Logic and Set Theory Examples 1

PTJ Michaelmas 2012

Important Note: There are probably more questions on these example sheets than any one student will wish to attempt, and they are not intended to be all of the same level of difficulty. Some (marked by − signs) are mere five-finger exercises to ensure that you have understood the basic definitions, and may safely be omitted if you are **confident** that you have understood them. Others (marked by + signs) are more challenging and/or slightly off the syllabus, and are intended for students who wish to explore the subject a bit more widely than can be done in the course.

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `ptj@dpmms.cam.ac.uk`.

− **1**. Write down all possible Hasse diagrams for a poset with four elements. [There are 16 of them.] How many of them are complete?

− **2**. Which of the following posets are complete? And which are chain-complete?

(i) The set of all finite subsets of an arbitrary set, ordered by inclusion.

(ii) The set of cofinite subsets (that is, subsets with finite complements) of an arbitrary set, ordered by inclusion.

(iii) The set of all transitive relations on an arbitrary set A, regarded as subsets of $A \times A$ and ordered by inclusion.

(iv) The set of all partial orderings of A, ordered by inclusion.

3. Let P and Q be posets. There are (at least) two possible ways of defining a partial ordering on $P \times Q$: the *pointwise order* is defined by $(a, c) \leq (b, d)$ if and only if $a \leq b$ and $c \leq d$, and the *lexicographic order* by $(a, c) \leq (b, d)$ if and only if either $a < b$ or ($a = b$ and $c \leq d$). Verify that both of these are partial orders. For each of the following properties, determine whether $P \times Q$ has the property (a) in the pointwise ordering, and (b) in the lexicographic ordering, if both P and Q have the property:

(i) being complete;

(ii) being totally ordered;

(iii) being chain-complete.

4. (i) Show that the fixed point of the function f constructed in the proof of the Knaster–Tarski Theorem given in lectures is in fact the unique largest fixed point of f. Modify the proof so that it produces the smallest fixed point of f instead.

(ii) More generally, show that if P is a complete poset and $f: P \to P$ an order-preserving map, then the set F of all fixed points of f is a complete poset in the ordering it inherits from P. [Warning: joins and meets in F need not coincide with those in P.]

− **5**. For each of the following functions $\Phi: [\mathbb{N} \rightharpoonup \mathbb{N}] \to [\mathbb{N} \rightharpoonup \mathbb{N}]$, determine ($a$) whether Φ is order-preserving, and (b) whether it has a fixed point:

(i) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, undefined otherwise.

(ii) $\Phi(f)(n) = f(n) + 1$ if $f(n)$ is defined, $\Phi(f)(n) = 0$ otherwise.

(iii) $\Phi(f)(n) = f(n-1) + 1$ if $f(n-1)$ is defined, $\Phi(f)(n) = 0$ otherwise.

6. Let P be a chain-complete poset with a least element, and $f: P \to P$ an order-preserving map. Show that the set of fixed points of f has a least element and is chain-complete in the ordering it inherits from P. Deduce that if $f_1, f_2, \ldots, f_n$ are order-preserving maps $P \to P$ which commute with each other (i.e. $f_i \circ f_j = f_j \circ f_i$ for all i, j), then they have a common fixed point. Show by an example that two order-preserving maps $P \to P$ which do not commute with each other need not have a common fixed point.

+ **7**. We call a poset P *Bourbakian* if every order-preserving map $P \to P$ has a least fixed point. Let P and Q be Bourbakian posets, and let $h\colon P \times Q \to P \times Q$ be a map which is order-preserving with respect to the pointwise ordering (cf. question 3). We denote the two components of the ordered pair $h(x, y)$ by $h_1(x, y)$ and $h_2(x, y)$ respectively.

(i) Show that, for each fixed $x \in P$, the mapping $g_x\colon Q \to Q$ defined by $g_x(y) = h_2(x, y)$ is order-preserving. Let $m(x)$ denote its least fixed point.

(ii) Show that m is an order-preserving map $P \to Q$. [Hint: if $x_1 \le x_2$, consider the effect of g_{x_1} on the set $\{y \in Q \mid y \le m(x_2)\}$.]

(iii) Show that the map $f\colon P \to P$ defined by $f(x) = h_1(x, m(x))$ is order-preserving. Let x_0 denote its least fixed point.

(iv) Show that $(x_0, m(x_0))$ is the least fixed point of h. [Thus a product of two Bourbakian posets is Bourbakian.]

8. A poset $(P, \le)$ is said to be *inductive* if each chain in P has an upper bound (but not necessarily a least upper bound). The usual statement of Zorn's Lemma says that every inductive poset (and not merely every chain-complete poset) has enough maximal elements.

(i) Give an example of a poset which is inductive but not chain-complete.

(ii) If $(P, \le)$ is any poset, let C denote the set of all chains in P, ordered by inclusion. Show that C is chain-complete.

(iii) If M is a maximal element of the poset C just defined, show that any upper bound for M in P is (a) a member of M, and (b) a maximal element of P.

(iv) Deduce the usual statement of Zorn's Lemma from the version proved in lectures.

9. Use Zorn's Lemma to prove

(i) that every partial ordering on a set can be extended to a total ordering;

(ii) that, for any two sets A and B, there exists either an injection $A \to B$ or an injection $B \to A$ [hint: consider the set of bijections from subsets of A to subsets of B, with a suitable ordering].

− **10**. Which of the following posets are lattices? Of those that are lattices, which are distributive?

(i) The set of all subspaces of a vector space, ordered by inclusion.

(ii) The set of natural numbers, ordered by divisibility.

(iii) The set $\Sigma*$ of all words over an alphabet Σ (that is, finite strings of members of Σ), ordered by the relation 'is a prefix of' (that is, $w \le x$ iff $x = wz$ for some z).

(iv) The unit square $[0, 1] \times [0, 1]$ with the lexicographic ordering (cf. question 3).

11. (i) Let L be the (five-element) lattice of subgroups of the non-cyclic group of order four. Show that there are no lattice homomorphisms $L \to 2$.

(ii) Find an example of a finite lattice L which admits at least one homomorphism $L \to 2$, but does not have enough such homomorphisms to separate its elements. [There is one with five elements.]

+ **12**. A lattice L is called a *Heyting algebra* if, for any two elements $a, b \in L$, there is a unique largest $c \in L$ with $c \wedge a \le b$. (The largest such c is usually denoted $a \Rightarrow b$; this defines a binary operation $\Rightarrow$ on L, which is called *Heyting implication.*)

(i) Show that every Boolean algebra is a Heyting algebra.

(ii) Show that every Heyting algebra is a distributive lattice.

(iii) Show that a complete lattice L is a Heyting algebra if and only if it satisfies the 'infinite distributive law'

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for all $a \in L$ and $S \subseteq L$. [Hint: consider $\bigvee \{c \in L \mid c \wedge a \le b\}$.] Deduce that every finite distributive lattice is a Heyting algebra.

(iv) If X is a topological space, show that the lattice $\mathcal{O}(X)$ of open subsets of X is a Heyting algebra. Under what conditions on X is it a Boolean algebra?

Logic and Set Theory Examples 2

PTJ Michaelmas 2012

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `ptj@dpmms.cam.ac.uk`.

− **1**. Which of the following propositional formulae are tautologies?
 (i) $((p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r)))$;
 (ii) $(((p \Rightarrow q) \Rightarrow r) \Rightarrow ((q \Rightarrow p) \Rightarrow r))$;
 (iii) $(((p \Rightarrow q) \Rightarrow p) \Rightarrow p)$;
 (iv) $((p \Rightarrow (p \Rightarrow q)) \Rightarrow p)$.

− **2**. Use the Deduction Theorem to show that the converse of the third axiom (i.e. the formula $(p \Rightarrow \neg\neg p)$) is a theorem of the propositional calculus.

3. Let t be a propositional formula not involving the constant $\perp$, and let $t' = t[\perp/p]$ be the formula obtained from t by substituting $\perp$ for all occurrences of a particular propositional variable p in t. Suppose that t' is a tautology but t is not; show that any proof of t' in the propositional calculus must involve an instance of the third axiom. Does this result remain true (a) if t is allowed to contain occurrences of $\perp$, or (b) if $\perp$ is replaced by $\top$?

4. Write down a proof of $(\perp \Rightarrow q)$ in the propositional calculus [hint: observe the result of question 3], and hence write down a deduction of $(p \Rightarrow q)$ from $\{\neg p\}$.

5. Show that if there is a deduction of t from $S \cup \{s\}$ in n lines (that is, consisting of n consecutive formulae), then $(s \Rightarrow t)$ is deducible from S in at most $3n + 2$ lines. Show further that there is a deduction of $\perp$ from $\{((p \Rightarrow q) \Rightarrow p), (p \Rightarrow \perp)\}$ in 16 lines [hint: use question 4], and hence calculate an upper bound for the length of a proof of the tautology of question 1(iii).

6. The beliefs of each member of a finite non-empty set I of individuals are represented by a consistent, deductively closed set of propositional formulae (in some fixed language $\mathcal{L}(P)$). Show that the set $\{t \in \mathcal{L}(P) \mid \text{all members of } I \text{ believe } t\}$ is consistent and deductively closed. Is the set $\{t \mid \text{over half the members of } I \text{ believe } t\}$ deductively closed or consistent?

7. A group G is called *orderable* if there exists a total ordering $\leq$ on G such that $g \leq h$ implies $gk \leq hk$ and $kg \leq kh$ for all k. Write down a propositional theory whose models are orderings of a given group G, and use the Compactness Theorem to show that G is orderable if and only if all its finitely-generated subgroups are orderable. Using the structure theorem for finitely-generated abelian groups, deduce that an abelian group is orderable if and only if it is torsion-free (i.e. it has no non-identity elements of finite order).

+ **8**. Let P be a set of primitive propositions. By a *Heyting valuation* of P we mean a function $v \colon P \to H$ from P to (the underlying set of) a Heyting algebra H (see sheet 1, question 12, for the definition). We extend v to a function $\overline{v} \colon \mathcal{L}(P) \to H$ in the obvious way: that is, $\overline{v}(\perp)$ is taken to be the least element 0 of H, and $\overline{v}(s \Rightarrow t)$ is the Heyting implication $\overline{v}(s) \Rightarrow \overline{v}(t)$ in H. A formula t is said to be a *Heyting tautology* if $\overline{v}(t) = 1$ for all Heyting valuations v (in arbitrary Heyting algebras H) of the primitive propositions involved in t.

(i) Verify that the axioms (K) and (S) are Heyting tautologies, and deduce that any formula which is provable in the propositional calculus without using the third axiom is a Heyting tautology.

(ii) Show that $(\perp \Rightarrow q)$ is a Heyting tautology.

(iii) By considering a suitable valuation $\{p, q\} \to T$ where T is a three-element chain, show that the formula of question 1(iii) is not a Heyting tautology.

(iv) Is the formula $(((p \Rightarrow q) \Rightarrow r) \Rightarrow (((q \Rightarrow p) \Rightarrow r) \Rightarrow r))$ a Heyting tautology?

9. Describe sets of axioms in suitable first-order languages (to be specified) for the following theories:

(i) the theory of ordered groups (i.e., groups with a compatible total ordering, cf. question 7);

(ii) the theory of fields;

(iii) the theory of fields of order p, for a fixed prime p [hint: take a signature with plenty of constants];

(iv) the theory of infinite-dimensional vector spaces over a (given) finite field F [hint: first express the assertion '$\{x_1, \ldots, x_n\}$ is linearly independent' as a finite conjunction];

(v) the theory of algebraically closed fields of characteristic zero;

(vi) the theory of posets in which every element lies below some maximal element;

(vii) the theory of totally ordered sets which are densely ordered (i.e. between any two elements there lies a third one) and have neither greatest nor least elements.

10. By a *substructure* of an (Ω, Π)-structure A, we mean a subset B of the underlying set of A which is closed under the operations in Ω (that is, $b_1 \ldots, b_n \in B$ implies $\omega_A(b_1, \ldots, b_n) \in B$ for each $\omega \in \Omega$), made into an (Ω, Π)-structure by taking ω_B to be the restriction of ω_A to B^n, and $[\pi]_B$ to be $[\pi]_A \cap B^n$ for each $\pi \in \Pi$.

(i) Show that if B is a substructure of A and ϕ is a quantifier-free formula of $\mathcal{L}(\Omega, \Pi)$ (with n free variables, say), then $[\phi]_B = [\phi]_A \cap B^n$. Give an example to show that this equality may fail if ϕ contains quantifiers.

(ii) A first-order theory T is called *universal* if its axioms all have the form $(\forall \vec{x})\phi$ where $\vec{x}$ is a (possibly empty) string of variables and ϕ is quantifier-free. Show that if T is universal, then every substructure of a T-model is a T-model.

(iii) Similarly, T is called *inductive* if its axioms have the form $(\forall \vec{x})(\exists \vec{y})\phi$ where ϕ is quantifier-free. Show that if T is inductive, and A is an (Ω, Π)-structure, then the set of substructures of A which are T-models is closed under unions of chains.

(iv) Which of the theories of question 9 are (axiomatizable as) universal theories? And which are inductive?

+ **11**. Let T be a first-order theory over a signature Σ, and let $T_\forall$ denote the set of all universal sentences over Σ which are derivable from T. Let M be a model of $T_\forall$. Let Σ' be the signature obtained from Σ by adjoining one new constant c_m for each element m of M, and let D_M be the theory over Σ' consisting of all sentences $\phi[c_{m_1}, \ldots, c_{m_k}/x_1, \ldots, x_k]$ where ϕ is a quantifier-free formula over Σ with free variables $x_1, \ldots, x_k$, and $(m_1, \ldots, m_k) \in [\phi]_M$. Show that $T \cup D_M$ is consistent, and deduce that there is a T-model $\widehat{M}$ having a substructure isomorphic to M. Hence show that the converse of question 10(ii) holds, i.e. that if T is a first-order theory for which every substructure of a T-model is a T-model, then there is a universal theory (over the same signature) having the same models as T.

[Method: suppose $T \cup D_M \vdash \perp$. Let F be a finite subset of this theory which is inconsistent; let ψ be the conjunction of the members of D_M which occur in F, and suppose $c_{m_1}, \ldots, c_{m_r}$ are the constants which appear in ψ. Let ψ' be the formula obtained from ψ on replacing the c_{m_i} by variables x_i; show that $(\forall x_1, \ldots, x_r)\neg\psi'$ is derivable from T, but not satisfied in M.]

12. Show that the sentences $(\forall x, y)((x = y) \Rightarrow (y = x))$ and $(\forall x, y, z)((x = y) \Rightarrow ((y = z) \Rightarrow (x = z)))$ are theorems of the predicate calculus with equality. [There is no need to write out formal proofs in full; but you shouldn't expect your supervisor to be satisfied with an argument based on the Completeness Theorem (further exercise: why not?).]

13. Show that every countable model of the theory of question 9(vii) is isomorphic to the ordered set of rational numbers. Is every countable model of first-order Peano arithmetic isomorphic to the set of natural numbers? [Hint: compactness.]

Logic and Set Theory Examples 3

PTJ Michaelmas 2012

Comments on and/or corrections to the questions on this sheet are always welcome, and may be e-mailed to me at `ptj@dpmms.cam.ac.uk`.

1. Is the collection of sets z satisfying

$$\neg(\exists u_1, \ldots, u_n)((z \in u_1) \wedge (u_1 \in u_2) \wedge \cdots \wedge (u_n \in z))$$

a set for any n? [Try to answer this **without** using the axiom of Foundation.]

$-$ **2**. Show that the Pair-Set axiom is deducible from the axioms of Empty-Set, Power-Set and Replacement.

$-$ **3**. Show that if x is a transitive set, then $\mathcal{P}x$ and $\bigcup x$ are also transitive. Are the converses true?

4. Use $\in$-induction to prove that the only automorphism of the structure $(V, \in)$ which is definable by a function-class is the identity.

5. Let $(V, \in)$ be a model of ZF, and let σ be a permutation of V (which you may assume to be given by a function-class). We define a new binary relation $\in^\sigma$ on V by $(x \in^\sigma y) \Leftrightarrow (x \in \sigma(y))$.

(i) Verify that the structure $(V, \in^\sigma)$ satisfies all the axioms of ZF except possibly for Foundation.

(ii) By taking σ to be the transposition which interchanges $\emptyset$ and $\{\emptyset\}$ (and fixes everything else), show that Foundation may fail.

(iii) More generally, let a be a set none of whose members is a singleton, and let σ be the permutation which interchanges x and $\{x\}$ for each $x \in a$. Show that $(V, \in^\sigma)$ satisfies a weak version of Foundation which says that every nonempty set x has a member y satisfying either $x \cap y = \emptyset$ or $y = \{y\}$.

6. Define S to be the smallest set a such that $(\emptyset \in a) \wedge (\forall x, y \in a)(x \cup \{y\} \in a)$.

(i) Explain how the axioms of ZF ensure that such a set exists and is unique.

(ii) Show that S is closed under $\cup$ (that is, $(\forall x, y \in S)((x \cup y) \in S)$).

(iii) Show that S is closed under $\bigcup$.

(iv) Show that S is the smallest set containing $\emptyset$ and closed under taking pairs and unions.

7. Use the $\in$-recursion theorem to show that there is a unique function-class $\overline{\mathrm{TC}}$ such that

$$(\forall x)(\overline{\mathrm{TC}}(x) = x \cup \bigcup\{\overline{\mathrm{TC}}(y) \mid y \in x\}) \ ,$$

and show that $\overline{\mathrm{TC}}$ coincides with the transitive closure operation as defined in lectures. Why is $\overline{\mathrm{TC}}$ unsatisfactory as a definition of transitive closure?

8. A class M is *transitive* if $(\forall x, y)(((x \in y) \wedge (y \in M)) \Rightarrow (x \in M))$ holds. Show that if M is a transitive class, then the structure $(M, \in)$ satisfies the axiom of extensionality, and that it satisfies each of the empty-set, pair-set and union-set axioms if and only if M is closed under the corresponding finitary operation on V. What more do you need to know about M to get a similar result for the power-set axiom?

9. If P is a property of sets, a set x is said to be *hereditarily* P if every member of $\mathrm{TC}(\{x\})$ has property P. Consider the classes HF, HC and HS of hereditarily finite, hereditarily countable and hereditarily small sets (where we call a set *small* if it can be injected into one of the sets $\omega, \mathcal{P}\omega, \mathcal{P}\mathcal{P}\omega, \ldots$): in each case determine which axioms of ZF hold, and which fail, in the structure obtained from the class as in the previous question.

10. Prove that each of the following is an alternative characterization of the set V_ω (the ωth stage in the von Neumann hierarchy):

(i) V_ω is the class HF of hereditarily finite sets (cf. question 9);

(ii) V_ω is the class $\{x \mid \mathrm{TC}(\{x\})$ is finite$\}$ of *strongly hereditarily finite* sets;

(iii) V_ω is the set S considered in question 6;

(iii) V_ω is the smallest set containing $\emptyset$ and closed under $\mathcal{P}$ and under formation of arbitrary subsets.

Deduce in particular that the class HF is a set. Is HC a set? If so, does it coincide with V_α for any α?

11. Consider the binary relation E on $\mathbb{N}$ defined by: $n\,E\,m$ iff the $(n+1)$st digit (counting from the right) in the binary expansion of m is 1. What can you say about the structure $(\mathbb{N}, E)$?

$+$ **12**. Let $(V, \in)$ be a structure satisfying all the axioms of ZF except Foundation, and also satisfying the weak Foundation axiom of question 4(iii). Suppose also that the *autosingletons* of V (that is, the elements x satisfying $x = \{x\}$) form an infinite set a.

(i) Show that we may define a 'rank function' $V \to \mathbf{On}$ such that each autosingleton has rank 0, and every other set x satisfies $\mathrm{rank}(x) = \bigcup\{\mathrm{rank}(y)^+ \mid y \in x\}$.

(ii) Let G be the group of all permutations of a; show that we may extend each $\pi \in G$ to a permutation π^* of V satisfying $\pi^*(x) = \{\pi^*(y) \mid y \in x\}$ for all x.

(iii) We define a set x to be *of finite support* if there exists a finite set $a' \subseteq a$ such that $\pi^*(x) = x$ whenever π fixes all the members of a'. Show that the class HFS of members of V which are hereditarily of finite support (in the sense defined in question 8) is a model of ZF minus Foundation. Show also that $a \in HFS$, but that

$$HFS \models \neg(\exists b \subseteq a \times a)(b \text{ is a total ordering of } a) \ .$$

(iv) Now suppose a has a dense total ordering without endpoints [if you wish, take $a \cong \mathbb{Q}$], and replace the group of all permutations of a by the subgroup H of order-preserving permutations. Defining the notions of (hereditary) finite support as before, but restricting to permutations in H, show that HFS is still a model of ZF minus Foundation, and that the given total ordering of a belongs to it. Show also that the set I of all *intervals* $(x, y) = \{z \in a \mid x < z < y\}$ with $x < y$ in a belongs to HFS, but that there is no function $f\colon I \to a$ in HFS with $f(i) \in i$ for all $i \in I$.

13. (i) Determine the rank of the set $\mathbb{R}$ of real numbers. [You may assume that a real number is an ordered pair of subsets of $\mathbb{Q}$ (a Dedekind section), that a rational number is an equivalence class of ordered pairs of integers, and so on.]

(ii) Show that there is a subset of $\mathbb{R}$ which (under the restriction of the usual ordering on $\mathbb{R}$) is order-isomorphic to $\omega + \omega$.

(iii) Show that all the axioms of ZF except for the scheme of Replacement hold in $V_{\omega+\omega}$. Why can we deduce from (ii) that Replacement does **not** hold?

14. Let r be a binary relation on a set a. Show that r is well-founded iff there exists a function $h\colon a \to \alpha$ for some ordinal α, such that $\langle x, y\rangle \in r$ implies $h(x) < h(y)$. Deduce that if every set can be well-ordered, then any well-founded binary relation on a set can be extended to a well-ordering. [Compare question 9(i) on sheet 1.]

Logic and Set Theory Examples 4

PTJ Michaelmas 2012

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− **1**. Prove or disprove the following statements:
(i) $\alpha + \beta$ is a limit ordinal if and only if β is a limit ordinal.
(ii) $\alpha \,.\, \beta$ is a limit ordinal if and only if either α or β is a limit ordinal.
(iii) Any limit ordinal can be written in the form $\omega \,.\, \alpha$ for some α.
(iv) Any limit ordinal can be written in the form $\alpha \,.\, \omega$ for some α.

2. Ordinal subtraction is defined synthetically by taking $\alpha - \beta$ to be the order-type of the set-difference $\alpha \setminus \beta$ (in particular, $\alpha - \beta = 0$ whenever $\alpha \leq \beta$). Prove the following identities:

$$(\alpha + \beta) - \alpha = \beta \quad ; \quad \alpha - (\beta + \gamma) = (\alpha - \beta) - \gamma \quad ; \quad \alpha \,.\, (\beta - \gamma) = \alpha \,.\, \beta - \alpha \,.\, \gamma \quad .$$

Show also that for any ordinal α there are only finitely many ordinals of the form $\alpha - \beta$. [Hint: consider the order-type of the set $\{\alpha - \beta \mid \beta \in \mathbf{On}\}$.]

3. A function-class $F \colon \mathbf{On} \to \mathbf{On}$ is called a *normal function* if it is strictly order-preserving (i.e. $\alpha < \beta$ implies $F(\alpha) < F(\beta)$) and continuous at limits (i.e. $F(\lambda) = \bigcup\{F(\alpha) \mid \alpha < \lambda\}$ for nonzero limits λ). Prove the following facts about a normal function F:
(i) $F(\alpha) \geq \alpha$ for all α;
(ii) $\bigcup\{0, F(0), F(F(0)), \ldots\}$ is the least ordinal β satisfying $F(\beta) = \beta$;
(iii) there is a normal function G whose range is exactly the class of ordinals β satisfying $F(\beta) = \beta$.
Find G when F is the function $\beta + (-)$ for some β, and when it is the function $\gamma \,.\, (-)$ for some nonzero γ.

4. Show that every ordinal α has a unique representation (its *Cantor Normal Form*) of the form

$$\alpha = \omega^{\alpha_1} \,.\, a_1 + \omega^{\alpha_2} \,.\, a_2 + \cdots + \omega^{\alpha_n} \,.\, a_n$$

where $n \in \omega$, $\alpha \geq \alpha_1 > \alpha_2 > \cdots > \alpha_n$, and $a_1, a_2, \ldots, a_n$ are nonzero natural numbers. [Hint: consider the least β such that $\omega^\beta > \alpha$; can it be a limit?] Describe how to calculate the Cantor Normal Form of $\alpha + \beta$ from those of α and β.

+ **5**. Consider the binary operation $\oplus$ on $\mathbf{On}$ defined recursively by setting

$$\alpha \oplus \beta = \bigcup(\{(\alpha \oplus \beta')^+ \mid \beta' < \beta\} \cup \{(\alpha' \oplus \beta)^+ \mid \alpha' < \alpha\}) \quad .$$

Prove the following facts about $\oplus$:
(i) $\oplus$ is commutative and associative, and has 0 as an identity element.
(ii) If $n < \omega$, then $\alpha \oplus n = \alpha + n$ for any α.
(iii) For any α, the set of ordinals less than ω^α is closed under $\oplus$.
(iv) If $\beta < \omega^\alpha$, then $\omega^\alpha \oplus \beta = \omega^\alpha + \beta$.
(v) $\omega^\alpha \,.\, m \oplus \omega^\alpha \,.\, n = \omega^\alpha \,.\, (m + n)$ for any $m, n < \omega$.
(vi) $\oplus$ coincides with the binary operation obtained by treating Cantor Normal Forms as if they were polynomials in ω, and applying the usual rules for adding polynomials.
[N.B.: You will need to prove (iii)–(vi) by a *simultaneous* induction.]

6. A choice function $g \colon \mathcal{P}^+ a \to a$ is said to be *orderly* if

$$g(b \cup c) = g(\{g(b), g(c)\})$$

for all nonempty sets $b, c \subseteq a$. Show that g is orderly iff it is induced (as in the proof given in lectures that the well-ordering theorem implies AC) by a well-ordering of a. [Hint: if g is so induced, we can recover the ordering by $(x \leq y) \Leftrightarrow (g(\{x, y\}) = x)$.]

$-$ **7**. Show that the assertion 'For any two sets a and b, there exists either an injection $a \to b$ or an injection $b \to a$' is equivalent to the axiom of choice. [One direction was done in question 9(ii) on sheet 1; for the converse, use Hartogs' Lemma.]

$+$ **8**. By a *selection function* for a set a, we mean a function $s\colon \mathcal{P}a \to \mathcal{P}a$ such that $\emptyset \subseteq s(b) \subseteq b$ for all $b \subseteq a$, both inclusions being strict unless b is empty or a singleton.

(i) Show that $\mathcal{P}\alpha$ has a selection function for any ordinal α. [Given a subset $b \subseteq \mathcal{P}\alpha$ with more than one element, consider the least β belonging to some but not all members of b.]

(ii) Conversely, suppose a has a selection function s. For any ordinal α and function $f\colon \alpha \to 2$, we define a subset $b(f)$ of a by the following recursion: if $\alpha = 0$, $b(f) = a$; if α is a limit, $b(f) = \bigcap\{b(f|_\beta) \mid \beta < \alpha\}$; if $\alpha = \beta^+$ and $f(\beta) = 1$, $b(f) = s(b(f|_\beta))$; if $\alpha = \beta^+$ and $f(\beta) = 0$, $b(f) = b(f|_\beta) \setminus s(b(f|_\beta))$. Show that, for each ordinal α, the nonempty members of the family $\{b(f) \mid \mathrm{dom}\, f = \alpha\}$ form a partition of a. Show also that $b(f)$ has at most one element for every f with domain $\gamma(\mathcal{P}a)$, and deduce that there is an injection from a to the power-set of an ordinal.

(iii) Deduce that the assertion 'Every set admits a selection function' implies that every set can be totally ordered.

9. (i) Show that the assertion 'Every set can be totally ordered' implies that every family of nonempty finite sets has a choice function.

(ii) By a *multiple-choice function* for a set a, we mean a function picking out a nonempty finite subset of each nonempty subset of a. Show that the assertion 'Every set has a multiple-choice function' implies that any totally orderable set can be well-ordered.

10 For each countable ordinal α, show that there is a subset of $\mathbb{R}$ which is well-ordered (in the usual ordering) and has order-type α. Is there a well-ordered subset of $\mathbb{R}$ (again, in the usual ordering) of order-type ω_1?

11. Suppose that no uncountable subset of $\mathcal{P}\omega$ can be well-ordered (equivalently, the Hartogs ordinal $\gamma(\mathcal{P}\omega)$ is ω_1). Let $a = \mathcal{P}(\omega \times \omega)$ be the set of all binary relations on ω, and b the quotient of a by the equivalence relation which identifies two relations r and s iff there is a permutation π of ω such that $(m, n) \in r \Leftrightarrow (\pi(m), \pi(n)) \in s$ for all $m, n \in \omega$. By considering equivalence classes whose members are equivalence relations, or otherwise, show that there is an injection $a \to b$ (as well as the evident surjection $a \to b$). By considering equivalence classes whose members are well-orderings, show that there is no bijection $a \to b$.

12 A subset x of an ordinal α is said to be *cofinal* if, for every $\beta \in \alpha$, there exists $\gamma \in x$ with $\beta \leq \gamma$. We define the *cofinality* $\mathrm{cf}(\alpha)$ of α to be the least ordinal β for which there exists an order-preserving map $\beta \to \alpha$ whose image is cofinal in α. Prove that

(i) for any α, $\mathrm{cf}(\mathrm{cf}(\alpha)) = \mathrm{cf}(\alpha)$;

(ii) for any limit ordinal α, $\mathrm{cf}(\alpha)$ is an initial ordinal [hint: given a bijection $f\colon \beta \to \gamma$ with $\beta < \gamma$, consider the function $g\colon \beta \to \gamma^+$ defined by $g(\delta) = \bigcup\{f(\delta') \mid \delta' \leq \delta\}$];

(iii) for any successor ordinal α, $\mathrm{cf}(\omega_\alpha) = \omega_\alpha$ [hint: try $\alpha = 1$ first];

(iv) for a nonzero limit ordinal λ, we have either $\mathrm{cf}(\omega_\lambda) < \omega_\lambda$ or $\omega_\lambda = \lambda$.

Show that there is a least ordinal α such that $\omega_\alpha = \alpha$. What is its cofinality?

$+$ **13**. (i) Show that any ordinal α may be made into a topological space by taking all subsets of the forms $\{\delta \in \alpha \mid \beta < \delta\}$ and $\{\delta \in \alpha \mid \delta < \gamma\}$, and pairwise intersections of these, as a basis for the open sets.

(ii) Show that, in this topology, α is compact iff it is either zero or a successor ordinal.

(iii) Suppose that α is a limit and $\mathrm{cf}(\alpha) > \omega$ (cf. the previous question). Show that α has the 'Bolzano–Weierstrass property' that every sequence has a convergent subsequence [hint: recall a proof from Analysis I that every sequence has a monotonic subsequence]. Show also that every continuous function $f\colon \alpha \to \mathbb{R}$ is bounded. [First show that there exists $\beta < \alpha$ such that $|f(\beta) - f(\gamma)| < 1$ for all $\gamma > \beta$.]

(iv) Show that the topology on α can be induced by a metric if and only if $\alpha < \omega_1$. [For 'if', use the first part of question 10; for 'only if', use parts (ii) and (iii).]