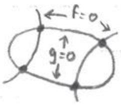
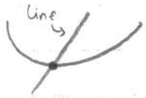
Algebraic Geometry.

Let k be a field. Take the affine space $\mathbb{A}^n_k = k^n$. (Often, $k = \mathbb{C}$)
Consider $\mathbb{A}^2_k$. An affine plane curve is a set $C = \{f(x,y) = 0\} \subset \mathbb{A}^2_k$, where $f \in K[X,Y]$. C has degree n if f has degree n.
A curve $D = \{f=0\}$ is a component of $C = \{g=0\}$ if $D \subset C$.

Theorem: Let $C, D \subset \mathbb{A}^2_k$, of degrees n, m, with $k = \bar{k}$. If C, D have $> mn$ points in common, then they have a component in common.

Example: $\begin{cases} f_2(x,y) = 0 \\ g_2(x,y) = 0 \end{cases}$  ,  - one intersection at ∞.

Let R be a ring. Let $f, g \in R[X]$, of degrees n, m. Say, $f(x) = a_0 + a_1X + \dots + a_nX^n$, $g(x) = b_0 + b_1X + \dots + b_mX^m$. Consider the $(n+m) \times (n+m)$ matrix:

$$A(f,g) = \left(\begin{array}{cccccccc} 0 & \cdots & 0 & a_0 & a_1 & \cdots & & a_n \\ & & & & & & & 0 \\ 0 & a_0 & a_1 & \cdots & a_n & & & \vdots \\ a_0 & a_1 & \cdots & a_n & 0 & \cdots & & 0 \\ b_0 & b_1 & \cdots & b_m & 0 & \cdots & & 0 \\ 0 & b_0 & b_1 & \cdots & b_m & & & \vdots \\ \vdots & & & & & & & 0 \\ 0 & \cdots & 0 & b_0 & b_1 & \cdots & & b_m \end{array} \right) \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} m \\ \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} n \end{array}$$

Definition: The k-resultant $r_k(f,g)$ is the determinant of the matrix obtained by deleting the first and last $k-1$ rows and columns of A. $[r_1(f,g) = r(f,g)]$.

Theorem: $f, g \in K[X]$ have a common divisor of degree $\geq k \iff r_i(f,g) = 0$, $1 \leq i \leq k$.

Proof: Remark: f, g have a common divisor of degree $\geq k \iff$ they have a common multiple of degree $\leq m+n-k$ ($f = \alpha\gamma$, $g = \beta\gamma \Rightarrow f\beta = g\alpha$)

We express $f\psi = g\varphi$ in terms of linear dependence in V_{n+m-k} = vector space of polynomials of degree $\leq n+m-k$.

$k=1$: $\psi = \sum_{i=0}^{m-1} \lambda_i x^i$, $\varphi = \sum_{j=0}^{n-1} \mu_j x^j$. Then, $\exists\, \varphi, \psi$ with $f\psi = g\varphi \iff$ the vectors $X^{m-1}f, \dots, Xf, f, g, Xg, \dots, X^{n-1}g$ are linearly dependent in V_{n+m-1}.

$k > 1$: $(\Rightarrow)$ exercise

$(\Leftarrow)$ By induction on k. Assume $r_1 = \dots = r_{k+1} = 0$. Let B be the matrix obtained by deleting the first and last k rows of A.
Since the last k columns of B are 0 and $r_{k+1} = 0$, there $\exists$ a linear combination of rows of B, having all zeroes except the first k entries $\Rightarrow \exists\, \varphi, \psi$ of degrees $\leq m-k-1$, $n-k-1$ such that $\deg(\psi f + \varphi g) < k$.
By induction, $f = \gamma\alpha$, $g = \gamma\beta$, $\deg\gamma \geq k$. So $\psi f + \varphi g = \gamma(\psi\alpha + \varphi\beta)$
$\Rightarrow \psi\alpha + \varphi\beta = 0 \Rightarrow \psi f + \varphi g = 0$.

Remark: The same statement holds for $f, g \in R[X]$ if R is a UFD.

Define <u>projective n-space</u>, $\mathbb{P}^n_k = (K^{n+1} - \{0\})/K^\times$, where $K^\times \ni \lambda : (a_0,..,a_n) \mapsto (\lambda a_0,.., \lambda a_n)$.

<u>Definition</u>: A <u>(projective) plane curve</u> is $C = \{ F(x_0, x_1, x_2) = 0 \}$ where $F(x_0, x_1, x_2) \in k[x_0, x_1, x_2]$ is a <u>homogeneous</u> polynomial.

<u>Remark</u>: F is homogeneous of degree n $\Leftrightarrow F(\lambda a) = \lambda^n F(a)$ for all $\lambda \in K^\times$, $a \in K^{n+1}$
$\Rightarrow \{F = 0\}$ is well-defined $\subset \mathbb{P}^2_k$.

Can think of $\mathbb{A}^2_k \hookrightarrow \mathbb{P}^2_k$ by identifying $\mathbb{A}^2$ with $\{x_0 \neq 0\}$, and $(x,y) \longmapsto (1; x; y)$, with 'inverse' $(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}) \leftarrow\!\dashv (x_0 : x_1 : x_2)$.
There is a correspondance: $\{f \in k[x,y]\} \longleftrightarrow \{F \in k[x_0, x_1, x_2], \text{ homogeneous}\}$ via
$f(x,y) \longmapsto F(x_0, x_1, x_2) = x_0^{\deg f} \cdot f(\frac{x_1}{x_0}, \frac{x_2}{x_0})$, and $F(x_0, x_1, x_2) \mapsto f(x,y) = F(1, x, y)$.

If $C = \{ f(x,y) = 0 \} \subset \mathbb{A}^2_k$, then $C = \{ F(x_0, x_1, x_2) = 0 \} \cap \mathbb{A}^2_k$.

Recall earlier theorem on degrees of curves. We prove a modified version:

<u>Theorem</u>: Let $C = \{ F(x_0, x_1, x_2) = 0 \}$, $D = \{ G(x_0, x_1, x_2) = 0 \} \subset \mathbb{P}^2_k$ be plane curves, and $n = \deg F$, $m = \deg G$. Then, $\# C \cap D > mn \Rightarrow C, D$ have a common component. (<u>Bezout's Theorem</u> - weak form).
<u>Proof</u>: Choose a coordinate system on $\mathbb{P}^2_k$ such that (i) $(0:0:1) \notin C \cup D$,
(ii) all the ratios $x_0(P_i)/x_1(P_i)$ are distinct, where $P_1, \dots, P_{n+m-1} \in C \cap D$.
By (i), $F(x_0, x_1, x_2)$, $G(x_0, x_1, x_2)$ are monic in x_2.
Regard $F, G \in R[x_2]$, $R = k[x_0, x_1]$. Then, $r(x_0, x_1) = r_{x_2}(F, G) \in R$, is a homogeneous polynomial of degree nm with nm+1 roots.
$(x_0(P_i) : x_1(P_i)) \in \mathbb{P}^1_k \Rightarrow r(x_0, x_1) = 0 \Rightarrow$ by Remark on Theorem with $R = k[x_0, x_1]$, F, G have a common factor of degree ≥ 1 in x_2.
So C and D have a common component.

<u>Idea</u>: For $p \in \mathbb{P}^2$, define $(C.D)_p = \text{mult}_{(x_0(p) : x_1(p))} r_{x_2}(F, G)$, in any coordinate system satisfying (i), (ii) above. Then, $\sum_{p \in \mathbb{P}^2} (C.D)_p = nm$.

<u>Nullstellensatz</u>

Let $I \subset k[x_1,.., x_n]$, an ideal. Let $V(I) = \{ a \in \mathbb{A}^n_k : f(a) = 0 \ \forall f \in I \} \subset \mathbb{A}^n_k$, the variety of I
For $V = V(I)$, let $I(V) = \{ f \in k[x_1,.., x_n] : f(a) = 0 \ \forall a \in V \}$, the ideal of V.

<u>Example</u>: $C = \{ x^2 = 0 \} \subset \mathbb{A}^2_k$. $C = V(x^2)$. But $I(C) = (x)$.

<u>Theorem</u>: $I(V(I)) = \text{rad } I = \{ f : f^n \in I, \text{ some } n \}$ - Hilbert's <u>Nullstellensatz</u>.

Theorem (weak Nullstellensatz): If $V(I) = \emptyset$, $I = k[X_1,\dots,X_n]$, for $I \subset k[X_1,\dots,X_n]$, $k = \bar{k}$.

Special case: Take $f, g \in k[X,Y]$. $\{f=0\} \cap \{g=0\} = V(f,g) = \emptyset \Rightarrow (f,g) = 1$.

Theorem: $f, g \in R[X]$. $r = r_1(f,g) \in (f,g)$.
Proof: Let B be the adjoint of A. So, $BA = r.I_{n+m}$. Let $v = (\lambda_{m-1},\dots,\lambda_1,\lambda_0,\mu_0,\dots,\mu_{n-1})$ be the first row of B, and α_j, $j = 1,\dots,n+m$ be the columns of A.
So $v.\alpha_1 = r$, $v.\alpha_j = 0$, $j \geq 2$. $r = \sum_{j=1}^{n+m} v.\alpha_j X^j = (\lambda_0 a_0 + \mu_0 b_0) + (\lambda_1 a_0 + \lambda_0 a_1 + \mu_0 b_1 + \mu_1 b_0)X + \dots = (\sum \lambda_i X^i) f + (\sum \mu_j X^j) g \in (f,g)$.

So, in the special case above, $r(f,g) = \text{const.} > 0$, so $\text{const.} \in (f,g)$, so $1 \in (f,g)$

Let R be a ring, and $f_1,\dots,f_s \in R[X]$. When do the f_i's have a common divisor (deg ≥ 1).
Let $\tilde{R} = R[\lambda_1,\dots,\lambda_s,\mu_1,\dots,\mu_s]$; $F = \sum \lambda_i f_i$, $G = \sum \mu_i f_i$, $\in \tilde{R}[X]$. $\tilde{R} \ni r(F,G) = \sum_{I,J} r_{IJ} \lambda^I \mu^J$, $r_{IJ} \in R$.
Definition: (r_{IJ}) is the resultant system of $f_1,\dots,f_s$.

Theorem: If R is a UFD, then $f_1,\dots,f_s$ have a common divisor of degree ≥ 1 $\Leftrightarrow$ all $r_{IJ} = 0$.
Proof: ($\Rightarrow$) F, G have a common divisor $\Rightarrow 0 = r \in \tilde{R} \Rightarrow r_{IJ} = 0$.
($\Leftarrow$) $r = 0 \Rightarrow F, G$ have a common divisor $\varphi \in \tilde{R}[X]$. $\varphi \mid F$, but F has no μ's, so $\varphi \in R[\lambda_1,\dots,\lambda_s][X]$. $\varphi \mid G$, so $\varphi \in R[X]$.

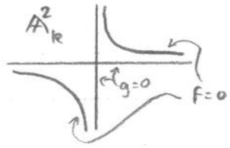
Theorem: $(r_{IJ}) \subset (f_1,\dots,f_s) \cap R$
Proof: We know that $r \in (F,G) \cap \tilde{R}$, ie, $\exists\, K, H \in \tilde{R}[X]$ such that $r = FK + GH$, say $K = \sum K_{IJ} \lambda^I \mu^J$, $H = \sum H_{IJ} \lambda^I \mu^J$. If $I = (i_1,\dots,i_s)$, let $I_h = (i_1,\dots,i_h - 1,\dots,i_s)$, and $J = (j_1,\dots,j_s)$, let $J_t = (j_1,\dots,j_t - 1,\dots,j_s)$
$\sum r_{IJ} \lambda^I \mu^J = \sum_{\substack{I,J \\ h,t}} (K_{I_h J} f_h + H_{I J_t} f_t) \lambda^I \mu^J$.

Lemma: Let K be an infinite field, $f \in k[X_1,\dots,X_n]$ of degree d. There exists a linear change of coordinates, $X = AY$, $A \in GL(n,k)$ such that $f(AY)$ has degree d in Y_n.
Proof: Write $f = \sum_{m \leq d} f_m$, with f_m homogeneous of degree d. Then, f has degree d in $X_n \Leftrightarrow f_d(0,\dots,0,1) \neq 0$. $f_d \not\equiv 0$, so $\exists\, (a_1,\dots,a_n) \in k^n$ such that $f_d(a) = 0$. Complete the vector a to a basis of k^n.

Exercise: Let R be an integral domain. The natural map $\varphi: R[X_1,\dots,X_n] \to (f: R^n \to R)$ is injective $\Leftrightarrow$ R is infinite.

Proof of weak Nullstellensatz: By induction on n. Let $I = (f_1,\dots,f_s)$ with $d = \deg f_1 \geq \deg f_i$. Change coordinates so that f_1 has degree d in X_n. Let $R = k[X_1,\dots,X_{n-1}]$, and regard the $f_i \in R[X_n]$. Let $(r_{IJ}) \in R$ be the resultant system of the f_i's.
Suppose $\exists\, b \in k^{n-1}$, $b \in V(r_{IJ})$. Evaluating each r_{IJ} at b gives 0, and evaluating each f_i at b gives polynomials in $k[X_n]$, say $\tilde{f}_i$. Now, $(\tilde{f}_1,\dots,\tilde{f}_s) \neq 0$, as f_1 has degree d in X_n, and the resultant system of $\{\tilde{f}_i\}$ vanishes, so $\exists\, g \in k[X_n]$, $g \mid \tilde{f}_i \ \forall i$. Let α be a root of g ($k = \bar{k}$, so α must exist). Then $f_i(b,\alpha) = 0 \ \forall i$, and so $(b,\alpha) \in V(I)$. #
So $V(r_{IJ}) = \emptyset$, thus $1 \in (r_{IJ})$, by induction. Hence, by above, $1 \in I$, so $I = (1)$

Example: $f = xy + 1$, $g = xy$



Consider $f, g \in R[y]$, $R = k[x]$.
$res_y(f, g) = \det\begin{vmatrix} 1 & x \\ 0 & x \end{vmatrix} = x$.

Proof of Nullstellensatz: $I = (f_1, .., f_s) \subset k[X_1, .., X_n]$; let $f \in I(V(I))$.

$J = (f_1, .., f_s, X_{n+1} f - 1) \subset k[X_1, .., X_{n+1}]$.

Claim: $V(J) = \phi$. $\Rightarrow 1 \in J$, by weak Nullstellensatz.

So, $\exists\ \sum g_{ij} X_{n+1}^j$, $\sum f_j X_{n+1}^j$ such that $\sum g_{ij} f_i X^j + \sum f_j X_{n+1}^j (X_{n+1} f - 1) = 1$.

Can regard this as an identity in $k(X_1, .., X_n)[X_{n+1}]$.

Thus, set $X_{n+1} = 1/f$, get $1 = \sum_{i,j=0}^{d} g_{ij} f_i / f^j \Rightarrow f^d \in I$.

Plane Curves. (k an algebraically closed field).

Definition: $C = \{F(X_0, X_1, X_2) = 0\} \subset \mathbb{P}^2$, F homogeneous, $F = \prod F_i^{n_i}$ with F_i homogeneous and prime. C is reduced if $n_i = 1$, and irreducible if only 1 of the n_i's is non-zero.

Can write $C = \sum n_i C_i$; $C_i = \{F_i = 0\}$ are the irreducible components of C, with multiplicities n_i.

Let $C = C_n$ be a curve of degree n, $L \subset \mathbb{P}^2$ a line, then: $n = \sum_{p \in \mathbb{P}^2} (L.C)_p$. If $p \in L \cap C$, choose $Q \in L$, then parametrize L by $t \to P + tQ$, then define $(L.C)_p = ord_{t=0} F(P + tQ)$.
If $P \in C$, $\mu_p C = \min_{L \ni p} (L.C)_p =$ degree of first nonvanishing homogeneous term in Taylor expansion of F at p. [Ie, $F(P+X) = \sum_{m=\mu_p C}^{n} F_m(P, X)$, F_m homogeneous, degree m in X].

Definition: A point $P \in C$ is nonsingular iff $\mu = 1$, and singular iff $\mu > 1$.

Convention: When studying $C \cap D$, we assume to be in a coordinate system such that: (i) $(0,0,1) \notin C \cup D$, (ii) if $p \neq q \in C \cap D$, $(X_0(p) : X_1(p)) \neq (X_0(q) : X_1(q))$.

Then define $(C.D)_p = ord_{(X_0(P) : X_1(P))} r_{X_2}(F, G)$, where $C = \{F = 0\}$, $D = \{G = 0\}$, degrees n, m.
Note: If C, D have no common component, $n.m = \sum_{p \in \mathbb{P}^2} (C.D)_p$.
(We will prove that $(C.D)_p$ is independent of coordinate choice, but we do not logically need this yet).

Lemma: If $\mu_p C \geqslant a$, $\mu_p D \geqslant b$, then $(C.D)_p \geqslant ab$.
Proof: Assume $P = (1, 0, 0)$. Set $x = \frac{X_1}{X_0}$; $y = \frac{X_2}{X_0}$. $f(x, y) = y^n + \cdots + y x^{a-1} P_1(x) + x^a P_0(x)$,

$g(x, y) = y^m + \cdots + y x^{b-1} Q_1(x) + x^b Q(x)$.

$$r_y = \det \begin{pmatrix} O & \cdots & \cdots \\ x^a P_0 & x^{a-1} P_1 & \cdots \\ x^b Q_0 & x^{b-1} Q_1 & \cdots \\ O & \cdots & \end{pmatrix} \begin{matrix} x^{b-1} \\ x^b \\ x^a \\ x^{a-1} \end{matrix}$$

$\leftarrow$ (multiply rows by this). Need $x^{ab} \mid r_y$.

Multiplying the rows as indicated gives new det $= x^{\frac{b(b+1)}{2} + \frac{a(a+1)}{2}} . r$
$= x^{(a+b)(a+b+1)/2}$ (something) $\Rightarrow x^{ab} \mid r$.

What if $P \neq (1,0,0)$? Hint: do a translation $T: P \to (1,0,0)$. Need to prove that $(C.D)_P$ is invariant under translation.

Theorem: $C = \{F_n = 0\} \subset \mathbb{P}^2$

(i) If C is reduced, then $\sum_{P \in C} \mu_P C(\mu_P C - 1) \leq n(n-1)$.

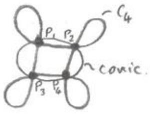
(ii) If C is reduced and irreducible, then $\sum_{P \in C} \mu_P C(\mu_P C - 1) \leq (n-1)(n-2)$.

$\{C$, curve of degree n, $\subset \mathbb{P}^2\} \leftrightarrow \{F_n(x_0, x_1, x_2)$, homogeneous, degree $n\}/K^* = \mathbb{P}^{\binom{n+2}{2}-1} = \mathbb{P}^{\frac{n(n+3)}{2}}$.

A linear system of curves of degree n is a projective subspace $\Lambda \subset \mathbb{P}^{\frac{n(n+3)}{2}}$.

Eg, $\{$conics (degree 2)$\} \leftrightarrow \mathbb{P}^5$, $\{$curves of degree 3$\} \leftrightarrow \mathbb{P}^9$.

Let $C = C_4$, reduced, with 4 singular points. This implies C is reducible.



Let $P_1, \dots, P_m \in \mathbb{P}^2$ points, and $\mu_1, \dots, \mu_m \geq 1$ multiplicities.

$\Lambda = \{C = C_n : \mu_{P_i}(C) \geq \mu_i\}$. $\dim \Lambda \geq \frac{n(n+3)}{2} - \sum \frac{\mu_i(\mu_i+1)}{2}$.

$\{$conics through $P_1, \dots, P_4\}$ has dimension ≥ 1. Choose $q \in C_4$, $q \neq P_1, \dots, P_4$.

Let Q be a conic through $P_1, \dots, P_4$ and q.

If Q, C_4 have no component in common, then $8 = 4 \cdot 2 = \sum_{P \in \mathbb{P}^2} (Q.C)_P \geq 8 + 1$.

$\Rightarrow Q, C$ have a common component.

Proof of Theorem above: (i) Let $F_{x_2} = \frac{\partial F}{\partial x_2}$. F_{x_2} is homogeneous of degree $n-1$.

Notice: (a) F, F_{x_2} have no common component, as C is reduced.

(b) For any $p \in \mathbb{P}^2$, $\mu_p(F_{x_2} = 0) \geq \mu_p(C) - 1$.

Let $D = \{F_{x_2} = 0\}$. So, $n(n-1) = C.D = \sum_p (C.D)_p \geq \sum \mu_i(\mu_i - 1)$.

Example: $C =$  $\}$ n lines. For this curve, $n(n-1) = \sum \mu_i(\mu_i - 1)$.

Question: Is this the only example?

Proof continued: (ii) Let C_n be a curve of degree n, with multiplicities μ_i at P_i.

Let $\Lambda = \{C_{n-2} : \mu_{P_i}(C_{n-2}) \geq \mu_i - 1\}$. $\dim \Lambda \geq \frac{(n-2)(n+1)}{2} - \sum \frac{\mu_i(\mu_i-1)}{2} = \frac{(n-2)(n-1)}{2} - \sum \frac{\mu_i(\mu_i-1)}{2} + n - 2$.

Assume that $(n-2)(n-1) - \sum \mu_i(\mu_i - 1) = -2k$.

Choose additional points $q_1, \dots, q_{n-k-2} \in C_n$. $\exists C_{n-2}$ such that $\mu_{P_i}(C_{n-2}) \geq \mu_i - 1$, and $q_i \in C_{n-2}$ also.

$n(n-2) = C_n \cdot C_{n-2} \geq \sum \mu_i(\mu_i - 1) + n - k - 2 = (n-1)(n-2) + 2k + n - k - 2 = n(n-2) + k$.

Contradiction if $k \geq 1$. (Proof valid for k not too large, eg. $k \geq n-2$)

Definition: Let $C = \{f(x,y) = 0\} \subset \mathbb{A}^2_k$ be reduced and irreducible. C is rational if $\exists\, \varphi(t), \psi(t)$ such that:

(i) $\mathbb{A}^1_k \ni t \to (\varphi(t), \psi(t)) \in \mathbb{A}^2_k$ is injective on $\mathbb{A}^1 - \{$finite set$\}$

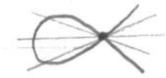
(ii) $f(\varphi(t), \psi(t)) = 0$.

Theorem: Suppose C is reduced, irreducible. Then $\sum \mu_i(\mu_i - 1) = (n-1)(n-2) \Rightarrow C$ rational.

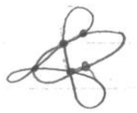
Remark: $\Leftrightarrow$ actually holds.

Examples: (i) Conic:  $t =$ slope of these lines.

(ii) Find all solutions to $X^2 + Y^2 = Z^2$ in $\mathbb{Q}$.

(iii) Cubics. $y^2 = x^3$:  $\}\ m = t$ $\quad$ $y^2 = x^2(x+1)$: 

(iv) C_4: quartic with 3 double points.



Number of intersections is 8, for conic and quartic. So a conic through the three double points also passes through two other points in C_4. Choose $q \in C_4$, a random point.

$\Lambda = \{$ conics Q containing p_1, p_2, p_3 and $q \} = \mathbb{P}^1$.

$\mathbb{P}^1 \ni \lambda \mapsto$ the unaccounted intersection $Q_\lambda \cap C$.

Exercise: $a(X_0^2 X_1 X_2 + X_0 X_1^2 X_2 + X_0 X_1 X_2^2) + b(X_0^2 X_1^2 + X_0^2 X_2^2 + X_1^2 X_2^2) = 0$.
Find a rational parametrization for this.

Proof of theorem: As in proof of (ii) in previous theorem, Λ has dimension $n-2$, as $k=0$. Choose $q_1, \dots, q_{n-3} \in C_n$. $\{ C_{n-2} : \mu_{p_i}(C_{n-2}) \geq \mu_i - 1,\ q_i \in C_n \} = \mathbb{P}^1$.
Use this to parametrise the curve.

Smooth Plane Cubics.

Theorem 1: No smooth cubic $C_3 \subset \mathbb{P}^2_k$ is rational.

Theorem 2: Assume $K = \bar{K}$, char $K \neq 2, 3$. Let $C \subset \mathbb{P}^2_k$ be a smooth cubic. In suitable coordinates, $C = \{F = 0\}$, for $F = X_0 X_2^2 + X_1(X_1 - X_0)(X_1 - \lambda X_0)$

$f(x,y) = y^2 + x(x-1)(x-\lambda)$, $\quad \lambda \in K$, $\lambda \neq 0, 1$.

Proof of 1: (Assume 2). Let $\varphi = p/q \in K(t)$, $p, q \in K[t]$, $(p,q) = 1$ and $\psi = r/s \in K(t)$, $(r,s) = 1$.
Assume $\psi^2 = \varphi(\varphi - 1)(\varphi - \lambda)$, in $K(t)$. We will show $\varphi, \psi \in K$.
So, $q^3 r^2 = s^2 . p(p-q)(p-\lambda q) \in K[t]$. As $(r,s) = 1$, we have $s^2 \mid q^3$. This implies that q is a square in $K[t]$. For, $q^3 \mid s^2$ as well, since $q^3 \mid s^2 p(p-q)(p-\lambda q)$, but $(p,q) = 1$. So we actually have $q^3 = \alpha s^2$, $\alpha \in K^*$.
So we get: $\alpha r^2 = p(p-q)(p-\lambda q) \Rightarrow p, p-q, p-\lambda q$ are all squares in $K[t]$, as p, q are coprime. We are then done, by the following lemma.

Lemma: $p, q \in K[t]$, $(p,q)=1$. Assume $\alpha p+\beta q \in K[t]$ is a square for 4 distinct ratios $(\alpha:\beta) \in \mathbb{P}^1_K$. Then $p, q \in K$.

Proof: Changing coordinates, ratios are $(1,0)$, $(0,1)$, $(1,-1)$, $(1,-\lambda)$. Then, p, q, $p-q$, $p-\lambda q$ are all squares, say $p=a^2$, $q=b^2$. Then $p-q=(a+b)(a-b)$, $p-\lambda q=(a+\mu b)(a-\mu b)$, both of which are squares. a and b are coprime, as p and q are. So, $a+b$, $a-b$, $a+\mu b$, $a-\mu b$ are all squares. But these are of lower degree, so continuing would lead to a contradiction, unless $p, q \in K$.

Definition: Let $C \subset \mathbb{P}^2_K$ be a plane curve. A smooth point p is inflectional iff $(C \cdot T_p C)_p \geqslant 3$.

Theorem 3: Assume $\operatorname{char} K \neq 2$. Let $p \in C_n$ be a smooth point, for $C_n = \{F=0\}$.
(i) If $\operatorname{char} K \nmid n-1$, then p is inflectional $\Leftrightarrow$ $H(x) = \det(F_{x_i x_j})$ vanishes at p.
(ii) If $\operatorname{char} K \mid n-1$, as in (i), but with $\Rightarrow$ only.

Proof of 2: (Assume 3). C has an inflectional point (as $H(x)$ has degree 3, so $\{H=0\}$ and $\{F=0\}$ intersect). Choose coordinates so that $(0,0,1)$ is a flex, $T_{(0,0,1)} C = (x_0=0)$, $C=(F=0)$. Then $F(0,x_1,x_2) = x_1^3$, so $F = a_0 x_0^3 + x_0^2 \varphi_1(x_1,x_2) + x_0 \varphi_2(x_1,x_2) + x_1^3$. Assuming $(0,0,1)$ is a smooth point $\Rightarrow$ φ_2 has degree 2 in x_2.
In affine space, $f(x,y) = a_0 + a_1 x + b_1 y + a_2 x^2 + cxy + y^2 + x^3$.
Define $y' = y - \frac{c}{2}x - \frac{b_1}{2}$. New equation: $f' = y'^2 + a_0' + a_1' x + a_2' x^2 + x^3$.
Exercise: C non-singular $\Rightarrow$ $P(x) = x^3 + a_2' x^2 + a_1' x + a_0 = 0$ has 3 distinct solutions.
$\Rightarrow$ in suitable coordinates, $P' = x(x-1)(x-\lambda)$.

Proof of 3: Let $p \in C$ be a smooth point, $q \in \mathbb{P}^2$, $C=\{F=0\}$.
Then, $F(p+tq) = t\sum F_{x_i}(p) q_i + t^2 \sum F_{x_i x_j} q_i q_j + \text{h.o.t.}$ (as $F(p)=0$).
Let $T = \{\sum F_{x_i}(p) X_i = 0\}$, $Q = \{\sum F_{x_i x_j} X_i X_j = 0\}$. Then $p \in C$ is inflectional $\Leftrightarrow$ $T \subset Q$ $\Rightarrow$ $H=0$ at p, as the quadratic form must then be degenerate.
Assume $H=0$ at p.
Remarks: (i) $p \in Q$. Indeed, $\sum F_{x_i x_j}(p) p_i p_j = n(n-1)F(p) = 0$, using that if F is homogeneous of degree n, then $F = n\sum F_{x_i} x_i$.
(ii) $T = T_p Q$ (if $\operatorname{char} K \nmid n-1$). $T_p(Q) = \{\sum F_{x_i x_j}(p) p_i x_j = 0\}$ $= (n-1)\sum F_{x_j}(p) x_j$.

Theorem: (Weierstraß). $f \in k[[z_1,\dots,z_n]] =: R$, $s = \operatorname{ord} f$. Assume $f_s = z_n^s + \dots$. Then there exists a unique unit $u \in R^\times$ such that $u.f = z_n^s + \sum_{i=1}^{s} \varphi_i(z_1,\dots,z_{n-1}) z^{s-i}$, $\operatorname{ord}\varphi_i \geqslant i$, where $\varphi_i \in k[[z_1,\dots,z_{n-1}]]$. (Assume $\operatorname{char} k = 0$).

Remark: Same is true for $\mathbb{C}\{z_1,\dots,z_n\}$, power series convergent in some neighbourhood of the origin.

Proof: Have $R = k[[z_1,\dots,z_n]]$ Let $\tilde{R} = k[[z_1,\dots,z_n,z_n^{-1}]]$, $R' = k[[\zeta_1,\dots,\zeta_{n-1},z_n]]$, $z_n\zeta_i = z_i$.
So, $R \subset R' \subset \tilde{R}$. (Reason: $z_n^s \mid f(z_n\zeta_1,\dots,z_n\zeta_{n-1},z_n)$.) So $f = z_n^s(1+r)$, $r \in R' \subset \tilde{R}$.

Since $r \in \tilde{R}$, write $r = \sum_{i \in \mathbb{Z}} r_i z_n^i$, $r_i \in k[[z_1, \dots, z_{n-1}]]$.
Now, $\ln(1+r) = \sum \frac{(-1)^{i+1} r^i}{i} = \sum w_i z_n^i = w$. $f = z_n^s e^w = z_n^s e^{w_+ + w_-}$, $w_\pm = \sum_{\mathbb{Z}^\pm} w_i z_n^i$.
So, $e^{-w_+} f = z_n^s . e^{w_-}$. Then $u = e^{-w_+} \in R$ is a unit.
$e^{w_-} = 1 + \sum_{i>0} \varphi_i z_n^{-i}$. $uf = z_n^s + \sum_{i=1}^{s} \varphi_i z_n^{s-i}$, since the sum must end at s, as LHS $\in R$

Exercise: (i) Think of char $k = p$. - 'log' is not valid.
(ii) We have used $\mathbb{C}\{z_1, \dots, z_n\}$. Consider just $R[[z, 1/z]]$ - not necessarily a ring. (Eg: $(1 + z + z^2 + \cdots) \times (1 + z^{-1} + z^{-2} + \cdots) = (1 + 1 + 1 + \cdots) + \cdots$).
Maybe use $R[[z_1, \dots z_n]][1/z_n]$.

Theorem: Take $k = \bar{k}$. Write $k(x)^* = \bigcup_{n \geq 1} k((x^{1/n}))$ - fractional formal power series.
$k(x)^*$ is algebraically closed.

Will prove this for convergent fractional power series.

$C = (f(x,y) = 0)$, take $p \in C$. Assume C has no multiple components.
$O \in (y^s + \sum \varphi_i(x) y^{s-i} = 0)$, $A'_y \times \Delta_x$, φ_i holomorphic. Choose Δ_x small enough so that $(f_y = 0) \cap C = \phi$.
Then consider the function

$$\begin{array}{c} C - \{0,0\} \\ \downarrow x \\ \Delta_x - \{0\} \end{array}$$

This is a smooth covering map. Every connected holomorphic covering of $\Delta - 0$ is isomorphic to $\Delta - \{0\} \longrightarrow \Delta - \{0\}$
$t \longmapsto t^d$

So $C - \{0,0\} \cong \coprod_{j=1}^{m} \Delta_j$, $t_j \mapsto t_j^{n_j}$. $[\sum n_j = s]$.
We have:

$$\begin{tikzcd} \coprod \Delta_j \arrow[r, phantom, "\cong"] & C - \{0,0\} \arrow[d, "x"] \arrow[r, phantom, "\subset"] & A'_y \times \Delta_x \\ t_j \arrow[rr, mapsto] \arrow[dr, mapsto] & & (t_j^{n_j}, \gamma_j(t_j)) \\ & t_j^{n_j} \in \Delta_x - \{0\} & \end{tikzcd}$$

Know that $f(t_j^{n_j}, \gamma_j(t_j)) \equiv 0$. This way, get distinct meromorphic parametrizations,
$D_j - \{0\} \ni t_j \longmapsto (t_j^{n_j}, \gamma_j(t_j))$.

Lemma: $\gamma_j(t_j) \to 0$ as $t_j \to 0 \Rightarrow \gamma_j$ is holomorphic, order ≥ 1.

These parametrisations γ_j are called branches of C at p.
(i) $y^s + \sum \varphi_i(x) y^{s-i} = \prod_{j=1}^{m} \prod_{\varepsilon^{n_j}=1} [y - \gamma_j(\varepsilon x^{1/n_j})]$.

(ii) Let $f = 0$, $g = 0$ be 2 curves. Call them C, D. Define $(C.D)_p = \sum_{\substack{\gamma_j, \\ \text{branches of } C}} \operatorname{ord}_{t=0} g(t^n, \gamma_j(t))$

Return to:

Theorem (Weierstrass): Let $f \in k[[X_1,..,X_n]]$; $\operatorname{ord} f = s$, $X_n^s \in f$. There exists a unique unit $u \in k[[X_1,..,X_n]]$ such that $uf = X_n^s + \sum_{i=1}^{s} \varphi_i X_n^{s-i}$, with $\varphi_i \in k[[X_1,..,X_{n-1}]]$, $\operatorname{ord} \varphi_i \geq i$.

Proof: (i) division by X_n^s: for all $g \in k[[X_1,..,X_n]]$, $\exists\, q_g, r_g$ uniquely characterised by: $g = q_g X_n^s + r_g$, with $q \in k[[X_1,..,X_n]]$ and $r = \sum_{i<s} r_i X_n^i$, $r_i \in k[[X_1,..,X_{n-1}]]$.

(ii) Do this for f: $f = vX_n^s + r$, v a unit. Let u be a unit: $uf = uvX_n^s + ur$
$= uvX_n^s + q_{ur} X_n^s + r_{ur}$.
Want unit u such that $uv + q_{ur} = 1$.

(iii) Let $\alpha = uv$. Then wish $\alpha + q_{\alpha \underset{\sim}{r}} = 1$, ie $\alpha = 1 - q_{\alpha \underset{\sim}{r}}$.
Define $T: k[[X_1,..,X_n]] \to k[[X_1,..,X_n]]$ by $\varphi \longmapsto -q_{\varphi \underset{\sim}{r}}$. (Exercise: $T: m^i \mapsto m^{i+1}$, where $m = (X_1,..,X_n)$)
So want $\alpha = 1 + T(\alpha)$. So take $\alpha = \sum_{i=0}^{\infty} T^i(1)$, which converges in $k[[X_1,..,X_n]]$ as it is a complete metric space. Or, $T^i(1) \in m^i$ each i, so this makes sense.

Suppose we have $f(x,y) = y^s + \sum_{i=1}^{s} \varphi_i(x)\, y^{s-i}$, $\varphi_i \in k[[x]]$.
$$= \prod_{\substack{\text{branches}\\ t \mapsto (t^{s_j}, \gamma_j(t))}} \prod_{\varepsilon^{s_j}=1} \left(y - \gamma_j(\varepsilon x^{1/s_j})\right).$$

Definition: Take f as above, $C = (f=0)$, $D = (g(x,y)=0)$. Let $(C.D)_{\text{origin}} = \sum_{\text{branches}} \operatorname{ord}_t g(t^{s_j}, \gamma_j(t))$.

Theorem: $(C.D)_{\text{origin}} = \operatorname{order}_x r_y(f,g)$.

Lemma: Let $R = k[a_1,..,a_n, b_1,..,b_m]$. Consider $F(x) = \prod(x-a_i)$, $G(x) = \prod(x-b_j)$, $\in R[x]$.
$R_x[F,G] = \alpha \prod_{i,j}(a_i - b_j)$, some $\alpha \in K^*$, $= \alpha \prod_i G(a_i)$.

Proof: $R(F,G) \in R$ is: (i) homogeneous of degree nm, (ii) 0 if $a_i = b_j$.
So $(a_i - b_j) | R$ for all i,j, so $\prod(a_i - b_j) | R$ and they have the same degree.

Take $k = \bar{k}$, and let $\mathcal{O} = K[x]_{(x)}$, so $\mathcal{O}$ is a Local Euclidean Domain with maximal ideal (x). Let M be a f.g. $\mathcal{O}$-module.

Definition: (i) M has finite dimension $\Leftrightarrow$ it has finite dimension as a vector space over k.

(ii) Let $\varphi: M \to M$ be a $\mathcal{O}$-linear map, and assume $K_\varphi = \ker \varphi$, $C_\varphi = \operatorname{coker} \varphi$ have finite dimension. Then, let $e(\varphi, M) = \dim_k C_\varphi - \dim_k K_\varphi$.

Lemma: If M has finite dimension then $e(\varphi, M) = 0$ for all M.
Proof: Exercise.

Lemma: $e(\varphi, \mathcal{O}^n) = e(\det\varphi, \mathcal{O})$.

Proof: $\exists$ coordinate changes $\psi_1, \psi_2 : \mathcal{O}^n \to \mathcal{O}^n$ such that $\psi_1 \varphi \psi_2^{-1} = \begin{pmatrix} x^{i_1} & & \\ & \ddots & \\ & & x^{i_n} \end{pmatrix}$ with $i_1 \le i_2 \le \dots \le i_n$.

Then $e(\psi_1 \varphi \psi_2^{-1}, \mathcal{O}) = e(\varphi, \mathcal{O})$, $e(\det\psi_1 \det\varphi \det\psi_2^{-1}, \mathcal{O}) = e(\det\varphi, \mathcal{O})$.

So we may assume φ is diagonal. Continue as exercise.

Theorem: $F, G \in \mathcal{O}[Y]$, F monic. Consider $\varphi = m_G : \frac{\mathcal{O}[Y]}{(F)} \to \frac{\mathcal{O}[Y]}{(F)}$, by multiplication by G. φ is $\mathcal{O}$-linear, and $\frac{\mathcal{O}[Y]}{(F)}$ is an fg $\mathcal{O}$-module, $\cong \mathcal{O}^n$.

Then $R_y(F,G) = \alpha \det(m_G)$, some $\alpha \in K^*$.

Proof: Uses above two lemmas.

(i) $G = Y - b$. Exercise.

(ii) $G = G_1 G_2$. Then, $m_G = m_{G_1} \circ m_{G_2}$, so $\det(m_G) = \det(m_{G_1})\det(m_{G_2})$.
So must show: $r(F, G_1 G_2) = r(F, G_1) \cdot r(F, G_2)$, but this is true by the earlier lemma ($F = \prod(x - \alpha_i) \Rightarrow r(F,G) = \prod G(\alpha_i)$).

(iii) Calculate in $\mathrm{Frac}(\mathcal{O})$, hence $G = \prod(Y - b_i)$.

Let $A = k[X_1, \dots, X_n]$. $\mathbb{A}^n_k = k^n$. We have: $\left\{ \begin{matrix} \text{ideals} \\ J \subset A \end{matrix} \right\} \underset{I}{\overset{V}{\rightleftarrows}} \left\{ \begin{matrix} \text{subsets} \\ X \subset \mathbb{A}^n_k \end{matrix} \right\}$

$V(J) := \{ p \in \mathbb{A}^n_k : f(p) = 0 \ \forall f \in J \}$, $I(X) := \{ f \in A : f(p) = 0 \ \forall p \in X \}$

Definition: $X \subset \mathbb{A}^n_k$ is <u>algebraic</u> iff $X = V(J)$, some J.

Properties:
(i) $V(0) = \mathbb{A}^n$, $V(1) = \phi$.
(ii) $I \subset J \Rightarrow V(I) \supset V(J)$
(iii) $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$
(iv) $V(\Sigma I_\lambda) = \cap V(I_\lambda)$.

$\Rightarrow$ Zariski topology.

$\mathbb{A}^2_k = \mathbb{A}^1_k \times \mathbb{A}^1_k$, but $\mathbb{A}^2_k$ does not have the product topology of the Zariski topology with itself.

Eg: $V(I_1 \cap I_2) \subset V(I_1) \cup V(I_2)$. $p \notin V(I_1) \cup V(I_2) \Rightarrow \exists f_i \in I_i$, $f_i(p) \neq 0$.
$f = f_1 f_2 \in I_1 \cap I_2$, so $f(p) \neq 0$.

Proposition:
(i) $X \subset Y \Rightarrow I(X) \supset I(Y)$.
(ii) $X \subset V(I(X))$, with '$=$' iff X algebraic.
(iii) $J \subset I(V(J))$.

Definition: $X \subset \mathbb{A}^n_k$ is <u>irreducible</u> $\Leftrightarrow$ $X = X_1 \cup X_2$, with X_i algebraic, implies $X_1 = X$ or $X_2 = X$.
Otherwise, X is <u>reducible</u>

Proposition:
(i) X irreducible $\Leftrightarrow$ $I(X)$ is prime.
(ii) If X algebraic, can write: $X = X_1 \cup \dots \cup X_N$, with X_i irreducible.

Proof: (ii) If $X = X_1 \cup X_2$, work with X_1, X_2. If this process never stops, find infinite chain $Y_1 \supsetneq Y_2 \supsetneq \dots \Rightarrow I(Y_1) \subsetneq I(Y_2) \subsetneq \dots \Rightarrow A$ not Noetherian.

Nullstellensatz: (i) $V(I) = \phi \Rightarrow I = (1)$ (Recall $k = \bar{k}$).

(ii) $I(V(J)) = \operatorname{rad} J$.

There is a 1-1 correspondance:

$$
\begin{array}{ccc}
\left\{ \begin{array}{c} \text{algebraic subsets} \\ X \subset \mathbb{A}^n_k \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} J \subset k[x_1,\dots,x_n] \\ J = \operatorname{rad} J \end{array} \right\} \\
\cup & & \cup \\
\left\{ \begin{array}{c} \text{irreducible} \\ \text{algebraic subsets} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{prime ideals} \\ J \subset A \end{array} \right\} \\
\cup & & \cup \\
\left\{ \begin{array}{c} \text{points} \\ \text{of } k^n \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{maximal ideals} \\ \text{in } A \end{array} \right\}.
\end{array}
$$

Corollary of Nullstellensatz: $(k = \bar{k})$ Every maximal ideal of $k[X_1,\dots,X_n]$ is $(X_1 - a_1, \dots, X_n - a_n)$.

Example: $C = \{(t^3, t^4, t^5) : t \in k\} \subset \mathbb{A}^3_k$. C is an algebraic subset. Use $k[u,v,w]$.
$I(C) = (uw - v^2,\ u^3 - vw,\ u^2 v - w^3)$ - exercise.
$I(C)$ cannot be generated by two polynomials.

Definition: $A \hookrightarrow B$. B is an f.g. A-algebra if $B = A[b_1,\dots,b_n]$
$(\Leftrightarrow A[X_1,\dots,X_n] \twoheadrightarrow B,\ X_i \mapsto b_i)$.
B is a finite A-algebra $\Leftrightarrow B = \sum_{i=1}^{n} A b_i$.

Theorem: Let A be a f.g. k-algebra (k any field). Then $\exists\ \varphi : k[X_1,\dots,X_n] \hookrightarrow A$ such that A is finite over $k[X_1,\dots,X_n]$. (Assume k is an infinite field).

Proof: By induction on the number of generators. $k[X_1,\dots,X_n] \xrightarrow{\varphi} A$, n minimal.
Say $a_i = \varphi(X_i)$. If φ is injective, then done. Otherwise, let $I = \ker \varphi$, $0 \neq F \in I$, $d = \deg F$. Let F_d be the homogeneous part of degree d.
If $F_d \ni X_n^d$, $F = X_n^d + \sum \psi_i(X_1,\dots,X_{n-1}) X_n^{d-i}$.
Let $B \subset A$ be the subring generated by $\varphi(X_1),\dots,\varphi(X_{n-1})$. Then $\psi_i(a_1,\dots,a_{n-1}) = b_i \in B$.
Clearly, (i) $A = B[a_n]$.
(ii) a_n satisfies $X^d + \sum b_i X^{d-i} = 0$.
But B is finite over a polynomial ring, and A is finite over B, so done.
And, we can change coordinates to ensure $F_d \ni X_n^d$, so always done.

Definition: Let $V \subset \mathbb{A}^n_k$ be an algebraic subset. $f : V \to \mathbb{A}^1_k$ is polynomial if $f = F|_V$, $F \in k[X_1,\dots,X_n]$.
$k[V] = \{\text{polynomial functions } f : V \to \mathbb{A}^1_k\} = k[X_1,\dots,X_n]/I(V)$
$=$ coordinate ring of V.

Zariski topology of V: $\{\text{closed subsets } W \subset V\} \leftrightarrow \{\text{algebraic } W \subset \mathbb{A}^n_k,\ W \subset V\}$
$\leftrightarrow \{I \subset k[V],\ \operatorname{rad} I = I\}$.

Definition: $V \subset \mathbb{A}^n_k$, $W \subset \mathbb{A}^n_k$. $f : V \to W$ is polynomial if $\exists\ f_1,\dots,f_m \in k[V]$ such that

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[dr, "(f_1,\dots,f_m)"'] & W \arrow[d, phantom, "\cap"] \\
& \mathbb{A}^m_k
\end{tikzcd}
$$

commutes.

$\{f: V \to W, \text{ polynomial}\} \xleftrightarrow{1-1} \{f^{\#}: k[W] \to k[V], \text{ ring homomorphism}\}$.

Let $\varphi \in k[W]$, $\varphi: W \to \mathbb{A}^1_k$, $V \xrightarrow{f} W$. Then $f^{\#}\varphi = \varphi \circ f \in k[V]$.
Check this is an isomorphism.

A polynomial function $V \to W$ is also called a morphism.

Definition: An affine algebraic variety is an isomorphism class of irreducible algebraic subsets $V \subset \mathbb{A}^n_k$.

Exercise: Let $V \subset \mathbb{A}^n_k$, algebraic. The following are equivalent.
(i) V is irreducible.
(ii) $U \subset V$ Zariski-open $\Rightarrow$ $U \subset V$ Zariski-dense.
(iii) $U', U'' \subset V$ Zariski-open $\Rightarrow U' \cap U'' \neq \phi$.

Definition: V be an affine variety. $K(V) = \operatorname{Frac} K[V]$ = fraction field
= Field of rational functions.
Note $K[V]$ is an integral domain.

Definition: $\operatorname{Dom}(f) = \{p \in V : f = g/h,\ g, h \in k[V],\ h(p) \neq 0\}$.

Exercise: Construct an affine variety V such that $k[V]$ is not a UFD

Theorem: (i) $D(f)$ is Zariski-open.
(ii) $D(f) = V \Leftrightarrow f \in k[V]$.
(iii) for $h \in k[V]$, define $V_h = \{p \in V : h(p) \neq 0\}$. Then, $D(f) \supset V_h \Leftrightarrow f \in k[V]_h$.

Proof: Define the ideal I of denominators of f, $I = \{h \in k[V] : hf \in k[V]\}$. Then:
(i) $V - D(f) = V(I)$. So, $D(f)$ is Zariski-open.
(ii) $D(f) = V \Leftrightarrow V(I) = \emptyset \Leftrightarrow I = (1) \Leftrightarrow f \in k[V]$, as f is everywhere regular and use Nullstellensatz.
(iii) Use Nullstellensatz. $k[V]_h = \bigcup_{i \geq 0} h^{-i} k[V]$.

$V \subseteq \mathbb{A}^n_k$
$\cup$
$\underset{\text{Z-open}}{U} \mapsto \mathcal{O}(U)$ = set of all rational functions defined at every point in U.

Sheaves.

Let X be a topological space.

Definition: A presheaf of abelian groups is a datum:
$X \underset{\text{open}}{\hookleftarrow} U \mapsto \mathcal{F}(U)$, abelian groups.

$V \underset{open}{\subset} U$. $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$, $\rho^U_U = id_{\mathcal{F}(U)}$ ρ is a restrictiony-shaped beast
if $W \subset V \subset U$, $\rho^U_W = \rho^V_W \circ \rho^U_V$

Examples: (i) $C(X, \mathbb{R})$. $(U) = \{ f : U \to \mathbb{R}$, continuous$\}$.
(ii) $X \subset \mathbb{R}^n$, open. $C^\infty(X, \mathbb{R})(U) = \{ f : U \to \mathbb{R} : f$ is $C^\infty \}$.
$C^\omega(X, \mathbb{R})(U) = \{ f : U \to \mathbb{R} : f$ real analytic$\}$.

Definition: A presheaf $\mathcal{F}$ is a sheaf iff the following sequence is exact, for all coverings $\mathcal{U} = \bigcup U_\alpha$; for all open $U \subset X$.

$$0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U_\alpha) \xrightarrow{+-} \prod \mathcal{F}(U_\alpha \cap U_\beta)$$

So an element vanishing when restricted to each U_α is zero, and any element which is in $\prod \mathcal{F}(U_\alpha)$ which agrees on intersections of U_α, U_β can be "glued" to give an element of $\mathcal{F}(U)$.

Definition: A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves is a datum:

$$\begin{array}{lccc} X \underset{open}{\supset} U & \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \quad U \text{ open} & \downarrow \rho^U_V & & \downarrow \rho^U_V \\ \quad V & \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Definition: Let $\mathcal{F}$ be a (pre)sheaf, $x \in X$; stalk of $\mathcal{F}$ at x is $\mathcal{F}_x := \varinjlim_{U \ni x} \mathcal{F}(U)$.

An element $s_x \in \mathcal{F}_x$ is an equivalence class of pairs (s, U), where $x \in U$ is open, $s \in \mathcal{F}(U)$. $(s, U) \sim (t, V) \iff \exists\, x \in W \subset U \cap V$ such that $\rho^U_W s = \rho^V_W t$. (or, "$s|_W = t|_V$").

Example: $X = \mathbb{C}^\times = \mathbb{C} - \{0\}$. $\mathcal{O}$ = sheaf of holomorphic functions on X.
Have $X \underset{open}{\supset} U \mapsto \mathcal{O}(U) = \{ f : U \to \mathbb{C}$, holomorphic$\}$. Let $\mathcal{O}^\times(U) = \{ f \in \mathcal{O}(U) : f(x) \neq 0 \ \forall x \in U \}$
Define $\phi : \mathcal{O} \to \mathcal{O}^\times$, $f \mapsto e^{2\pi i f}$.

Remark: Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then $X \hookleftarrow U \mapsto \text{Kernel}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$ is a sheaf. (Exercise). This is called $\ker \varphi$.

In the above example, have $0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{\phi} \mathcal{O}$. By definition, $\mathbb{Z}(U) = \prod_{\substack{\text{connected} \\ \text{components } V \text{ of } U}} \mathbb{Z}$.

Remark; If $U \subset \mathbb{C}^\times$ is simply connected, then $\phi(U) : \mathcal{O}(U) \to \mathcal{O}(U)$ is surjective.
$\int_{x_0}^{x} \frac{df}{f} \leftarrow\!\shortmid (f : U \to \mathbb{C})$

However, if $U = \mathbb{C}^\times$, $\phi : \mathcal{O}(\mathbb{C}^\times) \to \mathcal{O}(\mathbb{C}^\times)^\times$ is not surjective (e.g. consider $\frac{1}{z} \in \mathcal{O}(\mathbb{C}^\times)^\times$)
So the image of the sheaf is a presheaf but not a sheaf (as every point in $\mathbb{C}$ will have a simply connected neighbourhood).

Remark: $X \underset{open}{\hookleftarrow} U \mapsto \text{Im}(\mathcal{F}(U) \xrightarrow{\varphi(U)} \mathcal{G}(U))$ is a presheaf (exercise), but not a sheaf.

Theorem: Given a presheaf $\mathcal{F}$, $\exists$ a unique sheaf $\mathcal{F}^+$ and $t: \mathcal{F} \to \mathcal{F}^+$ a morphism of presheafs such that for any sheaf $\mathcal{G}$,
$\mathrm{Hom}_{presheaf}(\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{sheaf}(\mathcal{F}^+, \mathcal{G})$
("the inclusion functor from the category of sheafs to that of presheafs has a left adjoint").

$\mathcal{F}^+(U) = \left\{ s: U \to \prod_{x \in U} \mathcal{F}_x \text{ such that (i) } s(x) \in \mathcal{F}_x \ \forall x \in U, \text{ (ii) for all } x \in U, \ \exists V \ni x, V \subset U \text{ and } t \in \mathcal{F}(V) \text{ with } s(y) = t_y \text{ for all } y \in V \right\}.$

Rest of proof - exercise.

Theorem: Sheaf of X, $\mathrm{Sh}X$, is an abelian category.

Proof: Let $\varphi: \mathcal{F} \to \mathcal{G}$ be a morphism. Let $\mathcal{I}$ be the presheaf: $\mathcal{I}(U) = \mathrm{Im}(\mathcal{F}(U) \to \mathcal{G}(U))$.
Define $\mathrm{Im}\,\varphi := \mathcal{I}^+$.

Theorem: Let $f: X \to Y$ be a continuous map. There are functors
$f^\circ: \mathrm{Sh}Y \to \mathrm{Sh}X$, $f_\circ: \mathrm{Sh}X \to \mathrm{Sh}Y$, such that
$\mathrm{Hom}_{\mathrm{Sh}X}(f^\circ \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathrm{Sh}Y}(\mathcal{G}, f_\circ \mathcal{F})$.

(i) $\mathcal{F}$ a sheaf on X. $Y \underset{open}{\supset} U \mapsto \mathcal{F}(f^{-1}U)$ is a sheaf, denoted $f_\circ \mathcal{F}$.

(ii) $\mathcal{G}$ a sheaf on Y. $X \underset{open}{\supset} U \mapsto \varinjlim_{\substack{V \supset f(U) \\ open}} \mathcal{G}(V)$. is a presheaf, whose $+$ is denoted $f^\circ \mathcal{G}$.

Remark: Special case of (ii) $U \overset{j}{\hookrightarrow} X$ $j^\circ \mathcal{F} = \mathcal{F}|_U$ is very easy.

Algebraic Varieties

Let $X \subset \mathbb{A}^n_k$ be an irreducible algebraic subset. The assignment
$X \underset{Zariski\text{-}open}{\hookleftarrow} U \longmapsto \mathcal{O}_X(U) = \{ f \in k(X) : f \text{ regular at all } x \in U \}$,
is a sheaf of rings (exercise), for the Zariski topology.

$x \in X \leftrightarrow m_x \in k[X]$, maximal. $\mathcal{O}_{X,x} = k[X]_{m_x}$. (exercise).

Definition: An affine algebraic variety is the pair $(X, \mathcal{O}_X)$ for an irreducible algebraic subset $X \subset \mathbb{A}^n_k$ and a sheaf of regular functions $\mathcal{O}_X$.

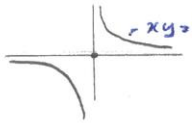
Definition: An algebraic pre-variety is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings $\mathcal{O}_X$ such that there exists a finite open covering $X = \bigcup U_\alpha$ and isomorphisms $\varphi_\alpha : (U_\alpha, \mathcal{O}_X|_{U_\alpha}) \xrightarrow{\sim} (X_\alpha, \mathcal{O}_{X_\alpha})$ for some affine varieties $(X_\alpha, \mathcal{O}_{X_\alpha})$.

[By isomorphism $\varphi_\alpha : (U_\alpha, \mathcal{O}_X|_{U_\alpha}) \xrightarrow{\cong} (X_\alpha, \mathcal{O}_{X_\alpha})$, mean that
(i) $\varphi_\alpha : U_\alpha \xrightarrow{\cong} X_\alpha$, a homeomorphism of topological spaces.
(ii) have isomorphism $\mathcal{O}_X|_{U_\alpha} \xrightarrow{\cong} \varphi_\alpha^\circ \mathcal{O}_{X_\alpha}$ (or, equivalently, $\varphi_{\alpha\circ}(\mathcal{O}_X|_{U_\alpha}) \cong \mathcal{O}_{X_\alpha}$)]

Examples: (i) If X is a pre-variety, $U \subset X$ open, then $U, \mathcal{O}_U := \mathcal{O}_X|_U$ is also a pre-variety.
[X affine $\Rightarrow X_f = \{x \in X : f(x) \neq 0\}$ form a basis for the Zariski topology of X as $f \in k[X]$].

Lemma: X affine $\Rightarrow X_f$ is affine.
Proof: If $X = V(I)$, $I \subset k[X_1, \ldots, X_n]$, then $X_f = V(J)$ where $J = (I, x_{n+1} f - 1) \subset k[X_1, \ldots, X_{n+1}]$
(cf proof that weak NSS $\Rightarrow$ NSS).

Eg: $\{x \in \mathbb{A}_k : x \neq 0\}$ is affine.



$xy = 1$, projects onto $X - \{0\}$.

Eg: $X = \mathbb{A}_k^2$, $U = \mathbb{A}_k^2 - \{0, 0\}$. Exercise: prove that U is not an affine variety.
(Hint: $\mathcal{O}(U) = k[\mathbb{A}_k^2]$).

Examples (ii): X algebraic prevariety, $Y \subset X$ irreducible algebraic subset, $\Rightarrow Y$ also a prevariety.
Idea: $Y \subset X$, $\mathcal{O}_Y \subset C(Y, \mathbb{A}_k^1)$
If $V \cap Y \underset{open}{\subset} Y$, $U \underset{open}{\subset} X$. $\mathcal{O}_Y(V) = \{\text{continuous } f : V \to \mathbb{A}_k^1 : f = \tilde{F}|_V, \text{ some } \tilde{F} \in \mathcal{O}_X(U)\}$.

Definition: A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of algebraic pre-varieties is a continuous map $f : X \to Y$ such that $\mathcal{O}_Y \circ f \subset \mathcal{O}_X$
[For all open subsets $V \subset Y$, all $\varphi \in \mathcal{O}_Y(V) \subset \{\text{continuous } \varphi : V \to \mathbb{A}^1\}$ this:
$\Rightarrow \varphi \circ f : f^{-1}(V) \to \mathbb{A}^1$ is in $\mathcal{O}_X(f^{-1}(V))$]

Examples: (i) $\mathbb{P}^1$ is obtained by gluing 2 copies of $\mathbb{A}^1$. $\mathbb{P}^1 = U_0 \cup U_1$, $U_0 = (x_0 \neq 0)$, $U_1 = (x_1 \neq 0)$. $U_0, U_1 \cong \mathbb{A}_k^1$, glued via $t \mapsto 1/t$.
(ii) $U_0, U_1 = \mathbb{A}^1$, $X = \mathbb{A}_k^1 \cup_{\mathbb{A}_k^1 - \{0\}} \mathbb{A}_k^1$

$$\begin{aligned} t &\longmapsto t \\ \mathbb{A}_k^1 - \{0\} &\to \mathbb{A}_k^1 \\ t &\longmapsto t. \end{aligned}$$

This is not ~~Haus~~ Hausdorff. We don't want this to be a variety. Since the Zariski topology is so coarse, insisting on Hausdorffness is not useful.

Definition: X is a variety $\Leftrightarrow$ for all prevarieties Y, given $Y \underset{g}{\overset{f}{\rightrightarrows}} X$ then $\{y \in Y : f(y) = g(y)\}$ is closed in Y, ie, it is separated.

Exercise: $\mathbb{P}^1$ is a variety, but X in (ii) is not.

Remark: $(f, g) : Y \to X \times X$. Then $\{y : f(y) = g(y)\} = (f, g)^{-1} \Delta \subset Y$
$\underset{X}{\uparrow} \Delta = (id_X, id_X)$.

X is a variety $\Leftrightarrow \Delta \subset X \times X$ is closed. (must define the product of varieties).

Example (ii) above: This is not a variety, for $\{y : f(y) = g(y)\} = \mathbb{A}^1 - \{0\}$ - not closed.

We wish to define the product of varieties.
For example, suppose $X = \mathbb{A}^{n_1}$, $Y = \mathbb{A}^{n_2}$, $X \times Y = \mathbb{A}^{n_1} \times \mathbb{A}^{n_2} = \mathbb{A}^{n_1+n_2}$, but we do not get the product topology, in general.

<u>Definition</u>: Z, with $p: Z \to X$, $q: Z \to Y$, is a <u>product</u> if for all W with $p': Z \to X$, $q': W \to Y$ $\exists$ unique morphism $\varphi: W \to Z$ such that $p\varphi = p'$, $q\varphi = q'$.

$$\begin{tikzcd} W \arrow[rr, dashed] \arrow[dr, "p'"'] \arrow[drr, "q'" near start] & & Z \arrow[dl, "p"] \arrow[d, "q"] \\ & X & Y \end{tikzcd}$$

<u>Exercise</u>: If it exists, the product is unique up to unique isomorphism.

<u>Theorem</u>: Products of prevarieties exist.

<u>Recall</u>: X is a variety $\Leftrightarrow$ $\Delta \subset X \times X$ is Zariski-closed.
$(\Leftarrow)$ $Y \underset{g}{\overset{f}{\rightrightarrows}} X \Rightarrow Y \xrightarrow{(f,g)} X \times X$, $\{f = g\} = (f,g)^{-1}\Delta \subset Y$.
$(\Rightarrow)$ $Y = X \times X$, $(f,g) = (p_1, p_2)$.

<u>Lemma</u>: products of affine varieties exist.

<u>Exercise</u>: (i) Let $X \subset \mathbb{A}^n$ be an affine variety, Z a prevariety. Then
$$\left\{ \begin{array}{c} \text{morphisms} \\ \varphi: Z \to X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{ring homomorphisms} \\ \varphi^\#: k[X] \to \Gamma(Z, \mathcal{O}_Z) \end{array} \right\}$$
(notation: if $\mathcal{F}$ is a sheaf on Z, $U \underset{\text{open}}{\subset} Z$, $\mathcal{F}(U) = \Gamma(U, \mathcal{F})$)

(ii) $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$, algebraic subsets. $X \times Y \subset \mathbb{A}^{m+n}$, irreducible $\Leftrightarrow$ X, Y irreducible.

<u>Proof of lemma</u>: $I(X) = (f_1, \dots, f_{m_1}) \subset \mathbb{A}^{n_1}$, $f_i = f_i(x_1, \dots, x_{n_1})$
$I(Y) = (g_1, \dots, g_{m_2}) \subset \mathbb{A}^{n_2}$, $g_j = g_j(y_1, \dots, y_{n_2})$

<u>Claim</u>: $I(X \times Y) = I = (f_1, \dots, f_{m_1}, g_1, \dots, g_{m_2})$.

<u>Proof</u>: Claim $\varphi = \varphi(\underline{x}, \underline{y}) \in k[x_1, \dots, x_{n_1}, y_1, \dots, y_{n_2}]$. $\varphi \equiv 0$ on $X \times Y \Rightarrow \varphi \in I$.
$\varphi = \sum_{i=1}^{s} P_i(x) Q_i(y)$. Use induction on S.
If all $P_i \equiv 0$ on X, nothing to prove. Otherwise, $\exists\, \xi \in X$ such that $P_s(\xi) \neq 0$. $\sum P_i(\xi) Q_i(y) \equiv 0$ on Y, hence $Q_s(y) \equiv -\frac{1}{P_s(\xi)} \cdot \sum_{i=1}^{s-1} P_i(\xi) Q_i(y) \mod I$.
$\Rightarrow \varphi \equiv \sum_{i=1}^{s-1} \left[P_i(x) - P_i(\xi)/P_s(\xi) \right] Q_i(y) \mod I$.
$\Rightarrow \varphi \in I$, by induction

To prove lemma, need to show that $X \times Y \subset \mathbb{A}^{n+m}$ with $X \times Y \to X$, $X \times Y \to Y$ is a categorical product.
By claim, $k[X \times Y] = k[X] \otimes_k k[Y]$. Then, the lemma follows by exercise (i), and the universal property of tensor product of rings.

$$\begin{tikzcd} & k[X] \otimes_k k[Y] = k[X \times Y] \arrow[d, dashed, "\exists!\text{, unique}"] & \\ k[X] \arrow[ur, "p^\#"] \arrow[r, "p'^\#"'] & \Gamma(W, \mathcal{O}_W) & k[Y] \arrow[ul, "q^\#"'] \arrow[l, "q'^\#"] \end{tikzcd}$$

Have $X = \bigcup U_\alpha$, $Y = \bigcup V_\alpha$. Want $X \times Y = \bigcup U_\alpha \times V_\alpha$.

Let $I \subset k[x_0, \dots, x_n]$ homogeneous prime ideal, ie generated by homogeneous elements. Want to define $X = V(I)$. (f is homogeneous of degree $d \iff f(\lambda \underline{x}) = \lambda^d f(\underline{x})$, $\lambda \in k^*$)
The ring $R = k[x_0, \dots, x_n]/I$ is graded, $= \bigoplus_{n \geq 0} R_n$.
$\mathbb{P}^n$ has a Zariski topology by $V(I)$, I homogeneous $\Rightarrow X = V(I)$ has a Z-topology.
$k(x) = \{ \frac{f}{g} : g \neq 0,\ f, g \in R_n, \text{ some } n \}$, the fraction field on X.

For $x \in X$, write $\mathcal{O}_{X,x} = \{ \varphi \in k(x) : \varphi = \frac{f}{g},\ g(x) \neq 0 \}$.
$\mathcal{O}_X(U) = \{ \varphi \in k(x) : \varphi \in \mathcal{O}_{X,x}, \text{ all } x \in U \}$. $m_x = \{ \varphi = \frac{f}{g} : f(x) = 0 \} \subset \mathcal{O}_{X,x}$.
$\mathcal{O}_X(U) \hookrightarrow \mathcal{C}(U, \mathbb{A}^1_k)$.

Exercise: $X \cap \{x_i \neq 0\}$ is an affine algebraic variety.

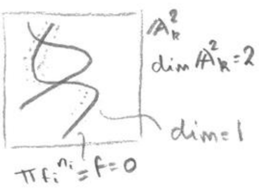
Dimension Theory

X an algebraic variety. $k(x) = \varinjlim_{U \underset{open}{\subset} X} \mathcal{O}_X(U)$.
$\dim X := \text{tr-deg}_k\, k(X)$. (Assume $k = \bar{k}$)

Proposition: If $Y \subsetneq X$ is a closed subvariety, then $\dim Y < \dim X$.
Proof: May assume X is affine. Let $A = k[X]$, $I(Y) = P \subset A$. $\text{tr.deg}\, A/P < \text{trdeg}\, A = n$.
Take $x_1, \dots, x_n \in A$. Let $\bar{x}_1, \dots, \bar{x}_n \in A/p$. Let $f \in P$ be anything. $x_1, \dots, x_n, f \in A$, so $\exists\, F \in k[Y_1, \dots, Y_{n+1}]$ such that $F(x_1, \dots, x_n, f) = 0$. Assume that F is irreducible.
$f \neq 0$, $k = \bar{k} \Rightarrow F(Y_1, \dots, Y_n, 0) \neq 0 \Rightarrow F(\bar{x}_1, \dots, \bar{x}_n, 0) = 0 \Rightarrow \bar{x}_1, \dots, \bar{x}_n$ are algebraically dependent.

Theorem: X an algebraically independent, $U \underset{open}{\hookrightarrow} X$. $g \in \Gamma(U, \mathcal{O})$ a function, $Z \subset U$ a component $\{x \in U : g(x) = 0\}$. $\dim Z = \dim X - 1$.
Proof: Immediate from Krull:

Theorem: A a f.g. k-algebra, $f \in A$. P a prime ideal, minimal among those $\ni f$
$\Rightarrow \text{trdeg}_k\, A/p = \text{trdeg}_k\, A - 1$.



P prime, $\ni f \iff$ (i) $P = (f_i)$, minimal prime.
(ii) M_x, $x \in (f = 0)$

Corollary 2: X an affine variety, $Z \subset X$ irreducible and closed. $r = \text{cod-}Z (= \dim X - \dim Z)$.
There are $f_1, \dots, f_r \in k[X]$ such that Z is a component of $V(f_1, \dots, f_k)$

Corollary 1: A chain of closed irreducible subsets: $(*)\ X \supsetneq Z^1 \supsetneq Z^2 \supsetneq \cdots \supsetneq Z^d \supsetneq \phi$
is saturated $\iff \forall i\ \nexists$ closed irreducible Y with $Z^i \supsetneq Y \supsetneq Z^{i+1}$. (definition)
If $(*)$ is saturated, $d = \dim X$.

Proof of 2: By induction on s. Given a saturated chain $X \supsetneq Z^1 \supsetneq \cdots \supsetneq Z^s$,
there are functions $f_1, \dots, f_s$ such that:
(i) Z^s is a component of $V(f_1, \dots, f_s)$
(ii) every component of $V(f_1, \dots, f_s)$ has codimension s.
Assume $f_1, \dots, f_{s-1}$ defined. $V(f_1, \dots, f_{s-1}) = Y_1 \cup \cdots \cup Y_k$, $Y_1 = Z^{s-1}$. $Z^s \subsetneq Z^{s-1} = Y_1$.
Need $f \in I(Z^s)$, $f \notin I(Y^i)\ \forall i$, ie $f \notin \bigcup I(Y^i)$.
Note that $I(Z^s) \not\subset I(Y^i)$ all i, so we are done, as $I(Y_i)$ are prime.

Relative Theory

$$\begin{array}{ccc} X & \supset & X_y = f^{-1}(y) \\ f\downarrow & & \downarrow \\ Y & \ni & y \end{array}$$

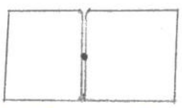
Definition: f is dominating if $\overline{f(X)} = Y$.

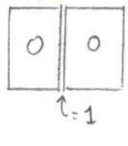
Theorem 1: Let $f: X \to Y$ be dominating, $r = \dim X - \dim Y$, $W \subset Y$ be closed, irreducible,
$Z \subset f^{-1}(W)$ a component, dominating W. Then $\dim Z \geq \dim W + r$ ~~Then $\dim Z \geq \dim W + r$.~~

Theorem 2: f, r as above. Then $\exists\ \phi \neq U \subset Y$ open such that
(i) $U \subset f(X)$
(ii) if $W \cap U \neq \phi$, $Z \cap f^{-1}(U) \neq \phi$, then $\dim Z = \dim W + r$.

Definition: $C \subset X$ is constructible $\iff C = \bigcup_{\text{finite}} C_i$, C_i locally closed. (= $\cap$ of open and closed)

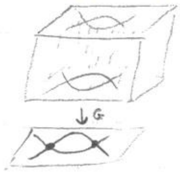
Corollary: (i) $C \subset X$ constructible $\Rightarrow f(C)$ constructible
(ii) $X \ni x \overset{\varphi}{\longmapsto} \max\{\dim Z : Z \ni x \text{ component of } f^{-1}(f(x))\}$ is Zariski upper semi-continuous. (ie, $\forall n \in \mathbb{N}$, $\varphi^{-1}\{a : a \geq n\}$ is closed).

Example: 
$\mathbb{A}^2$
$C = (\mathbb{A}^2 - \{x=0\}) \cup \{(0,0)\}$ is constructible

$f: \mathbb{A}^2 \to \mathbb{A}^2$, $f(x,y) = (x, xy)$. $f^{-1}(a,b) = (a, b/a)$ if $a \neq 0$.
f ie, $a \neq 0 \Rightarrow f^{-1}(a,b) = \{1 \text{ point}\}$.
$a = 0 \Rightarrow f^{-1}(0,b) = \phi$, unless $b = 0$. $a = b = 0$, $f^{-1}(0,0) = \{x = 0\}$.
$f(\mathbb{A}^2)$ is contructable. $U = \{a \neq 0\}$ as in Theorem 2.
Dimension of the fibre jumps at the origin.
φ: 

So φ is upper semi-continuous.

Proof of Theorem 1: May assume X, Y both affine (exercise). Let $\operatorname{cod} W = s$. So $\exists$ $f_1, \dots, f_s \in k[Y]$ such that (i) $W \subset V(f_1, \dots, f_s)$ is a component, (ii) every component of $V(f_1, \dots, f_s)$ has codimension s.



Claim: Let $g_i = f_i \circ f$. $Z \subset V(g_1, \dots, g_s)$ is a component.

Proof: $Z \underset{\text{irred.}}{\subset} \underset{\text{(closed)}}{f^{-1}(W)} \subset V(g_1, \dots, g_s)$

If Z is not an irreducible component of $V(g_1, \dots, g_s)$, then $Z \subsetneq Z'$, Z' a component.

~~In particular, $f(Z') \subset$ a component of $V(f_1, \dots, f_s)$. So $f(Z') \subset W$.~~

Now $W = \overline{f(Z)} \subset \overline{f(Z')} \subset V(f_1, \dots, f_s) \Rightarrow W = \overline{f(Z')}$. ie, $Z \subset Z' \subset f^{-1}(W)$.

So Z is a component of $f^{-1}(W)$, so $Z = Z'$.

So $\operatorname{cod} Z \leq s$, ie the statement.

Proof of Theorem 2: May assume X, Y affine (exercise). Immediate that we may assume Y is affine. For X: assume Y affine, let $X = \bigcup X_i$, an affine decomposition of X. $f \longleftrightarrow f_i : X_i \to Y$. Can find open subsets $U_i \subset Y$ such that theorem true for $f_i : X_i \to Y$. Claim: $U = \bigcap U_i$ works for f. Need a lemma...

Lemma: Let X, Y be affine, $f : X \to Y$ a morphism. There exists $g \in k[Y]$ and a decomposition:

$$\begin{tikzcd} X \arrow[d, "f"'] & f^{-1}(Y_g) = X_{g \circ f} \arrow[l, hook'] \arrow[d] \arrow[r, "\text{finite}"] & Y_g \times \mathbb{A}^r_k \arrow[dl, bend left, "p_1"] \\ Y & Y_g \arrow[l, hook'] & \end{tikzcd}$$

("Noether Normalisation for morphisms").

(continue next time).

Lemma (Noether normalisation for morphisms): $\varphi : X \to Y$ dominant morphism of affine varieties, $r = \dim X - \dim Y$. Then there exists $g \in k[Y]$ and a ~~unique~~ factorisation:

$$\begin{tikzcd} X_{g \circ \varphi} \arrow[d, "\varphi"] & X_{g \circ \varphi} \arrow[l, phantom, "\supset"] \arrow[d] \arrow[r, "\text{finite}"] & Y_g \times \mathbb{A}^r \arrow[dl, "p_1"] \\ Y & Y_g \arrow[l, phantom, "\supset"] & \end{tikzcd}$$

(A map $h : A \to B$ is finite if $h^*(k[B])$ is contained in $k[A]$ and $k[A]$ is finite over $h^*[k[B]]$.)

Proof: $\varphi^\# : k[Y] \hookrightarrow k[X]$ is injective. $\quad$ ($f = g$ on $\varphi(X) \subset Y \Rightarrow f = g$ on $\overline{\varphi(X)} = Y$).

$f \longmapsto f \circ \varphi$ $\quad$ (Or, if $\varphi^\#$ is not, $\exists$ a kernel which $\Leftrightarrow$ so φ not dominant.)

Let $K = k[Y]$, $A = S^{-1} k[X]$, $S = k[Y] - \{0\}$. $K \subset A$. A is a finitely generated K-algebra, $\operatorname{trdeg} A/K = r$.

By Noether, $K \subset K[T_1, \dots, T_m] \underset{\text{finite}}{\subset} A$.

Let $x_1, \dots, x_m \in k[X]$ be generators of $k[X]$ as a k-algebra. x_i satisfies an integral equation $P_i(t) = 0$, $P_i \in K[T_1, \dots, T_r][t]$.

If $g = \prod$ (denominators of coefficients of P_i) $\in k[Y]$, then $P_i(t) \in k[Y]_g [T_1, \dots, T_r][t]$

But then we have: $k[Y]_g \subset k[Y]_g [T_1, \dots, T_r] \underset{\text{finite}}{\subset} k[X]_g$

$$Y_g \xleftarrow{p_1} Y_g \times \mathbb{A}^r \xleftarrow{\text{finite}} X_g.$$

Theorem: $f : X \to Y$ dominant, $r = \dim Y - \dim X$. $\exists\ \phi \neq U \subset Y$ such that

(i) $U \subset f(X)$

(ii) $W \subset Y$, closed, irreducible, $W \cap U \neq \phi$, $Z \subset f^{-1}(W)$ is a component, $Z \cap f^{-1} U \neq \phi$.

Then $\dim Z = \dim W + r$.

Proof: (i) may assume that X, Y are affine.

(ii) use the lemma; $U = Y_g$ works. | by lemma, have reduced to either $X \to Y = Y \times \mathbb{A}^r \to Y$, or $X \to Y$ finite.

Corollary: (i) Any morphism $f : X \to Y$ is constructible.

(ii) $X \ni x \mapsto \max\{\dim Z : Z \text{ component of } f^{-1}f(x)\}$ is upper semi-continuous.

Proof: Enough to show that f dominant $\Rightarrow f(X)$ a constructible set.

$f(X) = U = Y - \Sigma W_i$.

Next look at $f|\cup f^{-1}W_i \to \cup W_i$

Use induction on $\dim Y$.

$$\begin{tikzcd} Z \arrow[r, phantom, "\subset"] \arrow[d, "f|_Z"] & X \arrow[d, "f"] \\ \overline{f|_Z(Z)} \arrow[r, hook, "g \circ \varphi"'] & Y \end{tikzcd},$$

so assuming Z = open $\cap$ closed.

For (ii), use theorems 1 and 2.

By theorem 2, $f(X) \supset U$, open. Let $W = Y - U$, $W = \cup W_i$, W_i closed, irreducible.

Let $Z_i = f^{-1}W_i$, $Z_i = \cup Z_{ij}$, Z_{ij} closed, irreducible.

Consider $f|_{Z_{ij}} : Z_{ij} \to W_i$. By induction (as $\dim W_i < \dim Y$),

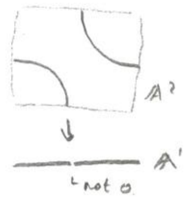
$f(Z_{ij}) \subset W_i$ is constructible, i.e. $f(Z_{ij}) \subset Y$ is constructible.

Key observation: $f(X) = U \cup f(Z_{ij}) \Rightarrow f(X)$ is constructible.

Definition: An algebraic variety X is proper $\Leftrightarrow p_2 : X \times Y \to Y$ is closed for all algebraic varieties Y.

Example: $\mathbb{A}^1$ is not proper because $p_2 : \mathbb{A}^2 \to \mathbb{A}^1$ is not closed.

$p_2(xy = 1) = \mathbb{A}^1 - \{0\}$, which is not closed.



Theorem: Projective $\Rightarrow$ proper.

Proof: (i) It is enough to prove that $\mathbb{P}^n$ is proper.

(Exercise: X proper, $Z \subset X$ closed $\Rightarrow Z$ proper)

(ii) X is proper $\Leftrightarrow p_2 : X \times Y \to Y$ is closed for Y affine.

(iii) X is proper $\Leftrightarrow p_2 : X \times \mathbb{A}^m \to \mathbb{A}^m$ is closed for all m.

By these, only need: $p_2 : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is closed. (Proved on page 1!)

(Fundamental Theorem of Elimination Theory).

Consequence: X connected projective variety, $\Gamma(X, \mathcal{O}_X) = k$

Proof: (i) $f : X \to Y$ morphism, X proper $\Rightarrow f(X)$ is proper.

$$\begin{tikzcd} \Gamma_f = \{(x,y) : y = f(x)\} \arrow[r, phantom, "\underset{\text{closed}}{\subset}"] \arrow[d, "p_1"] & X \times Y \arrow[d, "p_2"] \\ X & Y \end{tikzcd}$$

$f \in \Gamma(X, \mathcal{O}_X)$ is the same as saying $f : X \to \mathbb{A}^1_k$. Moreover, $f(X)$ is proper.

Corollary: $f : X \to Y$, X proper. Then $Y \ni y \mapsto \max\{\dim Z : Z \subset f^{-1}(y), \text{ component}\}$, (let $\dim \emptyset = -1$), is upper semi-continuous.

Theorem: Let $X \subset \mathbb{P}^3$ be a smooth (non-singular) cubic surface. Then X contains exactly 27 lines.

Lemma: Let $X \subset \mathbb{P}^3$ be a cubic surface (not necessarily smooth). Then X contains a line.

Grassmann Varieties.

$G(k,n) = \{ k\text{-dimensional linear subspaces } L_k \subset \mathbb{P}^n \}$.

So, for example, $G(1,3) = \{\text{lines in fixed } \mathbb{P}^3\}$.

Claim: $G(k,n)$ is a projective algebraic variety, in a natural manner, $\dim = (n-k)(k+1)$.

Proof: $G(k,n) = V_{k+1} \subset \mathbb{C}^{n+1}$. Let $M_{i_0 \cdots i_k}$ for $i_0 < \cdots < i_k$, be $\{(n+1)\times(k+1) \text{ matrices } V_{ij} : \det(v_{i_\iota, j}) \neq 0\}$

Eg: $G(1,3) = V_2 \subset \mathbb{C}^4$. $M_{01} = \begin{pmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \\ v_{20} & v_{21} \\ v_{30} & v_{31} \end{pmatrix} \cdot \underset{\substack{\Uparrow \\ G}}{\left(\begin{array}{c|c} a_{00} & a_{01} \\ \hline a_{10} & a_{11} \end{array}\right)} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ v_{20} & v_{21} \\ v_{30} & v_{31} \end{pmatrix}$ So $M_{01} \overset{G}{\cong} \mathbb{A}^4 \times G$.

$G \times M \to M \qquad G : (\mathbb{A}^4 \times G)$.

Remark: $G = GL(k+1)$ acts on $\bigcup_{i_0 < \cdots < i_k} M_{i_0 \cdots i_k} = M$. $G(k,n) = M/G$.

Note $M_{i_0 \cdots i_k}$ is G-invariant. In fact, $M_{i_0 \cdots i_k} \overset{G}{\cong} \mathbb{A}^{(n-k)\times(k+1)} \times G$.

$\Rightarrow$ quotient $M_{i_0 \cdots i_k}/G \xrightarrow{\cong} \mathbb{A}^{(n-k)\times(k+1)}$. These quotients are the affine pieces of the Grassmann. Write $U_{i_0 \cdots i_k}$ for $M_{i_0 \cdots i_k}/G$.

Exercise: show that the coordinate changes $U_{i_0 \cdots i_k} \to U_{j_0 \cdots j_k}$ are algebraic.

It is a projective variety: $G(k,n) \longrightarrow \mathbb{P}^{\binom{n+1}{k+1}-1}$

$V = \langle v_0, \ldots, v_k \rangle \longmapsto P_{i_0 \cdots i_k}(V) = v_{i_0} \wedge \cdots \wedge v_{i_k}$.

So in $G(1,3)$, $P \to (P_{01}, P_{02}, P_{03}, P_{12}, P_{13}, P_{23}) \subset \mathbb{P}^5$.

$G(1,3) = (P_{01}P_{23} - P_{02}P_{13} + P_{03}P_{12} = 0) \subset \mathbb{P}^5$ (exercise).

Proof of lemma: Let $\Lambda = \{\text{cubic surfaces in } \mathbb{P}^3\} = \mathbb{P}^N$, some N. ($N = 19$?)

Consider $G(1,3) \times \Lambda \supset I = \{(L,X) : L \subset X\}$. The projections: $G \xleftarrow{p_1} I \xrightarrow{p_2} \Lambda$. Need p_2 surjective.

If $L \in G$, $p^{-1}(L) = \{X : X \supset L\} = L_{N-4} \subset \Lambda$, ie, is of codimension 4 in Λ.

(Eg: $(x_2 = x_3 = 0) \subset X = \{f_3(x_0, \ldots, x_3) = 0\}$. Condition is $f(x_0, x_1, 0, 0) = 0$ - need coefficients $x_0^3, x_0^2x_1, x_0x_1^2, x_1^3$ to vanish. Ie, 4 conditions.)

The Grassmann has dimension 4, $\Rightarrow \dim I = N$.

I proper $\Rightarrow$ okay if p_2 is dominant. By dimension theory, okay if we can produce one cubic that has n lines, $1 \le n < \infty$. [By semi-continuity, $\dim I = \dim p_2(I)$]

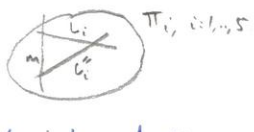
So here is a cubic: $x_0x_1x_2 - x_3^3 = 0$. This has three lines, so we are done.

Proof of Theorem: (a) Fix $m \subset X$. Idea: study lines $l \subset S$ such that $l \cap m \neq \emptyset$.

$\langle m, l \rangle = \Pi$, a 2-plane. $\Pi \cap S = m + l + l'$. Look that $\mathbb{P}^1 \ni \lambda \longmapsto \Pi_\lambda \cap X \setminus m = Q_\lambda$.

Want to study $\{\lambda : \det Q_\lambda = 0\}$.

(b) (Exercise): $\lambda \mapsto \det Q_\lambda$ is homogeneous of degree 5 in λ.



(c) Let $n \subset X$ be another line, $n \neq m, l_1, l_1'$.

$\Rightarrow n$ intersects either m or l_1 or l_1'. So each of m, l_1, l_1' intersects 10 lines, ie, 8 others, giving $3 + 8\times 3 = 27$ lines.

Zariski Tangent Space.

$f \in k[X_1,..,X_n]$, $p \in V(f)$, say $p = (a_i)$. $f^{(1)}{}_p = \Sigma \frac{\partial f}{\partial x_i}(p) \cdot (x_i - a_i)$.

Definition: If $X \subset \mathbb{A}^n_k$ is an affine algebraic variety, $p \in X$, Let $T_p X = \bigcap_{f \in IX} \{f^{(1)}_p = 0\}$.

Theorem: (i) The function $X \ni p \mapsto \varepsilon(p) := \dim T_p X$ is upper semi-continuous.
In particular, $\exists\ \emptyset \neq X_0 \subset X$, open, such that $\varepsilon(p) = r$ if $p \in X_0$
$\varepsilon(p) > r$ if $p \in X \setminus X_0$.

(ii) $r = \dim X$.

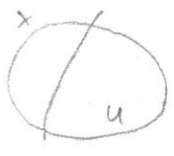
Proof: (i) $IX = (g_1,..,g_s)$. Matrix $(a_{ij}(x)) = \partial g_i / \partial x_j$. $\varepsilon(p) \geq r \iff \mathrm{rk}\,(a_{ij}(p)) \leq n-r$.
So all $(n-r)\times(n-r)$ determinants of $a_{ij}(p) = 0$, and so $\{\varepsilon(p) \geq r\}$ is closed.

Proposition: $T_p X = (m_p / m_p^2)^\vee$. $m_p \subset k[X]$ is the maximal ideal corresponding to p.

Proof: Assume wlog $p = (0,..,0)$. First consider $X = \mathbb{A}^n_k$. $M = M_0 \subset k[X_1,..,X_n]$.
$M \to (T_0 \mathbb{A}^n = k^n)^\vee$. This has kernel functions whose differential vanishes, ie M^2.
$f \mapsto df$ It is surjective.
So $M/M^2 \cong T_0 \mathbb{A}^n$
In general, if $0 \in X \subset \mathbb{A}^n$. $M \xrightarrow{\varphi} T_0 X^\vee$
$f \mapsto df|_{T_0 X}$.
Claim: $\mathrm{Ker}\,\varphi = IX + M^2$, $(\Rightarrow \tilde{\varphi}: M/M^2 \xrightarrow{\cong} T_0 X^\vee)$, if $df|_{T_0 X} = 0$, by definition, $df = dg$, some $g \in I$, ie. $f - g \in M^2$.

Proof of Theorem: (ii) Trivial if $X = V(f)$.
Claim: $X \underset{\text{open}}{\supset} U$, such that U is a hypersurface, or an open subset of one.
Proof: May assume that X is affine. By Noether, $k \subset k[Y_1,..,Y_n] \subset k[X]$, and the latter is a finite extension. May apply the primitive element theorem to $k(Y_1,..,Y_n) \subset k(X) \Rightarrow k(X) = k(Y_1,..,Y_n)(\alpha)$.
α satisfies $\Sigma \frac{f_i}{g_i} X^i = 0$, where $f_i, g_i \in k[Y_1,..,Y_n]$. Let $g = \Pi g_i$.
The hypersurface in question is: $F(Y_i, X) = \Sigma \frac{f_i g}{g_i} X^i = 0$.
Exercise: $X_g = V(F)_g$.

Remark: This claim implies the theorem (ii), by reduction to a hypersurface.



$X \supset U \subset Y$, hypersurface.

$X \supset X_0$, open, such that $\varepsilon|_{X_0} = r$
$Y \supset Y_0$, open, such that $\varepsilon|_{Y_0} = r$.

For Y, $\dim Y = r$. But $\dim Y = \dim U = \dim X$.

Definition: $f: X \dashrightarrow Y$ is a rational map if f is a morphism $U \to Y$ where $\emptyset \neq U \subset X$ is an open subset.

Example: $\{f: X \to \mathbb{A}^1_k\} = k(X)$.

Remark: If $f: X \dashrightarrow Y$ is dominant (ie, $\overline{f(U)} = Y$), then $g \circ f$ makes sense for all $g: Y \dashrightarrow Z$.

Exercise: $\{$ dominant rational maps $X \dashrightarrow Y\} = \{k$-homomorphisms $k(Y) \hookrightarrow k(X)\}$.

So, the claim above could be stated as "every variety is birationally equivalent to a hypersurface".

Example: $X = Bl_0 \mathbb{A}^2_k \to \mathbb{A}^2$. By definition, $X = (x\eta = y\xi) \subset \mathbb{A}^2 \times \mathbb{P}^1$

(x,y), coordinate on $\mathbb{A}^2$. $\quad \downarrow \pi_1$

$(\xi:\eta)$ coordinate on $\mathbb{P}^1$. $\quad \mathbb{A}^2$

$X = X_\xi \cup X_\eta$, X_ξ, X_η affine.

$X_\xi = (\xi \neq 0) \cong \mathbb{A}^2$, $(x, \eta/\xi = \eta') \xrightarrow{\pi} (x, x\eta')$.

$X_\eta = (\eta \neq 0) \cong \mathbb{A}^2$, $(\xi' = \xi/\eta, y) \mapsto (\xi' y, y)$.

Theorem: If $X \subset \mathbb{P}^3$ is a smooth cubic surface, then $X \dashleftarrow\dashrightarrow \mathbb{P}^2$ (birationally equivalent).

Proof: Let $l, n \subset X$ be disjoint lines. $X \setminus (l \cup n) \to l \times n$.

$p \mapsto (L_p \cap l, L_p \cap n)$,

where L_p is the unique line such that $p \in L_p$, $L_p \cap l \neq \emptyset$, $L_p \cap n \neq \emptyset$.

A. Coherent Sheaves.

Sheaves of modules: If $(X, \mathcal{O}_X)$ is a topological space, $\mathcal{O}_X$ a sheaf of rings of continuous functions, we can speak of sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules.

Exercise: $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, continuous, $\mathcal{O}_Y \circ f \subset \mathcal{O}_X$, then f produces functors:

$f_*: \{\mathcal{O}_X\text{-modules}\} \to \{\mathcal{O}_Y\text{-modules}\}$ (restriction of scalars)

$f^*: \{\mathcal{O}_Y\text{-modules}\} \to \{\mathcal{O}_X\text{-modules}\}$ (extension of scalars)

[Distinguish between $*$ and $\circ$: $f^*\mathcal{F} = f^\circ \mathcal{F} \otimes_{f^\circ \mathcal{O}_Y} \mathcal{O}_X$],

such that $\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$.

Examples: X be an affine algebraic variety, M a f.g. $k[X]$-module. We define a sheaf $M^\sim$ of $\mathcal{O}_X$-modules in the following way:

$X \supset U \mapsto M^\sim(U) := \{ s: U \to \prod_{x\in U} M_x : \forall\, y \in U\ \exists\, m \in M, f \in \mathcal{O}_{X,y}$ such that $s = m/f$ in a neighbourhood of $y \in U\}$,

where $M_x = S^{-1}M$, $S = k[X] - m_x$.

Ie, have X, $\{X_g : g \in k[X]\}$. M a $k[X]$-module. $M^\sim(X_g) = M_g = \{\frac{m}{g^n}\}/\sim$.

Definition: The fibre of $M^\sim$ at $x \in X$ is $M_x \otimes_{\mathcal{O}_{X,x}} k(x)$.

Theorem: Let X be an algebraic variety, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules.
The following are equivalent:

(i) There is an open covering $X = \bigcup U_i$ and surjective homomorphisms $g_i : \mathcal{O}_{U_i}^{s_i} \to \mathcal{F}|_{U_i}$, $s_i \in \mathbb{N}$.

(ii) $\exists\ X = \bigcup U_i$ and exact sequences $\mathcal{O}_{U_i}^{t_i} \xrightarrow{r_i} \mathcal{O}_{U_i}^{s_i} \xrightarrow{g_i} \mathcal{F}|_{U_i} \to 0$.

(iii) $\exists\ X = \bigcup U_i$, U_i affine, $k[U]$-modules M_i which are f.g., and isomorphisms $\mathcal{F}|_{U_i} \cong \tilde{M_i}$.

(iv) for all affine coverings $X = \bigcup U_i$, (iii) holds.

Definition: If (i) - (iv) hold, we say that $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules.

Warning: $f : X \to Y$ morphism. $\mathcal{F}$ $\mathcal{O}_Y$-coherent $\Rightarrow f^*\mathcal{F}$ coherent on X,
but $\mathcal{F}$ $\mathcal{O}_X$-coherent $\not\Rightarrow f_*\mathcal{F}$ coherent on Y.

Example: X affine, $\mathcal{F} = \mathcal{O}_X$, $f : X \to pt$. $f_*\mathcal{O}_X = \Gamma(X, \mathcal{O}_X) = k[X]$. This is not coherent because $k[X]$ is not a finite dimensional k-vector space.

Theorem: $f : X \to Y$ proper, $\mathcal{F}$ coherent on $X \Rightarrow f_*\mathcal{F}$ is coherent on Y.

Examples (i): X algebraic variety. Ω_X^1 a nice coherent sheaf on X.
For $\mathcal{F}$ an $\mathcal{O}_X$-module, define: a derivation with values in $\mathcal{F}$ is a sheaf homomorphism $d : \mathcal{O}_X \to \mathcal{F}$ such that
(i) d is k-linear: $d(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 d(f_1) + \lambda_2 d(f_2)$, $\lambda_i \in k$.
(ii) $d(f.g) = f dg + g\, df$.
Ω^1 is the target of the universal derivation. Ie, $\exists$ derivation $d : \mathcal{O}_X \to \Omega_X^1$, and for all derivations $d_{\mathcal{F}} : \mathcal{O}_X \to \mathcal{F}$ ($\mathcal{F}$ coherent), $\exists$ a unique $\mathcal{O}_X$-module homomorphism $\varphi : \Omega^1 \to \mathcal{F}$ such that the diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "d"] \arrow[dr, "d_{\mathcal{F}}"'] & \Omega^1 \arrow[d, "\varphi"] \\ & \mathcal{F} \end{tikzcd}$$

commutes.

Examples: (ii) $\Theta_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) = Der_k(\mathcal{O}_X, \mathcal{O}_X)$ is the tangent sheaf.

Exercise: Fibre of Θ_X at $x \in X$ is $(m_x/m_x^2)^\vee$.

B. Cohomology.

Let X be a topological space, $X = \bigcup U_i$ a covering, $\mathcal{F}$ a sheaf of abelian groups on X. We define a complex of abelian groups:
$C^\bullet(\{U_i\}, \mathcal{F})$, $\delta : C^p \to C^{p+1}$. Notation: if $i_0 < \dots < i_p$, $U_{i_0 \dots i_p} = U_{i_0} \cap \dots \cap U_{i_p}$.
$C^p(\{U_i\}, \mathcal{F}) = \prod_{i_0 < \dots < i_p} \mathcal{F}(U_{i_0 \dots i_p})$.

If $f \in C^p$ and $f_{i_0 \dots i_p}$ is the $i_0 \dots i_p$ - component, then
$(df)_{i_0 \dots i_{p+1}} = \Sigma (-1)^j \rho_j f_{i_0 \dots \hat{i}_j \dots i_{p+1}}$, where $\rho_j : \mathcal{F}(U_{i_0 \dots \hat{i}_j \dots i_{p+1}}) \to \mathcal{F}(U_{i_0 \dots i_{p+1}})$ is the restriction map.

Exercise: $C^\cdot$ is a complex (ie, $d^2 = 0$).

Definition: $\check{H}^p(\{U_i\}, \mathcal{F}) := H^p(C^\cdot(\{U_i\}, \mathcal{F}))$

Exercise: $\check{H}^0(\{U_i\}, \mathcal{F}) = \Gamma(X, \mathcal{F})$, all coverings.

Theorem: Let X be an affine algebraic variety, $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. For all Zariski coverings $\{U_i\}$ of X, $\check{H}^i(\{U_i\}, \mathcal{F}) = (0)$, all $i \geq 1$.

Corollary 1: X any algebraic variety, $\mathcal{F}$ a coherent sheaf, $\{U_i\}$ affine cover of X $\Rightarrow \check{H}^p(\{U_i\}, \mathcal{F})$ does not depend on $\{U_i\}$, and is denoted $H^p(X, \mathcal{F})$, or simply $H^p(\mathcal{F})$.

Corollary 2: X an algebraic variety, and $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent $\mathcal{O}_X$-modules, then $\exists$ long exact sequence
$\dots \to H^p(\mathcal{F}') \to H^p(\mathcal{F}) \to H^p(\mathcal{F}'') \xrightarrow{d} H^{p+1}(\mathcal{F}') \to \dots$

Proof: Exercise.

Definition: (i) A coherent sheaf on X is locally free of rank r $\Leftrightarrow \exists \{U_i\}$, a covering of X such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$.
(ii) A line bundle on X is a locally free sheaf of rank 1.

Definition: $\chi(X, \mathcal{F}) = \Sigma (-1)^i h^i(X, \mathcal{F})$, $h^i = \dim H^i$.

Theorem (Riemann-Roch): If C is a proper smooth connected curve, $\mathcal{L}$ a line bundle on C, then $\chi(\mathcal{L}) = \chi(\mathcal{O}_C) + \deg \mathcal{L}$.

Theorem (Serre Duality): $C, \mathcal{L}$ as above. There is a perfect duality pairing $H^0(\mathcal{L}) \otimes H^1(\Omega^1_C \otimes \mathcal{L}^\vee) \to k$. (ie, $H^1(\Omega^1_C \otimes \mathcal{L}^\vee) = H^0(\mathcal{L})^\vee$ as k-vector spaces).

Note: $\mathcal{F}$ is coherent on a curve C, then $H^i(C, \mathcal{F}) = (0)$ if $i \geq 2$.

Remark: $\{\text{line bundles on } X\}/\cong \; = \check{H}^1(X, \mathcal{O}_X^*) = \varinjlim_{\{U_i\}} \check{H}^1(U_i, \mathcal{O}^*)$

Let $\mathcal{L}$ be a line bundle on X, $\{U_i\}$ a covering such that $\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\cong} \mathcal{O}_{U_i}$, then $\varphi_i \varphi_j^{-1} : \mathcal{O}_{U_j}|_{U_{ij}} \xrightarrow{\cong} \mathcal{O}_{U_i}|_{U_{ij}}$. $\Rightarrow \varphi_i \varphi_j^{-1}(s) = g_{ij}(x) \cdot s$, $g_{ij} : U_{ij} \to k^*$, ie $g_{ij} \in \mathcal{O}(U_{ij})^*$.
Note that $\varphi_i \varphi_j^{-1} \varphi_j \varphi_k^{-1} = \varphi_i \varphi_k^{-1} \Rightarrow g_{ij} g_{jk} = g_{ik}$.
Ie, $g_{ij} \in \mathcal{O}(U_{ij})^*$ define an element $g \in C^1(\{U_i\}, \mathcal{O}^*)$, with $\delta g = 0$ in $C^2(\{U_i\}, \mathcal{O}^*)$
$\Rightarrow$ a cohomology class $(g_{ij}) \in \check{H}^1(\{U_i\}, \mathcal{O}^*)$.

Remark: What is $s \in \Gamma(X, \mathcal{L})$? $s_i = \varphi_i(s|_{U_i}) \in \mathcal{O}(U_i)$. So $s_j = \varphi_j \varphi_i^{-1} s_i = g_{ji} s_i$.
So $\Gamma(X, \mathcal{L}) \longleftrightarrow \{ s_i \in \mathcal{O}(U_i) : s_i = g_{ij} s_j \}$.

Definition: $\operatorname{Pic} X = \{ \text{line bundles on } X \} / \cong$.
This is a group under $\otimes$. $\mathcal{L} \leftrightarrow g_{ij}$, $\mathcal{L}' \leftrightarrow g'_{ij}$, $\mathcal{L} \otimes \mathcal{L}' \leftrightarrow g_{ij} \cdot g'_{ij}$.

Example: Coherent sheaves on a projective variety $X \subset \mathbb{P}^n$.
$R :=$ homogeneous coordinates of $X = k[X_0, \dots, X_n]/I(X) = \bigoplus_{n \geq 0} R_n$.
$M = \bigoplus_{n \geq 0} M_n$, a finitely generated R-module. $M \to M^\sim$ a coherent sheaf on X.
If $x \in X$, let $m_x \subset R$ be the maximal ideal corresponding to x.
Let $S_x = \{ \text{homogeneous } f : f \notin m_x \}$.
$M_x = S_x^{-1} M_{(0)} = \{ \frac{m}{s} : m \in M_k,\ s \in S_x, \text{ homogeneous of degree } k \}$.
$X \underset{\text{open}}{\hookleftarrow} U \longmapsto M^\vee(U) := \{ s : U \to \prod_{x \in U} M_x : \forall x \in U, \exists x \in V \subset U, \exists m \in M_k, t \in R_k$
such that $s(y) = m/t \ \forall y \in V \}$

Remark: Every sheaf on X arises in this way.

Example: $X = \mathbb{P}^n$. $R = k[X_0, \dots, X_n]$. Let $M = R(k)$ (ie, $M = R$, but $M_l = R_{k+l}$).
$\mathcal{O}(k) := R(k)^\sim$. $\mathcal{O}(k)$ is a line bundle, $\mathcal{O}(k_1) \otimes \mathcal{O}(k_2) = \mathcal{O}(k_1 + k_2)$.
$\Gamma(\mathbb{P}^n, \mathcal{O}(k)) = \{ \text{homogeneous polynomials of degree } k \}$.

Proposition: Let X be an algebraic variety. There is a 1-1 correspondance
$$\left\{ \begin{array}{l} \text{line bundles } \mathcal{L} \text{ on } X, \\ s_0, \dots, s_n \in \Gamma(X, \mathcal{L}) \\ \text{with no common zeroes} \end{array} \right\} \longleftrightarrow \{ \varphi : X \to \mathbb{P}^n \}$$

Proof: Given $\varphi : X \to \mathbb{P}^n$, define $\mathcal{L} = \varphi^* \mathcal{O}(1)$, $s_i = \varphi^*(X_i)$.
Given $X, \mathcal{L}$, $s_i \in \Gamma(X, \mathcal{L})$, $X \ni x \overset{\varphi}{\longmapsto} (s_0(x), \dots, s_n(x))$.

Divisors.

Definition: X is non-singular in codimension 1 iff $\mathcal{O}_{X,\Gamma}$ is a DVR for all $\Gamma \subset X$, irreducible, $\operatorname{codim}_\Gamma X = 1$.
($U_i \underset{\text{affine}}{\hookrightarrow} X$, $\Gamma \cap U_i \neq \phi$. $\mathcal{O}_{X,\Gamma} := k[U_i]_{I(\Gamma)} \subset k(X)$ - does not depend on $U_i \subset X$).

Examples: (i) C is a smooth curve $\Rightarrow$ C non-singular in codim 1.
(ii) If $\operatorname{Sing} X \subset X$ has codim ≥ 2, then X is non-singular in codim 1.

Will assume for now that X is non-singular in codim 1.

Definition: (i) $\operatorname{Div} X = \bigoplus_{\substack{\Gamma \hookrightarrow X \\ \operatorname{codim}_\Gamma X = 1}} \mathbb{Z}[\Gamma]$. $D = \sum_{\text{finite}} n_\Gamma \cdot \Gamma$ is called a Weil divisor.

(ii) There is an exact sequence: $k^* \to k(X)^* \to \operatorname{Div} X \to 0$
$$f \longmapsto \operatorname{div} f := \sum_{\Gamma \subset X} U_\Gamma(f) \cdot \Gamma.$$

Exercise: The sum in (ii) is finite.

D is principal $\iff$ $D = \operatorname{div}(f)$, some $f \in k(X)$.
$D_1 \equiv D_2$ - "linearly equivalent" $\iff$ $D_1 - D_2$ is principal.
$D \geqslant 0$ - "effective" $\iff$ $D = \sum n_\Gamma . \Gamma$, $n_\Gamma \geqslant 0$.
$Cl\, X := \frac{\operatorname{Div} X}{\equiv}$

Definition: X is factorial $\iff$ $\mathcal{O}_{X,x}$ is a UFD for all $x \in X$.

Exercise: (i) A smooth curve C is factorial.
(ii) A smooth variety is factorial.
(iii) $X = (xy = z^2) \subset \mathbb{A}^3$ is not factorial (as $z^2 = z.z = x.y$ - not unique).
(iv) $X = (x^2 + y^2 + z^2 + t^3 = 0) \subset \mathbb{A}^4$ is factorial.

Theorem 1: X factorial $\Rightarrow$ $Cl\,X = \operatorname{Pic} X$.
(Idea: $D = \sum n_\Gamma \Gamma \mapsto \mathcal{L}(D) = \{f : \operatorname{div} f + D \geqslant 0\}$)

Theorem 2: C a proper smooth curve. $f \in k(C)$, $\operatorname{div} f = \sum n_\Gamma . \Gamma \Rightarrow \sum n_\Gamma = 0$.

Definition: If $\mathcal{L} = \mathcal{L}(D)$, $D = \sum n_P P$, $\deg \mathcal{L} := \sum n_P$.

1st Chern Class.

X is non-singular in codimension 1.

Theorem: $\exists$ natural map $c_1 : \operatorname{Pic} X \to Cl\,X = \operatorname{Div} X/\equiv$

Proof: Let $K = k(X)$. We have: $0 \to \mathcal{O}^\times \to K^\times \to K^\times/\mathcal{O}^\times \to 0$.

[Notes: (i) if X is a topological space, S a sheaf of abelian groups on X, define $H^p(S) := \varinjlim_{\{U_i\}} \check{H}^p(\{U_i\}, S)$.
(ii) if $0 \to S' \to S \to S'' \to 0$ is exact, get a long exact sequence
$\cdots \to H^p(S') \to H^p(S) \to H^p(S'') \to H^{p+1}(S') \to \cdots$]

Now, $H^0(K^\times/\mathcal{O}^\times) \xrightarrow{\delta} H^1(\mathcal{O}^\times) \to H^1(K^\times)$

$\therefore$ surjective $\qquad$ $H^1(K^\times) = (0)$ - exercise

Let $\mathcal{L} \in \operatorname{Pic} X = H^1(\mathcal{O}^\times)$ be a line bundle. $\exists\, s \in \Gamma(X, K^\times/\mathcal{O}^\times)$ such that $\delta s = \mathcal{L}$
$s \leftrightarrow \{U_i\}$, $s_i \in K^\times(U_i) = K^\times$, $s_i|_{U_{ij}} = g_{ij}\, s_j|_{U_{ij}}$, $g_{ij} \in \mathcal{O}^\times(U_{ij})$. $[U_{ij} = U_i \cap U_j]$.
By definition of δ, g_{ij} is a cocycle defining $\mathcal{L}$, and $s = (s_i) \in \Gamma(X, \mathcal{L} \otimes K)$
$:= \{$meromorphic sections of $\mathcal{L}\}$.
$s \longmapsto \operatorname{div}(s) = D$, such that $D|_{U_i} = \operatorname{div}(s_i)$
Plan is to define $c_1(\mathcal{L}) = \operatorname{div}(s)$. Now, if $\delta(s') = s$, by the long exact sequence, $s' = \varphi s$, $\varphi \in K^\times$, $\operatorname{div} s' = \operatorname{div} s + \operatorname{div} \varphi$, iff $\operatorname{div} s' \equiv \operatorname{div} s$
Ie, $c_1 : \operatorname{Pic} X \to \frac{\operatorname{Div} X}{\equiv} = \operatorname{Pic} X$ is well-defined.

Theorem: X factorial $\Rightarrow c_1 : \mathrm{Pic}\, X \xrightarrow{\cong} \mathrm{Cl}\, X$.

Proof: Define $\mathrm{Cl}\, X \to \mathrm{Pic}\, X$

$D \mapsto \mathcal{L}(D)$, a sheaf.

$\mathcal{L}(D) \subset K$, $\Gamma(U, \mathcal{L}(D)) := \{ \varphi \in K^\times : (\mathrm{div}\, \varphi + D) \cap U \geq 0 \} \cup \{0\}$

X factorial $\Rightarrow \exists \{U_i\}$, covering of X, affine, $\varphi_i \in K^\times$ such that $D \cap U_i = \mathrm{div}\, \varphi_i$.

[X factorial, $\Gamma \subset X$, subvariety of codim 1, then $\exists$ neighbourhood U such that $x \in U$, $\Gamma \cap U = V(f)$, $f \in k[U]$]

$\Gamma(U_i, \mathcal{L}(D)) = \frac{1}{\varphi_i} \cdot \mathcal{O}(U_i) \subset K$. Ie, $\mathcal{L}(D)|_{U_i} \cong \mathcal{O}(U_i)$, so $\mathcal{L}(D)$ is a line bundle.

This map is an inverse to c_1 (exercise).

B. Linear Systems.

Assume X is factorial.

(i) $D \in \mathrm{Div}\, X$, define $|D| = \{ D' : D' \equiv D, D' \geq 0 \}$ - this is a complete linear system.

(ii) $|D| = \mathbb{P} H^0(X, \mathcal{L}(D))$.

Choose covering $\{U_i\}$, $s_i \in K^\times$ such that $D|_{U_i} = \mathrm{div}(s_i)$. Then $s_i|_{U_{ij}} = g_{ij} \cdot s_j|_{U_{ij}}$, $g_{ij} \in \mathcal{O}(U_{ij})^\times$, is a cocycle defining $\mathcal{L}(D)$. $s_i = g_{ij} s_j \Leftrightarrow s = (s_i) \in \Gamma(X, \mathcal{L} \otimes K)$.

if $D' \equiv D$, $D' = \mathrm{div}(s\varphi)$, $\varphi \in K^\times$, $D' \geq 0 \Leftrightarrow \sigma_i = s_i \varphi \in \mathcal{O}(U_i)$. $\Leftrightarrow \sigma = (\sigma_i) \in \Gamma(X, \mathcal{L})$.

This gives a well-defined 1-1 correspondance $|D| \leftrightarrow \mathbb{P} H^0(X, \mathcal{L})$

(iii) A linear system is a linear subspace, $\delta \subset |D|$. $\mathrm{Bs}\, \delta := \{ x \in X : x \in D' \text{ all } D' \in \delta \}$.

δ is free $\Leftrightarrow \mathrm{Bs}\, \delta = \phi$.

(iv) There is a 1-1 correspondance $\left\{ \begin{array}{c} \text{free linear} \\ \text{systems } \delta \text{ on } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{morphisms} \\ \varphi : X \to \mathbb{P}^1 \end{array} \right\}$

$\delta \longmapsto \varphi_\delta : X \to \delta^\vee$

$x \mapsto \{ D \in \delta : D \ni x \} \in \delta^\vee$

C. Riemann-Roch.

C = smooth connected proper algebraic curve.

Divisor $D = \sum n_i P_i$, say that $\deg D = \sum n_i$. At this stage, do not know that $D' \equiv D \Rightarrow \deg D' = \deg D$.

Theorem: $\chi(\mathcal{L}(D)) = \chi(\mathcal{O}) + \deg D$.

Corollary: (i) $D \equiv D' \Rightarrow \deg D = \deg D'$

(ii) Via $c_1 : \mathrm{Pic}\, C \xrightarrow{\cong} \mathrm{Cl}\, C$, can define $\deg : \mathrm{Pic}\, C \to \mathbb{Z}$.

(iii) $\chi(\mathcal{L}) = \chi(\mathcal{O}) + \deg \mathcal{L}$, any line bundle.

Proof of Theorem: Follows by induction, using the following lemma:

Lemma: Let $D_1 = D_2 + p$. Then $\chi(\mathcal{L}(D_1)) = \chi(\mathcal{L}(D_2)) + 1$.

Proof: Have $0 \to \mathcal{L}(D_1) \to \mathcal{L}(D_2) \to i_p * k \to 0$, $i_p : \{p\} \to C$.

"Super cool": $h^0(i_{p*} k) = 1$, $h^i(i_{p*} k) = 0$, $i \geq 1$.

So $\chi(i_{p*} k) = 1$

But $\chi(\mathcal{L}(D_2)) = \chi(\mathcal{L}(D_1)) + \chi(i_{p*} k)$. So done.

Theorem: X an algebraic variety, $\mathcal{F}$ a coherent sheaf on X. Then $H^p(X, \mathcal{F}) = 0$ for $p > \dim X$.

"Proof": (i) $H^p(X, \mathcal{F}) = \check{H}^p(\{U_i\}, \mathcal{F})$, for any affine cover $\{U_i\}$.

(ii) if $X \subset \mathbb{P}^n$, $\dim X = d$, choose coordinates x_i on $\mathbb{P}^n$ such that $X \cap \{x_{i_0} = \cdots = x_{i_p} = 0\} = \phi$, if $p > d$. $U_i := X - \{x_i = 0\}$ is an affine cover of X such that $U_{i_0 \cdots i_p} = \phi$ if $p > \dim X$, $\Rightarrow C^p(\{U_i\}, \mathcal{F}) = (0)$ if $p > \dim X$, any $\mathcal{F}$.

(iii) Question: For X an algebraic variety, does there exist an affine cover $\{U_i\}$ of X such that $U_{i_0 \cdots i_p} = \phi$ if $\dim X < p$?
(if X is proper, $\exists$ birational morphism $p: Y \to X$, Y projective - "Chow lemma").

Theorem (Riemann-Roch): C proper, smooth, connected algebraic variety of dimension 1 (ie, C is a curve), $\mathcal{L}$ a line bundle on C, then:
$h^0(C, \mathcal{L}) - h^1(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g$, where $g = h^1(\mathcal{O}_C)$ is the genus of C.

Theorem (Serre Duality): $H^1(\mathcal{L}) = H^0(\Omega_C \otimes \mathcal{L}^\vee)^\vee$

Theorem: X an algebraic variety, $x \in X$. The following spaces are canonically isomorphic:

(i) $T_{X,x} = \mathrm{Hom}(m_x/m_x^2, k)$

(ii) $\{k\text{-linear derivations } D: \mathcal{O}_{X,x} \to k(x)\}$ $= \mathcal{O}_{X,x}/m_x$, think of as a coherent sheaf

(iii) $\mathrm{Hom}_{X,x}(\Omega_{X,x}, k(x))$.

Proof: (i) <=> (ii): Take $D: \mathcal{O}_{X,x} \to k(x)$. So $D(fg) = f(x)Dg + g(x)Df$, all $f, g \in \mathcal{O}_{X,x}$.
$\Rightarrow D(m_x^2) = 0$. $\varphi \in m_x^2 \Leftrightarrow \varphi = \sum f_i g_i$, $f_i(x) = g_i(x) = 0$. In particular, D defines $l_D : m/m^2 \to k$.
Conversely, given $l: m/m^2 \to k$, define $D(f) = l(f - f(x))$.

(ii) <=> (iii): This is from the definition of Ω, by the universal property.

Exercise: X an algebraic variety, $\mathcal{F}$ a coherent sheaf on X. Define $\varphi(x) = \dim_k(\mathcal{F} \otimes k(x))$
$[\mathcal{F} \otimes k(x) = \mathcal{F}_x / m_x . \mathcal{F}_x]$.

(i) φ is upper semi-continuous.

(ii) $\varphi(x) = r$ is constant <=> $\mathcal{F}$ is locally free of rank r.

Proof: Use Nakayama.

Corollary: The following are equivalent:

(i) X is smooth

(ii) $\dim X = \dim_k m_x/m_x^2$, all $x \in X$. (Or just: $\dim_k m_x/m_x^2$ is constant)

(iii) Ω_X is locally free (rank $= \dim X$).

Note: (a) $\Omega_{S^{-1}A} = S^{-1}\Omega_A$

(b) If $A \twoheadrightarrow B$, $\exists$ exact sequence $I/I^2 \to \Omega_A \otimes B \to \Omega_B \to 0$.

These imply Ω is coherent.

Corollary to Theorem: $\Omega_{X,x} \otimes k(x) = m_x/m_x^2$

Applications of Riemann-Roch. (R-R)

(i) $g = h^1(\mathcal{O}) = h^0(\Omega)$, by Serre duality.

(ii) Apply R-R to $\mathcal{L} = \Omega$. $\left.\begin{array}{l} h^0(\Omega) - \underbrace{h^1(\Omega)}_{\overset{\shortparallel}{g}\,-1} = \deg \Omega + 1 - g \end{array}\right\} \Rightarrow \deg \Omega = 2g-2.$

[$\Rightarrow$ if C is over $\mathbb{C}$, then $g(C) = \#$holes. $\Omega = T^\vee$. Then $2g-2 = -\chi_{\text{topological}}$].

(iii) Hurwitz formula: $f: C' \to C$, finite, $d = [K':K]$. Ramification divisor, $R = \sum_{P \in C'} (\varepsilon_P - 1) \cdot P$, on C'. If $Q = f(P)$, $\mathcal{O}_{C,Q} \overset{f^\#}{\hookrightarrow} \mathcal{O}_{C',P}$ (DVR's). Let t, u be local parameters. $f^\# t = (\text{unit}) \cdot u^{\varepsilon_P}$.

Definition: The ramification is tame iff either (i) char $k = 0$

(ii) char $k \nmid \varepsilon_P$.

Theorem: If ramification is tame, then $2g'-2 = d(2g-2) + R$. $(d = \deg f)$.

Proof: $dt = \varepsilon_P u^{\varepsilon_P - 1}\, du$.

$(0 \to f^*\Omega \to \Omega' \to \overset{=R}{\Omega'/\Omega} \to 0)$

Example: If $f: C' \to \mathbb{P}^1$, then $R = 0$. So $d = 1$, $C' = \mathbb{P}^1$.

Serre Duality

Theorem: $H^1(C, \mathcal{L}) = H^0(C, \Omega_C \otimes \mathcal{L}^\vee)^\vee$

This can be proved in various ways:

(i) Hodge Theory (over $\mathbb{C}$).

(ii) Homological algebra (very general)

(iii) Adéles (number theory, "Tate Duality"). We will use (iii).

Let $K = k(C)$

Definition: $A = A_K = \coprod_{P \in C} K_P$, where:

(i) K_P is the completion of K wrt $\mathcal{O}_P \subset K$. (Here, can think of $K_P = K$).

(ii) $\coprod$ is the restricted direct product of the $\hat{\mathcal{O}}_P \subset K_P$.

("$\amalg \hookrightarrow \coprod \hookrightarrow \Pi$"). $\coprod K_P = \{(f_P)_{P \in C} : f_P \in K_P,\ f_P \in \hat{\mathcal{O}}_P, \text{almost all } P\}$.

Analogy: (iii) $\mathbb{Q} \hookrightarrow K$, number field. $\longleftrightarrow$ choosing $\pi: C \to \mathbb{P}^1_k$, $\pi^\#: k(t) \hookrightarrow k(C)$.

If $D = \sum_{p \in C} n_p P$, $A(D) = \{ (f_p)_{p \in C} \in A : \operatorname{ord}_p f_p \geq -n_p, \text{ all } p \}$.
$A = \bigcup_D A(D)$

Proposition: $H^1(C, \mathcal{L}) = A/_{A(D)+K}$.
Proof: Consider $0 \to \mathcal{L}(D) \to K \to K/_{\mathcal{L}(D)} \to 0$. This yields the long exact sequence:

$$H^0(K) \to H^0(K/_{\mathcal{L}(D)}) \to H^1(\mathcal{L}(D)) \to 0.$$
$$\overset{\shortparallel}{K} \qquad \overset{\cong\downarrow}{\textstyle\prod K_p/A(D) = A/A(D)}$$

So $H^1(\mathcal{L}) \cong A/_{A(D)+K}$.

Residues of Differentials.

Write Ω_K for $\Omega_C \otimes_{\mathcal{O}} K$, $= \Omega_{K/k}$ $(\cong K)$.
$K \xrightarrow{d} \Omega_K$
$t \mapsto dt$. After choosing $t \in K$, can think of $\Omega = K dt$.

$K \hookrightarrow K_p \cong k((t))$. $\Omega_K \hookrightarrow \hat{\Omega}_{K_p} = k((t))\, dt \xrightarrow{Res} k$
$$\sum_{n \geq -N} a_n t^n \longmapsto a_{-1}$$
Claim: $Res: \Omega_K \to k$ is well-defined, independent of the parameter t.

Properties of Residue. $Res_p : \Omega_K \to k$, for $p \in C$.
(i) k-linear
(ii) $Res\ \omega = 0$ if $v_p . \omega \geq 0$ $(\Leftrightarrow \omega \in \mathcal{O}_p . dt)$
(iii) $Res_p(d\varphi) = 0$, $\varphi \in K$.
(iv) $Res_p\left(\frac{d\varphi}{\varphi}\right) = v_p \varphi$, $\varphi \in K^\times$.

Theorem: $\sum_{p \in C} Res_p\, \omega = 0$.
Consider analogy with number fields. $\mathbb{Q} \hookrightarrow K$ $\updownarrow$ $k(t) \hookrightarrow k(C)$ $\deg \operatorname{div} \varphi = 0$.

For D a divisor, define the sheaf $\Omega(D) = \{ \omega \in \Omega \otimes K : \operatorname{div} \omega - D \geq 0 \}$.

Exercise: There is a non-canonical isomorphism $\Omega(D) \cong H^0(\Omega \otimes \mathcal{L}(D)^\vee)$.

Proof of Serre Duality: Want $\Omega(D) \times A/_{A(D)+K} \longrightarrow k$.
Take ω, $(f_p)_{p \in C}$. Define scalar product: $\langle \omega, (f_p)_{p \in C} \rangle = \sum_{p \in C} Res_p(\omega f_p)$.
Claim: this is a perfect duality.
This implies Serre Duality.

Further applications of Riemann-Roch.
(iv) $\deg \mathcal{L} \geq 2g-2 \Rightarrow H^1(\mathcal{L}) = (0)$ unless $\mathcal{L} \cong \Omega$.
$\deg \mathcal{L} > 2g-2 \Rightarrow H^1(\mathcal{L}) = (0)$.
$(H^1(\mathcal{L})^\vee = H^0(\Omega \otimes \mathcal{L}^\vee)$. $\deg(\Omega \otimes \mathcal{L}) = 2g-2-\deg \mathcal{L}$. $\deg(-) < 0 \Rightarrow h^0(-) = 0$.
$\deg(-) = 0 \Rightarrow \begin{cases} h^0(-) = 1, \text{ iff } (-) \cong \mathcal{O} \\ h^0(-) = 0, \text{ otherwise} \end{cases}$).

<u>Kodaira embedding</u>: if $\deg \mathcal{L} \geq 2g+1$, $\varphi_{\mathcal{L}} : C \hookrightarrow \mathbb{P}^{h_0(\mathcal{L})}$

<u>Proof:</u> (i) $\mathcal{L}$ is base-point free. $\Leftrightarrow$ $H^0(\mathcal{L}) \xrightarrow{\text{eval.}} k_p$ (surjective) for all $p \in C$.

$$0 \to \mathcal{L}(-p) \to \mathcal{L} \xrightarrow{\text{eval.}} k_p \to 0.$$

$H^0(\mathcal{L}) \to k_p \to H^1(\mathcal{L}(-p)) = (0)$ by (iv), so eval. is surjective.

So we get $\varphi_{\mathcal{L}} : C \to \mathbb{P}^{d+1-g}$, where the dimension comes from R-R.

(ii) φ is an embedding. $\Leftrightarrow$ $\forall\ p,q \in C$, $H^0(\mathcal{L}) \twoheadrightarrow k_p + k_q$.

Get $0 \to \mathcal{L}(-p-q) \to \mathcal{L} \to k_p \oplus k_q \to 0$

Need: $H^1(\mathcal{L}(-p-q)) = (0)$.

(iii) Need: $d\varphi : T_pC \hookrightarrow T_{\varphi(p)}\mathbb{P}$ $\Leftrightarrow$ $H^0(\mathcal{L}) \twoheadrightarrow (\mathcal{L}/\mathcal{L}(-2p))$

This again follows from $H^1(\mathcal{L}(-2p)) = (0)$.