# Partial Differential Equations.

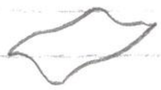
## Introduction.

Aim: give a taste of the techniques necessary for solving and analysing PDEs.

Laplace's equation: $\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}+\frac{\partial^2\varphi}{\partial x_3^2}=0$. Poisson's equation: $\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}+\frac{\partial^2\varphi}{\partial x_3^2}=h(x_1,x_2,x_3)$

Complex analysis: $f(z)$, analytic - solution to Cauchy-Riemann equations.

Wave equation: $c^2\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial t^2}=0$.

Geometry (differential):  $\underline{r}(u,v)$

1st fundamental form: $E\,du^2+2F\,du\,dv+G\,dv^2$.

Isothermal coordinates: $(x,y)$. So we have $u(x,y)$, $v(x,y)$.

$du=u_x dx+u_y dy$, etc. $\quad E\,du^2+2F\,du\,dv+G\,dv^2=H(x,y)(dx^2+dy^2)$

(via $w=u+iv$, $z=x+iy$)

$\lambda(dw+\mu d\bar{w})(d\bar{w}+\mu dw)=H\,dz\,d\bar{z}$

$dz=h(dw+\mu d\bar{w})$, $\frac{\partial z}{\partial w}=h$, $\frac{\partial z}{\partial\bar{w}}=\mu h$. So, $\frac{\partial z}{\partial\bar{w}}=\mu\frac{\partial z}{\partial w}$ - Beltrami equation

Isothermal coordinates exist $\Leftrightarrow$ solving this equation.

Curvature (Gaussian), $K$: $\frac{\partial^2\log H}{\partial z\partial\bar{z}}=-KH$.

Can we find a function $f$ such that $e^f(E\,du^2+F\,du\,dv+G\,dv^2)$ is a metric of constant curvature?

$\frac{\partial^2}{\partial z\partial\bar{z}}\log H+\frac{\partial^2 f}{\partial z\partial\bar{z}}=-cHe^f \Rightarrow \frac{1}{H}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)=-ce^f+K$ - non-linear differential equation.

Minimal surfaces: $z=u(x,y)$. $\quad(1+u_y^2)u_{xx}+\cancel{2x}\,2u_xu_yu_{xy}+(1+u_x^2)u_{yy}=0$.

In isothermal coordinates, $\underline{r}_{xx}+\underline{r}_{yy}=0$. $[\underline{r}_x\cdot\underline{r}_x=\underline{r}_y\cdot\underline{r}_y,\ \underline{r}_x\cdot\underline{r}_y=0]$.

Differential geometry yields systems of PDEs.

Will be concerned mainly with:
- linear equations for a scalar function
- existence of solutions
- uniqueness
- regularity

} require theory of distributions.

- emphasise the analogy with linear algebra.

Example: $\frac{\partial^2\varphi}{\partial x_1^2}+\frac{\partial^2\varphi}{\partial x_2^2}+\frac{\partial^2\varphi}{\partial x_3^2}=-4\pi\rho$ - distribution of mass. Point mass $\rightsquigarrow\frac{1}{r}$ potential.

$\varphi=\iiint\frac{\rho\,dV}{|\underline{r}-\underline{r}'|}$

- compare with $Ax=b$, $b=\sum b_ie_i$. $Ax_i=e_i$, $\sum b_ix_i$ - solution of $Ax=b$.

# 1. Ordinary Differential Equations.

**Examples:** (i) $\frac{dx}{dt} = x$, $x(0)=1$. $\Rightarrow x = e^t$, exists and is $C^\infty$ $\forall t$.

(ii) $\frac{dx}{dt} = x^2$, $x(0) = 1 \Rightarrow -\frac{1}{x} = t + c$, $x(0)=1 \Rightarrow c = -1 \Rightarrow x = \frac{1}{1-t}$, exists in $(-\infty, 1)$.

**System of ODEs:** $\frac{dx_i}{dt} = f_i(x_1, \dots, x_n, t)$, $1 \le i \le n$.

$$\frac{dx}{dt} = f(t,x), \quad x: (\alpha, \beta) \to \mathbb{R}^n, \quad f: (\alpha,\beta) \times U \to \mathbb{R}^n, \quad U \subset \mathbb{R}^n.$$

**Theorem 1.1 (Picard's Theorem):** Let $f(t,x)$ be continuous on $|t| < a$ and $\|x - x_0\| \le b$ and satisfy a Lipschitz condition: $\|f(t,x) - f(t,y)\| \le c\|x-y\|$. Then, if $h = \min(a, \frac{b}{m})$, where $m = \sup_{|t|<a, \|x-x_0\| \le b} \|f(t,x)\|$, the differential equation, $\frac{dx}{dt} = f(t,x)$ has a unique solution for $|t| < h$ with initial condition $x(0) = x_0$.

**Theorem 1.2 (Contraction Mapping Theorem):** Let $M$ be a complete metric space, and $T: M \to M$ such that $d(Tx, Ty) \le k\, d(x,y)$ $\forall x, y$, where $k < 1$. Then $T$ has a unique fixed point.

**Proof:** Choose $x_0 \in M$. $d(T^m x_0, T^n x_0) \le k^m d(x_0, T^{n-m} x_0)$, (wlog $n \ge m$)

$\le k^m \{ d(x_0, Tx_0) + d(Tx_0, T^2x_0) + \dots + d(T^{n-m-1}x_0, T^{n-m}x_0) \}$

$\le k^m \{ d(x_0, Tx_0) + k\, d(x_0, Tx_0) + \dots + k^{n-m-1} d(x_0, Tx_0) \} = k^m (1 + k + \dots + k^{n-m-1})\, d(x_0, Tx_0)$

$\le \frac{k^m}{1-k} \cdot d(x_0, Tx_0) \to 0$ as $m \to \infty$ $(k<1)$.

So, given $\varepsilon > 0$, $\exists N$ such that $m > N \Rightarrow d(T^m x_0, T^n x_0) < \varepsilon$ $\forall n > m$.

$\therefore \{T^n x_0\}$ is a Cauchy sequence, and converges to $x \in M$.

$x = \lim_{n\to\infty} T^n x_0$, $Tx = \lim_{n\to\infty} T^{n+1} x_0 = x$, so $x$ is a fixed point of $T$.

Uniqueness: suppose $Tx = x$, $Ty = y$. Then $d(x,y) = d(Tx, Ty) \le k\, d(x,y)$. But $k<1$, so this is impossible unless $d(x,y) = 0$.

**Remark:** The same result holds if $T^k$ satisfies the conditions of the theorem.

Suppose $T^k x = x$. Claim that $Tx$ is also a fixed point.

$T^k(Tx) = T(T^k x) = Tx$. Uniqueness $\Rightarrow$ $Tx = x$.

**Proof of Theorem 1.1:** Look for a fixed point of $(Tx)(t) = x_0 + \int_0^t f(s, x(s))\, ds$.

If $Tx = x$, then by differentiation, $\frac{dx}{dt} = f(t, x(t))$ and $x(0) = x_0$.

(i) Let $M = \{ x \in C([-h,h], \mathbb{R}^n) : x(0) = x_0, \sup_{|t| \le h} \|x(t) - x_0\| \le mh \}$.

Let $d(x(t), y(t)) = \sup_{|t| \le h} \|x(t) - y(t)\|$. $(M, d)$ is a complete metric space.

(ii) Does $T$ map $M$ to $M$?

Continuity of $f \Rightarrow$ $T$ maps continuous $\to$ continuous.

$Tx(0) = x_0$, so initial condition is okay. $\|Tx(t) - x_0\| \le \int_0^t \|f(s, x(s))\| ds \le mh$ $(= \sup\|f\|)$.

(iii) $\|T^k x(t) - T^k y(t)\| \le \int_0^t \|f(s, T^{k-1}x(s)) - f(s, T^{k-1}y(s))\| ds \le c \int_0^t \|T^{k-1}x(s) - T^{k-1}y(s)\| ds$ (Lipschitz)

(iv) Inductively, assume $\|T^l x(t) - T^l y(t)\| \le \frac{c^l |t|^l}{l!} \|x(t) - y(t)\|$.

$l = 0$ is okay.

from (iii), with $l = k-1$, $\|T^k(x) - T^k y(x)\| \le \frac{c^{k-1}}{(k-1)!} \cdot \|x - y\| \cdot c \int_0^t |s|^{k-1} ds \le \frac{c^k t^k}{k!} \|x-y\|$.

For any $x$, $\frac{x^k}{k!} \to 0$, so for sufficiently large $k$, $T$ is a contraction mapping.

Examples: (i) $\frac{d^n y}{dx^n} = f(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}})$.

Set $y_k = d^k y/dx^k$. Get: $\frac{dy}{dx} = y_1, \frac{dy_1}{dx} = y_2, \dots, \frac{dy_{n-1}}{dx} = f(x, y, y_1, \dots, y_{n-1})$

(ii) Geodesics in Riemannian geometry: $\frac{d^2 x_i}{dt^2} + \sum_{j,k} \Gamma^i_{j,k} \frac{dx_j}{dt} \cdot \frac{dx_k}{dt} = 0$, $x_i(0) = a_i$, $\frac{dx_i}{dt}(0) = b_i$.

Theorem 1.3 (Inverse Function Theorem): Let $U \subset \mathbb{R}^n$ be an open set and $f: U \to \mathbb{R}^n$ be a $C^1$ function such that $Df(x_0)$ is invertible for some $x_0 \in U$. Then, $\exists$ neighbourhoods $V, W$ of $x_0$, $fx_0$, such that $f: V \to W$ has a $C^1$ inverse $\varphi$.

Proof: First, normalise by setting $x_0 = fx_0 = 0$, $Df(0) = I$. (by translation, linear transformation).

Let $g(x) = x - f(x)$. [Recall that $\|A\| = \sup_{\|x\|=1} \|Ax\|$].

$Dg(0) = I - I = 0$, so by continuity, $\exists\, r > 0$ such that $\|x\| < 2r \Rightarrow \|Dg(x)\| < \frac{1}{2}$.

From MVT, $\|g(x)\| \le \frac{1}{2}\|x\|$ if $\|x\| < 2r$. So $g: \bar{B}(0,r) \to \bar{B}(0, \frac{1}{2}r)$.

Consider $g_y(x) = y + x - f(x)$. [Note that $g_y(x) = x$ iff $y = f(x)$].

If $\|y\| \le \frac{1}{2}r$ and $\|x\| < r$, then $\|g_y(x)\| \le \frac{1}{2}r + \|g(x)\| \le \frac{1}{2}r + \frac{1}{2}r = r$

So $g_y: \bar{B}(0,r) \to \bar{B}(0,r)$ - a complete metric space.

$\|g_y(x_1) - g_y(x_2)\| = \|g(x_1) - g(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|$, by MVT. So $g_y$ is a contraction mapping, and so has a unique fixed point in $\bar{B}(0,r)$.

So we have an inverse, $\varphi = f^{-1}$

(i) $\varphi$ continuous: $\|x_1 - x_2\| \le \|f(x_1) - f(x_2)\| + \|g(x_1) - g(x_2)\| \le \|f(x_1) - f(x_2)\| + \frac{1}{2}\|x_1 - x_2\|$.

So $\|x_1 - x_2\| \le 2\|f(x_1) - f(x_2)\|$, ie, $\|\varphi(y_1) - \varphi(y_2)\| \le 2\|y_1 - y_2\|$ - continuous, Lipschitz.

(ii) $\varphi$ differentiable: $\|\varphi(y_1) - \varphi(y_2) - (Df(x_2))^{-1}(y_1 - y_2)\| = \|x_1 - x_2 - (Df(x_2))^{-1}(f(x_1) - f(x_2))\|$

$\le \|Df(x_2)^{-1}\| . \|Df(x_2)(x_1 - x_2) - f(x_1) + f(x_2)\| \le A . \|x_1 - x_2\| . R$, where $R \to 0$ as $x_1 \to x_2$, since $f$ is differentiable. But this is $\le 2A\|y_1 - y_2\| R$, where $R \to 0$ as $y_1 \to y_2$.

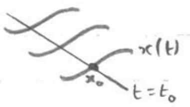
So $\varphi$ is differentiable, with derivative $(Df(x_2))^{-1}$, which is continuous.

Recall equation: $\frac{dx}{dt} = f(t,x)$, $x(0) = x_0$. Needed Lipschitz condition: $\|f(t,x) - f(t,y)\| \le c\|x - y\|$, $\forall\ |t| < a$, $\|x - x_0\| \le b$.

Remark: Suppose the equation is linear, $\frac{dx}{dt} = A(t)x$. Existence theorem requires $A(t)$ to be continuous. $\|A(t)x - A(t)y\| \le \|A(t)\| . \|x - y\|$, so Lipschitz.

Dependence of solution on $x_0$.

Consider $y(t,x)$, with $\frac{dy}{dt} = f(t,y)$, $y(0) = x$.




Lemma 1.4: Suppose $x_1(t)$, $x_2(t)$ are $C^1$ functions on $|t - t_0| \le a$ such that $\|\frac{dx_i}{dt} - f(t, x_i)\| < \varepsilon_i$ $(i = 1, 2)$, and suppose $f$ is Lipschitz with constant $K$. Then:

$$\|x_1(t) - x_2(t)\| \le \|x_1(t_0) - x_2(t_0)\| e^{K|t - t_0|} + \frac{\varepsilon}{K} e^{K|t - t_0|}, \text{ where } \varepsilon = \varepsilon_1 + \varepsilon_2. \quad - (*)$$

Proof: From assumptions, $\|x_1'(t) - x_2'(t) + f(t, x_2(t)) - f(t, x_1(t))\| \le \varepsilon_1 + \varepsilon_2 = \varepsilon$.

Set $\Psi(t) = \|x_1(t) - x_2(t)\|$, $\omega(t) = \|f(t, x_1(t)) - f(t, x_2(t))\|$.

Then, $|\Psi(t) - \Psi(t_0)| = \|\int_{t_0}^t (x_1'(s) - x_2'(s))\, ds\| \le \varepsilon(t - t_0) + \int_{t_0}^t \omega(s)\, ds \le \varepsilon(t - t_0) + K\int_{t_0}^t \Psi(s)\, ds$, by Lipschitz, so $\Psi(t) \le \Psi(t_0) + K\int_{t_0}^t (\Psi(s) + \frac{\varepsilon}{K})\, ds$.

(cont.)

Sublemma: If $l \geq g \geq 0$ such that $g(t) \leq A + K\int_{t_0}^{t} g(s)\,ds$, then
$$g(t) \leq A\left\{1 + K(t-t_0) + \dots + K^{n-1}(t-t_0)^{n-1}/(n-1)!\right\} + lK^n(t-t_0)^n/n!$$
Proof: By induction.

Set $g(t) = \Psi(t) + \varepsilon/K$. $\Psi(t) + \varepsilon/K \leq (\Psi(t_0) + \frac{\varepsilon}{K})e^{K(t-t_0)}$ - gives (*) by substitution.

Proposition 1.5: Let $y(t,x)$ be the solution on $J \times U$ to $\frac{dy}{dt} = f(t,y)$ with $y(t_0) = x$. Then $y$ on $J_0 \times U_0$ is continuous & satisfies a Lipschitz condition.

Proof: Let $x_i(t) = y(t,x_i)$ and use lemma 1.4 with $\varepsilon_1 = \varepsilon_2 = 0$.
$$\|y(t,x_1) - y(s,x_2)\| \leq \|y(t,x_1) - y(t,x_2)\| + \|y(t,x_2) - y(s,x_2)\| \leq \|x_1 - x_2\| e^{K|t-t_0|} + \sup_{J\times U}\|f\|.|t-s|,$$
from Lemma 1.4, and MVT for $y$ as a function of $t$.

This gives continuity. When $t=s$, get Lipschitz condition.

Proposition 1.6: If $f(t,y)$ is $C^1$, then the solution $y(t,x)$ to $\frac{dy}{dt} = f(t,y)$ with $y(t_0,x) = x$ is $C^1$.

Proof: $g(t,x) := D_2 f(t, y(t,x))$, where $D_2 \equiv$ derivative of $f(t,y)$ wrt $y$.

$g(t,x) : J\times U \to L(\mathbb{R}^n, \mathbb{R}^n) \cong \mathbb{R}^{n^2}$, the space of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$.

(i) Solve for $\lambda$ the equation $\frac{d\lambda}{dt} = g(t,x)\lambda$, with $\lambda(0,x) = I$, $\lambda : J\times U \to L(\mathbb{R}^n, \mathbb{R}^n)$.

[We can solve because this is linear in $\lambda$ and $g$ is continuous as $f$ is $C^1$]

(ii) Set $\theta(t,h) = y(t, x+h) - y(t,x)$.

$\frac{d\theta}{dt} = f(t, y(t,x+h)) - f(t, y(t,x))$.

So, $\|\frac{d\theta}{dt} - g(t,x)\theta\| = \|f(t,y(t,x+h)) - f(t,y(t,x)) - D_2 f(t, y(t,x))(y(t,x+h) - y(t,x))\|$

MVT: $\|f(z) - f(x)\| \leq \sup_{\text{segment}[z,x]} \|Df\|.\|z-x\|$

Put $g(x) = f(x) - Df(x_0)x$. Then, $\|f(z) - f(x) - Df(x_0)(z-x)\| \leq \sup\|Df - Df(x_0)\|.\|z-x\|$.

So, by MVT, $\|\frac{d\theta}{dt} - g(t,x)\theta\| \leq \|\theta\|.\sup_{z\in \text{segment}[y(t,x),\,y(t,x+h)]}\|D_2 f(t,z) - D_2 f(t, y(t,x))\| =: \|\theta\|.R(h)$

From Proposition 1.5, $\|\theta\| \leq K\|h\|$, Lipschitz condition.

So $\|\frac{d\theta}{dt} - g\theta\| \leq K.\|h\|.R(h)$. As $D_2 f$ is continuous on a compact set, $R(h) \to 0$ as $h \to 0$.

$\theta(0,x) = y(0,x+h) - y(0,x) = x+h-x = h$.

Referring to (i), $\lambda(0,x)h = Ih = h$. $\frac{d}{dt}(\lambda h) = g(\lambda h)$.

By lemma 1.4, with $\varepsilon_1 = 0$, $\varepsilon_2 = K\|h\|.R(h)$, $\|\theta(t) - \lambda(t)h\| \leq C.\|h\|.R(h)$.

Expand: $\|y(t,x+h) - y(t,x) - \lambda(t,x)h\| \leq C.\|h\|.R(h)$.

By definition of derivative, $y$ is differentiable with derivative $\lambda$. But $\lambda$ is continuous, from the existence theorem and proposition 1.5. $\therefore$ $y$ is $C^1$.

Consequences: $y(t,x)$ with $y(0,x) = x$, $D_2 y(0,x) = I$.

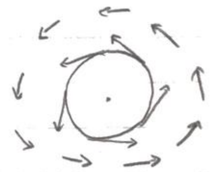
Therefore, by continuity of the derivative, $D_2 y(t,x)$ is invertible for $|t| < \delta$.

By inverse function theorem for fixed $t$, $x \mapsto y(t,x)$ has a $C^1$ inverse, on some open neighbourhood.

**Example:** $\begin{cases} dx_1/dt = x_2 \\ dx_2/dt = -x_1 \end{cases} \Rightarrow \frac{d^2x_1}{dt^2} = -x_1$, so $x_1 = A\sin(t+c) = x_2\sin t + x_1\cos t$
$x_2 = A\cos(t+c) = x_2\cos t - x_1\sin t$.

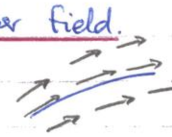
At $t=0$, $x_1 = A\sin c$, $x_2 = A\cos c$.

$\left.\begin{array}{l} y_1(t, x_1, x_2) = x_2\sin t + x_1\cos t \\ y_2(t, x_1, x_2) = x_2\cos t - x_1\sin t \end{array}\right\} = \varphi_t(x)$, rotation by angle $t$.



$$\varphi_{s+t}(x) = \varphi_s(\varphi_t(x))$$

Consider equations where $f$ is independent of $t$. $f: \underset{\subseteq \mathbb{R}^n}{U} \to \mathbb{R}^n$. - Vector field.
$\varphi_t(x)$ is the flow of the vector field.
A solution $y(t,x)$ is an integral curve of the vector field through $x$.



**Proposition 1.7:** Wherever they are defined, $\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$, if $\varphi_t(x)$ $[= y(t,x)]$ solves $\frac{dy}{dx} = f(y)$ with initial condition $\varphi_0(x) = x$.

**Proof:** $\varphi_{t+s}(x)$ is the solution ~~for~~ $y(t+s, x)$, $\varphi_t(\varphi_s(x))$ the solution $y(t, \varphi_s(x))$.
Follows by uniqueness of solutions with same initial condition at $t=0$.

**Remark:** The vector field vanishes at $x=a$ iff $\varphi_t(a) = a$ $\forall\ |t|<h$. (a is a fixed point $\forall\ \varphi_t$)
(i) suppose $\varphi_t(a) = a$, so $\frac{d\varphi_t}{dt} = 0$, so $f(a) = 0$.
(ii) suppose $f(a) = 0$, $y=a$ solves the equation $dy/dt = f(y)$ with initial condition $y(0) = a$. By uniqueness, $\varphi_t(a) = a$ $\forall\ |t|<h$.

## 2. First-order PDEs.

General form: $f(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}) = 0$.

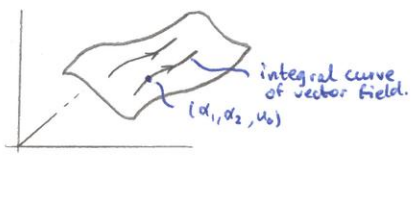
**Example:** $\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 = n(x_1, x_2, x_3)$, the refractive index of an inhomogeneous medium. The solution describes the wavefront of light.

**Definition 2.1:** A quasi-linear PDE is one of the form: $\sum_{i=1}^{n} a_i(x_1, \dots, x_n, u) \frac{\partial u}{\partial x_i} = b(x_1, \dots, x_n, u)$ - (i)

Key to solving is to study the ODE: $\left.\begin{array}{l} dx_i/dt = a_i(x_1, \dots, x_n, u) \\ du/dt = b(x_1, \dots, x_n, u) \end{array}\right\}$ (ii)

**Proposition 2.2:** Suppose $a, b$ are $C^1$ and $u = f(x_1, \dots, x_n)$ is a solution of (i) for $x \in U \subset \mathbb{R}^n$.
If $u_0 = f(\alpha_1, \dots, \alpha_n)$ for some $\alpha \in U$ and $(x(t), u(t))$ is the unique solution to (ii) with initial conditions $x(0) = \alpha$, $u(0) = u_0$, then $u(t) = f(x_1(t), \dots, x_n(t))$ $\forall\ |t|<h$.

**Proof:** $z(t) := u(t) - f(x_1(t), \dots, x_n(t))$. $dz/dt = du/dt - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} = b - \sum a_i\, \partial f/\partial x_i = 0$
$\Rightarrow z =$ constant, but $z(0) = 0$, so $z(t) \equiv 0$.




If we have a solution, then the integral curve of the vector field through any point on the solution surface lies in that surface.

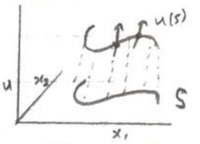
**Definition 2.3:** A submanifold of $\mathbb{R}^n$ of dimension $m$ is a subset $M \subset \mathbb{R}^n$ such that each point $x \in M$ has a neighbourhood $U \subset \mathbb{R}^1$ and a $C^1$ function $\varphi: U \to \mathbb{R}^{n-m}$ such that $U \cap M = \varphi^{-1}(0)$ and $D\varphi$ is surjective at all points $x \in M \cap U$.

A hyperplane is a manifold in $\mathbb{R}^n$ of dimension $n-1$.

By the implicit function theorem ($\cong$ inverse function theorem), a submanifold is really the image of a $C^1$ map $\Psi: V \to \mathbb{R}^n$ $[V \subset \mathbb{R}^m]$ whose derivative $D\Psi$ is injective.

**Definition 2.4:** The tangent space to a submanifold $M$ is the kernel of $D\varphi(x)$ $(= \operatorname{Im} D\Psi(y)$ where $\Psi(y) = x)$

**Theorem 2.5:** Let $S$ be a $C^1$ hypersurface in $U \subset \mathbb{R}^n$ and consider the initial value problem: $\sum_{i=1}^{n} a_i(x,u)\, \partial u/\partial x = b(x,u)$, $u = \varphi$ on $S$, for $C^1$ functions $a, b, \varphi$.
Suppose that for each $x \in S$, the vector field $a(x, \varphi(x))$ is not in the tangent space of $S$ at $x$. Then in some neighbourhood $V \subset U$, $\exists$ a unique solution to the initial value problem.




**Proof:** (i) Parametrise $S$. [$S$ is a submanifold of $\mathbb{R}^n$, so $S$ is defined as the set $(x_1(s_1,..,s_{n-1}),.., x_n(s_1,..,s_{n-1}))$, where $Dx$ is injective, $x: \underset{\subset \mathbb{R}^{n-1}}{U} \to \mathbb{R}^n$].

(ii) Solve ODE: $\begin{cases} dx_i/dt = a_i(x,u) \\ du/dt = b(x,u) \end{cases}$ with initial conditions: $x_i(0) = x_i(s_1,..,s_{n-1})$, $u(0) = \varphi(s_1,..,s_{n-1})$

From existence theorem for ODEs, $\exists$ such a solution for $|t| < h$.
From dependence on initial conditions, solution is $C^1$ in initial conditions.
Get: $x_i(t) = y_i(t, s_1,.., s_{n-1})$
$u(t) = w(t, s_1,.., s_{n-1})$

(iii) Consider the map: $(t, s_1,.., s_{n-1}) \mapsto (y_1(t,s_1,..,s_{n-1}),.., y_n(t,s_1,..,s_{n-1}))$. $-(*)$
Its derivative is given by the Jacobian matrix:

$$\begin{pmatrix} \partial y_1/\partial t & \cdots & \partial y_n/\partial t \\ \partial y_1/\partial s_1 & \cdots & \partial y_n/\partial s_1 \\ \vdots & & \vdots \\ \partial y_1/\partial s_{n-1} & \cdots & \partial y_n/\partial s_{n-1} \end{pmatrix} \xrightarrow{\text{at } t=0} \begin{pmatrix} a_1(x(s), \varphi(s)) & \cdots & a_n(x(s), \varphi(s)) \\ \partial x_1/\partial s_1 & \cdots & \partial x_n/\partial s_1 \\ \vdots & & \\ \partial x_1/\partial s_{n-1} & \cdots & \partial x_n/\partial s_{n-1} \end{pmatrix}$$

[The last $n-1$ rows are linearly independent from the fact that $S$ is a submanifold]
From the condition in the theorem, $(a_1,.., a_n)$ is not in the tangent space of $S$, and so is not a linear combination of the last $n-1$ rows, so the Jacobian has rank $n$ at $t=0$. Therefore, for small enough $t$, the Jacobian is invertible.
So, by the inverse function theorem, $\exists$ open sets such that map $(*)$ has a $C^1$ inverse, $F$.

(iv) Consider $w(t, s_1,.., s_{n-1})$. Then $w = u \circ F$ for some $C^1$ function $u(y_1,.., y_n)$.

(v) Claim $u(x_1,.., x_n)$ solves the equation: $\partial w/\partial t = b(y_1,.., y_n, u)$,
$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \cdot \frac{\partial y_i}{\partial t}(y_1,.., y_n, u) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} a$

(vi) Uniqueness follows from Proposition 2.2.

Examples: (i) $u\,\partial u/\partial x + y\,\partial u/\partial y = x$, $u = 2s$ on the curve $(x,y) = (s,1)$, $-\infty < s < \infty$.

$(a_1, a_2) = (u, y)$. On curve $S$, $(a_1, a_2) = (2s, 1)$.

Tangent space to $S$ is spanned by $(1,0)$, so $(a_1, a_2) \notin$ tangent space.

Solve $x_1 = x$, $x_2 = y$: $\frac{dx_1}{dt} = u$, $\frac{dx_2}{dt} = x_2$, $\frac{du}{dt} = x_1$

Get $x_2 = Ae^t$; $\frac{d^2x_1}{dt^2} = x_1$, so $x_1 = Be^t + Ce^{-t}$. $\therefore u = \frac{dx_1}{dt} = Be^t - Ce^{-t}$.

At $t=0$, $x_1 = s$, $x_2 = 1$ $\therefore s = B + C$, $A = 1$, $2s = B - C$ $\therefore 3s = 2B$, $-s = 2C$.

Hence $y_1(t,s) = \frac{3s}{2}e^t - \frac{s}{2}e^{-t}$

$y_2(t,s) = e^t$

$u = \frac{3s}{2}e^t + \frac{s}{2}e^{-t}$

So $u = \frac{s}{2}(3y_2 + y_2^{-1})$, $s(3y_2 - y_2^{-1}) = 2y_1$

$\therefore u = \frac{2y_1(3y_2 + y_2^{-1})}{2(3y_2 - y_2^{-1})} = \frac{y_1(3y_2^2+1)}{3y_2^2 - 1}$. So $u(x,y) = \frac{x(3y^2+1)}{3y^2-1}$

(ii) Birth process. Population size $n$. $P_n(t)$ = probability that population size at time $t$ is $n$.

Size $N$ at $t=0$.

Assume that in $(t, t+h)$ each individual has probability $\lambda h + R(h)$, where $R(h)/h \to 0$ as $h \to 0$, of creating a new member.

$P_n(t+h) = P_n(t)(1 - n\lambda h) + P_{n-1}(t)(n-1)\lambda h + o(h^2)$

$\quad \overset{\shortparallel}{P_n(t)} + P_n'(t)h + o(h^2)$.

So $P_n'(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t)$

Generating function, $G(t,s) = \sum_{n=0}^{\infty} s^n P_n(t)$.

$\frac{\partial G}{\partial t} = \sum s^n P_n'(t) = \sum -n\lambda s^n P_n(t) + \sum (n-1)\lambda s^n P_{n-1}(t) = -\lambda s\frac{\partial G}{\partial s} + \lambda s^2 \frac{\partial G}{\partial s}$.

Get: $\frac{\partial G}{\partial t} + \lambda s(1-s)\frac{\partial G}{\partial s} = 0$, $G(0,s) = s^N$. Exercise to solve this by method.

(iii) $\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} = ku$, $k$ constant. - Euler's equation for homogeneous functions.

Initial condition is: $u = h(x_1, \dots, x_{n-1})$ on $x_n = 1$.

$\left.\begin{aligned} dx_i/dt &= x_i \quad (1 \le i \le n) \\ du/dt &= ku \end{aligned}\right\} \Rightarrow \begin{aligned} x_i &= c_i e^t \\ u &= ae^{kt} \end{aligned}$

Let $x_i = s_i$ $(1 \le i \le n-1)$ $\therefore y_i = s_i e^t$, $1 \le i \le n-1$, $y_n = e^t$, $w = h(s_1, \dots, s_{n-1})e^{kt}$

Solution: $u = h\left(\frac{y_1}{y_n}, \dots, \frac{y_{n-1}}{y_n}\right) y_n^k$ - homogeneous of degree $k$.

## 3. Linear PDEs.

General PDE: $F\left(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \dots, \frac{\partial^m u}{\partial x^m}\right) = 0$

Linear PDE: $G(x) + a_0(x)u + \sum a_i(x)\,\partial u/\partial x_i + \dots$

Need more compact notation - multi-index notation (Schwarz)

Let $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{N}$. Define $|\alpha| = \sum_{i=1}^{n} \alpha_i$

Definitions: (3.1) $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, (3.2) $\alpha! = \alpha_1! \cdots \alpha_n!$

(3.3) $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, (3.4) $C_\alpha = C_{\alpha_1 \dots \alpha_n} \in \mathbb{C}$ (coefficient)

So, a polynomial of degree $n$ is: $p(x) = \sum_{|\alpha| \le n} C_\alpha x^\alpha$

Examples: (i) Taylor expansion of $f(x_1,\dots,x_n)$: $f(x) = \sum_{|\alpha|\le n} \frac{1}{\alpha!}\partial^\alpha f(0)\, x^\alpha$ + remainder.

(ii) Multinomial expansion: $(x_1+\dots+x_n)^m = \sum_{|\alpha|\le m} \frac{m!}{\alpha!} x^\alpha$ [Binomial: $(x_1+x_2)^m = \sum_{\alpha_1} \frac{m!}{\alpha_1!(m-\alpha_1)!} x_1^{\alpha_1} x_2^{m-\alpha_1}$]

Definition 3.5: A linear PDE of order $m$ is a differential equation of the form: $\sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha u = b(x)$

$P = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha$ is called a partial differential operator of order $m$.

[Linear: $P(\lambda_1 u_1 + \lambda_2 u_2) = \lambda_1 P(u_1) + \lambda_2 P(u_2)$]

Definition 3.6: The complete symbol of a partial differential operator $P$ is the function:

$\sigma(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)(i\xi)^\alpha$, $\xi = (\xi_1,\dots,\xi_m)$, where $P$ has order $m$.

Definition 3.7: The principal symbol of $P$ is the function: $\sigma_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)(i\xi)^\alpha$

Example: (i) Laplace operator: $P = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$.

$\sigma(x,\xi) = \sigma_2(x,\xi) = -(\xi_1^2+\xi_2^2+\xi_3^2)$.

(ii) Wave operator: $P = \frac{\partial^2}{\partial t^2} - c^2(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3)$

$\sigma(x,\xi) = \sigma_2(x,\xi) = -\tau^2 + c^2(\xi_1^2+\xi_2^2+\xi_3^2)$.

(iii) $\partial/\partial t = c(\partial^2/\partial x_1^2 + \partial^2/\partial x_2 + \partial^2/\partial x_3^2)$

$\sigma(x,\xi) = i\tau + c(\xi_1^2+\xi_2^2+\xi_3^2)$, $\sigma_2(x,\xi) = c(\xi_1^2+\xi_2^2+\xi_3^2)$

Can do the same for a system of PDEs: $Pu = \sum_{|\alpha|\le m} \underbrace{A_\alpha(x)}_{l\times k \text{ matrix of functions}} \partial^\alpha u$ ← function with values in $\mathbb{R}^k$

Examples: (i) $Pu = Du = (\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n})$ ($u$ a scalar function)

$\sigma(x,\xi): \xi \mapsto i\xi$, symbol $= i\times$(Identity matrix).

(In general, a differential operator with this symbol is called a connection)

(ii) Cauchy-Riemann Operator:

$P\binom{u}{v} = \begin{pmatrix} \partial u/\partial x - \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{pmatrix}$. $\sigma(x,\xi) = i\begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix}$

Proposition 3.8: If $f$ is a $C^\infty$ function and $P$ is a linear differential operator of order $m$, then $\sigma_m(x,\xi) = \lim_{t\to\infty} t^{-m}(e^{-itf} P e^{itf})$, where $\xi_i = \frac{\partial f}{\partial x_i}(x)$.

Proof: $P(e^{itf}u) = \sum_{|\alpha|\le m} a_\alpha(x)\partial^\alpha(e^{itf}u) = \sum_{|\alpha|=m} a_\alpha(x)\partial^\alpha(e^{itf}u)$ + {lower order terms}.

$= \sum_{|\alpha|=m} e^{itf} a_\alpha(x)(it)^m(\partial f)^\alpha$ + {lower order terms}.

So $e^{-itf}P(e^{itf}u) = t^m \sum a_\alpha(x)(i\xi)^\alpha u$ + {lower order terms}.

$\therefore \lim_{t\to\infty} t^{-m} e^{-itf} P(e^{itf}u) = \sum a_\alpha(x)(i\xi)^\alpha u$, where $\xi = (\partial f/\partial x_1,\dots,\partial f/\partial x_n)$.

Corollary: $P$ degree $m$, $Q$ degree $n$ $\Rightarrow$ $\sigma_{m+n}(PQ) = \sigma_m(P)\sigma_n(Q)$

Definition 3.9: Let $P = \sum_{|\alpha|\le m} A_\alpha(x)\partial^\alpha$ be a differential operator of order $m$. $P$ is said to be elliptic if $\sigma_m(x,\xi)$ is invertible for all $\xi \ne 0 \in \mathbb{R}^n$.

Examples: (i) Laplacian, $\sigma_2 = -(\xi_1^2+\xi_2^2+\xi_3^2) \ne 0$ if $\xi \ne 0$.

(ii) Cauchy-Riemann, $i\begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix}$. $\det = -(\xi_1^2+\xi_2^2) \ne 0$ if $\xi \ne 0$.

(iii) Wave operator: $\partial^2/\partial t^2 = c^2 \sum_{i=1}^{3} \partial^2/\partial x_i^2$.
$\sigma(x, \tau, \xi) = -\tau^2 + c^2(\xi_1^2 + \xi_2^2 + \xi_3^2)$ – not elliptic; hyperbolic.

Definition 3.10: A differential operator $P$ of order $m$ is said to be hyperbolic with respect to $x_1$ if, setting $\theta = (1, 0, .., 0)$, $\sigma_m(x, \theta)$ is invertible, and $\sigma(x\, \xi + t\theta)$ is invertible for all real $\xi$ and all $t$ with sufficiently large imaginary part.

If $\sigma(x, \xi)$ is homogeneous in $\xi$ (of degree $m$, so $\sigma_m(x,\xi) = \sigma(x,\xi)$), then $P$ is hyperbolic if $\det \sigma(x, \xi + t\theta) = 0$ has only real roots in $t$, and $\det \sigma(x, \theta) \neq 0$.

Examples: (i) $P = \sum_{i=1}^{n} a_i \partial/\partial x_i$. $\quad \sigma(x, \xi) = i \sum a_i \xi_i$, $\theta = (1, 0, .., 0)$.
$\sigma(x, \theta) = a_1$, $a_1 \neq 0$ iff hyperbolic.

(ii) wave-operator: $\partial^2/\partial t^2 = c^2 \sum_{i=1}^{3} \partial^2/\partial x_i^2$, $\sigma(x, \tau, \xi) = -\tau^2 + c^2(\xi_1^2 + \xi_2^2 + \xi_3^2)$.
$\sigma_2(\theta) = -1$, $\sigma(\tau + t, \xi_1, \xi_2, \xi_3) = -(t+\tau)^2 + c^2(\xi_1^2 + \xi_2^2 + \xi_3^2)$
$t + \tau = \pm \sqrt{c^2(\xi_1^2 + \xi_2^2 + \xi_3^2)}$

(iii) $\left.\begin{array}{l} \partial u/\partial x + \partial v/\partial y = 0 \\ \partial u/\partial y + \partial v/\partial x = 0 \end{array}\right\}$ $\sigma = \sigma_1 = i \begin{pmatrix} \xi_1 & \xi_2 \\ \xi_2 & \xi_1 \end{pmatrix}$. $\det \sigma = i(\xi_1^2 - \xi_2^2)$ – real roots.

Linear partial differential operator, $P$. $Pf = g$.
(cf. linear algebra, $Ax = b$. Need $x =$ "$A^{-1}$"$b$ with $\sum_j A_{ij}$ "$A^{-1}$"$_{jk} = \delta_{ik}$)
$PK = \delta_y$ – dirac delta function, $P = \sum a_\alpha(x) \partial^\alpha$
So, $\int (PK)\, \varphi(x)\, dx = \varphi(y)$.
Begs many questions:
(i) $K$ cannot be a $C^\infty$ function – need generalised functions, or distributions.
(ii) Solving $Pf = g$ – we have to be careful what class of function we are talking about.

Example: $P = d/dx$. $\quad K(x,y) = \begin{cases} 0 & x \leq y \\ 1 & x > y. \end{cases}$
$\int_{-\infty}^{\infty} \left(\frac{dK}{dx}\right) \varphi(x)\, dx = \int_{-\infty}^{\infty} -K\, d\varphi/dx\, dx = \int_y^\infty -\frac{d\varphi}{dx}\, dx = -[\varphi(x)]_y^\infty = \varphi(y)$.
[Supposing $\varphi(x)$ vanishes outside $[-N, N]$, some $N$].
Key is integration by parts.

For simplicity, focus on constant coefficient partial differential operators.
$P = \sum a_\alpha \partial^\alpha$, $a_\alpha$ constant, $\equiv P$ is translation invariant $(x_i \mapsto x_i + c_i \Rightarrow P \mapsto P)$
So, $f_c(x) = f(x+c)$ then $Pf = 0 \Rightarrow Pf_c = 0$.
In particular, fundamental solution: $K(x,y) = K(x+c, y+c) = K(x-y, 0) = k(x-y)$.
$\int K(x,y)\, \varphi(y)\, dy = \int k(x-y)\, \varphi(y)\, dy$ – convolution.

Fourier Transform works well for constant coefficient operators.
Recall: In $\mathbb{R}$, $\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx$, $\widehat{df/dx} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -f(x) \frac{d}{dx}(e^{-itx})\, dx = it\hat{f}$.

And, $\widehat{\sum a_k \left(\frac{d}{dx}\right)^k f} = \sum a_k (it)^k \hat{f}$ – the symbol of $P$.

## Fourier Transform in $\mathbb{R}^n$

Standard theory of Fourier Transform, $f \in L^1(\mathbb{R}^n)$.
$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int f(x) e^{-i\xi . x} dx$, exists, by dominated convergence theorem.
This is the Fourier Transform of $f$.

Note: if $f(x_1, .., x_n) = f_1(x_1) .. f_n(x_n)$, then $\hat{f} = \prod_{j=1}^{n} \hat{f}_j(\xi_j)$

By Riemann-Lebesgue, $|\hat{f}(\xi)| \to 0$ as $\|\xi\| \to \infty$, and moreover,
$\|\hat{f}\|_\infty = \sup_\xi |\hat{f}(\xi)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} |f(x)| = \frac{1}{(2\pi)^{n/2}} \|f\|_{L^1}$

We need a different space to which we can apply partial differential operators - a space preserved by Fourier Transform.

Definition 3.11: The space $\mathcal{S}(\mathbb{R}^n)$ of Schwarz functions on $\mathbb{R}^n$ is defined as:
$\mathcal{S}(\mathbb{R}^n) = \{ f \in C^\infty(\mathbb{R}^n) : \sup |x^\beta \partial^\alpha f| < \infty \ \forall \alpha, \beta \}$.
Ie, $f$ and its derivatives decay faster than any polynomial.

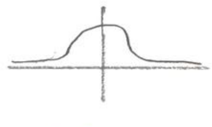
Note: if $f \in \mathcal{S}$, so do $x^\alpha f$ and $\partial^\beta f$.

Examples: (i) $e^{-\|x\|^2} \in \mathcal{S}$.
(ii) $p$ a polynomial $\Rightarrow p(x) e^{-\|x\|^2} \in \mathcal{S}$ (cf. Hermite polynomials)
(iii) $C^\infty$ functions of compact support.

$\int_{-\infty}^{\infty} \chi_{[0,1]} e^{-itx} dx = \int_0^1 e^{-itx} dx = \left[ \frac{e^{-itx}}{-it} \right]_0^1 = -\frac{e^{-it}}{it} + \frac{1}{it}$ - not of compact support.
Fourier Transform spreads out support of a function.
Bump function:

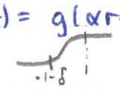
Lemma 3.12: Given $\delta > 0$, $\exists\, \varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi(x) = \begin{cases} 1 & \text{if } \|x\| \leq 1 \\ 0 & \text{if } \|x\| \geq 1+\delta \end{cases}$,
with $0 \leq \varphi \leq 1$ and $\varphi(-x) = \varphi(x)$.

Proof: (i) start with $f(x) = \exp\left(\frac{-1}{1-\|x\|^2}\right)$ if $\|x\| \leq 1$, $= 0$ otherwise.



Claim that this is $C^\infty$. $r = \|x\|$, $f(x) = f(r)$.
(ii) $g(r) = \int_{-\infty}^{r} f(t)\, dt \,/ \int_{-\infty}^{\infty} f(t)\, dt$. g:

(iii) $h(r) = g(\alpha r + \beta)$. Choose $\alpha, \beta \in \mathbb{R}$ such that $h(r) = \begin{cases} 0 & \text{if } r \leq -1-\delta \\ 1 & \text{if } r \geq -1 \end{cases}$
h:



Set $\varphi = h(r) h(-r)$ :

Take $f \in \mathcal{S}$. $\hat{f}$ is well-defined, since $\mathcal{S} \subset L^1$, $|f| < \frac{c}{(1+\|x\|^2)^N}$, any $N$.

$\widehat{\partial f / \partial x_j} = \frac{1}{(2\pi)^{n/2}} \int \frac{\partial f}{\partial x_j} e^{-ix.\xi} dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} \frac{\partial f}{\partial x_j} e^{-x_j \xi_j} dx_j \right) dx_1 \ldots$

$= \frac{-1}{(2\pi)^{n/2}} \int_{\mathbb{R}^{n-1}} \left( \int_{\mathbb{R}} -i \xi_j f e^{-ix_j \xi_j} dx_j \right) dx_1 \ldots = i \xi_j \hat{f}$

Also, $\frac{\partial \hat{f}}{\partial \xi_j} = \frac{1}{(2\pi)^{n/2}} \int (-ix_j) f e^{-ix.\xi} dx = \widehat{(-ix_j f)}$

**Lemma 3.13:** $\widetilde{\partial_x^\alpha f} = (i\xi)^\alpha \tilde{f}$, $\widetilde{x^\alpha f} = (i\partial_\xi)^\alpha \tilde{f}$, for $f \in \mathcal{S}$.

**Lemma 3.14:** If $f \in \mathcal{S}$, then $\tilde{f} \in \mathcal{S}$.

**Proof:** need to show that $(i\xi)^\alpha (i\partial_\xi)^\beta \tilde{f}$ is bounded. This is $\widetilde{\partial_x^\alpha(x^\beta f)}$, but since $f \in \mathcal{S}$, we have that $\partial_x^\alpha(x^\beta f)$ is bounded, so it is enough to show that $\tilde{f}$ is bounded. But $f$ is $L^1$, so $\hat{f}$ is bounded (or with stronger bounds).

**Lemma 3.15:** If $f \in \mathcal{S}$ and $f(0)=0$ then $f(x) = \sum_{j=1}^n x_j f_j(x)$, where $f_j \in \mathcal{S}$.

**Proof:** Consider $\int_0^1 \frac{\partial}{\partial t}(f(tx))\,dt = f(x) - f(0) \Rightarrow \int_0^1 \sum_{j=1}^n x_j \frac{\partial f(tx)}{\partial x_j}\,dt = f(x) = \sum x_j f_j(x)$.

$f_j(x)$ is $C^\infty$ but may not be in $\mathcal{S}$.

Put $g_j(x) = \frac{x_j}{\|x\|^2} f(x)$, singular at $x=0$ but has right behaviour as $\|x\| \to \infty$

And, $\sum x_j g_j(x) = \sum \frac{x_j^2}{\|x\|^2} f(x) = f(x)$.

Choose bump function, $\varphi$, such that $\varphi(x) = \begin{cases} 1 & \text{near } 0 \\ 0 & \text{outside some neighbourhood.} \end{cases}$

Then $\sum x_j \varphi(x) f_j(x) + \sum x_j (1-\varphi(x)) g_j(x) = \sum x_j \{\underbrace{\varphi f_j}_{C^\infty \text{ compact support}} + \underbrace{(1-\varphi) g_j}_{\in \mathcal{S}}\}$.

Take this as $f_j$ in lemma

**Lemma 3.16:** Suppose $T: \mathcal{S} \to \mathcal{S}$ is a linear transformation commuting with multiplication $x_j$ and differentiation $\partial/\partial x_j$ $\forall j$, then $T = c.\mathrm{Id}$ ($c$ = constant).

**Proof:** (i) If $f(a)=0$, then by lemma 3.15 (translated), $f(x) = \sum_{j=1}^n (x_j - a_j) f_j(x)$, $f_j \in \mathcal{S}$.

$Tf(x) = \sum_{j=1}^n (x_j - a_j) Tf_j(x) \Rightarrow (Tf)(a) = 0$.

(ii) Consider $T(f - fe^{-\|x-a\|^2})(a) = 0 \Rightarrow (Tf)(a) = f(a).T(e^{-\|x-a\|^2})(a) = c(a) f(a)$, $c$ independent of $f$.

(iii) Choose $f$ with no zeroes, then $c(a) = (Tf)(a)/f(a)$ is $C^\infty$ in $a$.

$$\left.\begin{aligned} \frac{\partial}{\partial x_j}(Tf) &= f\frac{\partial c}{\partial x_j} + c\frac{\partial f}{\partial x_j} \\ \shortparallel\quad & \\ T\frac{\partial f}{\partial x_j} &= c\frac{\partial f}{\partial x_j} \end{aligned}\right\} \Rightarrow \frac{\partial c}{\partial x_j} = 0, \text{ so } c \text{ is constant.}$$

**Theorem 3.17:** The Fourier Transform is an isomorphism from $\mathcal{S}$ to $\mathcal{S}$. Its inverse is:

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int \hat{f}(\xi) e^{ix.\xi}\,d\xi.$$

**Proof:** Take $F(f) = \hat{f}$ – Linear transformation : $\mathcal{S} \to \mathcal{S}$.

$F(\frac{\partial}{\partial x_j} f) = i\xi_j F(f)$, $F(x_j f) = i\frac{\partial}{\partial \xi_j} F(f)$, $F^2(\frac{\partial}{\partial x_j} f) = -\frac{\partial}{\partial x_j} F^2(f)$, $F^2(x_j f) = -x_j F^2(f)$

Set $R: \mathcal{S} \to \mathcal{S}$ to be: $(Rf)(x) = f(-x)$, so $R(\partial f/\partial x_j) = -\frac{\partial}{\partial x_j}(Rf)$, $R(x_j f) = -x_j(Rf)$.

By Lemma 3.16, $RF^2 = c.\mathrm{Id}$, so $(RF)F = c.\mathrm{Id}$.

If $c \neq 0$, inverse of $F$ is $\frac{1}{c}RF$.

Now, $((RF)\hat{f})(x) = \frac{1}{(2\pi)^{n/2}} \int \hat{f}(\xi) e^{ix.\xi}\,d\xi$ ← from R.

To evaluate $c$, take $f(x) = \exp(-\|x\|^2/2) = \exp(-\frac{1}{2}(x_1^2 + \dots x_n^2))$.

Note that $(\frac{\partial}{\partial x_j} + x_j) f = 0$, $1 \le j \le n$. So, $(i\xi_j + i\partial/\partial\xi_j)\hat{f} = 0 \Rightarrow \hat{f} = K\exp(-\frac{1}{2}\sum \xi_i^2)$.

To evaluate $K$, set $\xi = 0$.

$\therefore K = \frac{1}{(2\pi)^{n/2}} \int e^{-\|x\|^2/2}\,dx = \frac{1}{(2\pi)^{n/2}} \cdot \prod_{j=1}^n \int_{-\infty}^\infty e^{-x_j^2/2}\,dx = \frac{(\sqrt{2\pi})^n}{(2\pi)^{n/2}} = 1$.

So, $cf = RF^2(f) = Rf = f$ (as $f(-x) = f(x)$), so $c = 1$.

**Definition 3.18:** For $f, g \in \mathcal{S}(\mathbb{R}^n)$, define their convolution, $f*g$, by: $(f*g)(x) = \frac{1}{(2\pi)^{n/2}} \int f(x-y) g(y)\,dy$.

**Note:** $f*g = g*f$, $f*(g*h) = (f*g)*h$ – commutative algebra structure on $\mathcal{S}(\mathbb{R}^n)$.

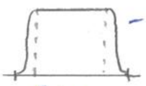
Theorem 3.19: For $f, g \in \mathcal{S}(\mathbb{R}^n)$, (i) $\int \hat{f} g \, dx = \int f \hat{g} \, dx$, (ii) $\int f \bar{g} \, dx = \int \hat{f} \bar{\hat{g}} \, dx$,
(iii) $\hat{f}\hat{g} = \widehat{f * g}$, (iv) $\widehat{fg} = \hat{f} * \hat{g}$.

Proof: (i) Both sides are equal to: $\frac{1}{(2\pi)^{n/2}} \iint f(x) g(\xi) e^{ix.\xi} \, dx \, d\xi$.
(ii) Put $h = \bar{\hat{g}}$. $\hat{h} = \int \bar{\hat{g}}(x) e^{-ix.\xi} d\xi$, so $\bar{\hat{h}} = \int \hat{g}(x) e^{ix.\xi} d\xi = g$.
From (i), $\int \hat{f} \bar{\hat{g}} \, dx = \int \hat{f} h \, dx = \int f \hat{h} \, dx = \int f \bar{g} \, dx$.
[Note: $f, g \in \mathcal{S}$, define a hermitian inner product: $\langle f, g \rangle = \int f \bar{g} \, dx$.
$F: \mathcal{S} \to \mathcal{S}$, $\langle Ff, Fg \rangle = \langle f, g \rangle$, i.e. $F$ is unitary]
(iii) Both sides equal $(2\pi)^{-n} \iint f(x) g(y) e^{-i(x+y).\xi} d\xi$.
(iv) Replace $f, g$ by $\hat{f}, \hat{g}$ and take Fourier Transform.

We have, $\hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-ix.\xi} dx$. If $f \in L^1$, Lebesgue theory $\Rightarrow$ $\hat{f}$ well-defined.
If $f \in L^2$, then ?...

Corollary 3.20: $F: \mathcal{S} \to \mathcal{S}$ extends to a unitary map of $L^2(\mathbb{R}^n)$ to itself.

Proof (for $n=1$): $\mathcal{S}$ is dense in $L^2$: given $f \in L^2$ and $\varepsilon > 0$, $\exists\, \varphi \in \mathcal{S}$ such that $\|f - \varphi\|_{L^2} < \varepsilon$. Why? Firstly, step functions are dense (by definition of Lebesgue integral). $\varphi$ a step function $\Rightarrow$ $\varphi = \sum_{i=1}^{n} c_i \chi_{I_i}$, where $\chi$ is the characteristic function of the integral.

 - bump function, $\psi \in \mathcal{S}$.

$$\int |\chi_{[0,1]} - \psi|^2 \, dx = \int_{-\delta}^{0} |\psi|^2 + \int_{1}^{1+\delta} |\psi|^2 < 2\delta \qquad (|\psi| \le 1)$$

Suppose $f \in L^2$, then $\exists$ a sequence $\varphi_n \in \mathcal{S}$ such that $\varphi_n \to f$ in $L^2$, so $\varphi_n$ is a Cauchy sequence, ie given $\varepsilon > 0$ $\exists\, N(\varepsilon)$ such that $\|\varphi_m - \varphi_n\| < \varepsilon \ \forall \ m, n \ge N(\varepsilon)$
Apply $F$, $\underbrace{\|F\varphi_m - F\varphi_n\|}_{\text{Cauchy, so converges in } L^2} = \|\varphi_n - \varphi_m\|$ (from (ii) of theorem 3.19)
Define $F(f) = \lim_{n \to \infty} F\varphi_n$.

## 4. Distributions.

$C_c^\infty(\mathbb{R}^n)$ - $C^\infty$ functions of compact support. $\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$
Recall, if $V$ is a vector space, its dual space $V'$ is the space of linear functions $f: V \to \mathbb{C}$.

Definition 4.1: If $f, \{f_n\} \in C_c^\infty$, then $f_n \to f$ as test functions if $f_1, \dots$ all have support within a fixed compact set $K$, and $\sup_{x \in K} |\partial^\alpha (f_n - f)| \to 0$ as $n \to \infty$ $\forall \alpha$.

Definition 4.2: A distribution $T$ is a linear map $f \mapsto \langle f, T \rangle$ from $C_c^\infty$ to $\mathbb{C}$ such that if $f_n \to f$ as test functions, then $\langle f_n, T \rangle \to \langle f, T \rangle$ in $\mathbb{C}$.

Examples: (i) Dirac $\delta$-function. Define $\langle f, \delta_a \rangle = f(a)$.
(ii) $\varphi$ a measureable function, $f \mapsto \langle f, T \rangle = \int f \varphi \, dx$ is a distribution.
(iii) $\langle T, f \rangle = (\partial^\alpha f)(a)$.

General Principle: Suppose $A: C_c^\infty \to C_c^\infty$ is a linear transformation, which is continuous (ie, $Af_n \to Af$ if $f_n \to f$) and suppose $\exists$ a continuous linear transformation $A': C_c^\infty \to C_c^\infty$ such that $\int (Af) g \, dx = \int f (A'g) \, dx$, $f, g \in C_c^\infty$ ($A'$ is the transpose of $A$), then $A$ can be extended to a linear map of distributions by: $\langle AT, f \rangle := \langle T, A'f \rangle$

Examples: (i) $A = \partial/\partial x_j : C_c^\infty \to C_c^\infty$. $\int \frac{\partial f}{\partial x_j} g \, dx = -\int f \frac{\partial g}{\partial x_j} dx$ (by parts). So $A' = -\partial/\partial x_j$:
So, define $\langle \partial T/\partial x_j, f \rangle = -\langle T, \partial f/\partial x_j \rangle$. $\therefore \langle \partial^\alpha T, f \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha f \rangle$.

(ii) Suppose $h \in C_1^\infty$, $A: C_c^\infty \to C_c^\infty$, $A(f) = hf$. $A' = A$. Multiply $T$ by $h$.
$\langle hT, f \rangle = \langle T, hf \rangle$.

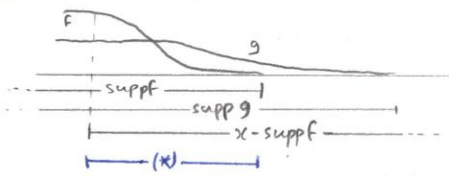
(iii) $(Rf)(x) = f(-x)$. $\langle RT, f \rangle = \langle T, Rf \rangle$.

(iv) $h \in C_c^\infty$. $A(f)$ = convolution with $h$, $(Af)(x) = \int h(x-y) f(y) \, dy =: h * f$.
(Note- omitted $(2\pi)^{-n/2}$ for convenience).
$\int (Af) g = \iint h(x-y) f(y) g(x) \, dx \, dy = \int f(y) [(Rh) * g](y) \, dy$
$\langle h * T, f \rangle = \langle T, (Rh) * f \rangle$.

Want to define convolutions more generally.

Definition 4.3: A continuous map $f: X \to Y$, $X, Y$ metric (topological) spaces is proper if, for each compact set $K \subseteq Y$, $f^{-1}(K)$ is compact.

We can take the convolution of two $C^\infty$ functions (not necessarily of compact support) if the map $\text{supp} f \times \text{supp} g \to \mathbb{R}^n$; $(x, y) \mapsto x + y$ is proper.
(Ie, if $\|x+y\| \le r$, then $\exists R$ such that $\|x\| \le R$, $\|y\| \le R$)
$\int f(x-y) g(y) \, dy \leftarrow$ integral over ~~supp f ∩ supp~~ $\text{supp} g \cap (x - \text{supp} f)$ - (*)
Check this is bounded if this support condition holds:




What is the support of a distribution? Recall that a distribution cannot be evaluated at a point.

Definition 4.4: A distribution vanishes on an open set $U \subseteq \mathbb{R}^n$ iff $\langle T, f \rangle = 0$ for all $f \in C_c^\infty$ such that $\text{supp} f \subset U$

Example: If $a \notin U$, then $\delta_a$ vanishes on $U$.

Definition 4.5: The support of a distribution $T$ is the complement of the set $\{ x \in \mathbb{R}^n : T \text{ vanishes in a neighbourhood of } x \} =: V$.

Note: $V$ is an open set. $\therefore$ $\text{supp} T$ is closed.

Examples: (i) $\operatorname{supp} \delta_a = \{a\}$.

(ii) if $\langle T, f\rangle = \int f g \, dx$, $g$ continuous, then $\operatorname{supp} T = \operatorname{supp} g$.

(iii) $\operatorname{supp} \partial T/\partial x_j \subseteq \operatorname{supp} T$.

Lemma 4.6: If $f \in C_c^\infty$ and $\operatorname{supp} f \cap \operatorname{supp} T = \phi$, then $\langle T, f\rangle = 0$.

Proof: If $x \in \operatorname{supp} f$, $\exists$ a ball $B_{2\delta}(x)$ on which $T$ vanishes. Supp $f$ is compact, so we can cover it with balls $\{B_\delta(x)\}$, $x \in \operatorname{supp} f$. By compactness, $\exists$ a finite subcovering $B_1, \dots, B_n$. Take a bump function $\varphi_i$ with $\varphi_i \equiv 0$ outside $\overline{B_i}$, and $\varphi_i > 0$ in $B_i$. Let $f_i := f\varphi_i / \sum_{j=1}^n \varphi_j$. $\operatorname{Supp} f_i \subseteq \overline{B_i} \subset B_{2\delta}(x)$. $\langle T, f_i\rangle = 0$.
But $\langle T, f\rangle = \langle T, \Sigma f_i\rangle = 0$. $(\Sigma f_i = f \Sigma \varphi_i / \Sigma \varphi_i = f)$

Corollary: if $f \in C^\infty$, $T$ a distribution, then $\langle T, f\rangle$ is defined wherever $\operatorname{supp} f \cap \operatorname{supp} T$ is compact.

Proof: $\langle T, f\rangle := \langle T, \varphi f\rangle$, where $\varphi$ is a bump function $\equiv 1$ on $\operatorname{supp} f \cap \operatorname{supp} T$.
(Use lemma to prove independent of $\varphi$)

Consequence: Define convolution of a distribution $T$ and a $C^\infty$ function $f$ such that $\operatorname{supp} T \times \operatorname{supp} f \to \mathbb{R}^n$ is proper. $(T * f)(x) := \langle T, f_x\rangle$, where $f_x(y) = f(x-y)$.
↑ satisfies conditions of corollary.

So, $\langle S * T, f\rangle = \langle S, (RT) * f\rangle$.

Properties of convolution:

(i) $T * f$ is a $C^\infty$ function, $S * T$ is a distribution.

(ii) $S * T = T * S$.

(iii) $S * (T * U) = (S * T) * U$.

(iv) $\operatorname{supp}(S * T) \subset \operatorname{supp} S + \operatorname{supp} T$

(v) $\partial^\alpha (S * T) = (\partial^\alpha S) * T = S * \partial^\alpha T$

(vi) $\delta_0 * T = T * \delta_0 = T$.

Proof of (vi): $\langle T * \delta_0, f\rangle = \langle T, (R\delta_0) * f\rangle = \langle T, \delta_0 * f\rangle. = \langle T, f\rangle$
[Using: $(Rf)(x) = f(-x) \Rightarrow R\delta_0 = \delta_0$, and $(\delta_0 * f)(x) = \langle \delta_0, f_x\rangle = f_x(0) = f(x)$].

Fourier Transform: $\hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) e^{-ix.\xi} dx$, $f \in C_c^\infty$.

Take one-variable case: $\hat{f}(\xi) = (2\pi)^{-1/2} \int f(x) e^{-ix\xi} dx$. Since $f \in C_c^\infty$, $\hat{f}(\xi)$ is well-defined for $\xi \in \mathbb{C}$.

$\lim_{h \to 0} \frac{\hat{f}(\xi+h) - \hat{f}(\xi)}{h} = (2\pi)^{-\frac{1}{2}} f(x)(-ix) e^{-ix\xi}$ - exists. So, $\hat{f}(\xi)$ is a holomorphic / analytic function, "Entire function."

"Identity theorem" in complex analysis $\Rightarrow$ if a holomorphic function vanishes on an open set, then it vanishes everywhere.

$\mathcal{S}$ - Schwarz space. Fourier Transform: $\mathcal{S} \to \mathcal{S}$ (isomorphically).
- need "convergence" definition.

$$\begin{array}{ccc} C_c^\infty & \subset & \mathcal{S} \\ \downarrow T & \swarrow & \\ \mathbb{C} & & \end{array}$$

Definition 4.7: $f_m \to f$ in $\mathcal{S}(\mathbb{R}^n)$ iff $\sup |x^\alpha \partial^\beta (f - f_m)| \to 0$ $\forall \alpha, \beta$. $(x \in \mathbb{R}^n)$

**Definition 4.8:** A tempered distribution is a linear map $T: \mathcal{S} \to \mathbb{C}$ such that $Tf_n \to Tf$ if $f_n \to f$ in $\mathcal{S}$.

$\hat{f}: \mathcal{S} \to \mathcal{S}$ is continuous ($f_m \to f \Rightarrow \hat{f}_m \to \hat{f}$). Let $T$ be a tempered distribution. Then $f \mapsto \langle T, \hat{f} \rangle$ is continuous - defines a distribution $\hat{T}$, ie $\langle \hat{T}, f \rangle = \langle T, \hat{f} \rangle$.
Note that since $\int f\hat{g}\,dx = \int \hat{f} g\,dx$, this is consistent with Transform of functions in $\mathcal{S}$.

**Remark:** $C_c^\infty \subset \mathcal{S}$ are dense, ie, if $f \in \mathcal{S}$ then given $\varepsilon > 0$ $\exists$ sequence $f_n \to f \in \mathcal{S}$ such that $f_n \in C_c^\infty$. $\varphi_n$, bump function on ball $B_n(0)$. $\varphi_n(f) \to f$.
$\therefore$ If $T$ is a tempered distribution, it is determined by its restriction to $C_c^\infty$.

**Aim:** Given a constant coefficient partial differential operator $P$, we want to solve the equation $Pu = f$. $P = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha$.

**Definition 4.9:** A fundamental solution for an operator $P$ is a distribution $K$ such that $PK = \delta_0$.

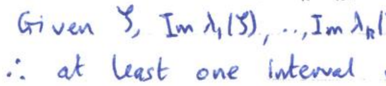
**Note:** If $f \in C_c^\infty$, then $P(K * f) = PK * f = \delta_0 * f = f$ - solved $Pu = f$ by putting $u = K * f$.

**Theorem:** every constant coefficient partial differential operator has a fundamental solution.
**Proof:** non-examinable. (Restatement later)

**Theorem 4.10:** Let $P = \sum_{|\alpha| \le k} a_\alpha \partial^\alpha$ be a constant coefficient partial differential operator (PDO).
Then, if $f \in C_c^\infty$, $\exists$ a $C^\infty$ $u$ such that $Pu = f$.

**Proof:** Solve $Pu = f$. Take Fourier Transform. $\left(\widehat{\partial f/\partial x_j} = i\xi_j \hat{f}\right)$ $\therefore \sigma(\xi)\hat{u}(\xi) = \hat{f}(\xi)$, so $\hat{u}(\xi) = \hat{f}(\xi)/\sigma(\xi)$
So, try to define $u(x) = (2\pi)^{-n/2} \int e^{ix.\xi} \hat{f}(\xi)/\sigma(\xi)\,d\xi$. ($\hat{f}$ entire - this is good)
Problem: $\sigma(\xi)$ may have zeroes - this is bad.
Point: the integrand is singular, but defined for complex $\xi$.
Idea: to change $\mathbb{R}^n$ to another contour. Choose a unit vector $\in \mathbb{R}^n$ such that $\sigma_k(\eta) = \sum_{|\alpha| = k} a_\alpha \eta^\alpha$. By rotation (an orthogonal transformation), assume $\eta = (0, .., 0, 1)$.
By multiplying by a constant, assume $a_{\alpha_0} = 1$, where $\alpha_0 = (0, .., 0, 1)$.
So, $\sigma(\xi) = \xi_n^k$ + terms of lower order in $\xi_n$.
For $\xi \in \mathbb{R}^n$, write $\xi = (\zeta, \xi_n)$, $\zeta \in \mathbb{R}^{n-1}$. $\sigma(\zeta, \xi_n) = 0$. Consider this as a polynomial of degree $k$ in $\xi_n$, with roots $\xi_n = \lambda_i(\zeta)$ - continuous in $\zeta$.
Order the roots: $\operatorname{Im}\lambda_i(\zeta) \le \operatorname{Im}\lambda_j(\zeta)$ if $i \le j$. If equality, then $\operatorname{Re}\lambda_i(\zeta) \le \operatorname{Re}\lambda_j(\zeta)$.
(continued later)

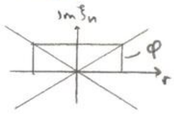
**Lemma A:** $\exists$ a bounded measureable function $\varphi(\zeta): \mathbb{R}^n \to \mathbb{R}$ such that $|\varphi(\zeta) - \operatorname{Im}\lambda_j(\zeta)| \ge 1$ $\forall j$

**Proof:** 

$\leftarrow$ $k+1$ intervals.
Given $\zeta$, $\operatorname{Im}\lambda_1(\zeta), .., \operatorname{Im}\lambda_k(\zeta)$ - $k$ real members.
$\therefore$ at least one interval must contain $[m, m+2]$. One of these: define $\varphi(\zeta) = m+1$.

Examples: (i) Hyperbolic PDO, $\sigma_k(\theta) \neq 0$, $\sigma(\xi + t\theta) \neq 0$ $\forall \xi$, $\forall$ and $\forall t$ with sufficiently large imaginary part. If $\operatorname{Im} \xi_n > m$, $\sigma(\xi, \xi_n) \neq 0$. Take $\varphi(\xi) = m+1$.

(ii) Laplace operator, $\sigma(\xi) = -\sum_{i=1}^{n} \xi_i^2$. $\xi_n^2 = -\sum_{i=1}^{n-1} \xi_i^2$, $\xi_n = \pm i \left(\sum_{i=1}^{n-1} \xi_i^2\right)^{1/2} = \pm i r$




Lemma B: Let $N = \{ w \in \mathbb{C}^n : \sigma(w) = 0 \}$ and for $\xi \in \mathbb{C}^n$ let $d(\xi)$ be the distance to $N$, ie $d(\xi) = \inf_{w \in N} \| \xi - w \|$. Then $|\sigma(\xi)| \geq \left(\frac{d(\xi)}{2}\right)^k$.

Proof: Take $\xi \notin N$. Define $g(z) = \sigma(\xi + z\eta)$, $\eta = (0, .., 0, 1)$. – polynomial in $z$ of degree $k$,
zeroes $\lambda_1, .., \lambda_k$: $g(z) = c(z-\lambda_1)..(z-\lambda_k)$. $\left|\frac{g(z)}{g(0)}\right| = \prod_{j=1}^{k} |1 - z/\lambda_j|$.
$\xi + \lambda_j \eta \in N$, so $|\lambda_j| \geq d(\xi)$, ($\eta$ is a unit vector)
So, if $|z| \leq d(\xi)$, $|z/\lambda_j| \leq 1$, so $|g(z)/g(0)| \leq 2^k$.
$|g^{(k)}(0)| = \left| \frac{k!}{2\pi i} \int_{|z| = d(\xi)} g(z) z^{-(k+1)} dz \right| \leq \frac{k!}{2\pi} \cdot \frac{2^k |g(0)|}{d(\xi)^{k+1}} \cdot 2\pi d(\xi) \leq k! \, d(\xi)^{-k} |g(0)| 2^k$
(using Cauchy's integral Formula)
Now, $g(0) = \sigma(\xi)$. $\partial^k \sigma / \partial \xi_n^k = k!$, $g^{(k)}(0) = k!$ Coefficient of $\xi_n^k$ is 1.
$k! \leq k! \cdot \frac{|\sigma(\xi)|}{d(\xi)^k} \cdot 2^k$ $\therefore$ $|\sigma(\xi)| \geq \left(\frac{d(\xi)}{2}\right)^k$.

Proof of Theorem 4.10 (continued): Consider $u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im} \xi_n = \varphi(\xi)} \overset{(*)}{e^{ix.\xi}} \cdot \frac{\hat{f}(\xi)}{\sigma(\xi)} d\xi_n d\xi$ $\hat{f}(\xi) \in \mathcal{S}$, $\sigma(\xi)$ bounded away from 0.

From Lemmas, $|\sigma(\xi)| \geq \left(\frac{d(\xi)}{2}\right)^k \geq \frac{1}{2^k}$ on $\operatorname{Im} \xi_n = \varphi(\xi)$
$(*)$: $e^{ix.\xi} = \exp\left\{ i \sum_{i=1}^{n} x_i \xi_i + i(x_n \operatorname{Re} \xi_n) - x_n \operatorname{Im} \xi_n \right\}$ bounded as $\varphi$ is bounded.
$\therefore$ $e^{ix.\xi}$ is bounded.
Integral converges uniformly, with all its derivatives, to a $C^\infty$ function.
This is because: (i) $|e^{ix.\xi}|$ is bounded, (ii) $\hat{f} \in \mathcal{S}$, (iii) $\frac{1}{\sigma(\xi)} \leq \frac{1}{2^k}$.
$Pu = (2\pi)^{-n/2} \int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im}(\xi_n) = \varphi(\xi)} \sigma(\xi) \cdot \frac{e^{ix.\xi}}{\sigma(\xi)} \cdot \hat{f}(\xi) \, d\xi_n d\xi$

$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix.\xi} \hat{f}(\xi) d\xi$, by Cauchy's Theorem, $= f(x)$. [fix $\xi$, compare $\operatorname{Im} \xi_n = 0$, $\operatorname{Im} \xi_n = \varphi(\xi)$]

Theorem 4.11: Every constant coefficient PDO has a fundamental solution.

Proof: Try $K(x) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im} \xi_n = \varphi(\xi)} \frac{e^{ix.\xi}}{\sigma(\xi)} d\xi_n d\xi$.
Problem – integral may not converge. Consider instead, $P_N = P(1-\Delta)^N$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$
$\sigma_N(\xi) = \sigma(\xi)\left(1 + \sum_{i=1}^{n} \xi_i^2\right)^N$, $\xi = (\xi_1, .., \xi_{n-1}, \xi_n + i\varphi(\xi))$
Because $\varphi(\xi)$ is bounded, $\sum_{i=1}^{n} \xi_i^2 \to \sum_{i=1}^{n} \xi_{0i}^2$ as $\xi \to 0$ on the conteur of integration.
Replace $\sigma$ by $\sigma_N$. $\sigma_N(\xi)^{-1}$ is integrable if $N > n/2$
($\sigma_N \sim \frac{1}{r^{2N}}$, polar coordinates, $\frac{1}{r^{2N}} \cdot r^{n-1} dr \, d\Omega$, $2N - n + 1 > 1$)
Claim $P_N K_N = \delta_0$ (distributions). Take $f \in C_c^\infty$.
Evaluate $\langle P_N K_N, f \rangle = \langle K_N, P_N' f \rangle$. ($P_N'$ – transpose. Replace $\partial/\partial x_j$ by $-\partial/\partial x_j$)
$= (2\pi)^{-n} \int_{\mathbb{R}^n} \underbrace{\int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im} \xi_n = \varphi(\xi)} e^{ix.\xi} \cdot \frac{(P_N' f)(x)}{\sigma_N(\xi)} d\xi_n d\xi}_{\equiv K_N} dx$.

$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im} \xi_n = \varphi(\xi)} \sigma_N(\xi)^{-1} d\xi_n d\xi \, . \, (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix.\xi} (P_N' f)(x) \, dx$.
Last part = Fourier Transform of $\Psi$, $\hat{\Psi}(-\xi)$, $\Psi(x) = P_N' f$.
$\hat{\Psi}(\xi) = \sigma_N'(\xi) \hat{f}(\xi) = \sigma_N(-\xi) \hat{f}(\xi)$, so $\hat{\Psi}(-\xi) = \sigma_N(\xi) \hat{f}(-\xi)$.
$\therefore$ What we had before $= (2\pi)^{-n/2} \int_{\mathbb{R}^{n-1}} \int_{\operatorname{Im} \xi_n = \varphi(\xi)} \hat{f}(-\xi) d\xi_n d\xi$
$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \hat{f}(-\xi) d\xi$, (by Cauchy), $= \left[(2\pi)^{-n/2} \int e^{ix.\xi} \hat{f}(-\xi) d\xi\right]_{x=0} = f(-x)|_{x=0} = f(0)$.
$\therefore$ $P_N K_N = \delta_0$, so $P(1-\Delta)^N K_N = \delta_0$ – a fundamental solution for $P$.

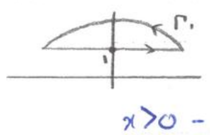
Remark: $K$ is not unique. For example, $P = \frac{\partial^2}{\partial x \partial y} \Rightarrow P(K + f(x) + g(y)) = PK = \delta_0$.
$(K_2 = K_1 + u$, where $Pu = 0)$

## 5. Laplace Operator.

$\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ - the Laplacian. $\sigma(\xi) = -\sum_{i=1}^{n} \xi_i^2$ - elliptic.

Find a fundamental solution, for $n=1$, so $\Delta = d^2/dx^2$

Method of section 4: $K(x) = (2\pi)^{-1} \int_{\operatorname{Im}\xi = \varphi} \frac{e^{ix\xi}}{-\xi^2}\, d\xi = -(2\pi)^{-1} \int_{\operatorname{Im}\xi = 1} \frac{e^{ix\xi}}{\xi^2}\, d\xi$



Set $\xi = Re^{i\theta}$. $e^{ix\xi} = e^{ix(R\cos\theta + i\sin\theta)} = e^{ixR\cos\theta} e^{-xR\sin\theta}$.

$x > 0$ - integral around semicircle $\to 0$.
$\int_{\Gamma_1} = 2\pi i x$ (residues) $= 0$, (no poles inside $\Gamma_1$)
$x < 0$ - complete with a semicircle below the line.
$\int_{\Gamma_2} \frac{e^{ix\xi}}{\xi^2}\, d\xi = -2\pi i .\ \operatorname{Res}(\xi = 0) = -2\pi i (ix) = 2\pi x$.

$\therefore K(x) = \begin{cases} 0 & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases}$. $K$ is unique up to addition of $ax+b$: 

( $\leftarrow$ fundamental solution of $d/dx$ )
Note: $\frac{d}{dx}\left(\frac{dK}{dx}\right) = \delta_0$. $\frac{dK}{dx} = \begin{cases} 0 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases}$. $\therefore \left(\frac{d^2}{dx^2} f\right)(-x) = \frac{d^2}{dx^2}(f(-x))$
Symmetry of equation: $(Rf)(x) = f(-x) \Rightarrow PR = RP$.

Find a fundamental solution invariant under symmetry $\Leftrightarrow$ $K$ an even function.
$(R\tfrac{1}{2}(f + Rf)) = \tfrac{1}{2}(f + Rf)$.
So, $K(x) = \tfrac{1}{2}|x|$ - symmetric.

In $\mathbb{R}^n$ $(n \geq 1)$, $\Delta$ is invariant under the action of the group $O(n)$ of orthogonal transformations: $x \mapsto Ax$, $AA^T = I$

$\frac{\partial}{\partial x_j}(f(Ax)) = \sum_k \left(A_{jk} \frac{\partial f}{\partial x_k}\right)(Ax)$ $\therefore \sum_j \frac{\partial^2}{\partial x_j^2}(f(Ax)) = \sum_{j,k,l} \left(A_{jl} A_{jk} \frac{\partial^2 f}{\partial x_l \partial x_k}\right)(Ax)$. $\left[\sum_j A_{jl} A_{jk} = (A^T A)_{lk} = \delta_{lk}\right]$

$\therefore \Delta(f(Ax)) = (\Delta f)(Ax)$ - $\Delta$ invariant under orthogonal transformations.

Seek a fundamental solution with orthogonal symmetry. Equivalently, a function $f(r)$, where $r^2 = \sum_{i=1}^{n} x_i^2$. If $\Delta K = \delta_0$, then $K$ satisfies $\Delta K = 0$ outside $0$, so consider:

$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}(f(r)) = \sum_j \frac{\partial}{\partial x_j}(f'(r)\, x_j/r) = \sum_j \left(f''(r)\, x_j^2/r^2 + f'(r)/r - f'(r) . \frac{x_j}{r^2} . \frac{x_j}{r}\right) = f''(r) + \frac{n}{r} f'(r) - \frac{1}{r} f'(r) = 0.$

So, $f''/f' = \frac{-(n-1)}{r}$, so $f' = cr^{1-n}$, so $f = \begin{cases} c_1 r^{2-n} + c_2 & n \neq 2 \\ c_1 \log r + c_2 & n = 2 \end{cases}$

Theorem 5.1: If $F(x) = \|x\|^{2-n}$ on $\mathbb{R}^n$ $(n \neq 2)$, then $\Delta F = (2-n)\sigma_n \delta_0$, where $\sigma_n$ = area of unit sphere in $\mathbb{R}^n$. (Hence $K = \frac{\|x\|^{2-n}}{(2-n)\sigma_n}$ is an (invariant) fundamental solution)

Proof: Think of $F$ as a distribution: $\langle F, f \rangle = \int \frac{f(x)}{\|x\|^{n-2}}\, dx$.
Near origin, $dx = r^{n-1}\, dr\, d\Omega$, $\|x\|^{n-2} = r^{n-2}$. $\lim \int_r^\infty$ exists, $\frac{r^{n-1}}{r^{n-2}}\, dr \sim r\, dr$.
Let $F_\varepsilon(x) = (\varepsilon^2 + \|x\|^2)^{\frac{2-n}{2}} \in C^\infty$. $\langle F_\varepsilon, f \rangle \to \langle F, f \rangle$ for each $f \in C_c^\infty$ as $\varepsilon \to 0$.
So $F_\varepsilon \to F$, in the sense of distributions. $\Delta F_\varepsilon \to \Delta F$ as distributions.

$$\Delta F_\varepsilon(x) = \sum_j \frac{\partial}{\partial x_j}\left((2-n)\left(\|x\|^2+\varepsilon^2\right)^{-n/2} x_j\right) = \sum_j \left\{(2-n)\left(\|x\|^2+\varepsilon^2\right)^{-\frac{n}{2}} + (2-n)\frac{(-n)}{2}\left(\|x\|^2+\varepsilon^2\right)^{-\frac{n}{2}-1}\cdot 2x_j x_j\right\}$$
$$= (2-n)\, n\, \varepsilon^2 \left(\|x\|^2+\varepsilon^2\right)^{-\frac{n}{2}-1} =: g_\varepsilon(x).$$

$g_1(x/\varepsilon) = (2-n)n.\left(\frac{\|x\|^2}{\varepsilon^2}+1\right)^{-\frac{n}{2}-1} = g_\varepsilon(x)\,\varepsilon^n.$

So, $\langle \Delta F_\varepsilon, f\rangle = \int g_\varepsilon(x) f(x)\,dx = \int_{\mathbb{R}^n} \varepsilon^{-n} g_1\left(\frac{x}{\varepsilon}\right) f(x)\,dx = \int \varepsilon^{-n}.\, g_1(y) f(\varepsilon y)\, \varepsilon^n dy \quad \left[y = \frac{x}{\varepsilon}\right]$

$\lim_{\varepsilon\to 0} \langle \Delta F_\varepsilon, f\rangle = \lim_{\varepsilon\to 0}\left(\int g_1(y) f(\varepsilon y)\,dy\right) = f(0)\int g_1(y)\,dy.$

$\therefore\ \Delta F = \left(\int g_1\,dy\right)\delta_0.$

Evaluate the constant: $\int g_1(x)\,dx = \sigma_n \int_0^\infty (2-n)(1+r^2)^{-\frac{n}{2}-1}.\, n r^{n-1}\,dr = \sigma_n.\, n.(2-n)\int_0^\infty (1+r^2)^{-\frac{n}{2}-1}.\, r^{n-1}\,dr.$

Put $r^2+1=s$: $\int = \frac{1}{2}\int_1^\infty s^{-\frac{n}{2}-1}.\,(s-1)^{\frac{n-1}{2}}\,ds = \frac{1}{2}\int_1^\infty s^{-2}\left(1-\frac{1}{s}\right)^{\frac{n}{2}-1} ds.$

Put $u=\frac{1}{s}$: $= -\frac{1}{2}\int_0^1 (1-u)^{\frac{n}{2}-1}\,du = 1/n.\quad \therefore \int g_1(x)\,dx = (2-n)\,\sigma_n.$

Remarks: (i) For $n=2$, $\frac{1}{2\pi}\log r$ is a fundamental solution.

(ii) Suppose $P = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$. $\bar{P} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$. $P\bar{P} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \Delta \quad \therefore P\left[\bar{P}\left(\frac{1}{2\pi}\log r\right)\right] = \delta_0.$

$\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{1}{2\pi}\log r\right) = \frac{1}{2\pi r}\left(\frac{x}{r} - \frac{iy}{r}\right) = \frac{\bar{z}}{2\pi z\bar{z}} = \frac{1}{2\pi z}.$

As a distribution, this is a fundamental solution of $P$.

(iii) $n=1$, $K = \frac{1}{1.\sigma_1.\|x\|^{-1}} = \frac{\|x\|}{\sigma_1}$. $\sigma_1 = 2$: 

$\Delta K = \delta_0.\quad K = \begin{cases} \frac{1}{(2-n)\sigma_n\|x\|^{n-2}}, & n \neq 2 \\ \frac{1}{2\pi}\log\|x\|, & n=2 \end{cases}$

Remark: $K$ is $C^\infty$ for $x \neq 0$. Any differential operator with this property is called hypoelliptic.

$Pf = g$, $C^\infty$. Is $f$ $C^\infty$ - Regularity.

Example: $P = \frac{\partial^2}{\partial x \partial y}$. $P(f(x)+g(y)) = 0$. $Pu=0$, if $f,g$ are $C^1$. P is not hypoelliptic.

Theorem 5.2: There is a distribution $E$ of compact support such that $\Delta E = \delta_0 + \Psi$, where $\Psi \in C_c^\infty$. (E is called a parametrix, or approximate inverse)

Proof: $K = \varphi K + (1-\varphi)K$, with $\varphi$ a bump function: $\varphi(r) = \begin{cases} 1 & \text{inside } B_{2\varepsilon}(0) \\ 0 & \text{outside } \bar{B}_{2\varepsilon}(0). \end{cases}$

$\Delta(\varphi K) = \delta_0 - \Delta((1-\varphi)K)$ - vanishes if $\|x\| > 2\varepsilon$. So $\Delta E = \delta_0 + \Psi$.

Theorem 5.3: Let $f,g$ be distributions such that $\Delta f = g$. If $g$ is $C^\infty$ near $a$, then $f$ is $C^\infty$ near $a$

Proof: $\langle g, \varphi\rangle$, where $\operatorname{supp}\varphi \subset B_\varepsilon(a)$, $= \int \tilde{g}\varphi\,dx$, where $\tilde{g} \in C^\infty(B_\varepsilon(a))$.

(i) Global version: suppose $g \in C_c^\infty(\mathbb{R}^n)$. $\delta_0 * f = f$. ($\delta_0$ has compact support)

$g = \Delta f = \Delta(\delta_0 * f) = (\Delta\delta_0) * f.$

$\Delta E = \delta_0 + \Psi, = (\Delta\delta_0) * E$. Hence, $E * g = E * \Delta\delta_0 * f = (\delta_0 + \Psi) * f = f + \Psi * f.$

$\therefore f = E*g - \Psi * f$. $g, \Psi\ C^\infty \Rightarrow f \in C^\infty(\mathbb{R}^n)$. (So, $\Delta u = 0 \Rightarrow u \in C^\infty(\mathbb{R}^n)$)

(ii) Local version: Take a bump function, $\varphi \equiv 1$ near a.

$\varphi g = \varphi\Delta f = \Delta f - (1-\varphi)\Delta f = \Delta f + h$, $h \equiv 0$ near a. $C^\infty$ as $g$ is. $C^\infty$ as $\Psi$ is.

$E * \varphi g - E*h = E * \Delta f = (\Delta E) * f = \delta_0 * f + \Psi * f.\quad \therefore f = E * \varphi g - E*h - \Psi * f$

Recall: $\operatorname{supp}(E*h) \subseteq \operatorname{supp}E + \operatorname{supp}h$. $\operatorname{supp}h \cap B_r(a) = \emptyset$. Choose $\varepsilon$ to define $E$ such that $\operatorname{supp}E \subset B_\varepsilon(a)$, $\therefore a \notin \operatorname{supp}(E*h)$ if $\varepsilon$ is small enough. $\therefore E*h$ is zero on a sufficiently small neighbourhood of a. $f$ is $C^\infty$ on this neighbourhood.

Dirichlet Problem for $\Delta$: Given $f: \partial\Omega \to \mathbb{C}$, solve $\Delta u = 0$ on $\Omega$, $u|_{\partial\Omega} = f$ on $\partial\Omega$


Note: a solution of $\Delta u = 0$ is called a harmonic function.

Proposition 5.4 (Divergence Theorem): Let $\Omega$ be an open bounded set in $\mathbb{R}^n$ with boundary $\partial\Omega$ $(=\bar{\Omega}\setminus\Omega)$ given by a $C^\infty$ hypersurface, and let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ function on $\bar{\Omega}$. Then, $\int_{\partial\Omega} \underline{f}\cdot\underline{n}\, d\sigma = \int_\Omega \sum_{j=1}^n \frac{\partial f}{\partial x_j}\, dx$

($\partial\Omega$ is locally defined by $f(x) = 0$ with $Df(x) \neq 0$; $\underline{n} = \frac{Df}{\|Df\|}$)

Proof (for ball): We shall prove that if $f \in C^1$ on $\|x\| < 1$ and continuous on $\|x\| \le 1$, then $\int_{\|x\|\le 1} \partial f/\partial x_j\, dx = \int_{\|x\|=1} f(x) \frac{x_j}{\|x\|}\, d\sigma$.

Take a bump function $\varphi_\varepsilon(r) = \begin{cases} 1 & \text{if } r < 1-\varepsilon \\ 0 & \text{if } r \ge 1 \end{cases}$ Then, $\int_{\|x\|\le 1} f\, dx = \lim_{\varepsilon\to 0} \int f \varphi_\varepsilon\, dx$.

So, $\int \frac{\partial f}{\partial x_j} \varphi_\varepsilon\, dx = -\int f \frac{\partial \varphi_\varepsilon}{\partial x_j}\, dx = -\int f \frac{d\varphi_\varepsilon}{dr}\cdot\frac{x_j}{\|x\|}\, dx = -\int g(r) \frac{d\varphi_\varepsilon}{dr}\cdot r^{n-1}\, dr$ (integrate over polars)

$= \int \frac{d}{dr}(g(r) r^{n-1}) \varphi_\varepsilon\, dr$. Take limit $\varepsilon \to 0$, $= \int_0^1 \frac{d}{dr}(g(r) r^{n-1})\, dr = g(1) = \int_{\|x\|=1} f(x)\, x_j/\|x\|\, dx$.

Then put $f = f_j$ and sum.

Consequences of Divergence Theorem.

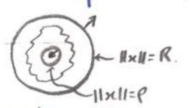
(i) $\int_\Omega \Delta u\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}\, d\sigma$. $\left[ f_i = \partial u/\partial x_i,\ \sum \frac{\partial u}{\partial x_i} n_i = \text{normal derivative} \right]$.

(ii) Green's Formula: $\int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) = \int_\Omega (u\Delta v - v\Delta u)\, dx$. $\left(\text{Take } f_i = u\, \partial v/\partial x_i - v\, \partial u/\partial x_i\right)$

$\partial f_i/\partial x_i = u\Delta v + \sum \frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i} - \sum \frac{\partial v}{\partial x_i}\cdot\frac{\partial u}{\partial x_i} - v\Delta u$.

Theorem 5.5 (Mean Value Property): Let $u$ be harmonic in $B(x_0, r)$. Then, for any $\rho < r$, $u(x_0) = \frac{1}{\sigma_n \rho^{n-1}} \int_{\|x-x_0\|=\rho} u(x)\, d\sigma$.

Proof: Take $x_0 = 0$. By regularity, $u$ is $C^\infty$ in $\bar{B}(0,\rho)$. Multiply $u$ by a bump function which $\equiv 1$ on $B(0,\rho)$, and $\equiv 0$ in a neighbourhood of $\{x : \|x\| = R\}$:




Call the resulting function $u$. $u \in C_c^\infty(\mathbb{R}^n)$, so apply $\delta_0$.

$u(0) = \langle \delta_0, u\rangle = \langle \Delta K, u\rangle = \langle K, \Delta u\rangle = \int_{R\ge\|x\|\ge\rho} \frac{\Delta u\, dx}{(2-n)\sigma_n r^{n-2}}$ (as $u = 0$ in $B(0,\rho)$)

$= \frac{1}{(2-n)\sigma_n} \cdot \int_{R\ge\|x\|\ge\rho} \left( \frac{\Delta u}{r^{n-2}} - \cancel{u\Delta\left(\frac{1}{r^{n-2}}\right)}^{=0} \right) dx = -\frac{1}{(2-n)\sigma_n} \cdot \int_{\substack{\|x\|=\rho,\\ \|x\|=R}} \left( \frac{1}{r^{n-2}}\cdot\frac{\partial u}{\partial n} - u\frac{\partial}{\partial r}\left(\frac{1}{r^{n-2}}\right)\right) d\sigma$ (Green's formula)

Now, $u \equiv 0$ in a neighbourhood of $\|x\| = R$, so $\int_{\|x\|=R} = 0$.

So, $u(0) = \dots = \frac{-1}{(2-n)\sigma_n} \cdot \frac{1}{\rho^{n-2}} \int_{\|x\|=\rho} \frac{\partial u}{\partial n}\, d\sigma + \frac{1}{(2-n)\sigma_n} \cdot \int_{\|x\|=\rho} u \frac{\partial}{\partial r}\left(\frac{1}{r^{n-2}}\right) d\sigma$

$= \frac{-1}{(2-n)\sigma_n} \cdot \frac{1}{\rho^{n-2}} \cdot \int_{\|x\|=\rho} \cancel{\Delta u\, dx}^{=0} - \frac{n-2}{(2-n)\sigma_n} \cdot \int_{\|x\|=\rho} u \cdot \frac{d\sigma}{\rho^{n-1}} = \frac{1}{\sigma_n \rho^{n-1}} \int_{\|x\|=\rho} u(x)\, dx$.

Theorem 5.6 (Dirichlet Problem for the Ball): Let $h$ be continuous on $\|x\| = 1$ and define $u(x)$ for $\|x\| < 1$ by: $u(x) = \frac{1}{\sigma_n} \cdot \int_{\|y\|=1} \frac{1-\|x\|^2}{\|x-y\|^n} h(y)\, d\sigma$. Then, $u(x)$ extends to a continuous function on $\|x\| \le 1$ which is harmonic in $\|x\| < 1$ and $u(x) = h(x)$ on $\|x\| = 1$

(Note: $x = 0$ gives the mean value property).

Proof: Poisson kernel $:= \frac{1}{\sigma_n} \cdot \frac{1-\|x\|^2}{\|x-y\|^n} =: P(x,y)$. Proof is a consequence of properties of $P$:

(i) $P(x,y)$ is harmonic as a function in $\|x\| < 1$ if $\|y\| < 1$.

Put $z = x - y$. $\sigma_n P = \frac{1 - \|y+z\|^2}{\|z\|^n} = \frac{1 - (y+z, y+z)}{\|z\|^n} = \frac{-2(y,z) - \|z\|^2}{\|z\|^n}$

Note: $\frac{1}{\|z\|^{n-2}}$ is harmonic, so is $\frac{\partial}{\partial z_i}\left(\frac{1}{\|z\|^{n-2}}\right) = -\frac{(n-2)}{\|z\|^{n-1}}\cdot\frac{z_i}{\|z\|} = -\frac{(n-2) z_i}{\|z\|^{n-2}}$. $\sigma_n P = -2\sum y_i \frac{z_i}{\|z\|^n} - \frac{1}{\|z\|^{n-2}}$.

(ii) $P(x,y) > 0$ if $\|x\| < 1$.

(iii) $\int_{\|y\|=1} P(x,y)\, d\sigma = 1$. Consider $f(x) = \int_{\|y\|=1} P(x,y)\, d\sigma$. $f(0) = \frac{\sigma_n}{\sigma_n} = 1$

Since $P$ is harmonic, so is $f$. (differentiating under the integral).

$f$ is also invariant under orthogonal transformations, $\|Ax\| = \|x\|$.

$f(Ax) = \int_{\|y\|=1} P(Ax, Ay)\, d\sigma = \int_{\|y\|=1} P(x,y)\, d\sigma = f(x)$. $\therefore$ $f$ is a function of $r$.

But $f = \begin{cases} c_1 r^{2-n} + c_2, & n \neq 2 \\ c_1 \log r + c_2, & n = 2 \end{cases}$. If $f(0) = 1 \Rightarrow c_1 = 0,\ c_2 = 1$ $\therefore f(r) = 1$.

Now consider $u(x) = \int_{\|y\|=1} P(x,y)\, h(y)\, d\sigma$. - Harmonic on $\|x\|<1$ by differentiating under the integral. All we need to show is $u(ry) \to h(y)$ as $r \to 1$ from below, if $\|y\| = 1$.

$$\|u(ry) - h(y)\| = \left| \int_{\|x\|=1} P(ry, x)\,(h(x) - h(y))\, d\sigma \right| \quad \text{[from (iii)]}$$
$$\leq \int P(ry,x)\,|h(x) - h(y)|\, d\sigma \quad \text{(From (ii))}$$
$$\leq \int_{\|y-x\| \leq \varepsilon} P(ry,x)\,|h(x) - h(y)|\, d\sigma + 2\sup|h| \int_{\|x-y\| \geq \varepsilon} P(ry,x)\, d\sigma.$$

First term: $h$ continuous on compact set, uniformly continuous, so given $\eta$, $\exists \varepsilon > 0$ such that $|h(x) - h(y)| < \eta$ if $\|x-y\| \leq \varepsilon$. $P \geq 0$, $\int P = 1$, so first integral $< \eta$.

Second term: $\sigma_n P(ry, x) = \frac{1-r^2}{\|x - ry\|^n}$

$\|x - ry\|^2 = r^2 + 1 - 2r(x,y)$. $\quad \varepsilon^2 \leq \|x-y\|^2 - 2 - 2(x,y)$

So, $\|x - ry\|^2 = r(2 - 2(x,y)) + (1 - r^2) \geq \varepsilon^2 r + (1-r^2)$

Hence, $P(ry,x) \leq \frac{1-r^2}{(r\varepsilon^2 + (1-r^2))^{n/2}} \to 0$ as $r \to 1$

$\therefore |u(ry) - h(y)| < \eta + \eta < 2\eta$, so $u(ry) \to h(y)$

**Theorem 5.8 (Maximum Principle):** If $u$ is continuous on $\bar{\Omega}$ and harmonic on $\Omega$, and $\Omega$ is connected, then $\sup_{x \in \bar{\Omega}} u = \sup_{x \in \partial\Omega} u$.

**Proof:** Suppose that it achieves its maximum $M$ at a point $a \in \Omega$.

Let $\Omega_1 = \{x \in \Omega : u(x) = M\}$ - non-empty, closed. $u(x) - M$ is harmonic.

$0 \leq \frac{\rho^{1-n}}{\sigma_n} \int_{\|x-a\|=\rho} (M - u(x))\, d\sigma = M - u(a)$, by M.V.P., $= 0 \Rightarrow u(x) = M$ on $\|x-a\| = \rho$.

Vary $\rho \Rightarrow u(x) = M$ on $B(a,\rho) \Rightarrow \Omega_1$ is open $\Rightarrow \Omega_1 = \Omega$ as $\Omega$ is connected.

**Corollary:** If $u, v$ solve Dirichlet problem for $h: \partial D \to \mathbb{R}$, then $u - v$ solves it for $0: \partial D \to \mathbb{R}$. By maximum principle, $u - v \leq 0$ in $\Omega \Rightarrow u \leq v$. But $v - u$ also solves the problem $\Rightarrow v \leq u \Rightarrow u = v$.

**Remark:** The proof works if we assume only $u(a) \leq \frac{\rho^{1-n}}{\sigma_n} \cdot \int_{\|x-a\|=\rho} u(x)\, dx$.

**Theorem 5.9 (Harnack's Inequality):** If $u(x)$ is continuous on $\|x\| \leq R$ and harmonic on $\|x\| < R$, and non-negative, then $\frac{R^{n-2}(R-r)}{(R+r)^{n-1}} u(0) \leq u(x) \leq \frac{R^{n+2}(R+r)}{(R-r)^{n-1}} u(0)$ $\quad [r^2 = \|x\|^2]$

**Proof:** use the Poisson kernel for a ball of radius $R$: $P(x,y) = \frac{1}{R\sigma_n} \cdot \frac{R^2 - \|x\|^2}{\|x-y\|^n}$ $\quad [\|y\| = R]$

if $\|y\| = R$, $\|x\| = r$: $R - r = \|y\| - \|x\| \leq \|x - y\| \leq \|x\| + \|y\| = R + r$. So, $\frac{1}{R\sigma_n} \cdot \frac{R^2 - r^2}{(R+r)^n} \leq P \leq \frac{1}{R\sigma_n} \cdot \frac{R^2 - r^2}{(R-r)^n}$.

But, $u(x) = \int_{\|y\|=R} P(x,y)\, u(y)\, dy$.

So, $\frac{1}{R\sigma_n} \cdot \frac{R^2 - r^2}{(R+r)^n} \cdot \int_{\|y\|=R} u(y)\, dy \leq u(x) \leq \frac{1}{R\sigma_n} \cdot \frac{R^2 - r^2}{(R-r)^n} \cdot \int_{\|y\|=R} u(y)\, dy$. MVP $\Rightarrow u(0) = \frac{R^{1-n}}{\sigma_n} \cdot \int u(y)\, dy$.

$\therefore \frac{R-r}{(R+r)^{n-1}} \cdot R^{n-2} \cdot u(0) \leq u(x) \leq \frac{R+r}{(R-r)^{n-1}} \cdot R^{n-2} \cdot u(0)$

Corollary: If $\{u_n\}_{n=1}^{\infty}$ is harmonic in $\Omega$ and $u_n \nearrow u$ pointwise (ie, $u_n(x)$ is monotonic increasing in $n$ and bounded above), then on each compact subset $K$, the convergence is uniform and the limit function is harmonic.

Proof: In a ball of radius $R$, if $n>m$: $0 \le u_n(x) - u_m(x) \le C(u_n(a) - u_m(a))$. Since $u_n(a)$ converges, $u_n(x)$ converges uniformly. $\therefore$ Since $u_n$ are continuous, $u(x)$ is continuous (uniform limit of continuous functions), and since $u_n(x) = \int_{\|y-a\|=\rho} P(x,y) u_n(y)\, dy$. Take limits: $u(x) = \int_{\|y-a\|=\rho} P(x,y) u(y)\, dy$. Since $u$ is continuous on the boundary, $u$ is harmonic by proof of Theorem 5.7 (Dirichlet Ball).

Aim now to solve the Dirichlet problem for more general $D$ - Perron's method.

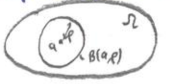
Definition 5.10: A continuous function $u$ on $\Omega$ is called subharmonic if, for each $x \in \Omega$ and sufficiently small $\rho$, $u(x) \le \frac{1}{\sigma_n \rho^{n-1}} \int_{\|x-y\|=\rho} u(y)\, dy$.

Examples: (i) Harmonic functions.

(ii) $C^2$ functions with $\Delta u \ge 0$. [Look at the last line of Theorem 5.5 - (MVP): $u(0) = \frac{-1}{(2-n)\sigma_n}\left\{\frac{1}{\rho^{n-2}} \int_{\|x\|=\rho} \Delta u\, dx - \frac{(2-n)}{\rho^{n-1}} \int_{\|x\|=\rho} u\, d\sigma\right]$, such as $u = \|x\|^2$.

Remark: As noted above, the maximum principle holds for subharmonic functions.

Suppose $u$ is subharmonic on $\Omega$, and $u \le h$ on $\partial\Omega$, and suppose $v$ is harmonic on $\Omega$, and $v = h$ on $\partial\Omega$. Then, $u - v$ is subharmonic, and $u - v \le 0$ on $\partial\Omega$. So, by maximum principle, $u \le v$ on $\Omega$. Idea of proof: define $v(x) = \sup\{u(x) : u$ subharmonic on $\Omega$, $u \le h$ on $\partial\Omega\}$



$u$ continuous on $\Omega$.

$u_{a,\rho}(x) = u(x)$ outside $B(a,\rho)$, $=$ solution of Dirichlet problem inside $B(a,\rho)$.

Lemma 1: If $u$ is subharmonic, then $u \le u_{a,\rho}$.

Proof: Outside the ball - equality. Inside the ball, $u$ is subharmonic, $u_{a,\rho}$ is harmonic and so subharmonic. $\therefore u - u_{a,\rho}$ subharmonic, $=0$ on boundary, so $u - u_{a,\rho} \le 0$, by maximum property.

Lemma 2: $u_{a,\rho}$ is subharmonic.

Proof: Required to ~~proof~~ prove that for sufficiently small $r$, $u_{a,\rho}(x) \le \frac{r^{1-n}}{\sigma_n} \int_{\|x-y\|\le r} u_{a,\rho}(y)\, dy$.

(i) $x \in \overline{B(a,\rho)}$, $u_{a,\rho} = u$, subharmonic. (Choose $r$ such that $B(x,r) \cap \overline{B(a,\rho)} = \phi$).

(ii) $x \in B(a,\rho)$, $u_{a,\rho}$ is harmonic.

(iii) $\|x - a\| = \rho$. Here, $u_{a,\rho}(x) = u(x)$. $\le \frac{r^{1-n}}{\sigma_n} \int_{\|x-y\|\le r} u(y)\, dy$, as $u$ is subharmonic,

$\le \frac{r^{1-n}}{\sigma_n} \int_{\|x-y\|\le r} u_{a,\rho}(y)\, dy$, by lemma 1.

Lemma 3: If $F(a,\rho) \subset \Omega$ and $u$ is subharmonic then $u(a) \le \frac{\rho^{1-n}}{\sigma_n} \int_{\|x-y\|=\rho} u(y)\, dy$.

Proof: $u(a) \le u_{a,\rho}(a) = \frac{\rho^{1-n}}{\sigma_n} \int_{\|y-a\|=\rho} u_{a,\rho}(y)\, dy$, as $u_{a,\rho}$ is harmonic on $B(a,\rho)$

$= \frac{\rho^{1-n}}{\sigma_n} \int_{\|y-a\|=\rho} u(y)\, dy$, as $u = u_{a,\rho}$ on $\|y - a\| = \rho$.

Lemma 4: If $u_1, \dots, u_n$ are subharmonic and $u_i \le h$ on $\partial\Omega$ then $v(x) := \max\{u_1(x), \dots, u_n(x)\}$ is subharmonic and $v \le h$ on $\partial\Omega$.

Proof: $v$ is continuous on $\bar\Omega$.

$$v(x) = \max\{u_i(x)\} \le \max\left\{\frac{r^{1-n}}{\sigma_n}\cdot\int_{\|x-y\|=r} u_i(y)\,dy\right\} \le \frac{r^{1-n}}{\sigma_n}\cdot\int \max\{u_i(y)\}\,dy.$$

$S := \{u$ subharmonic on $\Omega$, $u \le h$ on $\partial\Omega\}$.

Note: $S \ne \emptyset$, since $u(x) = m = \min_{x\in\partial\Omega} h(x)$ is in $S$.

Define, for each $x \in \bar\Omega$, $f_h(x) = \sup_{u\in S} u(x)$.

(i) $f_h(x)$ is well-defined, since by the maximum principle: $u(x) \le \sup_{y\in\partial\Omega} u(y) \le \max_{y\in\partial\Omega} h(y) = M$.

(ii) if $x \in \partial\Omega$, $\sup u(x) \le h(x) \Rightarrow f_h(x) \le h(x)$.

(iii) $f_h$ is subharmonic

Proof: Choose a sequence $x_1, x_2, \dots$ which is dense in $\partial\Omega$ (such as points with rational coordinates). For each $x_i$, $f_h(x_i) = \sup_{u\in S} u(x_i)$, so there exists a sequence of functions $u_n^{(i)} \in S$ such that $u_n^{(i)}(x_i) \nearrow f_h(x_i)$ as $h \to \infty$.

(Note that we can replace $u_n^{(i)}$ by any function $u \in S$ such that $u \ge u_n^{(i)}$).

Define $u_n(x) = \max\{u_n^{(1)}(x), \dots, u_n^{(n)}(x)\} \in S$, by lemma 4. Now, $u_n(x_i) \nearrow f_h(x_i)$ $\forall i$.

For each $a \in \Omega$, choose $\rho$ such that $\overline{B(a,\rho)} \subset \Omega$, and now by lemmas 1 to 3, replace $u_n$ by a subharmonic function harmonic in the ball.

If $x_i \in B(a,\rho)$, then $f_h(x_i) = \lim_{n\to\infty} u_n(x_i)$. $u_{n+1}(x_i) \ge u_n(x_i)$ $\forall i$

$\therefore u_{n+1}(x) \ge u_n(x)$ since $\{x_1, \dots\}$ is dense and $u_n$ is continuous.

$u_n(x) \nearrow$ on $\overline{B(a,\rho)}$ and is harmonic, so by the corollary to Harnack's theorem, the limit function $u$ is harmonic. So in $B(a,\rho)$, $f_h(x_i) = u(x_i)$, harmonic $\Rightarrow \lim_{x_{n(i)}\to x} f_h(x_{n(i)}) = u(x)$.

We need that $\lim f_h(x_{n(i)}) = f_h(x)$, but we could have added to the original dense set.

$\therefore f_h$ is a continuous function = harmonic on $B(a,\rho)$ $\therefore f_h$ is harmonic.

Theorem 5.10: Suppose for each point $y \in \partial\Omega$ there is a continuous function $g$ on $\bar\Omega$, subharmonic on $\Omega$ and such that $g(y) = 0$ and $g(x) < 0$ for all $x \in \partial\Omega$, $x \ne y$. ("barrier condition"). Then $f_h$ solves the Dirichlet problem on $\Omega$.

Proof: Note first that $f_h(x) \le -f_{-h}(x)$ for $\Omega$. Why? - if $u_1, u_2$ are subharmonic with $u_1 \le h$, $u_2 \le -h$ on $\partial\Omega$, then $u_1 + u_2$ is subharmonic and $u_1 + u_2 \le 0$ on $\bar\Omega$ $\therefore \sup(u_1+u_2) \le 0$.

$\therefore f_h + f_{-h} \le 0$.

Let $y \in \partial\Omega$ and $x_n \to y$ with $x_n \in \Omega$. Consider $f_h(x_n)$. Take $g$ subharmonic, with $g(y) = 0$ and $g < 0$ otherwise. Consider $u(x) = h(y) - \varepsilon + Kg(x)$ $(K > 0)$. This is subharmonic. $\exists\, r > 0$ such that if $\|x-y\| \le r$, then $u \le h$ independent of $K$, because $g(y) = 0$. If $\|x-y\| \ge r$, choose $K$ so large that $u(x) \le h(x)$ on $\partial\Omega$. Then since $u$ is subharmonic, $u \in S$: $h(y) - \varepsilon + Kg(x) \le f_h(x)$.

Similarly for $-h$: $-h(y) + \varepsilon + Kg(x) \le f_{-h}(x) \le -f_h(x)$. $\therefore |f_h(x) - h(y)| \le -Kg(x) + \varepsilon$

Let $x_n \to y$. $\therefore |f_h(x_n) - h(y)| < 2\varepsilon$ if $n > N(\varepsilon)$

$\therefore \lim_{n\to\infty} f_h(x_n) = h(y)$ solves the Dirichlet problem. "Perron's method".

## 6. The Heat Operator.

$P = \frac{\partial}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ - heat / diffusion operator. $(t,x) \in \mathbb{R}^{n+1}$.

$\sigma(\tau, \xi) = i\tau + \|\xi\|^2$, $\sigma_2(\tau,\xi) = \|\xi\|^2$. $P$ is not elliptic, as $\sigma_2(\tau, 0) = 0$ for $\tau \neq 0$.

Look for the fundamental solution. Compare with Theorem 4.10:

$K(x) = (2\pi)^{-(n+1)} \int_{\mathbb{R}^n} \int_{\operatorname{Im}\xi_{n+1} = \varphi(\xi)} \frac{e^{ix.\xi}}{\sigma(\xi)} \, d\xi_{n+1} \, d\xi$. - gives us a start. Take $\xi_{n+1} = \tau$.

$0 = \sigma(\tau, \xi) = i\tau + \|\xi\|^2$, $\tau = i\|\xi\|^2 = i \times$ (positive).

Try $K(x,t) = (2\pi)^{-(n+1)} \int_{\mathbb{R}^n} \int_{\operatorname{Im}\tau = -1} \frac{e^{i(t\tau + x.\xi)}}{i\tau + \|\xi\|^2} \, d\tau \, d\xi$ (not integrable on $\mathbb{R}^{n+1}$).

Formal (~~pa~~ for the moment). Put $\tau = -i + s$. $\int e^{ix.\xi} \int \frac{e^{i(s-i)t}}{is + 1 + \|\xi\|^2} ds = \int_{\mathbb{R}^n} e^{ix.\xi + t} \int_{-\infty}^{\infty} \frac{e^{ist}}{is + 1 + \|\xi\|^2} ds$

Want $\lim_{N \to \infty} \int_{-N}^{N} \frac{e^{ist}}{is + 1 + \|\xi\|^2} ds$. Put $s = N(\cos\theta + i \sin\theta)$. $e^{ist} = e^{itN\cos\theta} \cdot e^{-Nt\sin\theta}$.

If $t > 0$, complete contour in upper half plane; if $t < 0$, in lower half plane.

$\lim = 2\pi i \operatorname{Res}\left(\frac{e^{ist}}{is + 1 + \|\xi\|^2}, i(1 + \|\xi\|^2)\right) = \frac{2\pi i}{i} e^{i \cdot i(1 + \|\xi\|^2) t} = \begin{cases} 2\pi e^{-t(1 + \|\xi\|^2)}, & t > 0 \\ 0, & t < 0 \end{cases}$

$2\pi \int_{\mathbb{R}^n} e^{ix.\xi + t} \cdot e^{-t - t\|\xi\|^2} d\xi = 2\pi \int e^{ix.\xi - t\|\xi\|^2} d\xi \quad (t > 0)$

Try $K(x,t) = (2\pi)^{-n} \int e^{ix.\xi - t\|\xi\|^2} d\xi$, $t > 0$. (0 otherwise)

Recall, $\widehat{e^{-\|x\|^2/2}} = e^{-\|\xi\|^2/2}$ $\therefore (2\pi)^{-n/2} \int e^{iy.\eta - \|\eta\|^2/2} d\eta = e^{-\|y\|^2/2}$

Put $\eta = \sqrt{2t}\,\xi$, $y = \frac{x}{\sqrt{2t}}$. $e^{-\frac{\|x\|^2}{4t}} = (2\pi)^{-n/2} \int e^{ix.\xi - t\|\xi\|^2} d\xi \, (2t)^{n/2}$.

So, $K(x,t) = \begin{cases} (4\pi t)^{-n/2} \cdot e^{-\|x\|^2/4t}, & t > 0 \\ 0 & t < 0 \end{cases}$

**Theorem 6.1:** $K(x,t)$ is the fundamental solution to the heat operator $\partial/\partial t - \Delta$.

**Proof:** If $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$, need to define $\langle K, \varphi \rangle$: $\langle K, \varphi \rangle = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \cdot \int \frac{\hat{\varphi}(-\tau + i, \xi)}{i(\tau - i) + \|\xi\|^2} d\tau \, d\xi$.

This defines $K$ as a distribution... $= \frac{1}{(2\pi)^{\frac{n+1}{2}}} \cdot \int \frac{\widehat{e^t\varphi}(-\tau, \xi)}{1 + i\tau + \|\xi\|^2} d\tau \, d\xi$.

So, $\langle (-\frac{\partial}{\partial t} + \Delta) K, \varphi \rangle = \langle K, -\frac{\partial \varphi}{\partial t} + \Delta\varphi \rangle$. $\left[\text{Note: } \widehat{e^t \frac{\partial\varphi}{\partial t}} = \widehat{\frac{\partial}{\partial t}(e^t\varphi)} - \widehat{e^t\varphi} = i\tau \widehat{e^t\varphi} - \widehat{e^t\varphi}\right]$

$= \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int \frac{[(-i\tau + 1)\widehat{e^t\varphi} - \widehat{\Delta(e^t\varphi)}](-\tau, -\xi)}{\|\xi\|^2 + 1 + i\tau} d\tau \, d\xi$.

$= \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int \frac{(i\tau + 1 + \|\xi\|^2)\widehat{e^t\varphi}}{i\tau + 1 + \|\xi\|^2} d\tau \, d\xi = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int (\widehat{e^t\varphi})(-\tau, \xi) \, d\tau \, d\xi = (e^t\varphi)(0,0) = \varphi(0)$.

Thus $(\frac{\partial}{\partial t} - \Delta) K = \delta_0$, so $K$ is a fundamental solution.

**Interpretation:** Define $K$ as a distribution by the contour integral in the theorem. If $\varphi$ is $C_c^\infty$ such that $\operatorname{supp} \varphi \cap \{(x,t) : t = 0\} = \emptyset$, then $\langle K, \varphi \rangle = \iint K(x,t) \varphi(x,t) \, dt \, dx$.

**Remark:** $K(x,t)$ is $C^\infty$ if $t \neq 0$.

It is also $C^\infty$ for $x \neq 0$, putting $K(x, 0) = 0$ (cf. bump function construction)

Fundamental solution is $C^\infty$ outside the origin - hypoelliptic.

Consequence - if $\frac{\partial u}{\partial t} - \Delta u = f$, where $f$ is $C^\infty$, then $u$ is $C^\infty$ - regularity.


$\Omega \subset \mathbb{R}^n$. Time $t=0$, "temperature" $= f(x)$.

Solve $\frac{\partial u}{\partial t} = \Delta u$ on $\Omega \times [0,\infty)$, where $\lim_{t \to 0} u(x,t) = f(x)$

Initial value problem: take $\Omega = \mathbb{R}^n$ (cf. Dirichlet problem for $\Delta$ on the ball)

$$K_t(x,y) = \begin{cases} (4\pi t)^{-n/2} e^{-\|x-y\|^2/4t} &, t>0 \\ 0 &, t<0 \end{cases} \qquad PK = \delta_{(y,0)}$$

Theorem 6.2: Let $f(x)$ be continuous and bounded on $\mathbb{R}^n$. Then, $u(t) = \int K_t(x,y) f(y)\,dy$ $(= K_t * f)$ is $C^\infty$ for $t>0$ and satisfies $\frac{\partial u}{\partial t} = \Delta u$ for $t>0$. Moreover, $u$ extends to a continuous function on $\mathbb{R}^n \times [0,\infty)$ with $u(x,0) = f(x)$.

Proof: (Analogous to Dirichlet Problem on the ball for $\Delta$).

(i) $K_t(x,y)$ is $C^\infty$ for $t>0$.

(ii) $(\partial/\partial t - \Delta) K_t = 0$ for $t>0$ (fundamental solution)

(iii) $K_t(x,y) \geq 0$ for $t>0$.

(iv) $\int_{\mathbb{R}^n} K_t(x,y)\,dy = 1$.

(v) For any $\delta>0$, $\lim_{t\to 0^+} \int_{\|x-y\|\geq\delta} K_t(x,y)\,dy = 0$, uniformly in $x$, since:

This is: $\int_{\|y-x\|\geq\delta} e^{-\|x-y\|^2/4t} \cdot (4\pi t)^{-n/2}\,dy$. Let $(y-x) = (4t)^{1/2} z$.

$= \pi^{-n/2} \int_{\|z\|\geq \delta/\sqrt{4t}} e^{-\|z\|^2} dz = \pi^{-n/2} \int_{\frac{\delta}{\sqrt{4t}}}^{\infty} e^{-\|z\|^2} dz \to 0$ as $t \to 0^+$.

Also, if $\delta = 0$, $\int_{\mathbb{R}^n} K_t(x,y)\,dy = \pi^{-n/2} \int_0^\infty e^{-\|z\|^2} dz = \frac{1}{2}\pi^{-\frac{n}{2}} \int_{z_1} \cdots \int_{z_n} e^{-z_1^2} \cdots e^{-z_n^2}\, dz_1 \cdots dz_n = 1$

Suppose $(x,t) \to (a,0)$. Want $u(x,t) \to f(a)$. Then, given $\varepsilon>0$, $\exists\, \delta>0$ such that $|f(y)-f(a)| < \varepsilon$ if $\|y-a\| < 2\delta$ (continuity of $f$)

If $\|x-a\| \leq \delta$, $|u(x,t) - f(a)| = \left|\int K_t(x,y)[f(y)-f(a)]\,dy\right|$ (as $\int K_t = 1$)

$$\begin{aligned}
&\leq \int_{\|x-y\|<\delta} K_t(x,y)\,|f(y)-f(a)|\,dy + \int_{\|x-y\|\geq\delta} K_t(x,y)\,|f(y)-f(a)|\,dy \\
&\leq \int_{\|y-a\|<2\delta} K_t(x,y)\,|f(y)-f(a)|\,dy + 2\sup|f| \cdot \int_{\|y-x\|\geq\delta} K_t(x,y)\,dy \\
&< \varepsilon \int_{\|y-a\|<2\delta} K_t\,dy + 2\sup|f| \int_{\|y-x\|\geq\delta} K_t\,dy \\
&< \varepsilon + 2\sup|f| \int_{\|y-x\|\geq\delta} K_t\,dy \quad \text{(by (v))} \\
&< 2\varepsilon, \text{ if } \delta \text{ is sufficiently small.}
\end{aligned}$$

Theorem 6.3 (Maximum principle): Let $U \subset \mathbb{R}^n$ be an open bounded set and $\Omega = U \times (0,T)$. Let $u$ be continuous on $\bar{\Omega}$ and $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$ be continuous on $\Omega$ and satisfy $\frac{\partial u}{\partial t} - \Delta u \leq 0$. Then, $\sup\{u(x,t) : (x,t) \in \bar{\Omega}\} = \sup\{u(x,t) : x \in U \times \{0\} \cup \partial U \times [0,T]\}$.

[Note: $\partial(U \times [0,T]) = \partial U \times [0,T] \cup U \times \{0\} \cup U \times \{T\}$]

Proof: Assume $\partial u/\partial t - \Delta u < 0$. If $u$ continuous, then maximum is attained at $(x,t)$ on $\bar{U} \times [0, T-\varepsilon]$. Suppose it is an interior point. Then $\frac{\partial u}{\partial t} = 0$, $\frac{\partial u}{\partial x_i} = 0$ at $(x,t) \Rightarrow \Delta u > 0$. ※. If $\frac{\partial u}{\partial t} - \Delta u \leq 0$, put $v(x,t) = u(x,t) - kt$, $k>0$ - get strict inequality. Suppose $(x,t) \in U \times \{T\}$, $\partial u/\partial t \geq 0$. (If not, 'go back' by MVT to get a point with a larger value) $\Rightarrow \Delta u > 0 \ \forall \varepsilon > 0 \Rightarrow$ true on $\Omega$.

Corollary: Solution in Theorem 6.2 is unique (exercise).

$K_t \sim$ probability distribution - normal distribution with standard deviation $\sqrt{2t}$.
$K_t = (2\pi t)^{-n} \int e^{ix\cdot\xi - t\|\xi\|^2} d\xi$. Take convolution of $K_s$ and $K_t$. $(s,t>0)$
$K_s * K_t$: $\widehat{f * g} = \hat{f}\hat{g}$ (incorporating $(2\pi)^{-n/2}$ in the integral).
$(2\pi)^{-n/2} . K_t(x) = (e^{-t\|\xi\|^2})^\wedge (x)$ - symmetric, even in $x$.

$\therefore \widehat{(2\pi)^{-n/2} K_t * K_s} = \hat{K}_t \hat{K}_s = (2\pi)^{-n} . e^{-t\|\xi\|^2} . e^{-s\|\xi\|^2} = (2\pi)^{-\frac{n}{2}} . e^{-(t+s)\|\xi\|^2} = \widehat{K_{t+s}}$.
So $K_t * K_s = K_{t+s}$ - diffusion process, Brownian motion.

$\frac{\partial u}{\partial t} = \Delta u$. Compare with $\frac{\partial v}{\partial t} = Av$, $A$ a constant square matrix.
Solve $v(t) = \exp(tA) v(0) = (\sum_0^\infty \frac{t^n}{n!} A^n) v(0)$. Formally, $u(x,t) = \exp(t\Delta) u(x,0) = \sum_0^\infty \frac{t^n}{n!} \Delta^n u(x,0)$

Consider $u(x,t) = K_t * u(x,0) = K_t * f$. Suppose $f \in \mathcal{S}$.
$\hat{u} = \hat{K}_t . \hat{f} . (2\pi)^{n/2} = e^{-t\|\xi\|^2} \hat{f} = \sum_0^\infty \frac{(-1)^n t^n \|\xi\|^{2n}}{n!} \hat{f}$.

But, $\left(\sum_0^N \frac{(-1)^n t^n \Delta^n}{n!} f\right)^\wedge = \sum_0^N \frac{(-1)^n t^n (-\|\xi\|^2)^n \hat{f}}{n!} = \sum_0^N \frac{t^n \|\xi\|^{2n}}{n!} \hat{f}$.
- operator formalisation of heat equation

## 7. The Wave Operator.

$P = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial t^2} - \Delta$. $\sigma(\tau,\xi) = \sigma_2(\tau,\xi) = -\tau^2 + \sum_{i=1}^n \xi_i^2 = -\tau^2 + \|\xi\|^2$ - hyperbolic wrt $\tau$.

Define the distribution $K$, for $\varphi \in C^\infty$, by: $\langle K, \varphi \rangle = (2\pi)^{-\frac{n+1}{2}} \int_{\mathbb{R}^n} \int_{\operatorname{Im}\tau = 1} \frac{\hat{\varphi}(-\tau,-\xi)}{\sigma(\tau,\xi)} d\tau d\xi$.
$\varphi \in \mathcal{S}$, $\therefore \hat{\varphi}$ decays if $\operatorname{Im}\tau$ is bounded.

Proposition 7.1: $K$ is a fundamental solution of $P$.
Proof: $\langle PK, \varphi \rangle = \langle K, P'\varphi \rangle = (2\pi)^{-\frac{n+1}{2}} \iint \frac{(\sigma(-\tau,-\xi)\hat{\varphi})(-\tau,-\xi)}{\sigma(\tau,\xi)} d\tau d\xi = (2\pi)^{-\frac{n+1}{2}} \iint_{\operatorname{Im}\tau=1} \hat{\varphi}(-\tau,-\xi) d\tau d\xi$
$= (2\pi)^{-\frac{n+1}{2}} \iint_{\operatorname{Im}\tau=0} \hat{\varphi}(-\tau,-\xi) d\tau d\xi = \varphi(0)$, by Fourier inversion.

What is K?
$\langle K, \varphi \rangle = (2\pi)^{-\frac{n+1}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{\hat{\varphi}(-\tau+i,-\xi)}{\|\xi\|^2 - (\tau-i)^2} d\tau d\xi = (2\pi)^{-\frac{n+1}{2}} \iint \frac{\widehat{e^t\varphi}(-\tau,-\xi)}{2\|\xi\| i} \left( \frac{1}{i\tau+1-i\|\xi\|^2} - \frac{1}{i\tau+1+i\|\xi\|^2} \right) d\tau d\xi$.

[Now, $\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-at-it\tau} dt = \widehat{e^{-at}\chi_{[0,\infty)}} = \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{-at-it\tau}}{1-a-i\tau} \right]_0^\infty = \frac{-1}{\sqrt{2\pi}} \cdot \frac{1}{a+i\tau}$, $\operatorname{Re}(a) > 0$]

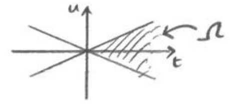
But, $\operatorname{Re}(1 \pm i\|\xi\|) > 0$, so: $\langle K, \varphi \rangle = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}} \frac{\widehat{e^t\varphi}(-\tau,-\xi)}{2\|\xi\| i} . \left\{ \left( e^{-(1-i\|\xi\|)t} - e^{-(1+i\|\xi\|t)} \right) \chi_{[0,\infty)} \right\}^\wedge dt d\xi$ ← wrt $t$
$= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \int_0^\infty \frac{e^t \hat{\varphi}_t(-\xi)}{2\|\xi\| i} (e^{-t} . 2i \sin\|\xi\| t) dt d\xi$ $[\varphi(x,t) = \varphi_t(x)]$
$= (2\pi)^{-n/2} \int_{\mathbb{R}^n} \int_0^\infty \hat{\varphi}_t(\xi) \cdot \frac{\sin\|\xi\|t}{\|\xi\|} dt d\xi$ - not integrable; consider as a distribution

**Note:** for the heat equation at this stage we had: $\iint \hat{\varphi}_t(-\xi)\, e^{-t\|\xi\|^2}\, dt\, d\xi$ - well-behaved for $t>0$.

**Case $n=1$:** $\frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ix\xi}\, dx = \frac{1}{\sqrt{2\pi}} \left[ \frac{e^{-ix\xi}}{-i\xi} \right]_{-a}^{a} = \frac{i}{\xi\sqrt{2\pi}} \left( e^{-ia\xi} - e^{ia\xi} \right) = \frac{-i.2i\sin a\xi}{\sqrt{2\pi}.\xi} = \frac{2\sin a|\xi|}{\sqrt{2\pi}.|\xi|}$

$\Rightarrow \hat{\chi}_{[-a,a]} = \frac{2}{\sqrt{2\pi}} \cdot \frac{\sin a|\xi|}{|\xi|}$. $\therefore \langle K, \varphi \rangle = (2\pi)^{-\frac{1}{2}} \int_0^\infty \int_{\mathbb{R}} \hat{\varphi}_t(-\xi)\, \hat{\chi}_{[-t,t]} \cdot \frac{\sqrt{2\pi}}{2}\, dt\, d\xi = \frac{1}{2} \int_0^\infty \int_{-t}^{t} \varphi(t,x)\, dt\, dx$.

Let $\Omega = \{ (x,t) \in \mathbb{R}^2 : |x| < t \}$. Then $K = K_\Omega$.



**Note:** $K$ is not $C^\infty$ on $\mathbb{R}^2 \setminus \{0\}$ - $P$ is not hypoelliptic.

If $u = f(x-t) + g(x-t)$, $f, g$ $C^2$, solves wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ - not necessarily $C^\infty$.

**Case $n=2$:** $\langle T, \varphi \rangle := \int_{\|x\| \le t} \frac{\varphi(x)}{\sqrt{t^2 - \|x\|^2}}\, dx$ in $\mathbb{R}^2$

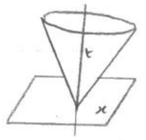
$$\hat{T}(\xi) = \frac{1}{2\pi} \int_{\|x\| \le t} \frac{e^{-ix.\xi}}{\sqrt{t^2 - \|x\|^2}}\, dx. \quad (\text{Put } \xi = \rho e^{i\psi},\ x = re^{i\theta})$$

$$= \frac{1}{2\pi} \int_0^t \int_0^{2\pi} \frac{e^{i\rho r\cos(\theta - \psi)}}{\sqrt{t^2 - r^2}} \cdot r\, dr\, d\theta = \frac{1}{2\pi} \int_0^t \int_0^{2\pi} \frac{e^{i\rho r\cos\theta}}{\sqrt{t^2 - r^2}} \cdot r\, dr\, d\theta \quad (\text{Put } r = t\sin\varphi)$$

$$= \frac{t}{2\pi} \int_0^{\pi/2} \int_0^{2\pi} e^{it\rho\cos\theta\sin\varphi} \cdot \sin\varphi\, d\theta\, d\varphi = \frac{t}{2\pi} \int_{\substack{\text{upper}\\ \text{hemisphere}}} e^{it\rho x_1}\, d\sigma \quad \left( \text{Spherical coordinates: } x = \begin{pmatrix} \cos\theta\sin\varphi \\ \sin\theta\sin\varphi \\ \cos\varphi \end{pmatrix} \right).$$

$$= \frac{1}{2} \cdot \frac{t}{2\pi} \int_{\text{sphere}} e^{it\rho x_1}\, d\sigma = \frac{1}{2} \cdot \frac{t}{2\pi} \int_{\text{sphere}} e^{it\rho x_3}\, d\sigma, \text{ by symmetry.}$$

$$= \frac{1}{2} \cdot \frac{t}{2\pi} \int_0^\pi \int_0^{2\pi} e^{it\rho\cos\varphi} \cdot \sin\varphi\, d\theta\, d\varphi = \frac{t}{2} \left[ \frac{-e^{it\rho\cos\varphi}}{it\rho} \right]_0^\pi = \frac{1}{2i} \left( \frac{-e^{-it\rho} + e^{it\rho}}{\rho} \right) = \frac{\sin\rho t}{\rho}$$



$\leftarrow$ forward light cone

$$\therefore \text{ For } n=2,\ \langle K, \varphi \rangle = \frac{1}{2\pi} \int_0^\infty \int_{\|x\| \le t} \frac{\varphi(t,x)}{\sqrt{t^2 - \|x\|^2}}\, dt\, dx$$

$$\operatorname{supp} K \subset \{ (x,t) \in \mathbb{R}^2 \times \mathbb{R} : \|x\| \le t \}$$

**Case $n=3$:** $\langle T, \varphi \rangle = \int_{\|x\| = t} \varphi(x)\, d\sigma$

$$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle = \int_{\|\xi\| = t} \hat{\varphi}(\xi)\, d\sigma_\xi = (2\pi)^{-\frac{3}{2}} \int_{\|\xi\| = t} \int_{\mathbb{R}^3} e^{-ix.\xi} \varphi(x)\, dx\, d\sigma_\xi.$$

(Symmetry). Consider: $\int_{\|\xi\| = t} e^{-ix.\xi}\, d\sigma_\xi = \int_{\|\xi\| = t} e^{-iAx.Ay}\, d\sigma_\xi = \int_{\|\xi\| = t} e^{-iAx.\xi}\, d\sigma_\xi$

$$\therefore \int_{\|x\| = t} e^{ix.\xi}\, d\sigma_\xi = \int_0^\pi \int_0^{2\pi} e^{-irt\cos\varphi} \cdot \sin\varphi\, d\theta\, d\varphi \quad [x = r(0,0,1),\ x.\xi = rt\cos\varphi].$$

$$= 2\pi \left[ \frac{e^{-rti\cos\varphi}}{irt} \right]_0^\pi = 2\pi \left[ \frac{e^{irt} - e^{-irt}}{irt} \right] = 4\pi \frac{\sin rt}{rt}.$$

$$\langle K, \varphi \rangle = (2\pi)^{-3/2} \int_{\mathbb{R}^3} \int_0^\infty \hat{\varphi}_t(-\xi) \cdot \frac{\sin\|\xi\|t}{\|\xi\|}\, dt\, d\xi$$

Comparing with $\hat{T}$: $\langle K, \varphi \rangle = \frac{1}{4\pi} \int_0^\infty \frac{1}{t} \int_{\|x\| = t} \varphi(x)\, dx$

$\nwarrow$ not $\|x\| \le t$

Remark: (i) for $n=3$, $K$ is supported on $\{(x,t): \|x\| = t\}$ (for 1, 2, $\|x\| \le t$) – Origin of Huygen's principle. Same is true in all odd dimensions, except $n=1$.

(ii) for all $n$, $\operatorname{supp} K \subseteq \{(x,t): \|x\| \le t\}$ – ie, the inside of the forward light cone.

(iii) $K_t$ has compact support, $\{x \in \mathbb{R}^n : \|x\| \le t\}$.

$P = \frac{\partial^2}{\partial t^2} - \Delta$. Initial value problem. $Pu = f(x,t)$, $u(x,0) = \varphi_0(x)$, $\frac{\partial u}{\partial t}(x,0) = \varphi_1(x)$ – (I)

Theorem 7.2: Let $K_t(x)$ be the fundamental solution of $P$. Then, $u(x,t) = K_t * \varphi_1(x) + \frac{\partial K_t}{\partial t} * \varphi_0(x) + \int_0^t K_{t-s} * f(x,s)\, ds$ solves the problem (I). – Kirchhoff's Formula.

convolution in $x$-variable & in $t$-variable

Proof: Assume for the moment that $\varphi_0, \varphi_1 \in \mathcal{S}(\mathbb{R}^n)$ and $t \mapsto f(-,t)$ is a $C^\infty$ to $\mathcal{S}(\mathbb{R}^n)$

Take $x$-Fourier Transform of (I):

$\underbrace{\left(\frac{\partial^2}{\partial t^2} + \|\xi\|^2\right)}_{\text{ODE}} \hat{u} = \hat{f}$, $\hat{u}(\xi, 0) = \hat{\varphi}_0$, $\frac{\partial}{\partial t}\hat{u}(\xi, 0) = \hat{\varphi}_1$.

Suppose we consider $v(t) = \int_0^t \hat{f}(s) \cdot \frac{\sin \|\xi\|(t-s)}{\|\xi\|}\, ds$.

$\frac{dv}{dt} = \left[\hat{f}(t) \cdot \frac{\sin\|\xi\|(t-t)}{\|\xi\|}\right]^{=0} + \int_0^t \hat{f}(s) \cos\|\xi\|(t-s)\, ds$, $\frac{d^2 v}{dt^2} = [\hat{f}(t)] - \int_0^t \|\xi\| . \hat{f}(s) . \sin\|\xi\|(t-s)\, ds$.

$\therefore \left(\frac{d^2}{dt^2} + \|\xi\|^2\right) v = \hat{f}$, so $v$ solves the ODE with initial conditions: $v(0) = 0$, $\frac{\partial v(0)}{\partial t} = 0$.

General solution: $\hat{u} = \int_0^t \hat{f}(s) . \frac{\sin(\|\xi\|(t-s))}{\|\xi\|}\, ds + \hat{\varphi}_0 \cos\|\xi\| t + \hat{\varphi}_1 \frac{\sin\|\xi\| t}{\|\xi\|}$.

Take inverse transform: $u = \int_0^t K_{t-s} * f(x,s)\, ds + K_t * \varphi_0 + \frac{\partial K_t}{\partial t} * \varphi_1$.

Note: $K_t$ has compact support, so $K_t * \varphi$ makes sense for any distribution $\varphi$

$\Rightarrow$ check formula works for $\varphi$ a distribution.