta = \sum (-1)^{d_i} a_i^* \otimes a_i$.

Proof: By Poincaré duality for $X \times X$, it is enough to see $\int_{X \times X} \varepsilon_\Delta \cdot (\alpha \otimes \beta) = \sum (-1)^{d_i} \int_{X \times X} (a_i^* \otimes a_i)(\alpha \otimes \beta)$, for all $\alpha, \beta \in H^*(X)$. Clearly sufficient to see when α, β are basis elements, ie, that $\int_{X \times X} \varepsilon_\Delta \cdot (a_j \otimes a_k^*) = \sum (-1)^{d_i} (a_i^* \otimes a_i)(a_j \otimes a_k^*)$.

RHS $= \sum (-1)^{d_i} \int_X (a_i^* a_j) \otimes \int_X (a_i \otimes a_k^*) = \delta_{jk}$, since many terms are zero

LHS $= \int_\Delta (\alpha \otimes \beta)|_\Delta = \int_X \alpha \cdot \beta = \delta_{jk}$, if $\alpha = a_j$, $\beta = a_k^*$.

Corollary: (i) $\int_{X \times X} \varepsilon_\Delta^2 = \sum_k (-1)^k \dim H^k(X) = \chi(X)$, the Euler number of X.

(ii)  $(x, x) \mapsto (x, x + \delta x)$.

$\int_{X \times X} \varepsilon_\Delta^2 = \int_{X \times X} \varepsilon_Z$, where Z = zeroes of a generic smooth tangent vector field.

So, $\int_{X \times X} \varepsilon_Z$ = algebraic number of points in Z.

"Hopf vector field theorem".

(iii) Lefschetz fixed point theorem.

$f: X \to X$. Γ_f = graph of $f \subset X \times X$. $\Gamma_f = (f \times id.)^{-1}\Delta$.

$\int_{X\times X} \varepsilon_\Delta \cdot \varepsilon_{\Gamma_f}$ = algebraic number of $\{x: f(x) = x\} = \sum_i (-1)^{d_i} \int_X f^*(a_i^*) a_i$

= $(i,i)^{th}$ matrix element of $f^*: H^*(X) \to H^*(X)$

= $\sum (-1)^k$. trace (f^* on $H^k(X)$).

Double Cochain Complexes.

$$\cdots \xrightarrow{\varphi} C^\bullet_{(k)} \xrightarrow{\varphi} C^\bullet_{(k+1)} \xrightarrow{\varphi} \cdots \qquad \varphi \circ \varphi = 0. \qquad C^q_{(p)} = C^{p,q}$$

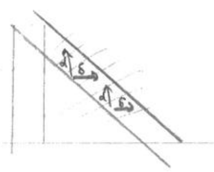
$$\xrightarrow{\varphi} \uparrow d \xrightarrow{\varphi} \uparrow d \xrightarrow{\varphi} \qquad \varphi: C^{p,q} \to C^{p+1,q}$$

$q \uparrow \ \overrightarrow{p}$

$\delta: C^{p,q} \to C^{p+1,q}$, $\delta = (-1)^q \varphi$.

$\delta^2 = 0$, $d\delta + \delta d = 0$, $d\varphi = \varphi d$

Assume $\exists\, p_0, q_0$ such that $C^{pq} = 0$, unless $p \geq p_0$, $q \geq q_0$.



Total complex of a double complex, $\hat{C}^\bullet$.

$\hat{C}^n = \bigoplus_{p+q=n} C^{pq}$.

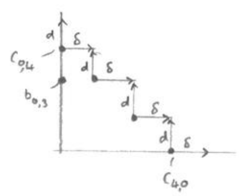
$\hat{\delta}: \hat{C}^n \to \hat{C}^{n+1}$, $\hat{d} = d + \delta$. $(d+\delta)^2 = 0$

Theorem: $\hat{C}^\bullet$ is acyclic <= each column $C^{p\bullet}$ is acyclic.

$H^k(\hat{C}^\bullet) = 0$ for all k. <= $H^q(C^{p\bullet}) = 0$ for all q.

Proof: Assume $p_0 = q_0 = 0$. Suppose we have a cocycle in $\hat{C}^n$,

$c = c_{0n} + c_{1,n-1} + \cdots + c_{n,0}$, $c_{p,q} \in C^{pq}$.



$dc_{0n} = 0$, $dc_{1,n-1} = -\delta c_{0n}$, $dc_{2,n-2} = -\delta c_{1,n-1}, \ldots,\ 0 = -\delta c_{n,0}$.

Choose b_0 such that $db_0 = c_{0n}$. Then $c_{1,n-1} - \delta b_0$ is closed, for $d(c_{1,n-1} - \delta b_0) = -\delta c_{0,n} + \delta d b_0 = -\delta c_{0n} + \delta c_{0n} = 0$.

Choose b_1 such that $db_1 = c_{1,n-1} - \delta b_0$. Note that $c_{2,n-2} - \delta b_1$ is closed.

Continue. We get $b = b_0 + b_1 + b_2 + \cdots$ such that $\hat{d} b = c$.

Consider a double complex $C^{\bullet\bullet}$ as above. Define $A^{pq} = \begin{cases} C^{pq} & \text{if } p \geq a \\ 0 & \text{if not} \end{cases}$

$B^{pq} = \begin{cases} C^{pq} & \text{if } p < a \\ 0 & \text{if not.} \end{cases}$

Obviously, $A^{\bullet\bullet}$ and $B^{\bullet\bullet}$ are both double complexes. $A^{\bullet\bullet}$ is a subcomplex of $C^{\bullet\bullet}$.

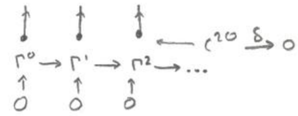
There is an obvious map of double complexes $C^{\bullet\bullet} \to B^{\bullet\bullet}$

We have an exact sequence of total complexes,

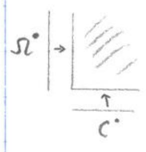
$$0 \to \hat{A}^\bullet \to \hat{C}^\bullet \to \hat{B}^\bullet \to 0$$

$$0 \to \bigoplus_{\substack{p+q=n \\ p \leq a}} C^{pq} \to \bigoplus_{p+q=n} C^{pq} \to \bigoplus_{\substack{p+q=n \\ p > a}} C^{pq} \to 0.$$

Suppose we have $C^{\bullet\bullet}$, with $C^{pq} = 0$ unless $p, q \geq 0$.
Suppose $H^k(C^{p\bullet}) = 0$ if $k \neq 0$, but $H^0(C^{p\bullet}) = \Gamma^p = \operatorname{Ker}(C^{p0} \xrightarrow{d} C^{p1})$.
Then $\begin{matrix} C^{\bullet\bullet} \\ \uparrow \\ \Gamma^{\bullet} \end{matrix}$ is a double complex with acyclic columns.



Have $0 \to \hat{C}^{\bullet} \to \operatorname{Tot}\left(\begin{matrix} C^{\bullet\bullet} \\ \uparrow \\ \Gamma^{\bullet} \end{matrix}\right) \to \Gamma^{\bullet} \to 0$
So $H^*(\Gamma^\bullet) \xrightarrow{\cong} H^*(\hat{C}^\bullet)$. Indeed, $H^k(\Gamma^\bullet) \xrightarrow{\cong} H^k(\hat{C}^\bullet)$



de Rham cohomology. X, smooth manifold, convex covering $\{U_\alpha\}_{\alpha \in S} = \mathcal{U}$
$\Omega^k(Y)$ = smooth k-forms on Y.
Define double complex $C^{pq} = \bigoplus_{\alpha_0, \dots, \alpha_p \in S} \Omega^q(U_{\alpha_0 \dots \alpha_p})$
$U_{\alpha_0 \dots \alpha_p} = U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$

$\omega_{\alpha_0 \dots \alpha_p} \in \Omega^k(U_{\alpha_0 \dots \alpha_p})$.
vertical differential = de Rham d.
horizontal differential δ: $(\delta\omega)_{\alpha_0 \dots \alpha_{p+1}} = \sum (-1)^k (\omega_{\alpha_0 \dots \hat{\alpha}_k \dots \alpha_{p+1}} | U_{\alpha_0 \dots \alpha_{p+1}})$

$\Omega^\bullet(U_{\alpha_0 \dots \alpha_p})$ has no cohomology except in degree 0. In degree 0 we have the constant functions $\mathbb{R}_{(\alpha_0, \dots, \alpha_p)}$
We can "put in a bottom row" (Γ^p from before), $\Gamma^p = \check{C}^p(\mathcal{U}) = \bigoplus_{\substack{\alpha_0 \dots \alpha_p \text{ such that} \\ U_{\alpha_0} \cap \dots \cap U_{\alpha_p} \neq \emptyset}} \mathbb{R}$.

Čech cohomology of covering $\mathcal{U} = H^*(\hat{C}^\bullet)$
Now look at a row. $\bigoplus_\alpha \Omega^q(U_\alpha) \xrightarrow{\delta} \bigoplus_{\alpha, \beta} \Omega^q(U_{\alpha\beta}) \xrightarrow{\delta} \cdots$
$\omega_\alpha \longmapsto (\delta\omega)_{\alpha,\beta} = (\omega_\beta | U_{\alpha\beta}) - (\omega_\alpha | U_{\alpha\beta})$.
Ker of δ on C^{0q} is $\Omega^q(X)$.
Thus $H^*(\hat{C}^\bullet) \cong H^*(\Omega^\bullet(X))$, if we show the rows are exact.

To prove the rows acyclic: $h: C^{pq} \to C^{p-1,q}$
$(h\omega)_{\alpha_0 \dots \alpha_{p-1}} = \sum_\alpha \lambda_\alpha \underbrace{\omega_{\alpha\alpha_0 \dots \alpha_{p-1}}}_{\text{defined on } U_\alpha \cap U_{\alpha_0} \cap \dots \cap U_{\alpha_{p-1}}}$ extended by zero.
where $\{\lambda_\alpha\}$ is a partition of unity subordinate to $\{\mathcal{U}\}$, ie, $\lambda_\alpha : X \to \mathbb{R}_+$, $\operatorname{supp} \lambda_\alpha \subset U_\alpha$, $\sum \lambda_\alpha = 1$.
We want to show $h\delta + \delta h = id.$, except in degree 0.

$(\delta h \omega)_{\alpha_0 \dots \alpha_p} = \sum_{\alpha, k} (-1)^k \lambda_\alpha \omega_{\alpha\alpha_0 \dots \hat{\alpha}_k \dots \alpha_p}$.
$(h\delta\omega)_{\alpha_0 \dots \alpha_p} = \sum_\alpha \lambda_\alpha (\delta\omega)_{\alpha\alpha_0 \dots \alpha_p}$
Adding these gives $\sum_\alpha \lambda_\alpha \omega_{\alpha_0 \dots \alpha_p} = (\sum_\alpha \lambda_\alpha) \omega_{\alpha_0 \dots \alpha_p} = \omega_{\alpha_0 \dots \alpha_p}$.

To show $H^*_{AS}(X) \cong H^*(\mathcal{U})$, where $\mathcal{U}$ is a contractible covering, coefficients in $\mathbb{Z}$.
$C^{pq} = \bigoplus_{\alpha_0, \dots, \alpha_p} C^q(U_{\alpha_0, \dots, \alpha_p})$  . Similar argument - see printed notes.

$C^\bullet_{\mathcal{U}}(X) \to C^\bullet(X)$. $C^\bullet_{\mathcal{U}}(X)$ has same cohomology as $C^\bullet(\mathcal{U})$

$C_{sing}(X): \bigoplus C^q_{sing}(U_\alpha) \to \bigoplus_{\alpha,\beta} C^q_{sing}(U_{\alpha\beta})$

Everything is fine except for the first term. Need lemma.

Prove $C^\bullet_{sing,\mathcal{U}}(X) \subset C^\bullet_{sing}(X)$ - prove by subdivision.

X, $\mathcal{U}$ open covering.

$$\begin{array}{ccc} C^\bullet_{\mathcal{U}}(X) & \to & C^{\bullet\bullet} \\ & & \uparrow \\ & & C^\bullet(\mathcal{U}) \end{array}$$

$C^{pq} = \prod_{\alpha_0,\dots,\alpha_p} C^q(U_{\alpha_0,\dots,\alpha_p})$

Proof: Show that multiplication in H^* is anticommutative.

$T: C^\bullet(X) \to C^\bullet(X)$, $(Tc)(x_0,\dots,x_p) = (-1)^{\frac{1}{2}p(p+1)} c(x_p,\dots,x_0)$

$Tc_1 \cdot Tc_2 = (-1)^{pq} T(c_2 c_1)$. want to show T induces id on H^*.

Enough to prove it on $C^\bullet_{\mathcal{U}}(X)$

$$\begin{array}{ccc} \circlearrowleft_T\, C^\bullet_{\mathcal{U}}(X) & \to & C^{\bullet\bullet} \circlearrowright_T \\ & & \uparrow \\ & & C^\bullet(\mathcal{U}) \circlearrowright \mathrm{id}. \end{array}$$

T induces a map from this picture to itself, commuting with all maps in the picture

We have:

$$\begin{array}{ccccc} C^\bullet_{\mathcal{U}}(X) & \longrightarrow & \mathrm{Tot}(C^{\bullet\bullet}) & \longleftarrow & C^\bullet(\mathcal{U}) \\ \circlearrowleft_T & & \circlearrowleft_T & & \circlearrowleft_{id} \\ H(C^\bullet_{\mathcal{U}}(X)) & \xrightarrow{\cong} & H^*(\mathrm{Tot}) & \xleftarrow{\cong} & H^*(\mathcal{U}) \\ \circlearrowleft_T & & \circlearrowleft_T & & \circlearrowleft_{id}. \end{array}$$

- since this commutes, must have $T = id$.

Suppose we have open covering U_1, U_2.

$$C^\bullet(X) \longrightarrow C^\bullet(U_1) \oplus C^\bullet(U_2) \to C^\bullet(U_{12}) \longrightarrow 0. \qquad U_{12} = U_1 \cap U_2.$$

$\prod_{\alpha_0,\dots,\alpha_p} C^q(U_{\alpha_0,\dots,\alpha_p})$ has terms $\alpha\beta, \beta\alpha, \alpha\alpha, \beta\beta$, etc. - "wasteful".

To improve, impose that: $c_{\alpha_0\dots\alpha_p} = 0$ if $\alpha_i = \alpha_j$, $i \neq j$.

$c_{\beta_0\dots\beta_p} = (-1)^\pi c_{\alpha_0\dots\alpha_p}$ for $\{\beta_i\}$ is a permutation of $\{\alpha_i\}$.

Theorem: $H^*(C^\bullet_{\mathcal{U}}(X)) \cong H^*$(alternating Čech cochains).

U_1 U_2

$$C^\bullet(X) \to C^\bullet(U_1) \oplus C^\bullet(U_2) \to C^\bullet(U_{12}) \to 0$$
$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

Suppose we have $X \overset{f_0}{\underset{f_1}{\rightrightarrows}} Y$, inducing $H^*(X) \overset{f_0^*}{\underset{f_1^*}{\leftleftarrows}} H^*(Y)$. If $f_0 \simeq f_1$, $f_0^* = f_1^*$. We proved this for the compact case; for others, say singular cohomology, see the handouts. Proof is an application of double complexes.

G. Hirsch: Suppose $p: Y \to X$ is a locally trivial fibration. Suppose

(i) $H^q(Y_x)$ is a finitely generated free abelian group for each x. $(Y_x = p^{-1}(x))$

(ii) $\exists$ elements $\{\eta_i\}$ in $H^*(Y)$ which restrict to a basis of $H^*(Y_x)$ for each x.

Then $H^*(Y)$ is a free module over $H^*(X)$ with basis $\{\eta_i\}$.

$p^*: H^*(X) \to H^*(Y)$ is a ring homomorphism.

Application 1: Künneth Theorem. Suppose $Y = X \times Z$, p = projection to X. Suppose each $H^q(Z)$ is finitely generated and free with basis $\{e_i\}$. Then take $\eta_i = q^* e_i$, where $q: X \times Z \to Z$. $H^*(X) \otimes H^*(Z) \cong H^*(X \times Z)$.

$$\sum_i p^*(e_i)\,\eta_i.$$

Application 2: Suppose $E \to X$ is a complex vector bundle. Let $Y = \mathbb{P}(E)$, ie $Y_x = \mathbb{P}(E_x)$. There is an obvious tautological line bundle on Y whose fibre at $L \subset E_x$ is L.

We know that $H^*(\mathbb{P}(E_x))$ is $\mathbb{Z} \oplus \mathbb{Z}e \oplus \mathbb{Z}e^2 \oplus \cdots \oplus \mathbb{Z}e^{n-1}$, where $e \in H^2(\mathbb{P}(E_x))$ is the Euler class of the tautological line bundle.

But clearly e is the restriction of $e_L \in H^2(Y)$.

Similarly e^k is the restriction of e_L^k.

So $H^*(Y) =$ free $H^*(X)$-module with basis $1, e_L, \ldots, e_L^{n-1}$.

ie, $H^*(Y) = H^*(X)[e_L]/(\text{relations})$, with relations $e_L^n - c_1 e_L^{n-1} + \cdots + (-1)^n c_n = 0$, with $c_i \in H^*(X)$, in fact, $c_i \in H^{2i}(X)$.

This $c_i = c_i(E)$ is the ith Chern class - "Grothendieck definition of c_i".

Proof of Hirsch: Let $\mathcal{U}$ be a contractible open covering of X, $\mathcal{U} = \{U_\alpha\}$, such that $Y|_{U_\alpha}$ is trivial for each α. Consider two double complexes,

$C^{\cdot\cdot}$, $C^{pq} = \prod_{\alpha_0, \ldots, \alpha_p} C^q(U_{\alpha_0 \ldots \alpha_p})$

$\tilde{C}^{\cdot\cdot}$, $\tilde{C}^{pq} = \prod C^q(p^{-1}U_{\alpha_0 \ldots \alpha_p})$.

We have a map of double complexes, $\underbrace{C^{\cdot\cdot} \oplus \cdots \oplus C^{\cdot\cdot}}_{\text{one copy for each } \eta_i} \longrightarrow \tilde{C}^{\cdot\cdot}$

$\{c_i\} \longmapsto \sum p^*(c_i)\hat{\eta}_i$,

where $\hat{\eta}_i$ is a cocycle representing η_i.

Claim that if  then the map is an isomorphism.

cut off to get finite columns - use induction.

A a ring. M a right A-module, N a left A-module. Can form $M \otimes_A N$.

Define Torsion Product, $\mathrm{Tor}_i^A(M,N)$, such that $\mathrm{Tor}_0^A(M,N) = M \otimes_A N$, as follows.

$N \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots$ - exact sequence. By tensoring, form

$M \otimes F_0 \leftarrow M \otimes F_1 \leftarrow \cdots$

Let $\mathrm{Tor}_i^A(M,N) = H_i(M \otimes F_\cdot)$. Does this depend on F_i?

Suppose we had $M \leftarrow \Phi_0 \leftarrow \Phi_1 \leftarrow \cdots$

Can form double complex $\{\Phi_p \otimes F_q\}$. $H^*(\Phi_\cdot \otimes N)$ is an alternative definition.

Homology: If we have a vector space over a field, $H_p(X)^* = H^p(X)$.

In general, $H_p(X) \cong H^{n-p}_{cpt}(X)$. (See question sheet 4)

As a hint, show $\exists$ exact sequence $C^\cdot_{cpt}(X) \leftarrow \bigoplus_\alpha C^\cdot_{cpt}(U_\alpha) \leftarrow \bigoplus_{\alpha,\beta} C^\cdot_{cpt}(U_{\alpha\beta}) \leftarrow \cdots$

This forms a double complex. We get a map ∂, going the other way. We get homology.