E9

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METHODS OF MATHEMATICAL PHYSICS. Revision Document

The following is a summary of material that will be assumed to be familiar. The exposition is not intended to teach, only to remind. The material appears in detail in many texts. A small set of revision examples follows. They can be used to check familiarity with the material, but they should not be allowed to take up supervision time. It is assumed that the reader is already familiar with real variable analysis. The symbols C1, P3, etc refer to the Part IA and IB courses denoted by them.

Chapter 0. Revision of Complex variable

0.1 Elementary properties

0.1.1 Algebraic definitions (See C1 and C2)

Define $\mathbb{C}$ as $\mathbb{R}\times\mathbb{R}$, with notation $z=(x,y),\ z\in\ \mathbb{C},\ x,y\in\ \mathbb{R}$, with $\mathbf{0}=(0,0)$ and forming a 2-dimensional vector space with usual axioms.

Multiplication: $z_1z_2 \stackrel{\text{def}}{=} (x_1x_2-y_1y_2,\ x_1y_2+x_2y_1)$, with $\mathbf{1}=(1,0)$.

Verify distributive, associative and commutative laws (*hint:* identify z with $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ and complex multiplication with matrix multiplication),

Define
$$z^{-1} \stackrel{\text{def}}{=} \left(\frac{x}{x^2+y^2},\ \frac{-y}{x^2+y^2}\right) \qquad (z\neq\mathbf{0})$$

giving an inverse. Thus $\mathbb{C}$ is a commutative field. The subset $\{z : y=0\}$ forms a subfield isomorphic with $\mathbb{R}$.

Change of notation write $(0,1)=i$ and (by abuse of notation) $(1,0)=1$. Then (x,y) becomes $x+iy$, and $i^2=-1$. Introduce the terms *real, imaginary, complex plane, Argand diagram.*

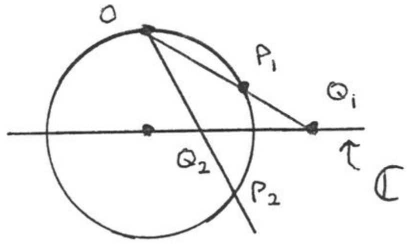
$\mathbb{C}$ has the same algebraic properties as $\mathbb{R}$, but the order property is lost [Attempts at similar generalisations to >2 dimensions result in loss of more properties, e.g. commutative. For example, vector product in $\mathbb{R}^3$]

0.1.2 Modulus, complex conjugate

$$|z| \stackrel{\text{def}}{=} (x^2+y^2)^{\frac{1}{2}} \geq 0 \text{ with equality only if } z=0\ ; \qquad \overline{z} \stackrel{\text{def}}{=} x - iy\ .$$

0.1.3 Elementary functions (see C_1 and C_2)

A function of a complex variable will mean $f : \mathbb{C} \to \mathbb{C}$, written $f(z)$. Polynomials, remainder theorem, Rational functions: $R(z) = P(z)/Q(z)$, P, Q being polynomials. (So far all similar to real case).



The Riemann Sphere. (see D1) Project point P on the Globe from O to Q on the plane, giving a bijection from the sphere to the plane provided an additional point, $\{\infty\}$ is added to $\mathbb{C}$ to correspond to O itself.
Möbius transformations $\zeta = (az+b)/(cz+d)$, are bijections $\mathbb{C} \to \mathbb{C}$ provided $\{\infty\}$ is included. Can map any 3 points onto any other three, and map circles or lines onto circles or lines. The cross ratio $(z_1, z_2; z_3, z_4) \stackrel{\text{def}}{=} \frac{z_1-z_3}{z_3-z_2} \cdot \frac{z_2-z_4}{z_4-z_1}$ is preserved. Note $(0, \infty; z, 1) = z$. Möbius transformations form a group (generated by translations and inverses). The 6 values of cross ratio resulting from permuting $\{z_i\}$ also form a group: $\zeta, 1-\zeta, 1/\zeta$, $1/(1-\zeta), (\zeta-1)/\zeta$.

0.1.4 Metric Properties, limits, continuity (see C5 and C6, also C9).

Introduce a "distance" between z_1, z_2

$$d(z_1 - z_2) \stackrel{\text{def}}{=} |z_1 - z_2|$$

Satisfies axioms for *metric*:

$$d(z_1, z_2) = d(z_2, z_1)\ ; \quad d(z_1 - z_2) = 0 \iff z_1 = z_2$$

$$d(z_1, z_2) + d(z_2, z_3) \geq d(z_1, z_3) \text{ (triangle inequality, verify directly).}$$

This can be used to define convergence of sequences $\{z_n\}$ $(n \in \mathbb{N})$. Converges to a if $|z_n - a| < \epsilon$ for $n > N(\epsilon)$.
Cauchy sequences. Completeness: every Cauchy sequence has limit point. Convergence of sums $\sum a_n$ by convergence of sequence of partial sums. Absolute convergence, uniform convergence of functions, etc. Continuity of functions of a complex variable by ϵ, δ characterization or by open sets mapped onto open sets. So far these properties introduce nothing novel compared to real case; they are equivalent to same statements about u and v where $f = u(x,y) + iv(x,y)$.

0.1.5 Power series (see C5 and C6)

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

Converges for $|z-a| < R$, diverges $|z-a| > R$, needs further investigation on the circle of convergence, $|z-a| = R$. Here R is the radius of convergence, where

$$R^{-1} \stackrel{\text{def}}{=} \lim_{n\to\infty} \sup |a_n|^{1/n} \ .$$

Note any of $0 \le R \le \infty$ is possible. Where convergent, in $|z-a| < R, f(z)$ is analytic (see below). Technical theorems about multiplication, uniformity of convergence, continuity of sum function, etc.

0.1.6 The exponential and related functions (see C5 and C6)

$$\exp(z) \stackrel{\text{def}}{=} \sum_{0}^{\infty} z^n/n! \ . \qquad R = \infty \text{ (use ratio test)}$$

Fundamental property $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$. Write $\exp(1) = e$. Periodicity of $\exp(i\theta)$ for $\theta \in \mathbb{R}$, period 2π. The functions ln, cosh, sinh, cos, sin, etc. and their properties. Polar representation $z = r\exp(i\theta)$.
The exponential limit:

$$\lim_{k\to\infty} \left(1 + \frac{z}{k}\right)^k = \exp(z)$$

0.2 Differentiation

0.2.1 Differentiability and the Cauchy–Riemann Equations (C5 and C6)

$f(z) = u(x,y) + iv(x,y)$, assumed differentiable wrt (x,y) in the sense of real analysis. In complex analysis $f'(z) \stackrel{\text{def}}{=} \lim\{f(z+h) - f(z)\}/h$ is required to be independent of route along which $h \to 0$. This has far-reaching consequences (see below). It implies

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \ ,$$ hence the Cauchy–Riemann (CR) equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ , \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

These are necessary for $f(z)$ to be differentiable; to be sufficient the four partial derivatives must also be continuous. (see Copson p.41)

0.2.2 Terminology

To classify functions in a *domain*, D, it is best to make D open (i.e. excludes the boundary) otherwise differentiability for z on the boundary gets messy as not all directions of h are available.

A function $f(z)$, $z \in D$, is *analytic* if $f'(z)$ exists wherever $f(z)$ is defined. The exceptional points are called *singularities*. If there are no singularities, $f(z)$ is *regular*. Ex: $\sin z$ is regular in $\mathbb{C}$, $1/z$ is analytic with a singularity at $z = 0$.

A singularity is *removable* if it can be removed by a sensible definition of $f(z)$ there. Ex: If $f(z) = z^{-1} \sin z$, then $z = 0$ is a singularity, which can be removed by defining $f(0) = 1$. We shall always assume that removable singularities have been removed!

A singularity $z = a$ is *isolated* if $f(z)$ is regular in $0 < |z - a| < \delta$ for some $\delta > 0$.

The behaviour at $z = \infty$ can be classified by the transformation $\zeta = 1/z$.

0.2.3 Harmonic Functions (C5 and C6; P3)

Assuming u, v are twice differentiable (as they must be: see §0.3.2 below)
CR equations give

$$\nabla^2 u = \nabla^2 v = 0$$

Also

$$\nabla u \cdot \nabla v = 0$$

Each analytic $f(z)$ automatically provides 2 solutions of Laplace equation in $\mathbb{R}^2$, such that the curves $u =$ const and $v =$ constant are everywhere orthogonal (conjugate harmonic functions). This is extremely useful in classical applied mathematics, e.g. fluid dynamics, electrostatics (see course on Fluid Dynamics for the main applications). The method can be extended by making successive analytical transformations, e.g. to $g(f)$ where g is analytic, known as *conformal transformations*. (see P3, O5)

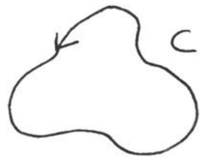
0.2.4. Stationary values of harmonic functions (C5 & C6)

Where $f'(z) = 0$ we have $\nabla u = \nabla v = 0$ so both u and v are stationary. They cannot be maximum or minimum. At such a point, rotate the axes to make the matrix of second derivatives of u (or v) diagonal. As $\nabla^2 u = 0$, the two diagonal entries must be of opposite sign. Such a point is a *saddle-point*. Identifying these is important in the method of steepest descent (see Chap. 3 below). We try to distort a contour to pass through a saddle point, with $v =$ constant along it. This ensures that, on the contour u is maximum at the saddle point and diminishes on rapidly as possible as we leave it on either side. Ex. Draw the curves $u =$ const and $v =$ const for $f(z) = z^2$.

0.3 Integration and Cauchy's Theorem (P3; C12)

0.3.1 Cauchy's Theorem (P3)

Given $f(z)$ within a domain D, if "integration" means the inverse of differentiation, we need $F(z)$ such that $F'(z) = f(z)$. If it exists we have an *indefinite* integral. A *definite* integral, $\int_a^b f(z)dz$ can be defined by developing the Riemann integral from the real case, defining a *contour*, C, as a path running from a to b $(C \subset D)$, parametrised by a real t, $z(t)$. (Subject to conditions, has to be rectifiable, of finite length). If $F(z)$ exists, the definite integral must be $[F(z)]_a^b$ and is independent of the path, so (going out and back along different paths) $\oint f(z)dz = 0$ around a closed loop. Cauchy's theorem is a statement of conditions under which this holds. Weak version (P3): the theorem holds if $f(z)$ is defined and analytic in the interior of C, *and* the four derivatives $\partial u/\partial x$ etc. are continuous there. This can be proved from the CR equations using Green's theorem (conditions for Green's theorem require the continuity).



As an example where the theorem fails because $f(z)$ has a singularity, consider

$$\int_c \frac{dz}{z}$$

where C is a circle centred on O. By direct calculation this is $2\pi i$. Idea of contour distortion in a domain of analyticity. Clearly $\int \frac{dz}{z} = 2\pi i$ for any C encircling O once anticlockwise, and vanishes if O is outside C. More generally

$$\frac{1}{2\pi i}\int \frac{dz}{z-a} = \text{number of times } C \text{ encircles } a \text{ anticlockwise (the winding number)}$$

(as $1/z$ does have continuous derivatives, these results follow from the weak version).

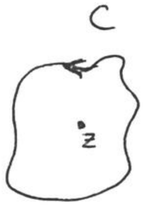
0.3.2 The strong version (C12)

These results can be obtained, at the cost of harder proofs, without assuming continuity of derivatives. Needs more careful preparation (Jordan curve theorem, winding numbers, etc.)

0.3.3 Cauchy Integral Formula

For $f(z)$ regular,

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} d\zeta$$



for z inside C ($= 0$ for z outside)
As this can be legitimately differentiated wrt z

$$f'(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

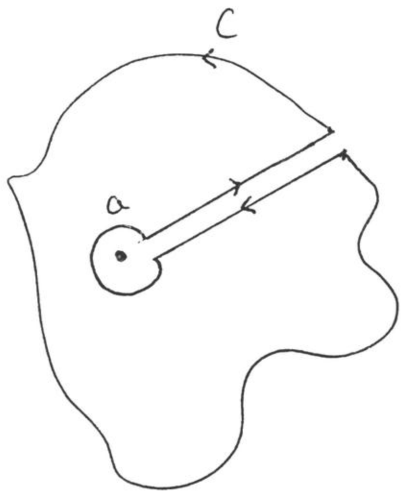
and so on to higher derivatives; thus $f'(z)$ it itself analytic, and so on, with

$$f^{(n)}(z) = \frac{1}{2\pi i} n! \oint \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

0.3.4 Taylor's Theorem obtained direct from previous

$$f(z) = f(a) + \sum_{1}^{\infty} c_n (z-a)^n \text{ where } c_n = f^{(n)}(a)/n!$$

0.3.5 Laurent's Theorem



If $z = a$ is an *isolated* singularity, use contour shown to get the Laurent expansion

$$f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n \quad \text{with}$$

$$c_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-a)^{n+1}} dz$$

In particular

$$c_{-1} = \frac{1}{2\pi i} \oint f(z) dz \qquad \text{is the } \textit{residue} \text{ of } f(z) \text{ at } z = 0$$

0.3.6. Terminology

If in the above, for some $k > 0$, $c_{-k} \neq 0$ but $c_n = 0$ for $n < -k$, the singularity at $z = a$ is called a *pole of order* k, in which case $(z-a)^k f(z)$ is regular. If $k = 1$, it is a *simple pole.* But if the series fails to terminate for negative n we have an *essential singularity* (e.g. $\exp(1/z)$).

There can also be *non-isolated singularities*, for example at $z = a$ if a is the limit of a sequence of singularities $\{a_n\}$, $a_n \to a$ as $n \to \infty$. Ex: $f(z) = \{\sin(1/z\}^{-1}$. A branch point (see §1.6 of course) is another form of non-isolated singularity. There is no Laurent expansion about a non-isolated singularity, and no residue there!

$f(z)$ is *meromorphic* in D if the only singularities are poles. $f(z)$ is *entire* or *integral* if it is regular on the whole of $\mathbb{C}$, excluding ∞.

0.4 Applications of Cauchy's Theorem

0.4.1 The Residue Theorem (P3)

$\oint f(z)dz = 2\pi i$ {sums of residues of poles inside C} (shrink contour to small loops surrounding poles, assumed isolated). This provides methodology for evaluating many definite integrals by crafty choice of $f(z)$ and C. Note Jordan's Lemma.

0.4.2 Liouville's Theorem (C12)

A bounded function in $\mathbb{C}$ must be constant.

0.4.3 The Fundamental Theorem of Algebra (C12)

Every polynomial in $\mathbb{C}$ has a zero.

0.4.4 The principle of the argument (C12)

Use of $\oint \ln f(z)dz$ to count zeros and poles.

Introductory Examples

You should be able to do these unaided - they are not intended for supervision use. If you have difficulties, some revision of Part IB work is indicated.

1. Plot the curves u constant and $v =$ constant where $f = u + iv = z^2$
2. If $f(z) = g(z)/h(z)$, with $g(a) \neq 0$, and $h(z)$ has a *simple* pole at $z = 0$, show that the residue of $f(z)$ at $z = a$ is $g(a)/h'(a)$. (This is a useful trick, but bear in mind that is works only for a simple pole)
3. Evaluate for real a

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2 + a^2} \qquad \text{(answer : } \frac{\pi e^{-|a|}}{|a|} \text{)}$$

4 Evaluate for real a

$$\int_0^{2\pi} \frac{\cos n\theta d\theta}{a^2 - 2a\cos\theta + 1} \qquad \text{(answer : } \frac{\pi}{2a^n(a^2 - 1)} \text{)}$$

[Use unit circle as a contour. What values of a are acceptable?]

5. Evaluate

$$\int_0^{\infty} \frac{dx}{(x^2+1)^2 (x^2+4)} \qquad \text{(answer : } \frac{\pi}{18} \text{)}$$

6. Evaluate

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx \qquad \text{(answer : } \frac{\pi}{2} \text{)}$$

(*hint:* $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$)

7. Evaluate

$$\int_0^{\infty} \frac{dx}{1 + x^n}$$

where $n \geq 2$. n an integer (answer: $\frac{\pi}{n}$ cosec $\frac{\pi}{n}$)