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Mathematical Tripos Part IIB — Dr J.P.Dougherty

METHODS OF MATHEMATICAL PHYSICS: — Michaelmas Term 1995

Example Sheet I

1. Evaluate the following integrals:

$$(i)\ \mathcal{P}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx \qquad (ii)\ \mathcal{P}\int_{-\infty}^{\infty}\frac{dx}{x-i} \qquad (iii)\ \mathcal{P}\int_{-\infty}^{\infty}\frac{e^{-x^2}}{x}dx\ .$$

2. Show that the function

$$f(z)=\sum_{n=0}^{\infty} z^{n!}$$

cannot be continued analytically beyond $|z|=1$.

3. The function $I(z)$ is defined for $\text{Im}z>0$ by

$$I(z)=\int_{-\infty}^{\infty}\frac{e^{-t^2}}{z-t}dt\ .$$

Show how an analytic continuation of $I(z)$ may be constructed to extend to any given z with $\text{Im}z\le 0$.

By a manipulation including integration by parts, show that $I(z)$ satisfies

$$I'+2zI=2\pi^{\frac{1}{2}}$$

and show also that $I(0)=-\pi i$. Hence obtain

$$I(z)=\exp\left(-z^2\right)\left\{2\pi^{\frac{1}{2}}\int_0^z\exp\left(s^2\right)ds-\pi i\right\}$$

4. What condition must z satisfy in order for the integral

$$F(z)=\int_{-\infty}^{\infty}\frac{du\ e^{uz}}{1+e^u}$$

to converge? If, in addition, $\text{Im}z>0$, show by closing the contour in the upper half plane that

$$F(z)=\pi\ \text{cosec}\ \pi z \qquad (\text{Im}z>0)\ .$$

How would this result differ if $\text{Im}z<0$?

For what region of the z-plane does the integral for $F(z)$ define a holomorphic function? Explain carefully how the principle of analytic continuation can be used to deduce the result for $\text{Im}z<0$ from the result for $\text{Im}z>0$.

5. Let $f_1(z)$ be the branch of $(z^2-1)^{\frac{1}{2}}$ defined by branch cuts in the z-plane along the real axis from -1 to $-\infty$ and from 1 to ∞, with $f_1(z)$ real and positive just above the latter cut. Let $f_2(z)$ be the branch of $(z^2-1)^{\frac{1}{2}}$ defined by a cut along the real axis from -1 to $+1$, with $f_2(x)$ real and positive for $(x-1)$ real and positive. Show that $f_1(z) = f_1(-z)$ but $f_2(z) = -f_2(-z)$.

Discuss briefly how to set up branch cuts for

$$f_3(z) = (z^3-1)^{\frac{1}{3}}$$

if (a) f_3 is to be single valued in $|z| < 1$, (b) f_3 is to be single valued, and approximately z, at large z.

6. Evaluate by contour integration

$$\int_1^\infty \frac{dx}{x(x-1)^{\frac{1}{2}}} \qquad \int_1^\infty \frac{dx}{x(x^2-1)^{\frac{1}{2}}}.$$

7. By integrating along a contour consisting of the positive real axis together with a line inclined at an angle $2\pi/q$ to it, show that (for a range of real values of p and q to be stated)

$$\int_0^\infty \frac{x^p}{x^q+1}dx = \frac{\pi}{q\sin\left(\frac{p+1}{q}\pi\right)}.$$

8. Show that the residue of $\Gamma(z)$ at $z=-n$ $(n=0,1,2,\ldots)$ is $(-1)^n/n!$.

9. By using a contour consisting of the boundary of a quadrant, indented at the origin, show that (for a range of z to be stated)

$$\int_0^\infty t^{z-1}e^{-it}dt = e^{-\frac{1}{2}\pi i z}\Gamma(z)$$

Hence evaluate

$$\int_0^\infty t^{z-1}\cos t\, dt, \qquad \int_0^\infty t^{z-1}\sin t\, dt.$$

10. Show that for n real and positive and $\operatorname{Re} z > 1$

$$n^{-z} = \frac{1}{\Gamma(z)}\int_0^\infty e^{-n\tau}\tau^{z-1}d\tau$$

and deduce that for $\operatorname{Re} z > 1$

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{\tau^{z-1}d\tau}{e^\tau - 1}. \qquad (*)$$

By taking first the case $\operatorname{Re} z > 1$, show that for $z \neq 1, 2, \ldots$

$$\frac{1}{2\pi i}\Gamma(1-z)\int_{-\infty}^{(0+)} \frac{t^{z-1}}{e^{-t}-1}\, dt = \zeta(z), \qquad (**)$$

where the path of integration is the Hankel path which here satisfies $|\operatorname{Im} t| < 2\pi$. (Why this restriction?)

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Mathematical Tripos Part IIB — Dr J.P. Dougherty

METHODS OF MATHEMATICAL PHYSICS: — Michaelmas Term 1995

Example Sheet II

1. Airy's equation is

$$\frac{d^2w}{dz^2} - zw = 0$$

What singular points $z \in \mathbb{C}$ does it have, if any? Classify the point $z = \infty$.

Find series solutions in powers of z , such that (i) $w(0) = 1$, (ii) $w'(0) = 1$.

2. Legendre's equation and Laguerre's equation are, respectively

$$(1 - z^2)\, w'' - 2z\, w' + n(n+1)w = 0 \,, \qquad (1)$$

$$zw'' + (1 + \mu - z)w' + nw = 0 \qquad (2)$$

where in the first instance, n and μ are arbitrary, for each equation, find the singularities in the finite complex plane, and the indices there. Classify the point at infinity.

Show how to construct series solutions of (2) in positive powers of z, and that one of the series terminates (yielding polynomials) if n is a nonnegative integer.

3. Show that the most general linear second order ordinary differential equation which has two regular singular points, at $z = A$ and $z = B$, is

$$w'' + \left[\frac{1-M}{z-A} + \frac{1+M}{z-B}\right] w' + \frac{N(A-B)^2}{(z-A)^2(z-B)^2} w \;=\; 0 \,, \qquad (3)$$

where M and N are arbitrary constants; we also define α, α' by $\alpha + \alpha' = M$, $\alpha\alpha' = N$. Write down and solve the equation when the two singular points are at 0 and ∞, in the two cases $\alpha \neq \alpha$ and $\alpha = \alpha'$. Use a Möbius transformation to deduce the general solution of (3). What is the significance of the two constants α and α' ?

4. By expanding $(1 - tz)^{-a}$, show that

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt = \frac{\Gamma(c-b)}{\Gamma(a)} \sum_0^\infty \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!},$$

where $(1 - tz)^{-a}$ takes its principal value, provided $\mathrm{Re}\, c > \mathrm{Re}\, b > 0$, and $|z| < 1$. You should explain the reason for these conditions. State the regions of the complex z-plane in which (i) the integral and (ii) the sum define an analytic function.

Explain how the integral provides an analytic continuation in z of the function defined by the sum.

5. Legendre's equation is as in Question 1 equation (1). Show that for $n = 0, 1, 2, \ldots$ the integral

$$P_n(z) = C_n \frac{1}{2\pi i} \oint \frac{(t^2-1)^n}{(t-z)^{n+1}} dt = \frac{C_n}{n!} \left(\frac{d}{dz}\right)^n \left[(z^2-1)^n\right]$$

satisfies the equation, where the contour is a loop encircling z, and C_n is a constant inserted to ensure that $P_n(1) = 1$. Show also that $C_n = 2^{-n}$.

6. Show that the equation

$$w'' + \frac{2}{z} w' + \left[1 - \frac{n(n+1)}{z^2}\right] w = 0$$

has solutions $z^{-\frac{1}{2}} F_{n+\frac{1}{2}}(z)$ where $F_{n+\frac{1}{2}}$ is any Bessel function of order $n + \frac{1}{2}$. [These are known as spherical Bessel functions, and one defines $f_n = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} F_{n+\frac{1}{2}}$ where F stands for $J, H^{(1)}, H^{(2)}$ and f stands for $j, h^{(1)}, h^{(2)}$].

Show also that $$j_0(z) = \frac{\sin z}{z}$$

7. Show that the confluent hypergeometric equation $zw'' + (c-z)w' - aw = 0$ has solutions of the form

$$w(z) = \int_\gamma t^{a-1}(1-t)^{c-a-1} e^{tz}\, dt,$$

provided $[t^a(1-t)^{c-a} e^{tz}]_\gamma = 0$. Taking $\mathrm{Re}\, z > 0$ without loss of generality, find paths which provide two independent solutions in each of the following cases (where m is a positive integer):

(i) $a = -m$, $c = 0$;
(ii) $\mathrm{Re}\, a < 0$, $c = 0$, a is not an integer;
(iii) $a = 0$, $c = m$;
(iv) $\mathrm{Re}\, c > \mathrm{Re}\, a > 0$, a and $c - a$ are not integers;
(v) $\mathrm{Re}\, c < \mathrm{Re}\, a < 0$, a and $c - a$ are not integers.

8. Transform Bessel's equation by writing $w(z) = z^\nu u(z)$ and solve the resulting equation by Laplace's method. Hence show that for $R(\nu) > -\frac{1}{2}$,

$$J_\nu(z) = \frac{1}{\pi^{\frac{1}{2}} (\nu - \frac{1}{2})!} \left(\frac{z}{2}\right)^\nu \int_{-1}^{1} e^{izt} (1-t^2)^{\nu - \frac{1}{2}}\, dt\,.$$

9. Find two independent solutions of the Airy equation (see question 1) in the form $w(z) = \int_\gamma e^{zt} f(t)\, dt$, where γ is to be specified in each case. Show that γ can be chosen so that $w(z)$ remains bounded for real z as $z \to \infty$, and that in this case

$$w(0) = iA\, 3^{-\frac{1}{6}} \Gamma(\tfrac{1}{3}) \qquad \text{and} \qquad w'(0) = -iA\, 3^{\frac{1}{6}} \Gamma(\tfrac{2}{3}),$$

where A is a constant.

Long Question

10. Laguerre's equation is (2) of Question 1. Obtain the solution (for $n = 0, 1, 2, \ldots$)

$$w = L_n^{(\mu)}(z) = \frac{(n+\mu)!}{\mu!\, n!} \Phi(-n;\ 1+\mu;\ z) = \frac{(-1)^n}{2\pi i} \int_C \frac{(1-t)^{n+\mu}}{t^{n+1}} e^{tz} dt$$

where C is a simple loop enclosing $t = 0$.

Show that $$L_n^{(\mu)} = \frac{e^z z^{-\mu}}{n!} \left(\frac{d}{dz}\right)^n (e^{-z} z^{n+\mu})$$

(*Hint:* put $\zeta = z(1-t)$ in the previous integral.) Deduce that $L_n^{(\mu)}$ is a polynomial.

Show also that $$\sum h^n L_n^{(\mu)}(z) = \frac{e^{-zh/(1-h)}}{(1-h)^{\mu+1}}\,.$$

Please send any corrections and comments by e-mail to jpd2@amtp.cam.ac.uk

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Mathematical Tripos Part IIB — Dr J.P. Dougherty

METHODS OF MATHEMATICAL PHYSICS: — Michaelmas Term 1995

Example Sheet III

1. Use the method of integration by parts to obtain asymptotic expansion of

$$E_\nu(z) = \int_1^\infty t^{-\nu} e^{-zt} dt \quad (\nu > 0) \; ;$$

for z real and positive.

2. The function $I(z)$ is defined for $-\pi < \arg z < \pi$ by

$$I(z) = \int_0^\infty \frac{e^{-t}}{t+z} dt.$$

Show that as $z \to \infty$

$$I(z) \sim \sum_0^\infty (-1)^n \frac{n!}{z^{n+1}}$$

in $|\arg z| < \pi - \delta$, for positive δ. Explain carefully whether or not exactly this result could have been obtained by changing variable to, for example $\tau = t/z$, expanding the integrand binomially and using Watson's lemma.

How do you reconcile the fact that the above asymptotic expansion is identical just above and just below the negative real axis with the fact that $I(z)$ has a branch cut on the negative real axis, across which its value jumps by $2\pi i e^z$?

Let $\tilde{I}(z)$ be the analytic continuation of $I(z)$ to $0 < \arg z < 2\pi$ which satisfies

$$\tilde{I}(ze^{2\pi i}) = I(z) - 2\pi i e^z \qquad \text{for} \quad -\pi < \arg z < 0.$$

Does the asymptotic expansion for $\tilde{I}(z)$ exhibit Stokes' phenomenon?

3. A function occurring in the Debye theory of specific heats is

$$D(x) = \frac{3}{x^3} \int_0^x \frac{t^3 dt}{e^t - 1} \left(= 1 - \frac{3x}{8} + \frac{x^2}{20} \dots \text{ for } x \ll 1 \right)$$

by writing $\int_0^x = \int_0^\infty - \int_x^\infty$, obtain

$$D(x) \sim \frac{\pi^4}{5x^3} - 3e^{-x}\left[1 + 0\left(\frac{1}{x}\right)\right] + 0\left[e^{-2x}\right] + \dots$$

(Consult material on the ζ function for the $\int_0^\infty$ part.)

4. The function already investigated on Sheet I Question 3, is defined by

$$f(z) = \int_{-\infty}^{\infty} \frac{e^{-t^2}dt}{z-t} \qquad \mathrm{Im}(z) > 0$$

and by the analytic continuation of this for $\mathrm{Im}\,(z) \leq 0$.

Use Watson's lemma to obtain the asymptotic expansion

$$f(z) \sim \frac{\pi^{\frac{1}{2}}}{z}\left(1 + \frac{1}{2z^2} + \frac{3}{4z^4}\cdots\right) - \epsilon i \pi e^{-z^2}$$

where $\epsilon = 0, 1, 2$ according as $\mathrm{Im}(z) > = < 0$ respectively.

By considering carefully the behaviour of e^{-z^2}, determine the positions of the Stokes lines.

5. Show that

$$\int_0^{\frac{1}{4}\pi^2} \exp\left[x \cos\sqrt{t}\right] dt \;\sim\; e^x\left(\frac{2}{x} + \frac{2}{3x^2} + \cdots\right)$$

as $x \to +\infty$.

Explain carefully how the asymptotic expansion would differ if the upper limit of the integral were replaced by $4\pi^2$.

6. The function $f(\theta)$ is defined for θ real and positive by

$$f(\theta) = \frac{1}{2\pi i}\int_\gamma e^{\theta(t+\frac{1}{3}t^3)}dt,$$

where the path γ begins at ∞ in the sector $-\frac{\pi}{2} < \arg t < -\frac{\pi}{6}$ and ends at ∞ in the sector $\frac{\pi}{6} < \arg t < \frac{\pi}{2}$. Find the two saddle points and show that the two paths of steepest *descent* through these points are

$$x = +[(2+y)/3y]^{\frac{1}{2}}(y-1); \quad (y>0) \qquad \text{and} \qquad x = -[(y-2)/3y]^{\frac{1}{2}}(y+1); \quad (y<0),$$

where $t = x + iy$. You should carefully justify your choice of signs for the square roots.

Prove that as $\theta \to +\infty$

$$f(\theta) \sim (\pi\theta)^{-\frac{1}{2}} \cos(\tfrac{2}{3}\theta - \tfrac{1}{4}\pi).$$

7. The Bessel function $J_n(x)$ $(n \in \mathbb{N},\ x \geq 0)$ is given by

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)dt$$

Use the method of stationary phase to obtain

$$J_n(x) \sim \left(\frac{2}{x\pi}\right)^{\frac{1}{2}} \cos(x - \frac{n\pi}{2} - \frac{\pi}{4}) \quad \text{as} \quad x \to \infty$$

8. Show that as $x \to +\infty$

$$\int_0^\pi e^{ix(t-\sin t)}\,dt \;\sim\; e^{i\pi/6}(6/x)^{\frac{1}{3}}\Gamma(4/3).$$

How would this result differ if the lower limit of the integral were $-\pi$?

9. Find asymptotic solutions of

$$xy'' - (x+2)y = 0\ .$$

10. A pendulum consists of a weight and a string which passes through a small ring, so enabling the length to be slowly varied. Obtain an approximate solution to the governing equation, and show that the energy of oscillation (kinetic plus potential) is proportional to the frequency. (The oscillations should be assumed to be of small amplitude, so that the usual linearized equation applies).

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METHODS OF MATHEMATICAL PHYSICS: — Michaelmas Term 1995

Example Sheet IV

1. Obtain the Laplace transforms of $\sinh(\alpha t)$ and $t^{-1}\sinh(\alpha t)$ where α is real and positive. Confirm the relation between the behaviour of the transforms as $p \to \infty$ and the behaviour of the functions as $t \to 0$.

2. If $f(t)$ for $t > 0$ is periodic with period T (i.e. $f(t+T) = f(t)$) show that its Laplace transform is

$$\frac{1}{1-e^{-pT}} \int_0^T f(t)e^{-pt}\,dt$$

Deduce the Laplace transform of $|\sin t|$.

3. Find the Laplace Transforms of the following, and verify the inversion formula

$$\text{(i) } t\cos at\ , \quad \text{(ii) } t^2 \cos at\ , \quad \text{(iii) } \sum_{n=1}^{\infty} H(t-n)$$

where $H(x)$ is the Heaviside function. Consider also

$$\text{(iv) } \sum_{n=0}^{\infty} H\left(t-2^n\right)$$

What goes wrong with the inversion?

4. Use the inversion formula to find the functions whose Laplace transforms are (a, b being real and positive, n a positive integer)

$$\text{(i) } \frac{p}{(p^2+a^2)\ (p^2+b^2)}\ (a \neq b) \qquad \text{(ii) } \frac{e^{-ap}}{p}$$

$$\text{(iii) } \frac{e^{-ap}}{(p-b)^n} \qquad \text{(iv) } \frac{e^{-ap^{\frac{1}{2}}}}{p^{\frac{1}{2}}} \qquad \text{(v) } \ln\left(\frac{p+1}{p}\right)$$

5. Use the Laplace transform to solve the simultaneous equations

$$\dot{x} + x + 2y = e^{2t}$$
$$2\dot{x} + \dot{y} - x = 0$$

subject to $x = y = 0$ at $t = 0$.

6. The temperature $T(r,t)$ at radius r and time t in a sphere of radius a satisfies the heat equation:

$$\frac{\partial T}{\partial t} = \frac{k}{r}\frac{\partial^2 (rT)}{\partial r^2},$$

where k is a positive constant. Initially, the temperature is zero, but for $t > 0$, the surface of the sphere is maintained at a constant temperature $T_\circ$. Show that

$$T(r,t) = \frac{T_\circ}{2\pi i}\frac{a}{r}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\sinh(r\sqrt{p/k}\,)}{\sinh(a\sqrt{p/k}\,)}\frac{e^{pt}}{p}\,dp,$$

for $t > 0$ and $0 \le r \le a$, where γ is real and positive.

Discuss carefully the location and nature of the singular points of the integrand, and evaluate the integral in the form of a series.

7. The variable $u(x,t)$ of a passive physical system is defined for $x \ge 0$, and satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2c\frac{\partial u}{\partial x},$$

with initial and boundary conditions $u(x,t) = 0$ for $t \le 0$ and $u(0,t) = U$ for $t > 0$, where c, and U are constants. Show, by taking Laplace transforms with respect to t, that

$$u(x,t) = Ue^{-2cx} - Ue^{-c(x+ct)}\int_0^\infty \frac{2s\sin(xs)}{\pi(s^2+c^2)}e^{-ts^2}ds.$$

Find the asymptotic behaviour of $u(x,t)$ in each of the limits $x \to \infty$ and $t \to \infty$, and compare your results with the corresponding behaviour in the case $c = 0$.

8 The function $y(t)$ satisfies the difference-differential equation

$$\dot{y}(t) + y(t) - y(t-1) = f(t),$$

where $f(t)$ vanishes for $t < 0$, and has a Laplace transform. Show that the solution which satisfies $y(t) = 0$ for $t \le 0$ is

$$y(t) = \int_0^t g(t-p)f(p)\,dp, \qquad \text{where} \qquad g(t) = \sum_0^\infty \frac{1}{n!}(t-n)^n e^{n-t}H(t-n),$$

and $H(t)$ is the Heaviside function.

Long Question

9. Using the Fourier fransform for space variables, and the Laplace transform with respect to time, show that the causal Green's function for the inhomogeneous Klein–Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(\mathbf{x}t) = \delta^3(\mathbf{x})\delta(t)$$

is

$$\phi(\mathbf{x},t) = \frac{1}{(2\pi)^4}\int \cdots \int \frac{e^{i(\mathbf{k}.\mathbf{x}-\omega t)}}{k^2 - \omega^2 + m^2}d^3k\,d\omega,$$

where the range or contours of integrations are to be specified.

Carry out the integrations over $\mathbf{k}$ to obtain

$$\phi(\mathbf{x},t) = \frac{1}{8\pi^2 r}\int_C \exp[i(\omega^2 - m^2)^{\frac{1}{2}}r - i\omega t]d\omega,$$

where $r = |\mathbf{x}|$, the contour C runs along the real axis indented above $\omega = \pm m$, and the branch of the square root is the one which is approximately $+\omega$ when ω is real and $|\omega|$ is large.

Show that ϕ vanishes for $r > t$, and that for $r < t$

$$\phi(\mathbf{x},t) = -\frac{m}{4\pi\rho}J_1(m\rho),$$

where $\rho = (t^2 - r^2)^{\frac{1}{2}}$.

[Hint: write $r = \rho\sinh u, t = \rho\cosh u$, $\omega = m\sin v$ and change the variable of integration successively to v and $\psi = v - iu$, with suitable changes of contour. It may be assumed that

$$J_1(z) = \frac{1}{2\pi}\int_0^{2\pi} e^{iz\sin\psi - i\psi}d\psi.]$$