Basic Algebraic Geometry, Michelmas 1996

ALESSIO CORTI

Problem Sheet 1

1) Let R be an integral domain. Prove that the ring homomorphism

$$\psi : R[x_1, ..., x_n] \to \{\text{functions } f : R^n \to R\}$$

(sending a polynomial to the corresponding polynomial function) is injective if and only if R is infinite.

2) The linear system of conics passing through 4 points is 1-dimensional unless the 4 points all lie on a line.

3) Let C, D be 2 plane cubics intersecting in 9 distinct points. Assume that 3 of these points lie on a line L. Conclude that the remaining 6 points of intersection lie on a conic [Hint: 1 curve in the linear system $\lambda C + \mu D$ contains the line L].

Generalize to 2 curves of degree n.

4) A smooth cubic curve has 9 distinct flexes and every line containing 2 flexes must also contain a 3rd. In suitable coordinates the 9 flexes are $(0, 1, -1)$, $(-1, 0, 1)$, $(1, -1, 0)$, $(0, 1, \alpha)$, $(\alpha, 0, 1)$, $(1, \alpha, 0)$, $(0, 1, \beta)$, $(\beta, 0, 1)$, $(1, \beta, 0)$ where α and β are the 2 solutions of $x^2 - x + 1 = 0$. Any cubic containing those 9 points has the form

$$x_0^3 + x_1^3 + x_2^3 + 3mx_0x_1x_2 = 0$$

5) Consider the plane 4ic:

$$\lambda_0(x_0^2x_1x_2 + x_0x_1^2x_2 + x_0x_1x_2^2) + \lambda_1(x_0^2x_1^2 + x_0^2x_2^2 + x_1^2x_2^2) = 0$$

For general values of the ratio $\lambda_0 : \lambda_1$ the 4ic is reduced and irreducible (prove it!). Find an explicit rational parametrization.

6) Prove that if $u, v \in \mathbb{Z}$: $u^2 + v^2$, $u^2 - v^2$ both squares implies $v = 0$. Try hard to do this yourself, then look at the hints in UAG, pg. 41–42.

7) Let

$$C = (y^4 - y^2 + x^4 = 0)$$
$$D = (y^4 - 2y^3 + (1 - x)y^2 - 2x^2y + x^4 = 0)$$

Compute the intersection points of the 2 curves, find local parametrisations for the 2 curves at those points, compute intersection multiplicities both using the resultant and the local parametrisations.

8) Let C be a plane curve.

Let $m(C)$ be the number of tangent lines to C passing through a point $q \in \mathbb{P}^2$, counted with multiplicity, and $i(C)$ the number of inflectional points of C, also counted with multiplicity.

An ordinary node is a singular point $p \in C$ of multiplicity 2 and such that C has 2 distinct tangent lines at p (make sense of these definitions).

If C has degree n and only ordinary nodes as singularities, prove that:

$$m = n(n - 1) - 2\delta$$
$$i = 3n(n - 2) - 6\delta$$

where $\delta(C)$ is the number of these nodes.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Basic Algebraic Geometry, Michelmas 1996

Alessio Corti

Problem Sheet 2

1) Let X, Y be topological spaces and $f : X \to Y$ a continuous map.
If $\mathcal{F}$ is a sheaf on X, define the sheaf $f_\bullet \mathcal{F}$ on Y.
Similarly, for a sheaf $\mathcal{G}$ on Y, define the sheaf $f^\bullet \mathcal{G}$ on X.
Prove the formula:

$$Hom_X(f^\bullet \mathcal{G}, \mathcal{F}) = Hom_Y(\mathcal{G}, f_\bullet \mathcal{F})$$

2) Let $(X, \mathcal{O}_X)$ be an algebraic prevariety. Recall that an open subprevariety of X is a (Zariski) open subset $U \subset X$ with the sheaf of functions:

$$\mathcal{O}_U := \mathcal{O}_X|U$$

[recall that, for any sheaf $\mathcal{F}$ on X, and denoting $j : U \hookrightarrow X$ the inclusion, we use the notation

$$\mathcal{F}|U := j^\bullet \mathcal{F}$$

The reason for this is that $j^\bullet$ for the inclusion $j : U \hookrightarrow X$ of an open subset is much easier than $f^\bullet$ for an arbitrary continuous map f.]

Given a finite collection X_α of algebraic prevarieties, and open subprevarieties $X_{\alpha\beta} \subset X_\alpha$ and *isomorphisms*:

$$\psi_{\alpha\beta} : X_{\alpha\beta} \to X_{\beta\alpha}$$

satisfying

$$\psi_{\alpha\gamma} = \psi_{\beta\gamma} \circ \psi_{\alpha\beta}$$

(whenever both sides are defined), construct an algebraic prevariety gluing the X_α. Check that the ensuing object is an algebraic prevariety as pedantically as you can at the same time using up no more than 5 handwritten pages.

3) Prove that $\mathbb{A}^2 \setminus \{0, 0\}$ is not (isomorphic to) an affine variety.

4) Let X be an algebraic variety, U and V open subvarieties. Assume that U and V are affine (i.e., isomorphic to affine varieties). Prove that $U \cap V$ is also affine [hint: if $i : U \subset X$, $j : V \subset X$ are the inclusions and $(i, j) : U \times V \to X \times X$ is their product, $U \cap V = (i, j)^{-1}\Delta$]. Show by example that the statement is wrong if X is a prevariety.

5) In class we discussed the following:

Theorem. *Let $f : X \to Y$ be a dominating morphism of varieties and $r = dim X - dim Y$. For all irreducible closed subsets $W \subset Y$ and all components Z of $f^{-1}W$ dominating W:*

$$dim Z \geq dim W + r$$

Show by example that the statement is false if $Z \to W$ is not dominating.

6) Prove that the product of 2 projective varieties is again a projective variety [hint: it is enough to prove that $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety. Think of mapping $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$ via $(x_i; y_j) \to (x_i y_j)$]. Conclude that a projective "variety" is a variety.

7) Do problem 5.12 on page 92 of Reid's UAG, following the hints given there. When you are done with it, remember that projective varieties are proper and appreciate.

8) Do problem 5.13 on page 93 of Reid's UAG.

9) Let X be an algebraic variety, $U \subset X$ an open subvariety. If U is proper, $U = X$.

Basic Algebraic Geometry, Michelmas 1996

Alessio Corti

Problem Sheet 3–4

1) Let $C \subset \mathbb{P}^2$ be a plane curve. A line L is a *bitangent* to C iff L is tangent to C at exactly 2 (distinct) points. Show that if $\deg C \geq 4$, C has a finite > 0 number of bitangents [hint: a constant count. Keep in mind the Fermat curve $x^m + y^m = z^m = 0$]. Show that a "general" curve of any degree has no tritangents.

2) Show that a general surface $S \subset \mathbb{P}^3$ of degree $m \geq 4$ contains no lines.

3) Do all the problems to §7 in UAG [pg. 111–113].

4) Consider 6 points $p_i \in \mathbb{P}^2$, $i \in \{1, 2, 3, 4, 5, 6\}$ in general position. By this we mean that no 3 of the points lie on a line, no 6 of them on a conic. Let S be the surface obtained blowing up the 6 points on $\mathbb{P}^2$. Let $H \in \mathbb{P}^2$ be a cubic curve passing through the 6 points, $H' \in S$ the strict transform. Prove that the linear system $|H'|$ defines an embedding $S \subset \mathbb{P}^3$, and S is a (smooth) cubic surface. Prove that every smooth cubic surface in $\mathbb{P}^3$ arises in this way [hint: use UAG 7.4].

5) Let $C \in \mathbb{P}^2$ be a smooth plane curve. What is the genus $g(C)$ of C? [there are great many ways to do this, many of which probably accessible to your imagination. It would spoil your fun if I were to give you any hints at this point].

6) Let $C \in \mathbb{P}^2$ be a plane curve of degree d and $C^* \subset \mathbb{P}^{2*}$ the *dual* curve (by definition this is the locus of tangent lines to C. Prove that C^* is an algebraic curve and

$$C \ni p \to T_pC \in C^*$$

a morphism.

a) Let $L \in \mathbb{P}^2$ be a line, not tangent to C. Define $\varphi : C \to L$ mapping $p \to T_pC \cap L$. Show that φ is ramified at p iff $p \in L$ or $p \in C$ is a flex.

b) If L is tangent to C at $p_1, \ldots, p_r$ and none of the p_i is a flex, then $L \in C^*$ is an ordinary r-fold point (i.e., by definition, a point of multiplicity r with r distinct tangents, and in particular r distinct smooth branches).

c) Let $p \in \mathbb{P}^2$ be a point not lying on C nor on any inflectional or multiple tangent to C, L a line not containing p, $\varphi : C \to L$ the projection from p. Use Hurwitz's formula to compute the degree of C^* (you should get $d(d-1)$).

d) A sufficiently general point $p \in C$ lies on $(d+1)(d-2)$ tangents of C (not counting the tangent at p.

e) Calculate the degree of the morphism φ in a) and use Hurwitz to count the flexes of C.

f) Assume that C^* has only ordinary nodes and cusps as singularities (this is true for sufficiently general C). Show that C has

$$\frac{1}{2}d(d-2)(d-3)(d+3)$$

bitangents [this may be quite hard, but should be fun to try]. In particular a plane quartic has 28 bitangents. Do these have anything to do with the 27 lines on a cubic surface? [hint: 28=27+1. A more constructive hint would be to choose a point $p \in S$ on the cubic surface and project down to $\mathbb{P}^2$. This is a finite morphism of degree 2, branched along a 4-ic in $\mathbb{P}^2$. Try and see where do the 27 lines go...]

7) Prove that a smooth algebraic curve C over a field of characteristic zero has at most $84(g-2)$ automorphisms. The idea is to show that C/G is a smooth curve, and apply the Hurwitz formula to the map

$$f : C \to C/G$$

8) Let X be a proper and smooth algebraic surface. Define a suitable *intersection product*:

$$ClX \times ClX \to \mathbb{Z}$$

on the class group of X, by generalising intersections of curves in $\mathbb{P}^2$ (don't be afraid, you can do it!). Then prove the Riemann-Roch theorem on X:

$$\chi\mathcal{L}(D) = \frac{1}{2}D \cdot (D - K) + 1 + p_a$$

where $K := \wedge^2\Omega$ is the canonical line bundle and $p_a := h^2\mathcal{O} - h^1\mathcal{O}$ [hint: generalise the proof for curves. Don't be afraid, you can do this one too].

Now go and read Beauville's "Complex Algebraic Surfaces", Chapter I, where the intersection product is *defined* via the Riemann-Roch formula!