PARTIAL DIFFERENTIAL EQUATIONS

Part IIB Course 1995

Example sheet 1

1. Find the general solution of the system of equations

$$\frac{dx_1}{dt} = x_1 x_2, \qquad \frac{dx_2}{dt} = -x_1 x_2$$

Consider the solution with initial condition $(x_1, x_2)(0) = (1, 1)$. What is the largest interval on which the solution is defined?

2. Show that the function $f(t, x) = \sqrt{|x|}$ does not satisfy a Lipschitz condition with respect to x in any rectangle centred at the origin in the (t, x) plane. Consider the differential equation

$$\frac{dx}{dt} = 2\sqrt{|x|}$$

with initial condition $x(0) = 0$. Show that for *any* nonnegative value of the constant c, the function

$$x(t) = (t-c)^2 \quad \text{for} \quad t \geq c \quad \text{and} \quad x(t) = 0 \quad \text{for} \quad t < c$$

is a solution to the problem. Does this contradict the existence theorem for ordinary differential equations?

3. Consider the equation

$$u\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x$$

and the initial curve $S : (x, y) = (s, s)$ for $s > 0$. Decide whether there is a unique solution, no solution or infinitely many solutions in a neighbourhood of $(1, 1)$ for each of the following initial conditions on S: (a) $u = 2s$ (b) $u = s$ (c) $u = \sin(\pi/2)s$.

4. Derive the formula $u = f(x_1 - a_1(u)y, \ldots, x_n - a_n(u)y)$ for the solution of the equation

$$a_1(u)\frac{\partial u}{\partial x_1} + \ldots + a_n(u)\frac{\partial u}{\partial x_n} + \frac{\partial u}{\partial y} = 0$$

with initial condition $u(x_1, \ldots, x_n, 0) = f(x_1, \ldots, x_n)$.

5. Let u be a C^1 solution to

$$a(x, y)\frac{\partial u}{\partial x} + b(x, y)\frac{\partial u}{\partial y} = -u$$

on the disk $\{(x, y) : x^2 + y^2 \leq 1\}$, and suppose $a(x, y)x + b(x, y)y > 0$ on the boundary. By considering the conditions for a maximum on the boundary, deduce that u vanishes identically.

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Example sheet 2

1. For the multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, prove that

$$\alpha! \leq |\alpha|! \leq n^{|\alpha|}\alpha!$$

2. In multi-index notation, prove that

$$\partial^\beta x^\alpha = \frac{\alpha!}{(\alpha-\beta)!}x^{\alpha-\beta}$$

if $\alpha_i \geq \beta_i$ for all $1 \leq i \leq n$, and is zero otherwise.

3. Let P and Q be scalar linear differential operators of orders m and n respectively. Give an example where PQ has order strictly less than $m+n$. If P or Q is elliptic, show that the order is $m+n$.

4. Let V be the space of all scalar linear differential operators of order ≤ 1 on $\mathbf{R}^n$. Show that if $P, Q \in V$, then $[P,Q] = PQ - QP \in V$, and that $[P[Q,R]]+[R[P,Q]]+[Q[R,P]] = 0$. Let $P \in V$ be of order 1. If P is elliptic, show that $n = 1$. If $n = 1$ and P is elliptic, show that $[P,Q] = 0$ if and only if $Q = aP + b$ for constants a, b.

5. (a) Show that the differential operator

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial/\partial x_1 + \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\partial/\partial x_2 + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\partial/\partial x_3 + \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\partial/\partial x_4$$

is elliptic.

(b) Show that if a scalar operator of odd order on $\mathbf{R}^n$ is elliptic, then $n = 1$.

6. Show that the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ is the largest subspace of $L^1(\mathbf{R}^n)$ which is invariant under the operations $f \mapsto \partial f/\partial x_i$ and $f \mapsto x_i f$ for all $1 \leq i \leq n$.

7. Try and construct a function $f \in \mathcal{S}(\mathbf{R})$ which does not have exponential decay: i.e. for all $a, b > 0$, $e^{a|x|^b}f$ is unbounded.

8. Use contour integral methods to prove that the Fourier transform takes the function $e^{-x^2/2} \in \mathcal{S}(\mathbf{R})$ to itself.

9. The nth Hermite polynomial $H_n(x)$ is defined by the generating function

$$\sum_0^\infty H_n(x)t^n/n! = e^{-t^2+2tx}$$

Show that the Fourier transform of $H_n(x)e^{-x^2/2} \in \mathcal{S}(\mathbf{R})$ is a multiple of itself.

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Example sheet 3

1. Let $f : \mathbf{R} \to \mathbf{R}$ be a *continuous* function. Show that if $f_h(x) = (f(x+h) - f(x))/h$, then as distributions,

$$f_h \to df/dx$$

as $h \to 0$.

2. Compute the distributions $(d/dx)^m |x|^n$ for m, n positive integers.

3. Find the most general distribution T which satisfies the following equations:

$$xT = 0 \qquad xdT/dx = 0 \qquad x^2 T = \delta_0 \qquad xdT/dx = \delta_0$$

$$dT/dx = \delta_0 \qquad dT/dx + T = \delta_0 \qquad T - (d/dx)^2 T = \delta_0$$

4. Find all solutions of the biharmonic equation $\Delta^2 u = 0$ (where Δ is the Laplacian in $\mathbf{R}^n$) which are functions of r. Find a fundamental solution for the operator Δ^2.

5. Let $\mathbf{C}^*$ be the non-zero complex numbers and $f(\eta, \zeta)$ a function which is holomorphic in $(\eta, \zeta) \in \mathbf{C} \times \mathbf{C}^*$. Define $g(\zeta)$ by

$$g(\zeta) = f((x_1 + ix_2)\zeta^2 + 2x_3\zeta - (x_1 - ix_2), \zeta)$$

If C is a closed contour in $\mathbf{C}^*$ show that the function u defined by

$$u(x_1, x_2, x_3) = \int_C g(z)dz$$

satisfies $\Delta u = 0$ in $\mathbf{R}^3$. If $f(\eta, \zeta) = \sum_{k=0}^{n} a_k(\zeta)\eta^k$, show that u is a harmonic polynomial. Express the harmonic functions u corresponding to $f(\eta, \zeta) = \eta^n / \zeta^{n+1}$ in terms of Legendre polynomials.

6. Show that the Dirac delta δ_0 is a *tempered* distribution, and compute its Fourier transform.

7. Show by taking the Fourier transform that every solution u of $\Delta u = 0$, where u is a tempered distribution, is a harmonic polynomial. [You may assume the result that any distribution whose support is the origin is a finite linear combination of derivatives of δ_0].

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Example sheet 4

1. Consider the Poisson kernel in two dimensions:

$$P(x, y) = \frac{1}{2\pi} \frac{1 - \|x\|^2}{\|x - y\|^2}$$

If the continuous function $h(\theta)$ on the unit circle is the uniform limit of the series

$$\sum_0^\infty (a_n \cos n\theta + b_n \sin n\theta)$$

use the Poisson kernel to find the harmonic function in the disc whose boundary value is h.

2. If $f(x)$ is a harmonic function, show that $\|x\|^{2-n} f(x/\|x\|^2)$ is harmonic also. Show that the map $x \mapsto x/\|x\|^2$ transforms a sphere through the origin into a hyperplane. Deduce how to solve the Dirichlet problem for the Laplacian in the domain $\Omega = \{x \in \mathbf{R}^n : x_n \geq 0\}$ with suitable behaviour of the continuous function h on the hyperplane $x_n = 0$.

3. Show that an open set $\Omega \subset \mathbf{R}^n$ satisfies the barrier property (for each $p \in \partial\Omega$ there is a subharmonic function g such that $g(p) = 0$ and $g < 0$ on $\partial\Omega\backslash\{p\}$) if for each point on $\partial\Omega$ there is a closed ball B such that $B \cap \bar{\Omega} = \{p\}$.

4. Find all solutions of the heat equation for $n = 1$ of the form $u(x, t) = t^{-1/2} f(xt^{-1/2})$.

5. Let $u(x, t)$ be a positive solution to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $t > 0$, and set $\theta = -2u^{-1}\partial u/\partial x$. Show that θ satisfies *Burger's equation*

$$\frac{\partial \theta}{\partial t} + \theta \frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial x^2}$$

Find, for $\phi \in C_0^\infty(\mathbf{R})$, a solution of Burger's equation which satisfies $\theta(x, 0) = \phi(x)$ and $\theta(x, t) \to 0$ as $t \to \infty$.

6. Let $u(x, t)$ be a twice continuously differentiable solution of the wave equation for $n = 3$ which is a function of $r = \|x\|$ and t. By putting $w = ru$ deduce that

$$u(x, t) = f(r - t)/r + g(r + t)/r$$

7. If $f \in C_0^\infty(\mathbf{R}^n)$, use the Taylor series of the solution of the heat equation with $\lim u(x, t) = f(x)$ to prove the n-dimensional *Weierstrass approximation theorem*: for each compact set $K \in \mathbf{R}^n$ there exists a sequence of polynomials $p_m(x)$ such that p_m converges uniformly to f on K.

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