Methods of Mathematical Physics.

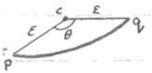
1. Complex Variable.

1.1: Cauchy principal value of improper integrals

Consider (real analysis), $I = \int_{-1}^{1} \frac{1}{x}\,dx$ - (1). It diverges and is meaningless.
Consider: $I_{mod} = \int_{-1}^{-\eta} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx$ $(\eta, \varepsilon > 0)$. So, $I_{mod} = \ln(\eta/\varepsilon)$. It can take any value.

Definition: Cauchy Principal Value of (1) is that with $\eta = \varepsilon$ as limit is taken $\varepsilon \to 0$.
So $I_{mod} = 0$, as would be naively expected by "symmetry."
More generally, the Cauchy Principal Value of $\int_a^b f(x)\,dx$, where $f(x)$ has a simple pole at $x = c$ $(a < c < b)$ is: $\lim_{\varepsilon \to 0} \left\{ \int_a^{c-\varepsilon} f(x)\,dx + \int_{c+\varepsilon}^{b} f(x)\,dx \right\}$, written $P\int_a^b f(x)\,dx$.

Connection with indented contour: Given $f: \mathbb{C} \to \mathbb{C}$, $f(z)$ with a simple pole at $z = c$.
Then we have a Laurent expansion: $f(z) = \frac{R}{z-c} + a_0 + a_1(z-c) + \dots$, where R = residue, valid for sufficiently small $|z-c|$.
Consider: $\int_p^q f(z)\,dz$ as shown:



- circle of radius ε, subtending angle ϑ.

$$\int_p^q f(z)\,dz = R\int \frac{e^{-i\varphi}}{\varepsilon} \cdot i\varepsilon e^{i\varphi}\, d\varphi \quad (\text{letting } z = c + \varepsilon e^{i\varphi})$$
$$+ \{\text{terms from the Taylor Series}\}.$$
$$= Ri[\varphi] = Ri\vartheta. \quad \underline{(2)}.$$ - for the residue term. The contribution from the other terms $\to 0$ as $\varepsilon \to 0$.

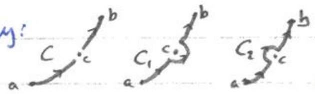
For a closed circle, this becomes $2\pi i R$, as is known from earlier courses, and the loop need not have been a circle nor need $\varepsilon \to 0$.
Our result holds only for a circular contour and in the limit $\varepsilon \to 0$.

In particular, consider $I_{indented} = \int_C f(z)\,dz$,

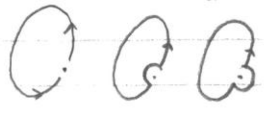
using the indented contour shown along a parted axis.
$\int_C = P\int_a^b + \pi i R$, (3), in limit $\varepsilon \to 0$.
Similarly, if C is:

$\int_C = P\int_a^b - \pi i R$

Still more generally:



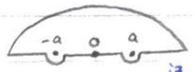
$\int_{C_1} = P\int_C + \pi i R$, $\int_{C_2} = P\int_C - \pi i R$.



(otherwise analytic). Here, $P\oint f(z)\,dz = \pi i R$.

For an integral of the type shown, with one simple pole on the contour, the CPV is obtained by including the residue there at $\frac{1}{2}$ value.

Eg: $I = P\int_{-\infty}^{\infty} \frac{e^{iz}}{a^2 - z^2}\,dz$. Use:



Close with upper $\frac{1}{2}$-circle as usual, and let its radius $\to \infty$. Residue at $z = \pm a$ is $\left.\frac{e^{iz}}{-2z}\right|_{z=\pm a} = \frac{e^{\pm ia}}{\mp 2a}$. $I = \pi i\left(\frac{e^{ia}}{-2a} + \frac{e^{-ia}}{2a}\right) = \frac{\pi \sin a}{a}$.

Taking real and imaginary parts: $P\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\,dx = \frac{\pi \sin a}{a}$, $P\int_{-\infty}^{\infty} \frac{\sin x}{a^2 - x^2}\,dx = 0$.

Easier example: $P\int_{-\infty}^{\infty} \frac{e^{iz}}{z}\,dz$ gives $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$.

Another kind of CPV: This occurs when $\int_{-\infty}^{\infty} f(x)\,dx$ diverges at both $\pm\infty$. Special attention is directed to $\int_{-R}^{R} f(x)\,dx$. If this converges, as $R\to\infty$, it is the CPV.

Eg: $P\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$. In the case of $\int_{-\infty}^{\infty} \frac{dx}{x}$, get a CPV in both senses.

1.2 Analyticity of functions as defined by integrals.

A common situation is for a function $f(z)$, $z\in D\subset\mathbb{C}$ to be defined in terms of an integral, over another variable $t\in\mathbb{R}$, of an integrand $g(z,t)$: $f(z)=\int_{t_1}^{t_2} g(z,t)\,dt$ ⊛.
Under what conditions is $f(z)$ analytic in D?

One answer: let $g(z,t)$ be analytic wrt $z\in D$ $\forall t\in[t_1,t_2]$, and let $g(z,t)$ be continuous wrt $(x,y,t)\in D\times[t_1,t_2]$. Under these conditions, f is analytic and its derivatives can be obtained by differentiating under the integral sign. (Pf: Copson P.108).

If one or both of t_1, t_2 is $\pm\infty$, obviously convergence of ⊛ is required. For $f(z)$ to fulfil the theorem, the convergence must be uniform (Copson P. 110)

Contour Integrals: Extend these results to integrals of the form $f(z)=\int_C g(z,\zeta)\,d\zeta$, where C is a fixed contour. This can be put into the form of ⊛ by parametrising. Still more generally, C might depend on z, $C(z)$. Eg: $\int_{z_0}^{z_1} g(z,\zeta)\,d\zeta$.
In this case we deal with things ad hoc.

1.3 Analytic Continuation.

In this section, the power series expansion will be a crucial property of analytic functions. Firstly though...

Lemma: the zeroes of a regular function (other than $f\equiv 0$) are isolated. If a regular function has an infinite sequence of zeroes with a limit point, then $f(z)\equiv 0$.

Proof: Assume $f(z)\not\equiv 0$; let $z=a\in D\subseteq\mathbb{C}$ be a zero. Then, $f(z)$ regular $\Rightarrow f(z)=\sum_{n=m}^{\infty} c_n(z-a)^n$, $c_m\neq 0$, $m\geq 1$. Within the circle of convergence of this series, $f(z)=\varphi(z)(z-a)^m$, where $\varphi(a)\neq 0$. By continuity, there is a neighbourhood of a in which $\varphi(z)\neq 0$.
In this neighbourhood, $f(z)\neq 0$, so the zero of f isolated.
Now suppose $f(z)$ has an infinite sequence of such isolated zeroes with a limit point $z=b$. Then $f(b)=0$ by continuity. Then b is a non-isolated zero, so $f(z)\equiv 0$.

Idea of analytic continuation: Given $f_1(z)$ regular in $z\in D_1\subseteq\mathbb{C}$, can we define $f_2(z)$ in D_2 such that $f_1(z)=f_2(z)$ in $D_1\cap D_2$ and $f_2(z)$ is regular for $z\in D_2$? Further condition is that $D_1\cup D_2$ is simply connected. There is no systematic answer; each case must be worked out. By the lemma, if f_2 exists it must be unique.

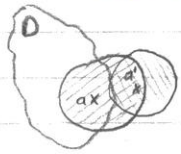
Continuation by Power Series: $f(z)$ regular in D. Although laborious we can do the following:

(i) Pick $a \in D$

(ii) Expand around a

(iii) Pick $a' \notin D$ but in expansion

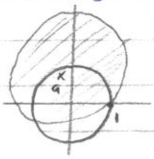
(iv) Expand around a'. etc.



Examples: (i) $f(z) = 1 + z + z^2 + \dots = \frac{1}{1-z}$, $D_1 = \{z : |z| < 1\}$.

Form a power series about $z = a$, $a \in D_1$: $f(z) = \sum_0^\infty \frac{(z-a)^n}{(1-a)^{n+1}}$.

This has a circle of convergence beyond the original one, reaching $z = 1$. Exceptionally, if $a \in \mathbb{R}$, $a \in (0,1)$, we gain nothing. [Remember, we must not include 1 in our new domain].



We could continue to $\mathbb{C}$, leaving a singularity at 1.

(ii) $f(z) = 1 + z + z^2 + z^4 + z^8 + \dots = \sum z^{2^n}$. This has radius of convergence 1.

Write $z = re^{i\theta}$. Then $z^{2^n} = r^{2^n} e^{i 2^n \theta}$. So, $\arg(z^{2^n}) = 2^n \theta$.

Consider $\theta = \frac{2\pi p}{2^q}$, $p, q \in \mathbb{N}$. Then $\arg(z^{2^n}) = 2\pi p . 2^{n-q}$.

For $n \geq q$, this is an integral multiple of 2π, so $e^{i 2^n \theta} = 1$. So $z^{2^n} = r^{2^n}$.

This looks the same as the example, but with $z = r$.

The series becomes: {early terms} $+ 1 + 1 + \dots \Rightarrow$ singularity at $r = 1$.

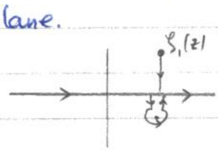
So the unit circle is densely populated with singularities. [The points of the form $p/2^q$ are dense in $[0,1]$]. So no analytic continuation by power series is possible. The unit circle is called a natural barrier to f.

(iii) $f(z) = \sum z^{n!}$

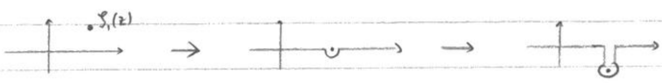
Analytic continuation of functions defined by an integral. (Refer back to §1.2.)

Consider $f(z) = \int_{-\infty}^{\infty} g(z, \zeta)\, d\zeta$.

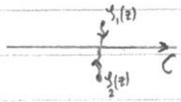
Suppose for each fixed z, $g(z, \zeta)$ has a singularity at $\zeta_1(z)$, whose position depends on z. Start with $f(z)$ in the domain of z where $\zeta_1(z)$ is in the upper half plane. There $f(z)$ will be regular. When z leaves this domain, $\zeta_1(z)$ crosses the real axis, and $f(z)$ as defined would be discontinuous. To achieve a definition of $f(z)$ that would provide the analytic continuation from the original domain, distort the contour as shown.



Example: $f(z) = \int_{-\infty}^{\infty} \frac{e^{-\zeta^2}}{z - \zeta}\, d\zeta$:



"Pinching Poles":



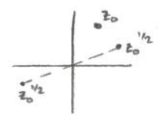
Singularity cannot be avoided

Example: $f(z) = \int_{-\infty}^{\infty} \frac{d\zeta}{z^2 + \zeta^2}$ $(= \frac{\pi}{z})$. Poles are at $\zeta = \pm iz$.

1.4 Multivalued Functions.

We have: function, $f: \mathbb{C} \to \mathbb{C}$. Need to generalise this to "multiple-valued" functions. We need these to invert maps.

Obvious example: $f(z) = z^{1/2}$. Half of the f-plane is mapped onto the whole z-plane. The other half maps it a second time. This appears to create two separate functions, $f_1(z)$, $f_2(z)$, and these appear to be regular functions. These are called branches.



They look locally like separate functions, but globally they get mixed up. Suppose we choose $f_1(z)$ to mean the positive square root for z real and positive. If we don't encircle $z=0$, the value reverts to its original value. If we do, the function changes sign. $z=0$ is called a branch point. At $z=0$, $f_1(z) = f_2(z)$

Other examples: (i) $f(z) = z^{1/3}$ - three branches.

(ii) $f(z) = \ln z$ - infinity of branches.

(iii) maps can be multivalued in either direction, eg $f^3 = z^2$, or $f = \exp(z^{1/2}) \Leftrightarrow z = (\ln f)^2$.

To handle multiple-valued functions globally, we have to alter the topology of $\mathbb{C}$. There are two methods: (i) insert branches

(ii) Riemann surfaces.

Branch cuts: propose curves across which $f(z)$ is not expected to be analytic or even continuous. Contours of integration may not cross them. Such curves normally begin and end at branch points, including $z=\infty$. Where there are placed is open to choice - to be specified. Finally, must specify which branch of the function is intended.

Examples: (i) $f(z) = z^{1/2}$, as above. Branch cut runs from $z=0$ to $z=\infty$, but may be placed anywhere. Define $f(z)$ at one point $(z \neq 0)$. The points on the cut are deleted from $\mathbb{C}$.

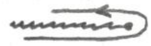
(ii) $f(z) = (z^2-1)^{1/2} = (z-1)^{1/2}(z+1)^{1/2}$. Branch points at $z = \pm 1$, but not at $z=\infty$.

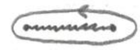
(a) 

Could choose to have $f(z) = +i$ at $z=0$, and continue analytically from there. Place cuts as shown.

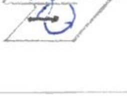
(b) 

Choose $f(z) \approx z$ for large $|z|$.
Just above $z=0$, $f(z) = i$.

A branch cut integral: Integrals along both sides of a branch cut 

Example: $\oint (z^2-1)^{1/2} dz$ as shown:  , with the definition of $(z^2-1)^{1/2}$ as in (b) above.
Put $z = \cos\theta$, $(0 \le \theta \le 2\pi)$, so $dz = -\sin\theta$. $\int_0^{2\pi} i\sin\theta(-\sin\theta)d\theta = -i\int_0^{2\pi} \sin^2\theta \, d\theta = -i\pi$.

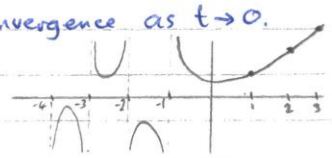
Riemann Surface: Alternative way to modify topology of $\mathbb{C}$; replace it with two or more copies.

Examples: $f(z) = z^{1/2}$:  . $f(z) = \ln z$: infinitely many sheets.

1.5. The Gamma Function

Define ! by: $0! = 1$, $n! = n[(n-1)!]$ - recurrence relation. $(n \in \mathbb{N})$.
Can we draw a smooth curve through these? Ask for $f(x)$ with $f(x) = x f(x-1)$ - (1).
The answer is given for $x > -1$ by Euler: $f(x) := \int_0^\infty t^x e^{-t}\, dt$ - (2); $f(0) = 1$; recurrence relation obtained by integration by parts.

Integral always converges at $t \to \infty$. Requires $x > -1$ for convergence as $t \to 0$.
Accept (1) as applying even for $x < -1$. Get the following:



Extension to $\mathbb{C}$: The gamma function, $\Gamma(z)$, defined by founders as $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ - (3), wherever the integral converges, and by analytic continuation elsewhere. t^{z-1} has as its principal value, $t^{z-1} = \exp\{(z-1)\ln t\}$, where $\ln t$ takes its real value.
We expect this to be analytic (by §1.2).
At limit $t \to \infty$, no difficulty about convergence (it is uniform).
At limit, $t \to 0$, convergence requires $\mathrm{Re}(z) > 0$.

The analytic continuation for $\mathrm{Re}(z) \le 0$ can be achieved by the recurrence relation:
$\Gamma(1) = \Gamma(2) = 1$, $\Gamma(z) = (z-1)\Gamma(z-1)$, $\Gamma(z-1) = \frac{\Gamma(z)}{z-1}$ - (4).
This, used repeatedly, provides continuation into a new vertical strip of width 1, working to the left. By our theorem on analytic continuation, it must be unique.
Note: $\Gamma(z) = (z-1)!$, $\Gamma(z+1) = z!$, $z \in \mathbb{N}$ - (5)

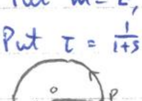
The Beta Function.

In (3), replace z with $m \in \mathbb{C}$ $(\mathrm{Re}\, m > 0)$, and t with x^2: $\Gamma(m) = 2\int_0^\infty x^{2m-1} e^{-x^2}\, dx$.
Multiply this by the same equation with n, y: $\Gamma(m)\Gamma(n) = 4\int_0^\infty x^{2m-1} e^{-x^2} dx \int_0^\infty y^{2n-1} e^{-y^2} dy$.
Change to polars: $\Gamma(m)\Gamma(n) = 4\int_0^\infty r^{2m+2n-1} e^{-r^2} dr \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta$.
Notice that first integral is $\frac{1}{2}\Gamma(m+n)$, by above.
So, via $\tau = \cos^2\theta$, get $\Gamma(m)\Gamma(n) = \Gamma(m+n)\int_0^1 \tau^{m-1}(1-\tau)^{n-1}\, d\tau$. - (6)

Define the Beta Function: $\beta(m,n) = \int_0^1 \tau^{m-1}(1-\tau)^{n-1}\, d\tau$ - (7), so $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ - (8).
This definition is extended to $m, n \in \mathbb{C}$ by analytic continuation.

Special cases of (8).

(i) $m = n = 1/2 \Rightarrow \Gamma(\frac{1}{2}) = \{\beta(\frac{1}{2},\frac{1}{2})\}^{1/2} = \pi^{1/2}$.
(ii) Put $m = z$, $n = 1-z$ $(0 \le \mathrm{Re}(z) < 1)$. Then, $\Gamma(z)\Gamma(1-z) = \int_0^1 \tau^{z-1}(1-\tau)^{-z}\, d\tau$.
Put $\tau = \frac{1}{1+s}$, so $d\tau = \frac{-ds}{(1+s)^2}$. Then the integral is $\int_0^\infty \frac{s^{-z}}{1+s}\, ds =: I(z)$.



Sector OP gives the required integral, $s^{-z} = \exp(-z\ln s)$. Sector QO gives $-e^{-2\pi i z} I$.
The contribution from the arc $\to 0$ as its radius $\to \infty$: a $\frac{1}{1+s}$ cancels the circumference and s^{-z} gives a decaying factor. The residue at $s = -1$ is $e^{-i\pi z}$.
So, $(1 - e^{-2\pi i z}) I(z) = 2\pi i e^{-\pi i z}$, so $I(z) = \frac{\pi}{\sin \pi z}$. $\therefore \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$. - (9)

Again extend by analytic continuation. This gives a way of computing $\Gamma(z)$ in $Re(z)<0$ without using successive strips.

(iii) $m=n=z$ $(Re(z)>0)$. $(7) \Rightarrow \beta(z,z) = \int_0^1 \tau^{z-1}(1-\tau)^{z-1}\,d\tau = \int_0^1 (\tau-\tau^2)^{z-1}\,d\tau$

Let $s=(2\tau-1)^2 = 1-4(\tau-\tau^2)$. The ranges of integration are: τ - , s -

$ds = 4s^{1/2}d\tau$. $\beta(z,z) = \int \left(\frac{1-s}{4}\right)^{z-1}\frac{ds}{4s^{1/2}} = 2^{1-2z}\int_0^1 (1-s)^{z-1}s^{-1/2}\,ds$

So, $\beta(z,z) = 2^{1-2z}\beta(\tfrac{1}{2},z)$.

Write in terms of Γ: $\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = 2^{1-2z}\cdot\frac{\Gamma(z)\Gamma(1/2)}{\Gamma(z+1/2)}$. But $\Gamma(\tfrac{1}{2}) = \pi^{1/2}$, so

$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = \pi^{1/2}\cdot 2^{1-2z}\cdot\Gamma(2z)$ – (10). This is Legendre's Duplication Formula.

(iv) Euler's Limit.

Set $m=z$ $(Re(z)>0)$, n replaced by $n+1\in\mathbb{N}$. $\beta(z,n+1) = \frac{\Gamma(z)\Gamma(n+1)}{\Gamma(z+n+1)} = \int_0^1 \tau^{z-1}(1-\tau)^n\,d\tau$.

Put $t=\tau/n$, and multiply both sides by n^z.

$\frac{\Gamma(z)\Gamma(n+1)n^z}{\Gamma(z+n+1)} = \int_0^n t^{z-1}(1-t/n)^n\,dt$.

Now, let $n\to\infty$. Then RHS $\to \int_0^\infty t^{z-1}e^{-t}\,dt = \Gamma(z)$. So $\Gamma(z) = \lim_{n\to\infty}\frac{\Gamma(z)\,n!\,n^z}{\Gamma(n+z+1)}$

Resist temptation to cancel $\Gamma(z)$, and then $\Gamma(z) = \lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\cdots(z+n)}$ – (11). This is Euler's Limit.

A modification of this is: $\Gamma(z) = \frac{1}{z}\prod_r\left\{(1+\tfrac{1}{r})^z(1+\tfrac{z}{r})^{-1}\right\}$. – (12).

Note: if you do cancel above, $\Gamma(z+n+1)\sim n!\,n^z$ as $n\to\infty$. LHS would have been $(z+n)!$

Euler's idea was $z\in\mathbb{R}$, $0<z<1$.

(v) Canonical Product (Weierstrass)

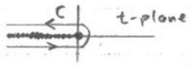
Start from (11), inverted: $\frac{1}{\Gamma(z)} = z\lim_{n\to\infty}\left\{\left[\prod_{r=1}^n(1+\tfrac{z}{r})\right]\exp(-z\ln n)\right\} = z\lim_{n\to\infty}\left\{\left[\prod_{r=1}^n(1+\tfrac{z}{r})e^{-z/r}\right]\exp\left(\sum_1^n\tfrac{z}{r}-z\ln n\right)\right\}$.

But $\left\{\sum_1^n\frac{1}{r}-\ln n\right\}\to\gamma$ as $n\to\infty$, where $\gamma=$ Euler's constant $\simeq 0.5772$.

Hence, $\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{r\in\mathbb{N}}\left[(1+\tfrac{z}{r})e^{-z/r}\right]$. – (13)

This displays $\frac{1}{\Gamma(z)}$ as an entire function, i.e. no poles in $\mathbb{C}$. It has zeroes at $z=0,-1,-2,\dots$ and nowhere else. $\Rightarrow \Gamma(z)$ has poles at $z=0,-1,-2,\dots$, and no zeroes.

(vi) Hankel's Contour Integral.

Consider $I(z) = \int_C e^t t^{-z}\,dt$, where C is: 

Interpret t^{-z} as $\exp(-z\ln t)$, where $\ln t$ is real for $t\in\mathbb{R}_+$.

On the upper side of the cut, $-z\ln t = -z(\log|t|+\pi i)$, so $t^{-z} = |t|^{-z}e^{-\pi i z}$ there.

On the lower side of the cut, $t^{-z} = |t|^{-z}e^{\pi i z}$.

Put $t=-s$ on the contour. Then $I(z) = (e^{\pi i z}-e^{-\pi i z})\int_0^\infty e^{-s}s^{-z}\,ds = 2i\sin\pi z\cdot\Gamma(1-z)$.

So far this holds for $Re(z)<1$, to get $I(z)$ convergent. Substituting from (9), $I(z) = \frac{2\pi i}{\Gamma(z)}$.

$\therefore \frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C e^t t^{-z}\,dt$ – (14). True $\forall z$ by analytic continuation.

Examples: (i) Calculate the volume of the unit sphere in $\mathbb{R}^n$.

Switch to polars, $r=|\underline{x}|$, Ω represents the collection of angular variables.

$V_n = \int_{|r|\le 1} dx_1\cdots dx_n = \int_\Omega\int_0^1 r^{n-1}\,dr\,d\Omega = \frac{1}{n}\int_\Omega d\Omega$.

Now consider, $\int_{-\infty}^\infty\cdots\int_{-\infty}^\infty e^{-\Sigma x_i^2}\,dx_1\cdots dx_n = (\sqrt{\pi})^n = \pi^{n/2}$. (It factorises easily).

In polars, $\int_\Omega\int_0^\infty e^{-r^2}r^{n-1}\,dr\,d\Omega = \frac{1}{2}\int_\Omega\int_0^\infty t^{\frac{1}{2}(n-2)}e^{-t}\,dt\,d\Omega = \frac{1}{2}\Gamma(\tfrac{n}{2})\int_\Omega d\Omega$.

We can now eliminate $\int d\Omega$. $V_n = \frac{\pi^{n/2}/n}{\frac{1}{2}\Gamma(n/2)} = \frac{\pi^{n/2}}{\Gamma(\frac{1}{2}n+1)} = \frac{\pi^{n/2}}{(n/2)!}$

(ii) Calculate $|\Gamma(iy)|$ for $y\in\mathbb{R}$.

Reflection principle: $\Gamma(\bar{z}) = \overline{\Gamma(z)}$, since $\Gamma(\mathbb{R}) = \mathbb{R}$.

So, $\Gamma(-iy) = \overline{\Gamma(iy)}$. From (9), $iy\Gamma(iy)\Gamma(-iy) = \frac{\pi}{\sin\pi(1+iy)} = \frac{-\pi}{\sin i\pi y}$.

$\therefore |\Gamma(iy)| = \left(\frac{\pi}{y\sinh\pi y}\right)^{1/2}$

1.6. The Riemann Zeta Function.

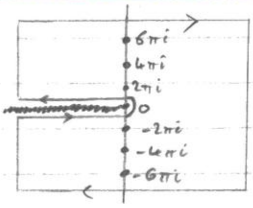
Define $\zeta(z) = \sum_{\mathbb{N}} n^{-z}$ - (15). Converges for $Re(z)>1$ and provides a regular function there. Clearly $\zeta(1)$ diverges. Can we extend the definition of $\zeta(z)$ to $Re(z) \le 1$?
For $z=2,4$, there are results like $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$. $\zeta(3/2)$, $\zeta(5/2)$ are required in statistical mechanics.

The Integral Formula

Recall (14): $\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C e^t t^{-z}\,dt$ where C is the Hankel contour.
Put $z \to 1-z$, $t \to u$: $\frac{2\pi i}{\Gamma(1-z)} = \int_C e^u u^{z-1}\,du = n^z \int_C e^{nt} t^{z-1}\,dt$, setting $u = nt$, $n \in \mathbb{N}$.
Then, $\frac{2\pi i}{\Gamma(1-z)} \sum_{\mathbb{N}} n^{-z} = \sum_{\mathbb{N}} \int_C e^{nt} t^{z-1}\,dt = \int_C \frac{e^t t^{z-1}}{1-e^t}\,dt$. (Move the sum inside - use GP).
Then $\zeta(z) = \frac{\Gamma(1-z)}{2\pi i}\int_C \frac{t^{z-1}}{e^{-t}-1}\,dt$ - (16)
Here, the integral converges $\forall z \in \mathbb{C}$ and so defines an entire function. So, (16) provides the analytic continuation of $\zeta(z)$ into $Re(z) \le 1$.

The only singularities of $\zeta(z)$ must be those of $\Gamma(1-z)$, unless cancelled by those of the integral. The poles of $\Gamma(1-z)$ occur at $z=1,2,3,\dots$, but we know already that $\zeta(z)$ does not have singularities at $z=2,3,\dots$, so poles there have been cancelled. That leaves $z=1$ as the only singularity of $\zeta(z)$ in $\mathbb{C}$. It is a simple pole, residue 1.

The Reflection Formula.



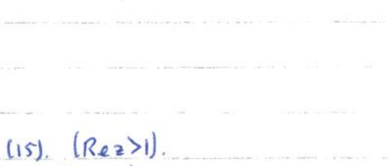
Poles occur in (16) at $t = 2\pi n i$ ($n \in \mathbb{Z}$). Add a rectangular contour, as shown. Vertices at $z = \pm K \pm (2K-1)\pi i$, $K \in \mathbb{N}$.
Let $K \to \infty$. The residue of $\frac{t^{z-1}}{e^{-t}-1}$ at $t = \pm 2\pi n i$ is $-(\pm 2\pi n i)^{z-1}$.

So, $\zeta(z) = \Gamma(1-z)\sum_{\mathbb{N}}\left[(2\pi n i)^{z-1} + (-2\pi n i)^{z-1}\right] = \Gamma(1-z)(2\pi)^{z-1}\sum_{\mathbb{N}} n^{z-1}(i^{z-1} + (-i)^{z-1})$.
$= \Gamma(1-z)(2\pi)^{z-1}\sum_{\mathbb{N}} n^{z-1}\, 2\cos(\frac{1}{2}\pi(z-1)) = \Gamma(1-z).\, 2^z.\, \pi^{z-1}.\, \cos(\frac{1}{2}\pi(z-1))\,\zeta(z-1)$.
This doesn't ~~relat~~ evaluate $\zeta(z)$ in terms of more elementary functions; it relates $\zeta(z)$ to $\zeta(1-z)$.
The result is more usually quoted with z and $1-z$ swapped: $\zeta(1-z) = 2^{1-z}.\pi^{-z}.\cos(\frac{1}{2}\pi z)\Gamma(z)\zeta(z)$ -(17).

From this we can identify the zeroes of $\zeta(z)$: at $z = -2n$ where $n \in \mathbb{N}$ (from the cos term).
Apply l'Hôpital's rule - gives $\zeta(0) = \frac{1}{2}$.

The Euler Product.

Return to (15), multiply by 2^{-z} and subtract the result from (15). ($Re z>1$).
Get $(1-2^{-z})\zeta(z) = \frac{1}{1^z} + \frac{1}{3^z} + \frac{1}{5^z} + \dots$ Similarly, $(1-2^{-z})(1-3^{-z})\zeta(z) = \sum_{2\nmid n, 3\nmid n} \frac{1}{n^z}$.
Let p_n be the nth prime, so $p_1 = 2$, $p_2 = 3$, etc. Continue with the above process, to the limit
$\prod_{\mathbb{N}}(1 - \frac{1}{p_n^z})\zeta(z) = 1$. So, $\zeta(z) = \prod_{\mathbb{N}}(1-\frac{1}{p_n^z})^{-1}$ - (18)



From this, $\zeta(z)$ has no zeroes in $Re(z)>1$, and by (17),
it has no zeroes in $Re(z)<0$ except $z = -2, -4, \dots$

Application to Number Theory.

(z is replaced by $s=\sigma+it$) Define $\pi(x) = \#$ of primes $\le x$. So, $\pi(x) = \sum_{\mathbb{N}} H(x-p_n)$, where H is the Heaviside function.

Now, $\ln \zeta(z) = -\sum_{\mathbb{N}} \ln\left(1-\frac{1}{p_n^z}\right) = -\sum_2^\infty \left\{ \left(\pi(n)-\pi(n-1)\right) \ln\left(1-{}^1/_{n^z}\right)\right\} = -\sum_2^\infty \pi(n)\left\{ \ln\left(1-\frac{1}{n^z}\right) - \ln\left(1-\frac{1}{(n+1)^z}\right)\right\}$

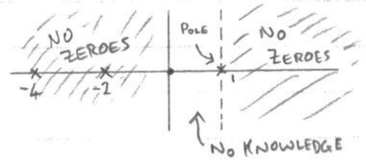
(use $\frac{d}{dx}\left[\ln\left(1-\frac{1}{x^z}\right)\right] = \frac{-z}{x(x^z-1)}$) $\ldots = -\sum_2^\infty \pi(n) \cdot \int_n^{n+1} \frac{z}{x(x^z-1)}\,dx = z\int_2^\infty \frac{\pi(x)}{x(x^z-1)}\,dx$.

So, $\frac{1}{z}\ln\zeta(z) = \int_2^\infty \frac{\pi(x)}{x(x^z-1)}\,dx$. – (19)

– looks like an integral transform. Can we invert? Yes, we get $\pi(x) \sim \frac{x}{\ln x}$.

This last bit is the Prime Number Theorem.

Picture of what we know about $\zeta(z)$:



Riemann noticed that there are zeroes on $\mathrm{Re}(z)=\frac{1}{2}$, and didn't appear to be any others in the strip $0<\mathrm{Re}(z)<1$.

The Riemann Hypothesis: there are no zeroes in the strip apart from $\mathrm{Re}(z) = {}^1/_2$.

It has been shown (by Hadamard) that there are no zeroes on $\mathrm{Re}(z)=1$. $\Rightarrow$ proof of prime number theorem, $\pi(x) \sim \frac{x}{\ln x}$.

If the Riemann Hypothesis is true, $\pi(x) - \mathrm{li}(x) = O(x^{1/2}\ln x)$, where $\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t} \sim \frac{x}{\ln x}$

2. Special Functions.

Γ and ζ do not arise from differential equations. Many of the famous "named" equations do arise from second order linear DEs. Much can be learned from looking at these in the complex plane. Many of these equations come from separation of variables in Cartesians, Polars, ...

2.1. Ordinary Differential Equations.

Discuss equations of type, for $w(z)$, $\frac{d^2w}{dz^2} + p(z)\frac{dw}{dz} + q(z)w = 0$ – (1).

In the context, $w(z)$ is analytic. So consider the case where $p(z)$, $q(z)$ are also analytic. We will assume that they are regular except at a small number of poles.

If $z=a$ is an ordinary point of (1), we mean that there is a neighbourhood of a in which both p, q are regular. In this case, $p(z) = \sum p_r (z-a)^r$, $q(z) = \sum q_r (z-a)^r$.

So, try $w(z) = \sum c_r (z-a)^r$ – (2)

Not surprisingly, substitution into (1) and equating powers of z leads to a solution with c_0, c_1 arbitrary, and $c_2, c_3, \ldots$ produced by recurrence relations.

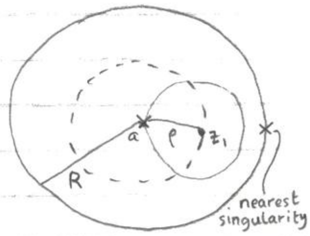
Theorem (no proof given): the radius of convergence of the series for w is at least equal to the minimum of the ROC's of $p(z)$, $q(z)$.

2.2. Nature of a Solution near a Singularity.

Ask about analytic continuation of (2) around poles of $p(z), q(z)$. We shall always assume that any singularities of $p(z), q(z)$ are <u>isolated</u>, otherwise the DE itself would be different after a trip around a singularity.

Can get an idea of what to expect from first order equation: $\frac{dw}{dz} + p(z)w(z) = 0$ - <u>(3)</u>
Solution: $\ln w = -\int^z p(\xi)\,d\xi$. w is regular wherever p is. Give a pole at $z=0$.
$p(z) = \{\text{regular part}\} + \frac{d_1}{z} + \frac{d_2}{z^2} + \frac{d_3}{z^3} + \dots$
$w(z) = \{\text{regular part}\} \times z^{-d_1} \exp\left\{\frac{d_2}{z^2} + \frac{d_3}{z^3} + \dots\right\}$.
So a simple pole implies a branch point at $w(0)$, unless $d_1 \in \mathbb{Z}$. A double pole in $p(z)$ implies an essential singularity in $w(z)$ $\propto \exp(d_2/z)$
An essential singularity in $p(z)$, eg $p(z) = (\frac{1}{z} - 1)\exp(\frac{1}{z})$, then $w(z) = \exp(z\exp(\frac{1}{z}))$, ie a very nasty singularity. So in (1), only very mild singularities of p, q will be tolerated.

Let $z=a$ be a singularity. Let R be the distance to the next nearest singularity. Start at $z=z_1$ on $|z-a| = \rho < R$. We have convergent power series for some solution. Continue this analytically via points $z_1, z_2, \dots$ on $|z-a| = \rho$.



We can then deal with analytic continuation of $w_1(z)$ and a linearly independent solution $w_2(z)$. Let $\{w_1(z), w_2(z)\}$ become $\{w_1^+(z), w_2^+(z)\}$ via this process.
A second-order DE has at most two L.I. solutions, so $w_i^+ = \alpha_{ij} w_j(z)$ - <u>(4)</u>, using summation convention. The numerical matrix (α_{ij}) is the continuation matrix.
We know that $\det(\alpha_{ij}) \neq 0$ since by going clockwise about $z=a$ we would have got $(\alpha_{ij})^{-1}$.
We use linear algebra to find the eigenvalues and eigenvectors to diagonalise (α_{ij})

<u>Note:</u> (α_{ij}) need not be hermitian, so if the roots are equal the subject is a little more difficult.

Write λ_1, λ_2 for the eigenvalues; find eigenvectors.

<u>Case $\lambda_1 \neq \lambda_2$:</u> Then there exist eigenvectors giving functions w_1, w_2.
$(\alpha_{ij}) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $w_j^+ = \lambda_j w_j$ - <u>(5)</u>. Now define $\sigma_j = \frac{1}{2\pi i}\ln\lambda_j$ - <u>(6)</u>
This leaves σ_j indeterminate for the moment (by addition of $m \in \mathbb{Z}$). Clearly, $\lambda_j = \exp(2\pi i\sigma_j)$.
Now consider $(z-a)^{\sigma_j} = \exp\{\sigma_j \ln(z-a)\}$. If these are continued analytically around $z=a$, they acquire the factors λ_j. Now write $w_j(z) = (z-a)^{\sigma_j} v_j(z)$ - <u>(7)</u>
The functions v_j are single-valued, so they have (at worst) an isolated essential singularity at $z=a$. The v_j have Laurent expansions: $v_j(z) = \sum_{\mathbb{Z}} c_n^{(j)}(z-a)^n$ - <u>(8)</u>
If, for a particular j, this series extends back to $n=-\infty$ then the ambiguity in σ_j merely amounts to the possibility of transferring $(z-a)^m$ between the two factors in (7)
But if our series does have only finitely many negative powers, we can choose our σ_j so that the expansion for v_j begins at $n=0$: $w_j(z) = (z-a)^{\sigma_j}\sum_{n\geq 0} c_n^{(j)}(z-a)^n$, $c_0 \neq 0$ - <u>(9)</u>

If this happens for both $j=1,2$, the singularity at $z=a$ is perversely called a ~~regular~~ regular singularity. In that case, $w(z) = A w_1(z) + B w_2(z)$ $-$ (10), is the general solution with w_i as in (9). Note, $\lambda_1 \neq \lambda_2 \Rightarrow \sigma_1 - \sigma_2 \notin \mathbb{Z}$

Case $\lambda_1 = \lambda_2 = \lambda$: Either we can still choose a base $w^{(j)}(z)$ such that $(\alpha_{ij}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, in which case all the above results still hold. In the case where both series terminate, $\sigma_1 - \sigma_2 \in \mathbb{Z}$. Or, we can choose a base such that $(\alpha_{ij}) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. As before, $w_1(z) = (z-a)^{\sigma_1} v_1(z)$ and the same discussion applies. To find the form of $w_2(z)$, recall the elementary idea of reduction of order applied to the form of (α_{ij}).
Set $w_2(z) = u(z) w_1(z) = u(z)(z-a)^{\sigma_1} v_1(z)$ $-$ (11)
Use (α_{ij}) to continue around $z=a$: $w_2^+(z) = w_1 + \lambda w_2 = (1+\lambda u) w_1 = (\lambda^{-1} + u) w_1^+$.
So, in a sense, $u^+ = \lambda^{-1} + u$
This means that $u(z)$ has the form $u(z) = \frac{1}{2\pi i \lambda} \cdot \ln(z-a) + s(z)$, where $s(z)$ is single-valued but need not be regular. $s(z)$ would be fixed by the original DE.
So, $w_2(z) = \frac{1}{2\pi i \lambda}(z-a)^{\sigma_1} \{ v_1(z) \ln(z-a) + v_2(z) \}$ $-$ (12).
At this stage, the constant factor $\frac{1}{2\pi i \lambda}$ can be dropped.
Anyway, $w(z) = A(z-a)^{\sigma_1} v_1(z) + B(z-a)^{\sigma_1} \{ v_1(z) \ln(z-a) + v_2(z) \}$ $-$ (13)

To qualify as a regular singularity in this case, v_1 and v_2 must both have terminating Laurent series. In this case, choose σ_1 so that v_1 is as before. Even then $v_2(z)$ may be left with a pole of finite order, and the contribution $(z-a)^{\sigma_1} v_2(z)$ can be rewritten as $(z-a)^{\sigma_2} v_3(z)$, where $v_3(z)$ is regular, $v_3(0) \neq 0$, $\sigma_1 - \sigma_2 \in \mathbb{Z}$.
If $z=a$ is not a regular singularity, it is called an ~~irreg~~ irregular singularity.

Form of coefficients for a regular singularity. Method of solution.
Return to (1). Wlog, put singularity at $z=0$: $w''(z) + p(z) w'(z) + w(z) = 0$.
Can we recognise whether $z=0$ is a regular singularity, and if so go on to find the solutions $\sigma_1, \sigma_2, v_1, v_2$?

Theorem (Fuchs): If $z=0$ is singular, it is a regular singularity iff $z p(z)$ and $z^2 q(z)$ are regular in a neighbourhood of $z=0$.
Proof (sketch): Necessary: start with (10). From it, form w' and w'' and form the DE satisfied ~~satisfying~~ by eliminating A, B. This leads to (1) with $p(z), q(z)$ in explicit form, having property stated. Do same for (13) if there are to be equal roots.
Sufficiency: start with Taylor expansions: $z p(z) = \sum_{n \geq 0} p_n z^n$, $z^2 q(z) = \sum_{n \geq 0} q_n z^n$ (not all of p_0, q_0, q_1 zero). Proceed...

Look for solutions of the form $w(z) = z^{\sigma} \sum_{r=0}^{\infty} c_r z^r$, $c_0 \neq 0$. Substitute into (1), multiply and equate powers: $c_0 F(\sigma) = 0$, where $F(\sigma) = \sigma(\sigma-1) + p_0 \sigma + q_0$ $-$ (14),
$c_n F(\sigma+n) = -\sum_{s=0}^{n-1} c_s \{ (\sigma+s) p_{n-s} + q_{n-s} \}$, $n \geq 1$ $-$ (15).
$F(\sigma)$ is the indicial equation with roots σ_1, σ_2. Adopt convention: $\mathrm{Re}(\sigma_1 - \sigma_2) \geq 0$.
For each root, solve (15) for $n = 1, 2, 3, \ldots$ in turn $\Rightarrow$ recurrence relation for $\{c_n\}$.
σ_1 always gives a solution.

Case 1: $\sigma_1 - \sigma_2$ not an integer - get 2 L.I. solutions.
Case 2: $\sigma_1 = \sigma_2$ - method gives only one solution. The second must be Logarithmic.
Case 3: $\sigma_1 - \sigma_2 \in \mathbb{Z}^+$. Attempt to use σ_2 is in trouble when we reach $n = m$ in (15), because $F(\sigma_2 + m) = F(\sigma_1) = 0$, so c_m appears to be ∞. If so, second solution must be Logarithmic. Still more exceptionally, the RHS of (15) might vanish at that point. If so, $c_n =$ "$0/0$"; can let c_n be arbitrary and continuing gives back w_1, and w_2 in above form still exists.
We still have to show how to find the Logarithmic solution.
3(a): 1 series, $\lambda_1 = \lambda_2$, and log solution required.
3(b): 2 series, $\lambda_1 = \lambda_2$, but $\exists$ 2 L.I. eigenvectors.

It still remains to show, how, in cases 2 and 3(a), how to get the second solution, hence $\sigma_2(z)$. Two methods: (i) put $w_2(z) = u(z) w_1(z)$ and get a first order equation for u', (ii) shift σ_2 slightly from its critical value and then do a clever limit with l'Hôpital's rule, amounting to looking at $\frac{\partial}{\partial \sigma}\{\text{1st solution}\}\big|_{\sigma=\sigma_2}$.

2.3. The point at ∞

Go back to (1): $w'' + p(z) w' + q(z) w = 0$.
Put $\zeta = \frac{1}{z}$ and transform: $\frac{d^2 w}{d\zeta^2} + \left\{\frac{2}{\zeta} - \frac{1}{\zeta^2} p(1/\zeta)\right\} \frac{dw}{d\zeta} + \frac{1}{\zeta^4} q(1/\zeta) w = 0$ – (16)
Classify $z = \infty$ according to behaviour at $\zeta = 0$.

Ordinary point: $\frac{2}{\zeta} - \frac{1}{\zeta^2} p(1/\zeta)$ and $\frac{1}{\zeta^4} q(1/\zeta)$ need to be regular at $\zeta = 0$.
Ie, $2z - z^2 p(z)$ and $z^4 q(z)$ must be bounded as $z \to \infty$. (17).
Regular singularity: $2 - \frac{1}{\zeta} p(1/\zeta)$ and $\frac{1}{\zeta^2} q(1/\zeta)$ must be regular at $\zeta = 0$.
Ie, $z p(z)$ and $z^2 q(z)$ must be bounded at $z \to \infty$. (18) If so, we expect solutions of the type $z^{-\sigma} \sum c_n z^{-n}$ near $z = \infty$. Indicial equation for σ is $\sigma^2 + (1 - p_0)\sigma + q_0 = 0$
Failing (18), we have an irregular singularity at ∞.

2.4. Example: Bessel's Equation.

$w'' + \frac{1}{z} w' + \left(1 - \frac{\nu^2}{z^2}\right) w = 0$ – (19), where, in general, $\nu \in \mathbb{C}$. Convention: $Re(\nu) \geq 0$.
Particular interest lies in case $\nu \in \mathbb{N}$, eg $\nabla^2 \varphi$ separated in cylindrical polars. Also, $n + \frac{1}{2} \in \mathbb{N}$ arises when we separate in spherical polars, eg in quantum theory of scattering.

We see that $z = 0$ is a regular singularity. $z \to \infty$ is an irregular singularity.
$z p(z) = 1$, $z^2 q(z) = z^2 - \nu^2$.
(14) $\Rightarrow F(\sigma) = \sigma^2 - \nu^2$. Indicial equation: $F(\sigma) = 0$, $\sigma^2 = \nu^2$, $\sigma = \pm \nu$.
(15) reduce to: $c_k F(\sigma + k) = -c_{k-2}$. $c_{-1} = c_{-2} = 0$; generally $c_0 \neq 0$. $c_k = 0$ for all odd k.

$$c_{2r} = \frac{-c_{2r-2}}{(\sigma + 2r)^2 - \nu^2} = \begin{cases} \frac{-c_{2r-2}}{2r(2r+2\nu)}, & \sigma = \nu \\ \frac{-c_{2r-2}}{2r(2r-2\nu)}, & \sigma = -\nu \end{cases}$$

Case 1: $\sigma_1 - \sigma_2 \notin \mathbb{N}$, so $2\nu \notin \mathbb{N}$.

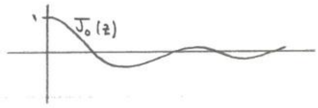
$$w_1(z) = c_0 z^\nu \left\{ 1 - \frac{z^2}{2^2(1+\nu)} + \frac{z^4}{2^4 . 2! (1+\nu)(2+\nu)} - \dots \right\} = c_0 z^\nu \sum_{r=0}^{\infty} \frac{(-z^2/4)^r}{r!\,\Gamma(\nu+r+1)} \cdot \Gamma(\nu+1).$$

Usual to define standard Bessel function: $J_\nu(z) = \left(\frac{1}{2}z\right)^\nu \sum_{r=0}^{\infty} \frac{(-z^2/4)^r}{r!\,\Gamma(\nu+r+1)}$

Similarly, second solution: $J_{-\nu}(z) = \left(\frac{1}{2}z\right)^{-\nu} \sum_{r=0}^{\infty} \frac{(-z^2/4)^r}{r!\,\Gamma(-\nu+r+1)}$ — (20)

General solution: $w(z) = A J_\nu(z) + B J_{-\nu}(z)$. Radius of convergence $= \infty$.

Case 2: $\nu = 0$, so $\sigma_1 = \sigma_2 = 0$. (20) $\Rightarrow J_0(z) = \sum_{r=0}^{\infty} \frac{(-z^2/4)^r}{(r!)^2}$ — (21)



We do not find any second solution.

Case 3(a): $2\nu \in \mathbb{N}$, in fact $\nu \in \mathbb{N}$, call it $\nu = n$.
$J_n(z)$ as in (20) - okay, standard Bessel function. $J_{-n}(z)$ is merely a multiple of J_n - nothing new. $J_{-n}(z)$ as in (20) appears to have its first n terms all 0, but this is an artefact of omitting the $\Gamma(\nu+1)$ factor. Logarithmic solution required.

Case 3(b): $2\nu \in \mathbb{N}$, but here odd.
$J_{-\nu}$ looks alright after all. Explanation: going back to $c_k F(\sigma+k) = -c_{k-2}$, we can set $c_0 = 0$, $c_{-1} = 0$; odd coefficients vanish initially. When we get to a certain point, the odd coefficients reappear. Resulting series gives $J_{-\nu}$ again. So J_ν, $J_{-\nu}$ are okay after all.

Exercise: $J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z$, $J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z$.

The Logarithmic solution: when $\nu = 0, 1, 2, \dots$
Weber's function: $Y_\nu(z) = \frac{\cos\nu\pi\, J_\nu(z) - J_{-\nu}(z)}{\sin\nu\pi}$ — (22).
Let $\nu \to 0, 1, 2$, or something, and $Y_\nu(z)$ survives as a non-trivial solution. We use L'Hôpital's rule and get an independent solution involving $\frac{\partial}{\partial\nu}$, $J_{-\nu}$, J_ν, etc. [See Copson, p.328-9]

2.5. Global Classification of Equations.

We want to classify any given $w'' + p(z)w' + q(z)w = 0$, according to the number and type of the singularities, including $z = \infty$. The domain is always regarded as $\mathbb{C} \cup \{\infty\}$. Möbius transformations map this onto itself and are invertible; they enable us to transform equations like (1) into standard form.

Recall for reference some known results:

- if $z = 0$ is a regular singularity of (1), indices σ_1, σ_2 then

$= 0$, if ordinary point

$$\left.\begin{aligned} p(z) &= \frac{1-\sigma_1-\sigma_2}{z} + O(1) \\ q(z) &= \frac{\sigma_1\sigma_2}{z^2} + O\left(\frac{1}{z}\right) \end{aligned}\right\} (23)$$

$\frac{2}{z}$, if ordinary point

- if $z = \infty$ is a regular singularity of (1) with indices σ_1, σ_2, then

$$\left.\begin{aligned} p(z) &= \frac{1+\sigma_1+\sigma_2}{z} + O(1/z^2) \\ q(z) &= \frac{\sigma_1\sigma_2}{z^2} + O(1/z^3) + O\left(\frac{1}{z^4}\right) \end{aligned}\right\} (24)$$

$= 0$, if ordinary point.

0 singularities.
This is impossible, for it would require p, q to be regular in $\mathbb{C}$, yet $p(z) = \frac{2}{z} + O(z^{-2})$, $q(z) = O(z^{-4})$ as $z \to \infty$ — (25). This is impossible by Liouville's Theorem.

1 regular singularity.

Put it at $z=0$. The only possibility is $w'' + \frac{2}{z} w' = 0$ $-(26)$.

This has indices $0, -1$ at $z=0$, $w(z) = A/z + B$.

Put it at $z=\infty$, then $w''=0$, with solution $w(z) = Az + B$.

2 singularities.

Put them at 0 and ∞. Various conditions force the equation to take the following form: $w'' + \frac{1-\sigma_1-\sigma_2}{z} w' + \frac{\sigma_1\sigma_2}{z^2} w = 0$ $-(27)$ This has indices σ_1, σ_2 at $z=0$, and $-\sigma_1, -\sigma_2$ at $z=\infty$. The solutions are $w = Az^{\sigma_1} + Bz^{\sigma_2}$ $-(28)$

If $\sigma_1 - \sigma_2 \in \mathbb{Z}$, this still works (case 3(b))

If $\sigma_1 = \sigma_2$, we have only one solution (case 2). To get the logarithmic second solution, start with indices $\sigma, \sigma+\delta$. Write the solution in the form $w = Az^{\sigma} + C\frac{z^{\sigma+\delta} - z^{\sigma}}{\delta}$ $-(29)$

L'Hôpital's rule leads $\frac{\partial}{\partial\sigma}(z^{\sigma}) = \ln z . z^{\sigma}$. So, in the limit, $w = (A + C\ln z)z^{\sigma}$.

See problem sheet for the case where singularities are at $z = A, B$.

Idea of confluence of singularities.

Digress back to the first order equation: $w' + \left(\frac{\sigma}{z} - \frac{\sigma}{z-\varepsilon}\right) w = 0$ $-(30)$ Ie, $w' - \frac{\sigma\varepsilon}{z(z-\varepsilon)} w = 0$.

Solution is: $w \propto z^{-\sigma}(z-\varepsilon)^{\sigma} = \left(1 - \frac{\varepsilon}{z}\right)^{\sigma}$ $-(31)$.

Consider the limit $\varepsilon \to 0$, but $\sigma\varepsilon$ staying constant, $\sigma\varepsilon = -\mu$.

The equation becomes $w' + \frac{\mu}{z^2} w = 0$, and the solution from (31) becomes $w(z) = \lim_{\sigma\to\infty} \left(1 + \frac{\mu}{\sigma z}\right)^{\sigma} = \exp(\mu/z)$, and this works!

Now return to (27) and shift the singularity from 0 to $z=A$:

$w'' + \frac{1-\sigma_1-\sigma_2}{z-A} w' + \frac{\sigma_1\sigma_2}{(z-A)^2} w = 0.$ $-(32)$.

Now let $A \to \infty$ to get a confluence as $z\to\infty$; we'll get an irregular singularity there. Try $\sigma_1 = \lambda_1 A$, $\sigma_2 = \lambda_2 A$, where λ_i stay constant. We get $w'' + (\lambda_1+\lambda_2)w' + \lambda_1\lambda_2 w = 0$ $-(33)$

$w(z) \propto (1 - z/A)^{\lambda_1 A} \to e^{-\lambda_1 z}$ (from original solution), and $w(z) = e^{-\lambda_1 z}, e^{-\lambda_2 z}$ from (33) directly.

In the case $\sigma_1 = \sigma_2$, we get a ze^{m} term, as expected.

3 regular singularities.

Provides many of the 'named' transcendental functions: Legendre, Chebyshev, Bessel, Laguerre, Hermite, Airy, etc. A big but manageable theory. It leads to:

(i) 3 distinct regular singularities;

(ii) confluence of two of the singularities;

(iii) confluence of all three singularities.

2.6. The Hypergeometric Equation and Functions.

Symmetric Formulation.

Put regular singularities at $z = A, B, C$ with indices (α,α'), (β,β'), (γ,γ'). For $z=\infty$ to be an ordinary point requires: $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma' = 1$ $-(34)$.

This completely determines $p(z), q(z)$. - very clumsy.

Standard Formulation

Put singularities at: $z=0$, indices $0, 1-c$; $z=1$, indices $0, c-a-b$; $z=\infty$, indices a, b — (35)

The equation is: $z(z-1)w'' + \{(a+b+1)z - c\}w' + abw = 0$. This is the standard hypergeometric equation. Rewriting, $w'' + \left\{\frac{c}{z} + \frac{1+c-a-b}{1-z}\right\}w' - \frac{ab}{z(1-z)}w = 0$ — (36)

[*Exercise:* check that the indices are as stated]

Any other equations of this class can be related to (36) by transformations:

(i) Möbius transformations on $\mathbb{C}\cup\{\infty\}$ of type $\{A,B,C\} \leftrightarrow \{0,1,\infty\}$. Indices are preserved.

(ii) Transformations of type $w_{old} = (z-A)^{\xi}(z-B)^{\eta}(z-C)^{\zeta} w_{new}$; σ, σ' at A would become $\sigma-\xi$, $\sigma'-\xi$, provided $\xi+\eta+\zeta=0$. We can then require two indices to vanish.

Rewrite (36) as: $\left(z\frac{d}{dz}+a\right)\left(z\frac{d}{dz}+b\right)w = \left(z\frac{d}{dz}+c\right)\frac{dw}{dz}$ — (37).

Look for a series solution near $z=0$ for index 0. $w(z) = \sum c_n z^n$, $c_0 \neq 0$.

Then (37) $\Rightarrow$ $(n+c)(n+1)c_{n+1} = (n+a)(n+b)c_n$, $n \geq 0$.

$w(z) = F(a,b;c;z) = 1 + \frac{a.b}{1.c}z + \frac{a.(a+1).b.(b+1)}{2!\,c(c+1)}z^2 + \dots$ — (38) — The Hypergeometric Function

If $c = 0,-1,-2,\dots$, this fails and we get the logarithmic case.

If a or $b = 0,-1,-2,\dots$, the series terminates and we get Jacobi polynomials.

Radius of convergence of (38) is 1, except in the polynomial case.

The second solution at $z=0$ is $w_2(z) = z^{1-c}.F(1+a-c, 1+b-c; 2-c; z)$ — (39)

The two solutions coincide at $z=1$ - case 2.

Any $c \in \mathbb{Z}$ makes one or other of the series fail - case 3.

The other pairs of solutions near $z=1$, $z=\infty$ can also be expressed in terms of F using Möbius transformations that permute $\{0,1,\infty\}$. the argument z becomes $1-z$ or $1/z$.

Example: Legendre's equation and functions: $(1-z^2)w'' - 2zw' + \left\{n(n+1) - \frac{m^2}{1-z^2}\right\}w = 0$ — (40)

This arises from separating $\nabla^2\psi$ in spherical polars. Get $\left\{\begin{matrix} r^m \\ r^{-m-1}\end{matrix}\right\} P_n^m(\cos\theta)e^{\pm im\varphi}$, $-n \leq m \leq n$. $n \in \mathbb{N}$. $P_n^m(z)$ is the non-singular solution of (40) - a polynomial.

Solutions are: $w(z) \propto (1-z^2)^{m/2}.F\left(n+m+1, m-n; m+1; \frac{1-z}{2}\right) \propto \frac{1}{2^n.n!}(z^2-1)^{\frac{m}{2}}\left(\frac{d}{dz}\right)^{n+m}(z^2-1)^n$

- associated Legendre polynomial.

2.7. The Confluent Hypergeometric.

Return to (36); replace z by ζ: $\zeta(\zeta-1)w'' + \{(a+b+1)\zeta - c\}w' + abw = 0$.

This has singularities at $0, 1, \infty$. Move $z=1$ singularity to $z=b$.

Put $\zeta = \frac{z}{b}$, $\frac{d}{d\zeta} = b\frac{d}{dz}$.

Equation becomes: $\frac{z}{b}\left(\frac{z}{b}-1\right)b^2\frac{d^2w}{dz^2} + \left\{(a+b+1)\frac{z}{b} - c\right\}b\frac{dw}{dz} + baw = 0$.

So, $z\left(\frac{z}{b}-1\right)\frac{d^2w}{dz^2} + \left\{\left(\frac{a+1}{b}+1\right)z - c\right\}\frac{dw}{dz} + aw = 0$.

Now let $b \to \infty$: $zw'' + (c-z)w' - aw = 0$ — (41)
This is the confluent hypergeometric equation. It has a regular singularity at $z=0$, indices $0, 1-c$, and an irregular singularity at $z=\infty$.

Do the same to our solution (38); replace z with z/b:
$w(z) = F(a,b;c;z/b) = 1 + \frac{ab}{c}\cdot\frac{z}{b} + \frac{a(a+1)b(b+1)}{2!\,c(c+1)}\cdot\left(\frac{z}{b}\right)^2 + \dots$
Now let $b \to \infty$: $w(z) = \Phi(a;c;z) = 1 + \frac{a}{c}z + \frac{a(a+1)}{2!\,c(c+1)}z^2 + \dots$ — (42)
— the confluent hypergeometric function.
Second solution: $w(z) = z^{1-c}.\,\Phi(1+a-c;\,2-c;\,z)$ — (43)

General properties: radius of convergence of (42) is ∞ — an entire function
Its behaviour at ∞: can be intuitively seen that for large $|z|$, $w''-w'=0$, approximately, near $z=\infty$. Solutions: $w = Ae^z + B$, showing an essential singularity at ∞.
Difficult to connect these two solutions with (42), (43).

Some particular cases.
(i) if $a=c$ we get $\exp(z)$.
(ii) if $c \in \mathbb{Z}$, either (42) or (43) fails. One solution has to be logarithmic.
(iii) if $-a \in \mathbb{N}$, (42) terminates to give a polynomial. Standard notation: $a=-n$, $c=1+\mu$.
Gives $zw'' + (1+\mu-z)w' + nw = 0$ — Laguerre's Equation. Get Laguerre polynomials:
$w(z) = L_n^{(\mu)}(z) = \frac{\Gamma(n+\mu+1)}{n!\,\Gamma(\mu+1)}\cdot\Phi(-n;\,1+\mu;\,z)$.
(iv) Hermite's Equation: $w'' - 2zw' + 2nw = 0$. Not of this class, but put $\zeta = z^2$. Transforms equation to confluent hypergeometric. $w(\zeta) = \begin{cases} \Phi(-\frac{1}{2}n;\frac{1}{2};\zeta) & \text{if } n \text{ is even} \\ \zeta^{1/4}\,\Phi(\frac{1}{2}-\frac{1}{2}n;\frac{3}{2};\zeta) & \text{if } n \text{ odd.} \end{cases}$

(v) Bessel functions. After some manipulation, $J_\nu \propto z^\nu e^{-iz}\,\Phi(\nu+\frac{1}{2};\,2\nu+1;\,2iz)$
For large $|z|$, Φ contains contribution 1 or e^{2iz}. $J_\nu \propto z^\nu(e^{iz}, e^{-iz})$ — wavy for real z.

Triple confluence.
Start with singularities at $K, Ke^{2\pi i/3}, Ke^{4\pi i/3}$, with indices $\frac{1}{6} \pm \frac{1}{3}K^{3/2}$. Let $K \to \infty$.
Equation becomes: $w'' - zw = 0$ — (45). — Airy's equation.
No singularities in finite $\mathbb{C}$. Big one at ∞.
Put $\zeta = \frac{2}{3}z^{3/2}$. $W(\zeta) = z^{-1/2}w(z)$. Then: $\frac{d^2W}{d\zeta^2} + \frac{1}{\zeta}\frac{dW}{d\zeta} - \left(1 - \frac{1}{9\zeta^2}\right)W = 0$. — Bessel's equation for $W(i\zeta)$. So, $w(z) = z^{1/2}\,J_{\pm\frac{1}{3}}\left(\frac{2i}{3}z^{3/2}\right)$

2.8. Contour Integral Solutions.

Another useful approach to supplement power series. Can get them: (i), direct from the equation, (ii) direct from power series.

(a) Laplace's method.
Look for a solution in the form $w(z) = \int_C e^{zt} f(t)\,dt$ — (47). Can choose C and $f(t)$ freely. Then, $w'(z)$ = same with an extra t inserted, $w''(z)$ = same with an extra t^2 inserted. $zw(z) = \int z e^{zt} f(t)\,dt = [e^{zt}f(t)]_C - \int_C e^{zt}\frac{\partial f}{\partial t}\,dt$. Similarly for $zw'(z)$, etc.

Now consider (temporary notation): $(a_0+a_1z)w'' + (b_0+b_1z)w' + (c_0+c_1z)w = 0$.
Substitute and get an equation involving f, $\partial f/\partial t$.
Note that not both a_0, a_1 vanish. If $a_1 \neq 0$, move origin to $z = -\frac{a_0}{a_1}$, so it is sufficiently illustrative to deal with the case $a_0 = 0$, $a_1 = 1$. If $a_1 = 0$, set $a_0 = 1$. These are sufficiently general. Apply method to our two standard equations - confluent hypergeometric and Airy.

(b) Confluent hypergeometric equation - (41)

$zw'' + (c-z)w' - aw = 0$. Substitute (47): $\int_C e^{zt}\{zt^2 + (c-z)t - a\}f(t)\,dt = 0$.

So, $\int_C f(t)\{\underbrace{(t^2-t)\frac{\partial}{\partial t}}_{\text{integrate by parts.}} + \underbrace{(ct-a)}_{\text{leave as is.}}\}e^{zt}\,dt = 0$.

Get $(*)$: $\left[f(t)(t^2-t)e^{zt}\right]_C + \int_C e^{zt}\left\{-\frac{\partial}{\partial t}\left[f(t)(t^2-t)\right] + (ct-a)f(t)\right\}dt. = 0$.

This is true for all z. To get the integral to vanish for all z, set $\{\cdots\} = 0$.

So, $\quad -(t^2-t)f' - (2t-1)f + (ct-a)f = 0$.

Ie, $\quad \frac{f'}{f} = \frac{-2t+1+ct-a}{t(t-1)} = \frac{a-1}{t} + \frac{c-a-1}{t-1} \Rightarrow f(t) \propto t^{a-1}(1-t)^{c-a-1}$.

$w(z) = \int_C t^{a-1}(1-t)^{c-a-1}e^{zt}\,dt$ -(48)

In $(*)$, $[\cdots]_C = 0$ condition is: $\left[t^a(1-t)^{c-a}e^{zt}\right]_C = 0$ (49)

If $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$, $\int_C$ can be $\int_0^1$, so $w(z) = \int_0^1 t^{a-1}(1-t)^{c-a-1}e^{zt}\,dt$ -(50).

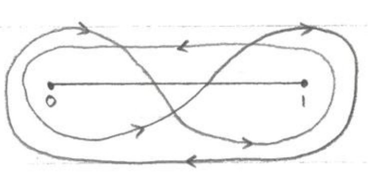
$0 \le t \le 1$ with "sensible" principal values.

Detailed calculation shows that $w(z) = \frac{\Gamma(c-a)\Gamma(a)}{\Gamma(c)} \cdot \Phi(a,c;z)$

Another choice for C is $[-\infty, 0]$. We have $u(a,c;z) = \int_0^\infty t^{a-1}(1-t)^{c-a-1}e^{zt}\,dt$.

This is the solution which is bounded as $z \to \infty$.

Pochhammer's Integral:



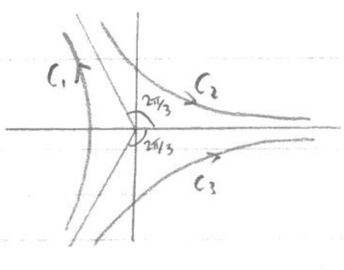
(c) Airy's equation.

$w'' - zw = 0$. So, $\frac{1}{wz}\cdot\frac{d^2w}{dz^2} = 1$.

We get $\left[-f(t)e^{zt}\right]_C + \int_C e^{zt}\{t^2f(t) + \frac{\partial f}{\partial t}\}dt = 0$ -(51).

So, $f(t) \propto \exp(-\frac{1}{3}t^3)$.

The standard solution: $w(z) = Ai(z) \equiv \frac{1}{2\pi i}\int_{C_1} e^{zt-\frac{1}{3}t^3}\,dt$ -(52)

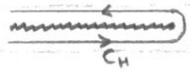


$Bi(z) = \frac{1}{2\pi}\left(\int_{C_2} - \int_{C_3}\right)$ -(53).

-More about $Ai(z)$, $Bi(z)$ later.

(d) Bessel functions; Schläfli's integral.

Recall Hankel's integral and contour: $\frac{1}{\Gamma(\gamma)} = \frac{1}{2\pi i}\int_{C_H} e^t t^{-\gamma}\,dt$.



Use this in Bessel series (20): $J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{r=0}^{\infty} \frac{(-1)^r (z/2)^{2r}}{\Gamma(\nu+r+1)\,r!}$

Use Hankel with $\gamma = \nu + r + 1$.... $= \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \int_{C_H} \exp\left(t - \frac{z^2}{4t}\right) t^{-\nu-1}\,dt$ - (54).

If $\nu = n \in \mathbb{Z}$ there is no need for the branch cut: $J_n(z) = \frac{1}{2\pi i}\oint \exp\left(\frac{1}{2}z\left(u - \frac{1}{u}\right)\right)\frac{du}{u^{n+1}}$ - (55), where, for $\oint$, simply go around the pole at $z=0$.

(55) is the formula for the nth coefficient in the Laurent expansion about $u=0$ of $\exp\left[\frac{1}{2}z\left(u-\frac{1}{u}\right)\right]$.

But this means, $\exp\left[\frac{1}{2}z\left(u-\frac{1}{u}\right)\right] = \sum_{-\infty}^{\infty} u^n J_n(z)$ - (56), so we have a generating function for $\{J_n(z)\}_{n\in\mathbb{Z}}$.

Take the contour in (55) to be the unit circle: $u = e^{i\theta}$.

Then, $J_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \exp(iz\sin\theta - in\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi} \cos(z\sin\theta - n\theta)\,d\theta$ - (57).

This is the Fourier coefficient in the series for $e^{iz\sin\theta}$, so

$e^{iz\sin\theta} = \sum_{-\infty}^{\infty} e^{in\theta} J_n(z) = J_0(z) + 2\sum_1^\infty J_{2n}(z)\cos 2n\theta + 2i\sum_1^\infty J_{2n+1}(z)\sin(2n+1)\theta$ - (58).

(57) and (58) are inverse Fourier Series formulae.

(e) Hypergeometric function.

$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \cdot \int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-z)^{-a}\,dt$ - (59).

This type of integral, involving $(t-z)^{-a}$ is called an Euler transform. This can be used in getting connection formula.

3. Asymptotic Expansions.

3.1. Introduction.

Refer to (50) of chapter 2, put $a=c=1$ - an integral for $u(1,1;z)$. Now call this $w(z)$.

$w(z) = \int_0^\infty \frac{e^{-zt}}{1+t}\,dt$, $Re(z)>0$. - (1). Clearly $w \to 0$ as $z \to \infty$.

$w(z) = \int_0^\infty e^{-zt}(1 - t + t^2 - t^3 + \dots)\,dt = \frac{1}{z} - \frac{1}{z^2} + \frac{2!}{z^3} - \frac{3!}{z^4} + \dots$ - (2).

There are problems: this is divergent $\forall z$; we have misused expansion for $(1+t)^{-1}$, as it is valid only for $|t|<1$.

But, take $w(10) = 0.1 - 0.01 + 0.002 - 0.0006 = 0.0914$, and the real answer, done numerically, is 0.0915.

Approach (2) more cautiously. $\frac{1}{1+t} = 1 - t + t^2 + \dots + (-t)^{n-1} + \frac{(-t)^n}{1+t}$ for any n (exact).

Now, with no divergence worries, $w(z) = \frac{1}{z} - \frac{1}{z^2} + \frac{2!}{z^3} - \dots + (-1)^{n-1}\cdot\frac{(n-1)!}{z^n} + \int_0^\infty \frac{(-t)^n e^{-zt}}{1+t}\,dt$.

The "remainder" term has modulus $< \left|\int_0^\infty t^n e^{-zt}\,dt\right| = \left|\frac{n!}{z^{n+1}}\right|$.

Using the first n terms of (2) gives an approximation of $w(z)$ such that the error is numerically less than the first term neglected. For large z, the terms at first decrease and then begin to increase when $n \simeq |z|$. Valuable information can be extracted from such an approximation, even though it eventually diverges. This is an example of an asymptotic expansion.

Exercise: Show that $w(z)$ satisfies $w' - w = -\frac{1}{z}$, so that $w(z) = e^{z}\int_z^\infty \frac{e^{-s}}{s}\,ds$.

Convergence vs. Asymptotics.

Let $S_n(z)$ be the nth partial sum of some series that claims to give information about some $f(z)$. Distinguish between:

(i) convergence; for fixed z, $|f(z) - S_n(z)| \to 0$ as $n \to \infty$ — (3)

(ii) asymptotics; for fixed n, $\left|\frac{f(z)}{S_n(z)}\right| \to 1$ as $z \to a$ — (4), where a is the limit point under consideration. Usually, but not necessarily, $a = \infty$, as in above example.

Usually $z = a$ is an essential singularity of $f(z)$, possibly even a branch point. It is very difficult to describe the behaviour of $f(z)$ as $z \to a$ from the Laurent series. Often, with power series, the result may hold only in sector $\alpha < r < \beta$ for some α, β. Idea is that $f(z)$ is unfamiliar, while the $S_n(z)$ are also essential singularities, but are familiar.

Recall: notation, O, o, $\sim$:
$$f(z) = O(g(z)) \Rightarrow \left|\frac{f(z)}{g(z)}\right| \to \text{const. as } z \to \infty.$$
$$f(z) = o(g(z)) \Rightarrow \left|\frac{f(z)}{g(z)}\right| \to 0 \text{ as } z \to \infty.$$
$$f(z) \sim g(z) \Rightarrow \left|\frac{f(z)}{g(z)}\right| \to 1 \text{ as } z \to \infty.$$

Poincaré's definition of an asymptotic expansion.

Definition: A sequence $\{\varphi_n\}$, $n = 0, 1, 2, \dots$ of functions is an asymptotic sequence as $z \to \infty$, in some sector (to be specified), if $\varphi_{n+1} = o(\varphi_n)$ $\forall n$.

Example: (i) $\varphi_n = z^{-n}$ - no sector required.

(ii) $\varphi_n = e^{-nz}$ - sector $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$

(iii) $\varphi_n = e^{-z^n}$ - sector shrinks to $\arg z = 0$.

(iv) $\varphi_n = \varphi_0 z^{-n}$ - φ_0 is prescribed - the ~~com~~ common case.

If $f(z)$ is a given function and $\{\varphi_n(z)\}$ is a given asymptotic sequence and $\exists$ constants $\{a_n\}$ such that $f(z) - \sum_0^N a_n \varphi_n(z) = o(\varphi_{N+1}(z))$ $\forall N$ — (5), then we write $f(z) \sim \sum_0^\infty a_n \varphi_n(z)$ as $z \to \infty$ — (6).

This definition makes no implication for the convergence of the RHS.

Say that $\sum a_n \varphi_n(z)$ is an asymptotic expansion of $f(z)$.

We can say something about the error committed if we stop after N terms, by replacing N by $N+1$ in (5): $f(z) - \sum_0^N a_n \varphi_n = a_{N+1}\varphi_{N+1} + o(\varphi_{N+1}) = O(\varphi_{N+1})$. — (7).

So the error at the Nth partial sum is of order of the first term omitted.

Given $f(z)$, usually implicitly in the sense of an integral, or the solution of a differential equation, the difficulty is to discover a suitable sequence $\{\varphi_n\}$. A clever choice of $\varphi_0(z)$ often leads to example (iv) above. Once the $\{\varphi_n\}$ have been selected, the coefficients exist and are unique and are determined recursively by

$$a_N = \lim_{z\to\infty} \frac{f(z) - \sum_{0}^{N-1} a_n \varphi_n(z)}{\varphi_N(z)} \quad -(8)$$

However, a series given by (6) does not uniquely specify the function $f(z)$. For if $f-g = o(\varphi_N)\ \forall N$, then f and g have the same asymptotic expansion.
Example: $g(z) = f(z) + e^{-kz}$, since $e^{-kz} = o(z^{-n})\ \forall n$.

Asymptotic power series: $f(z) \sim \varphi_0(z) \sum_{n=0}^{\infty} \frac{a_n}{z^n}$, $\frac{f(z)}{\varphi_0(z)} \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ $\quad -(9)$.

3.2. Elementary Properties.

Go back to (5). It is easy to show that adding two asymptotic expansions that use the same set $\{\varphi_n\}$ does as expected. Similarly for multiplication by scalars.
When we try multiplication, $\varphi_m(z)\varphi_n(z)$ will often not belong to $\{\varphi_n\}$. Similarly, with differentation, we don't know the relative order of $\{\varphi_n\}$.
So, from now on, restrict attention to case (9), with $\varphi_0(z)$ taken care of separately. Then there is a systematic theory of how asymptotic series can be manipulated.

If $f(z) \sim \sum a_n z^{-n}$ and $g(z) \sim \sum b_n z^{-n}$ as $z\to\infty$, then.
(i) the series can be added, and multiplied by scalars.
(ii) the series for $f(z)g(z)$ can be found by formal multiplication and rearrangement.
(iii) same applies to f/g, provided $b_0 \neq 0$.
(iv) the results from term-by-term differentiation hold.
(v) term-by-term Integration is valid in form: $\int_z^\infty \{f(z) - a_0 - \frac{a_1}{z}\}\,dz \sim \frac{a_2}{z} + \frac{a_3}{2z^2} + \frac{a_4}{3z^3} + \ldots$
Proofs of (i) – (v) are routine. (See Erdelyi, pp17-22).

Suppose that $f(z)$ is single-valued and analytic outside $|z|>R$, and $\exists$ a relation $f(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$ which holds for all $\arg z$. Then the series is a convergent series. For as $z\to\infty$, $f(z)\to a_0$, and $f(z)$ is thus non-singular, and has a power series expansion in powers of $1/z$. But this must be the series first quoted.
Similarly, if the power series starts not at $n=0$ but at some later power, it is simply a Laurent series of finite order.
Inevitably, genuine asymptotic expansions must work in restricted sectors. Usually, different series are valid in different sectors, with different $\varphi_0(z)$'s. This is called Stokes' Phenomenon. The frontiers between the sectors are called Stokes' Lines.

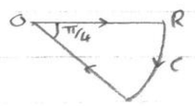
3.3. Expansion of Integrals.

Example (i): $w(z) = \int_0^\infty \frac{e^{-zt}}{1+t}\,dt$, $Re(z)>0$.
$w(z) = e^z f(z)$, where $f(z) = \int_z^\infty \frac{1}{s} e^{-s}\,ds$ (via: $1+t = \frac{s}{z}$).

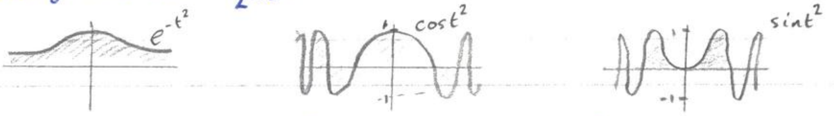
Do an integration by parts on $f(z)$:

$$f(z) = \left[-\frac{e^{-s}}{s}\right]_z^\infty - \int_z^\infty \frac{-e^{-s}}{-s^2}\,ds = \frac{e^{-z}}{z} - \int_z^\infty \frac{e^{-s}}{s^2}\,ds = \frac{e^{-z}}{z} - \left[-\frac{e^{-s}}{s^2}\right]_z^\infty + \int_z^\infty \frac{(-2)e^{-s}}{s^3}\,ds$$
$$= \frac{e^{-z}}{z} - \frac{e^{-z}}{z^2} - 2\int_z^\infty \frac{e^{-s}}{s^3}\,ds \sim e^{-z}\left\{\frac{1}{z} - \frac{1}{z^2} + \frac{2!}{z^3} - \frac{3!}{z^4} + \cdots\right\}, \text{ as before.}$$

It gives another form of the remainder which can be used to establish the asymptotic property.

Lemma: $\int_C e^{-s^2}\,ds$ on contour shown:  s-plane.

Show integral on curved part $\to 0$ as $R \to \infty$.
Put $s = e^{-i\pi/4}t$ on inclined part. $\int_0^\infty e^{it^2}\,dt = e^{i\pi/4}\int_0^\infty e^{-s^2}\,ds = e^{i\pi/4}\cdot\frac{\pi^{1/2}}{2} = \frac{(1+i)\pi^{1/2}}{2^{3/2}}$
$\int_0^\infty \cos t^2\,dt = \int_0^\infty \sin t^2\,dt = \frac{\pi^{1/2}}{2^{3/2}}$.



Example (ii): Fresnel integers: $C(z) = \int_0^z \cos t^2\,dt$, $S(z) = \int_0^z \sin t^2\,dt$.

Work with $f(z) = C(z) + i\,S(z)$.

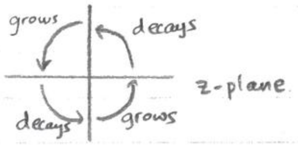
$f(z) = \int_0^z e^{it^2}\,dt = \int_0^\infty e^{it^2}\,dt - \int_z^\infty e^{it^2}\,dt$

$$\int_z^\infty e^{it^2}\,dt = \int_z^\infty \frac{1}{2it}\cdot 2it\,e^{it^2}\,dt = \left[\frac{e^{it^2}}{2it}\right]_z^\infty + \int_z^\infty \frac{e^{it^2}}{2it^2}\,dt = \frac{e^{iz^2}}{2iz} + \int_z^\infty \frac{-1}{4t^2}\cdot 2it\,e^{it^2}\,dt.$$

$$\cdots = -\frac{e^{iz^2}}{2iz}\left\{1 + \frac{1}{2iz^2} + \frac{1.3}{(2iz^2)^2} + \cdots + \frac{1.3\ldots(2n-1)}{(2iz^2)^n}\right\} + \underbrace{\frac{1.3\ldots(2n+1)}{(2i)^{n+1}}\int_z^\infty \frac{e^{it^2}}{t^{2n+1}}\,dt}_{\text{remainder}}$$

$f(z) \sim \frac{(1+i)\pi^{1/2}}{2^{3/2}} + \frac{e^{iz^2}}{2iz}\left\{1 + \frac{1}{2iz^2} + \frac{1.3}{(2iz^2)^2} + \cdots\right\}$, asymptotic expansion, for z real, positive.

For z real and negative, want $\int_0^z = \int_{-\infty}^z - \int_{-\infty}^0$, so do the same.
$f(z) \sim -\frac{(1+i)\pi^{1/2}}{2^{3/2}} + \frac{e^{iz^2}}{2iz}\cdot\{\text{same series}\}$.

For $z \in \mathbb{C}$, consider behaviour of e^{iz^2}:  z-plane.

The real and imaginary axes are both Stokes' lines.

Example (iii): Complementary error function: $\operatorname{erfc}(z) = \frac{2}{\pi^{1/2}}\int_z^\infty e^{-t^2}\,dt$. (See Olver, p.43, 67 for more).

Example (iv): Incomplete factorial function: $F(a,z) = \int_z^\infty e^{-t}\,t^a\,dt$.

Example (v): Debye function. (See example sheet 3).

3.4. Expansion of Integrals – Watson's Lemma.

This considers integrals of the form: $f(z) = \int_0^\infty e^{-zt^r} g(t)\,dt$, $r > 0$ $\quad -(10)$,
where $g(t)$ is regular in some neighbourhood of $t=0$ and has a Taylor series there: $g(t) = \sum_{n=0}^\infty c_n t^n$ $\quad -(11)$, not generally convergent in $[0,\infty)$.

Note: our introductory example in section 3.1 was of this form, with $r=1$ and $g(t) = (1+t)^{-1}$.

Watson's lemma is a formal statement of conditions under which the procedure we used in the example gives a valid asymptotic expansion.
Recall that we substituted (11) into (10) and integrated term by term (notwithstanding divergence). In the more general case (10), this gives $f(z) \sim \sum_{n=0}^{\infty} \frac{c_n}{r} \Gamma\left(\frac{n+1}{r}\right) z^{-(n+1)/r}$, $Re(z)>0$ -(12)
To do the integration, one needs $\int_0^\infty t^n e^{-zt^r} dt = z^{-(n+1)/r} \cdot \Gamma\left(\frac{n+1}{r}\right)$

(Other version: t^r is replaced with s, so exponential is e^{-zs})

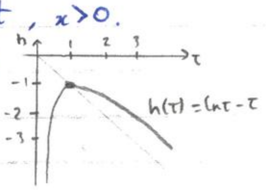
Examples: (i) $\frac{\pi^{1/2}}{2} \cdot \mathrm{erfc}(z) = \int_z^\infty e^{-t^2} dt = e^{-z^2} \int_0^\infty \underbrace{e^{-\frac{1}{4}\tau^2}}_{g(\tau) \text{ for Watson's Lemma.}} e^{-z\tau} d\tau$ (via $t = z + \frac{1}{2}\tau$)

(ii) Plasma dispersion function: $f(z) = \pi^{-1/2} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{z-t} dt$, $Im(z) > 0$, and its analytic continuation.
Leads to: $f(z) \sim \frac{1}{z}\left\{1 + \frac{1}{2z^2} + \frac{3}{4z^6} + \cdots\right\} - \varepsilon i \pi^{1/2} \cdot e^{-z^2}$,
where $\varepsilon = \begin{cases} 0 & \text{if } Im z > 0 \\ 1 & \text{if } Im z = 0 \\ 2 & \text{if } Im z < 0 \end{cases}$ (See example sheet 3, question 4)

3.5. Laplace's Method; Method of Steepest Descents. (saddle-point method)

Example (i): The factorial function, $x!$ $(=\Gamma(x+1))$. $f(x) := x! = \int_0^\infty t^x e^{-t} dt$, $x>0$.

Put $t = x\tau$: $f(x) = x^{x+1} \int_0^\infty e^{x(\ln\tau - \tau)} d\tau = x^{x+1} \int_0^\infty e^{xh(\tau)} d\tau$ -(13)



As $h(\tau)$ has a maximum at $\tau = 1$ with $h'(1) = 0$, $h''(1) = -1$,
write $\tau = 1 + s/x^{1/2}$
$x! = x^{x+1} e^{-x} \int_{-x^{1/2}}^{\infty} \exp\left\{x \ln\left(1 + \frac{s}{x^{1/2}}\right) - s x^{1/2}\right\} ds$ (no approximation so far).

For large x, "fit a parabola" to $\{\ldots\}$ and let $x \to \infty$.
$\frac{x!}{x^{x+\frac{1}{2}} e^{-x}} \simeq \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}s^2 + O(x^{-1/2})\right\} ds \to (2\pi)^{1/2}$ -(14)

So, $x! \sim (2\pi)^{1/2} \cdot x^{x+\frac{1}{2}} \cdot e^{-x}$ - Stirling's formula.

This looks like the leading term in an asymptotic expansion. To see how later terms can be found (in principle!), go back to (13), and take a different turn, by introducing a new definition of s, satisfying $\ln\tau - \tau = -1 - s^2$ -(15)
This turns (13) into: $f(x) = x^{x+1} \cdot e^{-x} \int_{-\infty}^{\infty} \underbrace{e^{-xs^2} \cdot \frac{d\tau}{ds}}_{\text{Watson's lemma.}} ds$

Equate terms: $\tau = 1 + 2^{1/2} s + \frac{2}{3}s^2 + \frac{1}{9 \cdot 2^{1/2}} s^3 - \frac{2}{135} s^4 + \ldots$
$\frac{d\tau}{ds} = 2^{1/2} + \frac{4}{3}s + \frac{1}{3 \cdot 2^{1/2}} s^2 - \frac{8}{135} s^3 + \ldots$

When substituted into the integral, only even powers of s contribute, giving (by Watson's lemma): $\Gamma(x+1) = x! \sim (2\pi)^{1/2} \cdot x^{x+\frac{1}{2}} \cdot e^{-x} \cdot \left\{1 + \frac{1}{12x} + \frac{1}{288x^2} + \cdots\right\}$. Exercise - fill in the details!

General case - Laplace's method.

Consider integrals of type: $f(x) = \int_\alpha^\beta g(t)\, e^{xh(t)}\, dt$ -(16), $x \in \mathbb{R}, x > 0$, $h(t)$ twice differentiable, $g(t)$ continuous.

Expect that for $x \to \infty$ the important contributions to (16) will be made near maxima of $h(t)$. This could occur at an end-point, with $h'(t) \neq 0$ - omit this possibility. Usually the maximum occurs at an internal point, with $h'(t) = 0$, and $h''(t) < 0$. If there are several of them, split $[\alpha, \beta]$ into subintervals, each containing one maximum. So, wlog, assume $\exists$ just one maximum, at $t = t_1$, so $h'(t_1) = 0$, $h''(t_1) < 0$. Follow procedure used above for $x!$

$$f(x) \sim g(t_1)\, e^{xh(t_1)} \int_{-\infty}^{\infty} \exp\left\{ \tfrac{x}{2}(t-t_1)^2 . h''(t_1) \right\} dt. \qquad \left[h(t) = h(t_1) + \tfrac{1}{2}(t-t_1)^2 h''(t_1) + \dots \right]$$

$$\text{So, } f(x) \sim g(t_1)\, e^{xh(t_1)} . \left\{ \frac{2\pi}{-xh''(t_1)} \right\}^{1/2} \quad -(17)$$

Example (ii). Tripos 1991/II/22.

$$I(a,b) = \int_0^\infty \exp\left\{ -\left(\tfrac{a}{t}\right)^{1/2} - \tfrac{t}{b} \right\} dt.$$

Intended that a be large and b be small.

Try $t = \lambda\tau$, then $\{..\} = -\left(\frac{a}{\lambda}\right)^{1/2} \tau^{-1/2} - \left(\frac{\lambda}{b}\right)\tau$. Choose λ to make the two coefficients the same, so $\lambda^3 = ab^2$. This turns I into:

$I = a^{1/3} b^{2/3} \int_0^\infty \exp\{-x(\tau^{-1/2} + \tau)\}\, d\tau$, where $x = (a/b)^{1/3}$

Let $h(\tau) = \tau^{-1/2} + \tau$. So $h'(t) = 0 \Rightarrow \frac{1}{2}\tau^{-3/2} - 1 = 0 \Rightarrow \tau = 2^{-2/3}$

$h(2^{-2/3}) = -2^{1/3} - 2^{-2/3} = -3.2^{-2/3}$

$h''(\tau) = -\frac{3}{4}\tau^{-5/2}$, so $h''(2^{-2/3}) = -\frac{3}{4}. 2^{5/3} = -3.2^{-1/3}$

So, $I \sim a^{1/3} b^{2/3} e^{-x.3.2^{-2/3}} . \left\{ \frac{2\pi}{x.3.2^{-1/3}} \right\}^{1/2}$, so $I \sim \left(\frac{16\pi^3 a b^5}{27} \right)^{1/6} . \exp\left\{ -\left(\frac{27a}{4b} \right)^{1/3} \right\}$

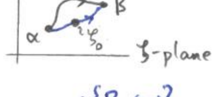
Method of Steepest Descent (Debye, 1908)

Extend Laplace's method to contour integrals.

Consider $z \in \mathbb{C}$. $f(z) = \int_C e^{zh(\zeta)} g(\zeta)\, d\zeta$ -(18), where a path C is prescribed, not closed, and we can't close it and apply residues.

Can we get an approximation as $|z| \to \infty$ in some direction, or in some sector? Also assume g and h are regular in such domain, as is needed in what follows.

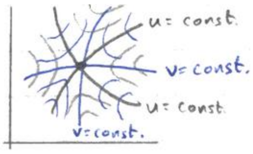
If we parametrise C by $t \in \mathbb{R}$, problem looks the same as Laplace's method. But we'd like to be able to distort C as follows:



Result appears to depend on choice of C, since $\max\{Re(z)\}$ could be different on different choices. Distort C to choose the case where the maximum is least.

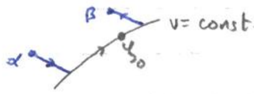
Start with z real and positive. Then $\max\{Re(h) : \zeta \in C\}$ will be different for each choice of C, but it will be least if h passes through a saddle-point. There, $h'(\zeta_0) = 0$, and so by the Cauchy-Riemann equations, both real and imaginary parts of h are stationary. Recall that u and v cannot have maxima or minima as they satisfy $\nabla^2 u = \nabla^2 v = 0$.

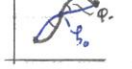
At $\zeta = \zeta_0$ we have $h'(\zeta_0) = 0$. Assume $h''(\zeta_0) \neq 0$.



The curves v= constant give the path of steepest decrease in u.

So when z is real and positive, $|e^{zh}| = e^{zu}$ and decreases as rapidly as possible either side of ζ_0 along the desired path. So the desired path is given by $v=$ const.

Example: 

So far, z is real and positive. Shift $\arg z$ a little from zero and repeat. Shift $e^{i\varphi}$ factor from z to h. The stationary point ζ_0 is unaltered. 

Result (18) is formally unaltered initially. But by the time the phase factor has reached π, hills $\leftrightarrow$ valleys! At some point, we would have to make a change in the selection of ζ_0. These would be a succession of discrete structural changes, at each stage requiring a new ζ_0, but finally reverting to the original ζ_0.
The changes produce Stokes' lines.
For examples, see section 3.7.

3.6. Method of Stationary Phase.

Note: These two methods coincide in $\mathbb{C}$. They differ in $\mathbb{R}$.

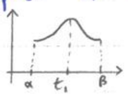
Return to real variable theory and consider: $f(x) = \int_\alpha^\beta e^{ixh(t)} g(t)\,dt$ -(19).
Looks similar to (16), except for factor i.
To get an idea, start with Fourier integral, so $h(t)$ becomes just t.
$I(x) = \int_\alpha^\beta e^{ixt} q(t)\,dt$, and ask about the limit $x \to \infty$.

Riemann-Lebesgue lemma. (Olver P.74).
Provided $q(t)$ is continuous, have $I(x) = o(1)$ as $x \to \infty$
If, further, $q(t)$ is differentiable, an integration by parts: $I(x) = [\cdots]_\alpha^\beta + \int_\alpha^\beta \cdots = O(x^{-1}) + O(x^{-1})$.
If twice differentiable, same again shows $I(x) = O(x^{-2})$. If infinitely differentiable, have an asymptotic expansion.

Return to (19). If h, g are continuous and h is monotonic, could convert (19) to:
$f(x) = \int_{h(\alpha)}^{h(\beta)} e^{ixh} G(h)\,dh$, where $G(h)$ incorporates $(h')^{-1}$.
Riemann-Lebesgue lemma tells us that $f(x) = o(1)$ as $x \to \infty$. Get stronger results if h is differentiable, etc. Rapid oscillations are cancelling out.

If $h(t)$ is not monotonic, the above would apply in sections, but larger contributions come from the neighbourhoods of turning points - $G(h)$ would be large.
Intuitively, the oscillations "stay in phase" near where $h'(t) = 0$. To work this out,

suppose h is at least twice differentiable and has just one stationary point.



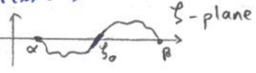
$h'(t_1)=0,\ h''(t_1)<0.$

As in approach to factorial function, expand $h(t)$ about $t=t'$, and let range of integration extend to $(-\infty,\infty)$. $f(x)\sim\int_{-\infty}^{\infty}\exp\{ixh(t_1)+\frac{x}{2}i(t-t_1)^2h''(t_1)\}\,g(t_1)\,dt.$

Let $s=t-t_1$, $f(x)\sim e^{ixh(t_1)}g(t_1)\int_{-\infty}^{\infty}\exp\{ixh''(t_1)s^2\}\,ds\sim\left\{\frac{2\pi}{xh''(t_1)}\right\}^{1/2}g(t_1)\exp\{ixh(t_1)+i\pi/4\}$ - (20),

using $\int_{-\infty}^{\infty}e^{i\theta s^2}ds=\left(\frac{\pi}{\theta}\right)^{\frac{1}{2}}.e^{i\pi/4}$. (See earlier). Outcome is $f(x)\propto x^{-1/2}$

If we looked at this in the complex domain:



Applications to Group Velocity.

Have a system governed in space and time by Pdes with constant coefficients. These have wavelike solution. (Keep to 1 spatial dimension).

$f(x,t)\propto e^{i(kx-\omega t)}$. ω is a function of k. This follows from the definition of the Pde.

Initial value problem, $f=f_0(x,0)$. We construct Fourier Transform:

$f_0(x,0)=\int_{-\infty}^{\infty}F(k)\,e^{ikx}\,dk.$

$f_0(x,t)=\int_{-\infty}^{\infty}\underbrace{F(k)}_{g(k)}\,\underbrace{e^{i(kx-\omega t)}}_{h(k)}\,dk.$ Note: this oversimplifies - there might be several nodes!

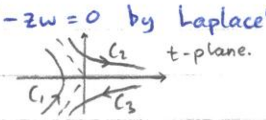
Ask about $f(x,t)$ for large x,t. Dominant contributions will be determined by $\frac{d}{dk}\{kx-\omega(k)t\}=0$. That is, $x=\frac{d\omega}{dk}t$ - (21). ($\frac{d\omega}{dk}\equiv$ group velocity).

3.7. Examples of the Method of Steepest Descents.

(a) Airy's Equation and Function.

Recall results from section 2.8 (c) for $w''-zw=0$ by Laplace's method:

$w(z)=Ai(z)=\frac{1}{2\pi i}\int_{C_1}\exp(zt-\frac{1}{3}t^3)\,dt.$ - (22)



Not suitable for saddle-point method in this form. Put $t=z^{1/2}\zeta$, giving

$Ai(z)=\frac{z^{1/2}}{2\pi i}\int\exp\{z^{3/2}(\zeta-\frac{1}{3}\zeta^3)\}\,d\zeta.$

This is now of the form required for (16), but with $z^{3/2}$ replacing z, and $g(\zeta)=1$, and $h(\zeta)=\zeta-\frac{1}{3}\zeta^3$.

Saddle-points occur at $h'(\zeta)=0$, ie $\zeta=\pm1$, at which $h(\zeta)=\pm\frac{2}{3}$, $h''(\zeta)=-2\zeta=\mp2$.

Start with z real, $z>0$. Then the contours in the ζ-plane are similar to the t-plane, and we can make C_1 go through $\zeta=-1$. Now apply (18) to get:

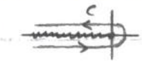
$Ai(z)\sim\frac{z^{1/2}}{2\pi i}\cdot e^{-\frac{2}{3}z^{3/2}}\cdot\left\{\frac{2\pi}{-2z^{3/2}}\right\}^{1/2}\sim\frac{z^{-1/4}}{2\pi^{1/2}}\cdot\exp(-\frac{2}{3}z^{3/2})$ - (23).

Now let z become complex, say $z=re^{i\theta}$. The ζ-plane differs from the t-plane by being rotated through $\frac{1}{2}\theta$, the saddle-point staying fixed. For small θ this makes no difference, but when θ reaches $2\pi/3$ the situation is such that C_1 has to be deformed to pass through the other saddle-point, and the exponent in (23) has to flip over to $(+\frac{2}{3}z^{3/2})$. Note that when $\theta=\pi$ is reached, the exponential in (23) is purely oscillatory, while at $\theta=0$ it was decaying.

(b) Bessel's Equation and Functions.

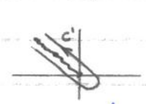
Start from Schläfli's integral, equation (54) in section 2.8(d).

$J_\nu(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^\nu \int_C \exp\left\{t - \frac{z^2}{4t}\right\} t^{-\nu-1}\, dt.$ - (24) :

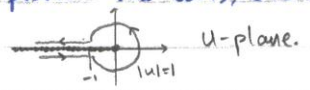


This needs preliminary transformations:

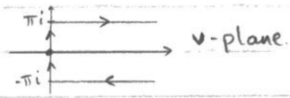
(i) $t = \frac{1}{2}zu$. $J_\nu(z) = \frac{1}{2\pi i}\int_{C'} \exp\left\{\frac{1}{2}z(u - \frac{1}{u})\right\} u^{-\nu-1}\, du$:



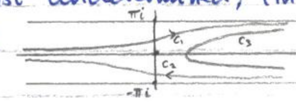
Branch cut is now inclined at angle $\arg(z)$ to the negative real axis. Take $\mathrm{Re}(z) > 0$ from now on; branch cut can be swung back to the negative real axis. (If $\mathrm{Re}(z) < 0$, it has to go to the positive real axis, etc.) Then redefine contour to include unit circle as shown:



(ii) $u = e^v$. $J_\nu(z) = \frac{1}{2\pi i}\int_C \exp(z\sinh v - \nu v)\, dv$ -(25).



But, on substituting integrals of form (25), with C at first undetermined, (in manner of Laplace's method), reveals several possibilities ($\mathrm{Re}(z) > 0$):



Clearly $J_\nu(z) = \frac{1}{2\pi i}\int_{C_3} \cdots$, but C_1 and C_2 give standard solutions called Hankel functions, $H^{(1)}_\nu(z)$ and $H^{(2)}_\nu(z)$, $H^{(i)}_\nu(z) = \frac{1}{\pi i}\int_{C_i} \cdots, (i = 1, 2)$. So $J_\nu(z) = \frac{1}{2}\left\{H^{(1)}_\nu(z) + H^{(2)}_\nu(z)\right\}$.

One can now apply the method of steepest descents. In the strips in which C_1 and C_2 are confined, available saddle-points are found from $h(v) = \sinh v$, $h'(v) = \cosh v$, so $\cosh v = 0$ is required, giving $v = \pm\frac{1}{2}\pi i$, at which $h(v) = h''(v) = \pm i$ respectively. The steepest descents paths are very much like those shown for C_1, C_2. Using the steepest descents formula for the leading term of the series then gives:

$H^{(1)}_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \exp\left\{i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right\}$, $H^{(2)}_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \exp\left\{-i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right\}$

from which, finally, $J_\nu(z) \sim \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$. -(26)

Detailed work on the above lines can yield the form of the asymptotic expansion for which (26) forms the "$\varphi_0(z)$" term. This is set out in several text-books, but the details sometimes seem confusingly different!

Another use for (25) is to find asymptotic formula for $J_\nu(z)$ where both z and ν are to be large, say $\nu = kz$ for fixed k. In this case, the whole exponent of (25) has to be regarded as $zh(v)$, so $h(v) = \sinh v - kv$. This gives a completely different set of results, which show different behaviour depending on whether $k <, =, > 1$. (See Watson's "Bessel Functions").

3.8. The Liouville-Green Solutions.

A new question - where functions arise as the solution of a differential equation, can we get directly to an asymptotic expansion without going via an integral?

Again consider $w'' + p(z)w' + q(z)w = 0$ -(26), where there is a singularity at $z \to \infty$.

One approach is to try an inverse power series combined with an exponential: $w(z) \sim e^{\lambda z} . z^{-\alpha} \sum_{r=0}^{\infty} c_r z^{-r}$.

Substitute, rearrange, get a quadratic equation for λ.

The Liouville-Green Solutions (also called the WKB method)

Step 1 (convenient but not necessary): Get rid of the first order term in (26) by changing variables: $w(z) = W(z)\exp\{-\frac{1}{2}\int^z p(\zeta)d\zeta\} \Rightarrow W''(z) + Q(z)W(z) = 0$, where $Q(z) = q - \frac{1}{2}p' - \frac{1}{4}p^2$. Assume this has been done, and revert to small letters: $w'' + q(z)w = 0$.

Step 2: Look for solutions of the form: $w(z) = a(z)e^{i\theta(z)}$ – (27).

If q were constant, this would work with $a(z) = $ const, $\theta(z) = \pm q^{1/2}z$. Suppose $q(z)$ varies slowly with z. Then $a(z)$ will also vary slowly (in $\mathbb{R}$). $q(z)$ varies only slightly over an interval $2\pi q^{1/2}$.

From (27), $w' = (a' + ai\theta')e^{i\theta}$, $w'' = (a'' + 2a'i\theta' + ai\theta'' - a\theta'^2)e^{i\theta}$.

The differential equation requires $a'' + 2a'i\theta' + ai\theta'' + (q - \theta'^2)a = 0$.

Impose the condition that $q - \theta'^2 = 0$.

In the case $q > 0$, have: $\theta(z) = \pm\int^z \{q(z')\}^{1/2}dz'$ – (28).

$a(z)$ must satisfy: $a'' + 2a'i\theta' + i\theta''a = 0$ – (29)

So far, no approximation! We still have a second order equation for $a(z)$.

The approximation consists of neglecting a'' (verify later)

If so, $2a'/a + \theta''/\theta' = 0$, so $a^2\theta' \simeq$ constant. So $a \propto \{\theta'\}^{-1/2} \propto \{q(z)\}^{-1/4}$

We have solutions: $w(z) = q^{-\frac{1}{4}}\{A\exp(i\int^z q^{1/2}d\zeta) + B\exp(-i\int^z q^{1/2}d\zeta)\}$ – (30),

with A, B constants.

These are the Liouville-Green/WKB/WKBJ solutions.

To check domain of validity, work with the case $q(z) = O(|z|^n)$ as $z \to \infty$.

Then, $\theta' = O(|z|^{n/2})$, $a = O(|z|^{-n/4})$. The first term of (29) is $O(|z|^{-2-n/4})$; the other two terms are $O(|z|^{\frac{1}{2}n-1-\frac{1}{4}n})$. The approximation is justified if $\frac{n}{2} - 1 > -2$, ie $n > -2$.

This is exactly the converse of the condition that the equation has a regular singularity at $z \to \infty$. For validity the equation must have an irregular singularity at $z \to \infty$.

Suggests that the status of (30) is as an asymptotic expansion as $z \to \infty$. (27) is approximate for $q > 0$. If $q < 0$, the method still works, with exponents real – use e^θ instead of $e^{i\theta}$.

Examples: (i) Bessel's equation: $w'' + \frac{1}{z}w' + (1 - \frac{\nu^2}{z^2})w = 0$.

Transform: $w(z) = W(z)z^{1/2}$. Get $W'' + \underbrace{\{1 + \frac{1/4 - \nu^2}{z^2}\}}_{q(z)}W = 0$.

$(28) \Rightarrow \theta(z) = \pm\int^z (1 + \frac{1/4 - \nu^2}{z^2})^{1/2}dz$.

For large $|z|$, $\frac{d\theta}{dz} \simeq \pm 1$, $a(z) = (1 + \frac{1/4 - \nu^2}{z^2})^{-1/4}$, $w(z) = z^{-\frac{1}{2}}(1 + \frac{1/4 - \nu^2}{z^2})^{-\frac{1}{4}}(Ae^{iz} + Be^{-iz})$

(ii) Hermite's Equation: $w'' - 2zw' + 2nw = 0$.

Transform: $w(z) = W(z)e^{\frac{1}{2}z^2}$. Get $W'' + (2n+1-z^2)W = 0$ (Quantum mechanics – simple harmonic oscillator).

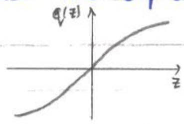
$q(z) = 2n+1-z^2$. For large $|z|$, $q(z) \simeq -z^2$, $\theta(z) \simeq \pm\frac{i}{2}z^2$. $\exp(i\theta) \simeq \exp(\mp iz^2)$, etc.

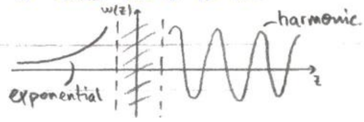
(iii) Airy's Equation: $w'' - zw = 0$.

$q(z) = -z$. For large z, $z > 0$, $w \sim z^{-\frac{1}{4}}\exp\{\pm\frac{2}{3}z^{3/2}\}$; $z < 0$, $w \sim z^{-\frac{1}{4}}\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\{\frac{2}{3}|z|^{3/2}\}$.

Nature of this asymptotic expansion (cf (30)).

One interpretation is as the leading term of an asymptotic expansion as $z \to \infty$.
Another approximation is to introduce a large parameter λ : $w'' + \lambda^2 q(z) w = 0$ -(32),
and conceive of the solution as $w(z;\lambda)$ as $\lambda \to \infty$.
(30) becomes $w(z;\lambda) = q^{-\frac{1}{4}} \{ A \exp(\lambda i \int^z q^{1/2} d\xi) + B \exp(-\lambda i \int^z q^{1/2} d\xi) \}$ -(33)
In this setting $w(z;\lambda)$ is being considered for a range of values of λ, over the full range of z, except that places where $q=0$ have to be excluded by a suitable interval. This interval gets smaller as λ increases. The expectation is that (33) is the leading term in an asymptotic expansion wrt λ.





Example: Construct the DE of which (33) is the solution.

It is: $w'' + \{ \lambda^2 q + \underbrace{\left(\frac{1}{4} \cdot \frac{q''}{q} - \frac{5}{16} \left(\frac{q'}{q} \right)^2 \right)}_{\text{The limit } \lambda \to \infty \text{ leaves this term negligible.}} \} w = 0$ -(34)

4. Transform Methods

4.1. The Fourier Transform.

Given $f(x)$, $-\infty < x < \infty$, define the Fourier Transform (FT) by: $F(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$ -(1)
This requires that the integral in (1) be convergent. If so, we have the Fourier Inversion formula: $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$ -(2)

Check list of properties.

(a) Duality: $f \leftrightarrow F$, $x \leftrightarrow k$.
(b) Shifting: FT of $f(x-a)$ is $e^{-ika} F(k)$, FT of $e^{i\lambda x} f(x)$ is $F(k-\lambda)$.
(c) Derivative: $f'(x)$ has FT $ikF(k)$.
(d) Convolution: $h = f * g$ is defined by $h(x) = \int_{-\infty}^{\infty} f(y) g(y-x) dy$. FT of $f*g$ is $F(k)G(k)$.
(e) Corollaries: eg, Parseval's Theorem.

4.2. The Laplace Transform.

The Laplace Transform (LT) of $f(t)$ is defined as: $F(p) = \int_0^{\infty} e^{-pt} f(t) dt$ -(3)
Note: t is often "time."
Here, p may be complex, but $Re(p) > \gamma$ is required, where γ is whatever is required for (3) to converge.
Many functions, such as t^2, t^n, e^{at}, have an LT but not an FT. Some functions do not have an LT, eg e^{t^2}.

Note that (3) only represents $f(t)$ in $t \geq 0$. Any $f(t)$ $(-\infty < t < \infty)$ is replaced by $H(t) f(t)$, where $H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$.

Examples: $f(t)=1$, $F(p)=\int_0^\infty e^{-pt}\,dt = 1/p$.

$f(t)=e^{at}$, $F(p)=\int_0^\infty e^{(a-p)t}\,dt = 1/p-a$, $Re(p) > Re(a) = $ "γ".

$f(t)=e^{i\omega t}$, $F(p)=1/p-i\omega$.

$f(t)=\cosh at$, $F(p) = p/p^2-a^2$; $f(t)=\sinh at$, $F(p)=a/p^2-a^2$.

$f(t)=\cos\omega t$, $F(p)= p/p^2+\omega^2$; $f(t)=\sin\omega t$, $F(p)=\omega/p^2+\omega^2$.

$f(t)=\delta(t-t_0)$, $t_0>0$, $F(p)=\int_0^\infty \delta(t-t_0)\,e^{-pt}\,dt = e^{-pt_0}$. If $t_0\to 0^+$, we have $F(p)=1$.

$f(t)=t^n$, $F(p)=\int_0^\infty t^n e^{-pt}\,dt = -p^{n-1}\int_0^\infty \tau^n e^{-\tau}\,d\tau = \frac{\Gamma(n+1)}{p^{n+1}}$, $Re(n)>-1$

General Properties of the Laplace Transform.

(i) Linearity.

(ii) change of scale: LT of $f(\alpha t)$ is $\frac{1}{\alpha}F(p/\alpha)$ } trivial.

(iii) derivative: $\mathcal{L}\left(\frac{df}{dt}\right) = \int_0^\infty e^{-pt}\cdot\frac{df}{dt}\,dt = \left[e^{-pt}.f\right]_0^\infty - \int_0^\infty (-pe^{-pt})f\,dt = pF(p) - f(0)$ - (4).

Assume $e^{-pt}f(t)\to 0$ as $t\to\infty$.

Similarly, $\mathcal{L}\left(\frac{d^2f}{dt^2}\right) = p^2F(p) - pf(0) - f'(0)$, $\mathcal{L}(f^{(n)}) = p^nF(p) - \{p^{n-1}f(0) + p^{n-2}f'(0) + \dots + f^{(n-1)}(0)\}$ - (5)

(iv) $\mathcal{L}\left(\int_0^t f(u)\,du\right) = \frac{F(p)}{p}$ - (6) [Converse of (4)].

(v) Shifting theorems: (a) $\mathcal{L}(e^{at}f(t)) = F(p-a)$. - (7)

(b) Let $g(t) = \begin{cases} f(t-a), & t>a>0 \\ 0 & 0<t<a.\end{cases}$ $\mathcal{L}(g(t)) = \int_a^\infty e^{-pt}f(t-a)\,dt = \int_0^\infty e^{-p(\tau+a)}f(\tau)\,d\tau = e^{-ap}.F(p)$. - (8)

(vi) $\mathcal{L}[t^nf(t)] = (-1)^n.\frac{d^n}{dp^n}F(p)$ - (9). $[(d/dp)^n$ on (4)].

(vii) Initial value theorem: $\lim_{t\to 0} f(t) = \lim_{p\to\infty} pF(p)$, if both limits exist. - (10). [Using (4) with $p\to\infty$]

(viii) Final value theorem: $\lim_{t\to\infty} f(t) = \lim_{p\to 0} pF(p)$, if both limits exist. - (11).

(ix) Convolution Theorem.

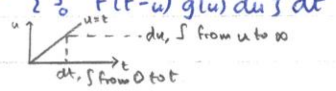
Given $f(t)$, $g(t)$ with LT's $F(p)$, $G(p)$, what function of t has LT $= F(p)\,G(p)$?

$$F(p)G(p) = \int_0^\infty e^{-pv}f(v)\,dv\int_0^\infty e^{-pu}g(u)\,du = \int_0^\infty\int_0^\infty e^{-p(v+u)}f(v)\,g(u)\,du\,dv.$$

$$= \int_0^\infty g(u)\left\{\int_0^\infty e^{-p(u+v)}f(v)\,dv\right\}du = \int_0^\infty g(u)\left\{\int_{t=u}^\infty e^{-pt}f(t-u)\,dt\right\}du$$

$$= \int_0^\infty e^{-pt}\left\{\int_0^t f(t-u)\,g(u)\,du\right\}dt = \mathcal{L}(f*g) \quad \text{- (12)}$$

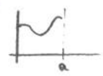
↑ using :



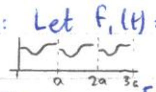
Range of $\int_0^t \cdots du$ is confined by requirement that the arguments of $f(t-u)$ and $g(u)$ must be ≥ 0.

(x) differentiation wrt a parameter: Given $f(t,a)$ with LT$=F(p,a)$, have $\mathcal{L}\left(\frac{\partial f}{\partial a}\right) = \frac{\partial F}{\partial a}$.

(xi) transform of a periodic function: Let $f_1(t) = 0$, except in $0\leq t<a$: f_1 is



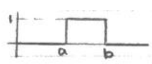
Let $f_2(t) = \sum_{n=0}^\infty f_1(t-na)$, so f_2 is



By shifting theorem, $F_2(p) = \sum_{n=0}^\infty e^{-nap}F_1(p) = \frac{F_1(p)}{1-e^{-ap}} = F_1(p).\underbrace{\frac{1}{1-e^{-ap}}}_{G(p)}$ - (13)

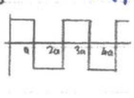
As suggested by the convolution theorem, $G(p)$ is LT of $\sum_{n=0}^\infty \delta(t-na)$.

Examples: (i) $f(t)$ as shown:



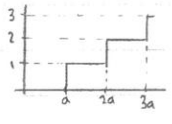
$F(p) = \frac{e^{-ap}-e^{-bp}}{p}$

(ii) square wave function:



$F(p) = \frac{\tanh(ap/2)}{p}$

(iii) staircase function:



$F(p) = \frac{e^{-ap}}{p(1-e^{-ap})}$

(iv) $f(t) = J_0(t) = \sum_{m=0}^{\infty} \frac{(-t^2/4)^m}{(m!)^2}$

$$F(p) = \sum_{m=0}^{\infty} \frac{(-1/4)^m}{(m!)^2} \int_0^{\infty} e^{-pt} \cdot t^{2m}\, dt = \sum_{m=0}^{\infty} \frac{(-1/4)^m\,(2m)!}{(m!)^2} \cdot \frac{1}{p^{2m+1}}$$

$$= \frac{1}{p} \sum \frac{(-1/2)^m\,(2m-1)(2m-3)\ldots}{m!\; p^{2m}} \quad \text{(Legendre's duplication formula)}$$

$$= \frac{1}{p}\left(1 + \frac{1}{p^2}\right)^{-1/2} \quad \text{(Binomial)}, \quad = (p^2+1)^{-1/2}.$$

$J_0(at)$ has LT $= (p^2+a^2)^{-1/2}$

$\partial/\partial a$ of this, then set $a=1$: can get LT of $J_0'(t)$, etc.

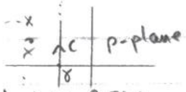
For $J_n(at)$, $F(p) = \frac{1}{a^n} \frac{\{(p^2+a^2)^{1/2} - p\}^n}{(p^2+a^2)^{1/2}}$.

4.3. The Bromwich Inversion Formula.

Given $F(p)$ which purports to be the LT of some $f(t)$, can we recover $f(t)$? This is often the final step in applications, eg DEs. If $f(t)$ exists, it is unique. [Lerch's Theorem: $F(p) \equiv 0 \Rightarrow f(t) = 0$]. One approach is to build up a "dictionary" of LTs, together with properties, eg shifting. Is there a more systematic method, as with Fourier Transform?

$f(t) = 0$ for $t<0$. As noted in section 4.1, if $f(t)$ has an LT, then $\exists\, \gamma \in \mathbb{R}$ such that $F(p)$ converges for $\mathrm{Re}(p) > \gamma$, and then $F(p)$ is regular for $\mathrm{Re}(p) > \gamma$. When presented with an $F(p)$, we must check whether this is so. If not, $F(p)$ cannot be an LT.

Now consider: $I = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} F(p)\, dp = \int_{\gamma-i\infty}^{\gamma+i\infty} \int_{t=0}^{\infty} e^{pt-pt'} f(t')\, dt'\, dp$. (Put $p = \gamma + iy$).



Singularities of $F(p)$ to left of contour.

So, $I = i e^{\gamma t} \int_{y=-\infty}^{\infty} \int_{t'=-\infty}^{\infty} e^{iy(t-t')} \{e^{-\gamma t'} f(t')\}\, dy\, dt'$,

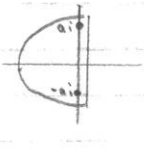
where we extend the t'-integration to run from $-\infty$ to ∞: $f(t') = 0$, $t'<0$.

The double integral is just the Fourier Transform and its inversion of $\{e^{-\gamma t} f(t)\}$ with FT variable y. So this gives $2\pi(e^{-\gamma t} f(t))$, so $I = 2\pi i f(t)$.

Finally, we have: $f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} F(p)\, dp$ — (14) This is the Bromwich Inversion Formula.

The contour is called the Bromwich contour. In applying (14) to $t<0$, we can close the contour in the right hand half plane, so we get $f=0$ as it should. To make use of (14) for $t>0$, we would like to continue $F(p)$ analytically into $\mathrm{Re}(p) < \gamma$. But this may not be possible. If impossible, (14) may be true but difficult to make use of. Commonly, $F(p)$ can be continued. If it is meromorphic (isolated poles only), we can use residues to compute $f(t)$. In other cases, branch points occur in $\mathrm{Re}(p) < \gamma$, and we have to do what we can.

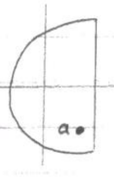
Examples: (i) $F(p) = \frac{p}{p^2+a^2}$. Simple poles: $p = \pm ia$, $\gamma = 0^+$.



Residues of $\frac{e^{pt} \cdot p}{p^2+a^2}$ at $\pm ia$ are $\frac{1}{2} e^{\pm iat}$.

$f(t) = \frac{1}{2} e^{iat} + \frac{1}{2} e^{-iat} = \cos at$.

(ii) $F(p) = \frac{1}{(p-a)^2}$. Double pole at $p=a$.



To get residue at $p=a$ of $\frac{e^{pt}}{(p-a)^2}$, use Taylor expansion of e^{pt} about $p=a$: $\frac{e^{pt}}{(p-a)^2} = \frac{e^{at}}{(p-a)^2} \left\{1 + (p-a)t + \frac{1}{2!}(p-a)^2 t^2 + \ldots\right\}$

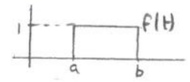
$\Rightarrow$ coefficient of $(p-a)^{-1}$ is $e^{at} \cdot t$, so $f(t) = te^{at}$

(iii) $F(p) = \dfrac{e^{-ap} - e^{-bp}}{p}$, $b > a > 0$, real. $F(p)$ has no singularities.

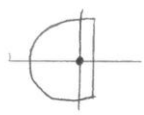
If $t < a$, close the contour to the right : $f(t) = 0$.

If $a < t < b$, close contour to the right for e^{-bp}/p, and to the left for e^{-ap}/p : gives residue 1, so $f(t) = 1$.

If $t > b$, close contour to left for both terms: 2 residues, ± 1 - cancel, so $f(t) = 0$.



(iv) $F(p) = p^{-n-1}$ ($n \in \mathbb{N}$). Close contour to the left.



Write $\dfrac{e^{pt}}{p^{n+1}} = \dfrac{1}{p^{n+1}} + \dfrac{t}{p^{n}} + \dfrac{t^2}{2p^{n-1}} + \dots + \dfrac{t^n}{n!\,p} + \dots$

$\mathrm{Res}\,(p^{-n-1}, 0) = t^n/n!$, as expected from section 4.2.

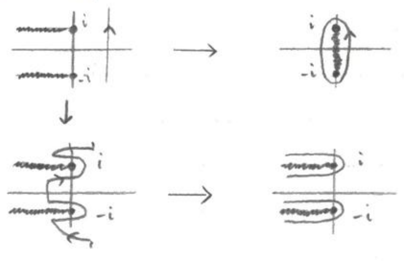
if $n \notin \mathbb{N}$, we get a branch point. Not able to close the contour.

$f(t) = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{e^{pt}}{p^{n+1}}\, dp$. Deform C as shown. Take t real and positive, $p = \xi/t$, ξ a new variable.

$f(t) = t^n . \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{e^{\xi}}{\xi^{n+1}}\, d\xi = \dfrac{t^n}{\Gamma(n+1)}$, by Hankel expression for $\dfrac{1}{\Gamma(z+1)}$.

This still works for $t > 0$, although t not real.

(v) $F(p) = (p^2+1)^{-1/2}$ (So $f(t) = J_0(t)$).



- asymptotic expansion of $J_0(t)$.

4.4. Use of Transforms in Differential Equations.

Example (i): $x'' - 3x' + 2x = 4e^{2t}$, where $x(0) = -3$, $x'(0) = 5$

Take the LT of the equation, including the initial condition:

$[p^2 X - p(-3) - 5] - 3[pX + 3] + 2X = \dfrac{4}{p-2}$.

Tidy up algebra: $X(p) = \dfrac{14-3p}{(p-1)(p-2)} + \dfrac{4}{(p-1)(p-2)^2}$

Find residues of $e^{pt} X(p)$: $X(p) = -7/_{p-1} + 4/_{p-2} + 4/_{(p-2)^2}$

Then $x(t) = -7e^{t} + 4e^{2t} + 4te^{2t}$

Alternatively, $\ddot{x} - 3\dot{x} + 2x = f(t)$. Can we get a Green's function with $f(t)$ unspecified, $x(0) = x_0$, $x'(0) = x_1$.

$(p^2 - 3p + 2)\, X = F(p) + (px_0 + x_1) - 3x_0$.

$X(p) = G(p)\,\{F(p) + (px_0 + x_1) - 3x_0\}$, where $G(p) = \dfrac{1}{p^2 - 3p + 2}$.

$x(t) = g(t) * \{\underbrace{f(t)}_{\text{P.I.}} + \underbrace{x_0\,\delta'(t) + (x_1 - 3x_0)\,\delta(t)}_{\text{C.F.}}\}$

Eg: $x'' + k^2 x = f(t)$. Get $g(t) = k^{-1} \sin kt$

Example (ii) : Simultaneous ODEs.

$$\left.\begin{aligned} \dot{x} + x + 2y &= e^{2t} \\ 2\dot{x} + \dot{y} - x &= 0 \end{aligned}\right\} \quad x = y = 0 \text{ at } t = 0.$$

Take LT: $X(p) = \dfrac{p}{(p-1)(p-2)^2}$, $Y(p) = \dfrac{1-2p}{(p-1)(p-2)^2}$

Example (iii): PDEs - the heat conduction equation.

$u(x,t)$ in $x, t \geq 0$. $\dfrac{\partial u}{\partial t} = \lambda \dfrac{\partial^2 u}{\partial x^2}$, with $u(x,0) = 0$, $u(0,t) = \varphi(t)$, u bounded as $x \to \infty$.

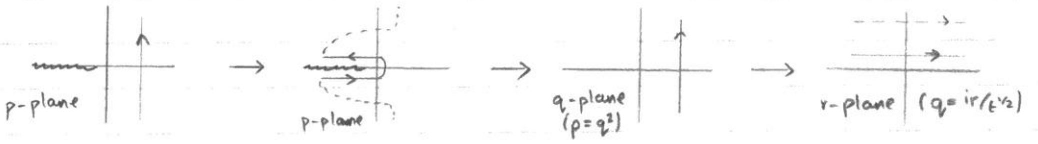
Take LT wrt t, leaving x unaltered. Call this $U(x,p)$. Let $\mathcal{L}\{\varphi(t)\} = \Phi(p)$.

Get: $pU = \lambda \dfrac{\partial^2 U}{\partial x^2}$, where $U(0,p) = \Phi(p)$.

This is an ODE wrt x, with p a parameter. Discard the growing solution as $x \to \infty$.

$U(x,p) = \Phi(p) \exp\{-(p/\lambda)^{1/2} x\}$. $\quad u(x,t) = \varphi(t) * g(x,t)$.

$G(x,p) = \exp\{-(p/\lambda)^{1/2} x\}$, $\quad g(t) = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \exp\{pt - (p/\lambda)^{1/2} x\}\, dp$



$$g(x,t) = \frac{1}{2\pi i} \int \exp\left\{q^2 t - \frac{qx}{\lambda^{1/2}}\right\} 2q\, dq \qquad (p = q^2) \quad \text{Let } q = ir/t^{1/2}$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \exp\left\{-r^2 - \frac{irx}{t^{1/2}\lambda^{1/2}}\right\} (-2r) \frac{dr}{t} = \frac{-1}{\pi i t} \cdot \frac{-t^{1/2}\cdot\lambda^{1/2}}{i} \cdot \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \exp\left\{-r^2 - \frac{irx}{t^{1/2}\cdot\lambda^{1/2}}\right\} dr$$

$$= \frac{\lambda^{1/2}}{\pi t^{1/2}} \cdot \frac{\partial}{\partial x} \left\{ e^{-x^2/4\lambda t} \underbrace{\int_{-\infty}^{\infty} \exp\left[-\left(r + \frac{ix}{2t^{1/2}\lambda^{1/2}}\right)^2\right] dr}_{\pi^{1/2}} \right\} = \frac{x}{2(\pi\lambda)^{1/2}} \cdot \frac{\exp\{-x^2/4\lambda t\}}{t^{3/2}}$$

Example (iv): The wave equation in 3 dimensions; the Causal Green's Function

$\left(\dfrac{1}{c^2} \cdot \dfrac{\partial^2}{\partial t^2} - \nabla^2\right) u(\underline{r}, t) = f(\underline{r}, t)$.

Take FT wrt $\underline{r}$ (new variable $\underline{k}$), LT wrt t (new variable p):

$U(\underline{k}, p) = \displaystyle\int_{t=0}^{\infty} \int_{\underline{r} \in \mathbb{R}^3} e^{-pt - i\underline{k}\cdot\underline{r}}\, u(\underline{r}, t)\, d^3\underline{r}\, dt$. $F(\underline{k}, p)$ defined similarly using $f(\underline{r}, t)$.

Take transforms of DE. F can also incorporate initial data.

$(p^2/c^2 + k^2) U = F$.

$U(\underline{k}, p) = G(\underline{k}, p) F(\underline{k}, p)$, where $G(\underline{k}, p) = \dfrac{1}{p^2/c^2 + k^2}$.

$u(\underline{r}, t) = g(\underline{r}, t) * f(\underline{r}, t)$. $g(\underline{r}, t)$ is the field generated if $f = \delta^3(\underline{r})\delta(t)$ - "instanton."

To invert $G(\underline{k}, p)$, start by doing the FT.

Consider $I = \dfrac{1}{(2\pi)^3} \displaystyle\int_{\mathbb{R}^3} \frac{\exp(i\underline{k}\cdot\underline{r})}{p^2/c^2 + k^2}\, d^3\underline{k}$ (remember $\mathrm{Re}(p) > 0$ at first).

Use polars, (k, θ, φ) in k-space, with $\theta = 0$ pointing along $\underline{r}$.

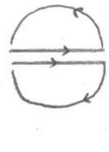
$$I = \frac{1}{(2\pi)^3} \int_{k=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} \frac{\exp(ikr\cos\theta)}{p^2/c^2 + k^2} \cdot k^2\, dk.\ \sin\theta\, d\theta\, d\varphi.$$

Integral over φ is trivial; over θ is easy: put $\cos\theta = \mu$, etc.

$$I = \frac{1}{(2\pi)^2} \int_{k=0}^{\infty} \frac{1}{ikr} \cdot \frac{e^{ikr} - e^{-ikr}}{p^2/c^2 + k^2}\, k^2\, dk = \frac{1}{(2\pi)^2} \cdot \frac{1}{2} \cdot \int_{-\infty}^{\infty} \frac{1}{ikr} \cdot \frac{e^{ikr} - e^{-ikr}}{p^2/c^2 + k^2} \cdot k^2\, dk.$$

- Simple poles at $k = \pm ip/c$.

Close with $\left\{\begin{matrix}\text{upper}\\ \text{lower}\end{matrix}\right\}$ half-circle for terms $\left\{\begin{matrix}e^{ikr}\\ e^{-ikr}\end{matrix}\right\}$ respectively.



The two contributions are the same.

$$I = \frac{1}{(2\pi)^2} \cdot \frac{1}{2} \cdot 2\pi i \cdot \frac{1}{ir} \cdot 2 \cdot \frac{e^{-rp/c}}{2} = \frac{e^{-rp/c}}{4\pi r}$$

$g(\underline{r},t) =$ inverse LT of $\frac{e^{-rp/c}}{4\pi r}$.

Recall that 1 is the LT of $\delta(t)$, and use the shifting theorem;

we get $g(\underline{r},t) = \frac{\delta(t - r/c)}{4\pi r}$. - retarded potential.

Exercise: $\left(\frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2} - \nabla^2 u + m^2\right) u = f$ - Klein-Gordon equation

Sheet 4, question 8: $\dot{y}(t) + y(t) - y(t-1) = f(t)$

and $x + \lambda x + C \int_0^t x(\tau) e^{-k(t-\tau)} d\tau = f(t)$.