Groups and Representation Theory 1

Here A_n is the alternating group of degree n, and p is a prime.

1. Let P be a finite p-group with $1 \neq Q \lhd P$. By considering the action of P on the elements of Q by conjugation, prove that $Q \cap Z(P) \neq 1$.

2. a. Show that any group of order p^2 is abelian. What are the possible isomorphism types of such groups?
b. Show that a non-abelian group G of order p^3 has centre of order p, and that G has $p^2 + p - 1$ conjugacy classes.

3. Describe the conjugacy classes of elements in the symmetric group S_n. Let $g \in A_n$. Prove that the S_n-class of g either is a single A_n-class, or splits into two A_n-classes of equal size. Prove further that the latter occurs precisely when all the cycles of g (in its disjoint cycle decomposition) have distinct odd lengths.
[Consider centralizers.]

4. Find a Sylow p-subgroup of A_5 and of A_6 for $p = 3$ and for $p = 5$. Find the corresponding normalizers.

5. Consider the symmetric group S_p. Show that the only elements which commute with a p-cycle are its powers. If P is a Sylow p-subgroup, show that $|P| = p$ and $|N(P)| = p(p-1)$.

6. Prove that any group G of order pqr, where p, q and r are prime numbers, has a normal Sylow subgroup.

7. (IB 75/4/5) Let H be a proper subgroup of finite index n in the group G. Show that G has a normal subgroup K, contained in H, of index at most $n!$ in G. Show further that, if G is also simple and non-Abelian, then $n \geq 5$ and G is isomorphic to a subgroup of the alternating group A_n on n symbols.
Prove that a simple non-Abelian group having a subgroup of index 6 must either be isomorphic to A_6 or have order 60.
[Sylow's theorems and the simplicity of A_6 may be assumed.]

8. Prove that any simple group of order < 60 must be cyclic of prime order.

9. (75/1/4) Let p be a prime and P a Sylow p-subgroup of the finite group G. Show that the index of the normalizer of P in G is congruent to 1 modulo p.
Prove that a simple group of order 1092 has a single conjugacy class of subgroups of index 14 but no subgroup of index 13.
[It may be assumed that in a finite group Sylow p-subgroups exist and are all conjugate.]

10. Let G be a finite group of even order with a cyclic Sylow 2-subgroup P. By considering the left regular action of P on G (or otherwise) show that G has a normal subgroup of index 2. Deduce that G has a normal 2-complement, that is, G has a normal subgroup of index $|P|$.

11. This exercise provides an alternative proof of the existence of Sylow subgroups in finite groups.

(a) Assume that the group G contains a Sylow p-subgroup P, and let H be a subgroup of G. Let H act on the set $(G : P)$ of the left cosets of P in G by left multiplication; there is an orbit Σ of size prime to p. If $xP \in \Sigma$, show that the stabilizer H_{xP} is a Sylow p-subgroup of H. [Note that $H_{xP} = H \cap G_{xP} = H \cap xPx^{-1}$.]

(b) Show that the subgroup of unitriangular matrices is a Sylow p-subgroup in the group $GL_n(p)$ of non-singular $n \times n$ matrices over the field of p elements.

(c) Deduce that the symmetric group S_n, and hence any finite group, has a Sylow p-subgroup.

12. Let $G = SL_2(3)$, the groups of all 2×2 matrices of determinant 1 over the field of 3 elements. Show that G has order 24. Show also that G has a unique element of order 2. Prove that G has A_4 as a quotient group but has no subgroup isomorphic to A_4.

13. Find the conjugacy classes in the group $GL_3(2)$ of non-singular 3×3 amtrices over the field of 2 elements (they have sizes 1, 21, 42, 56, 24, 24). Deduce that $GL_3(2)$ is a simple group.

Groups and Representation Theory 2

1. a. Prove that any group generated by two involutions is dihedral. More precisely, if G is generated by s and t of order 2 with st of order m, then $G \cong D_{2m}$, the dihedral group of order $2m$.

b. Deduce that any homomorphic image of a dihedral group is dihedral (possibly degenerate).

2. Let W be a reflection group on V with root system Φ. Prove that the following are equivalent:

i. W is essential (so fixes no non-zero vector),

ii. $V = \langle \Phi \rangle$,

iii. $\bigcap_{\alpha \in \Phi} H_\alpha = \{0\}$, where $H_\alpha = \alpha^\perp$.

3. Let G be the reflection group $I_2(m)$ isomorphic to the dihedral group D_{2m} of order $2m$.

a. Find all the reflections in G; show that they are all conjugate if m is odd, whereas there are two conjugacy classes if m is even.

b. Draw the root system of G with roots of length 1. Choose fundamental reflections s_1 and s_2 and indicate on your drawing the corresponding fundamental set $\Delta = \{\alpha_1, \alpha_2\}$ and the set Φ^+ of all positive roots containing it.

c. Find all the fundamental sets and verify that they are conjugate under G.

4. Let G be the symmetric group S_{n+1} acting as a reflection group of type A_n on the Euclidean space $V = \mathbf{R}^\mathbf{n}$. Show that the reflections are precisely the transpositions of G.

5. Let $G = B_n$ be the set of $n \times n$ signed permutation matrices - that is matrices with entries $0, \pm 1$, with precisely one non-zero entry in each row and each column. Prove that G is a subgroup of $\mathrm{GL}_n(\mathbf{R})$. Using a natural homomorphism from G to the group S_n of all permutation matrices prove that $|G| = 2^n n!$. Show that G is generated by the reflections on $\mathbf{R}^\mathbf{n}$ with roots $\pm\varepsilon_k$ and $\pm\varepsilon_i \pm \varepsilon_j$ ($1 \le k \le n$, $1 \le i < j \le n$). Show that $\varepsilon_1 - \varepsilon_2$, $\varepsilon_2 - \varepsilon_3$, ..., $\varepsilon_{n-1} - \varepsilon_n$, ε_n is a fundamental system for G. Show finally that $\varepsilon_1 + \varepsilon_2$ is the highest root (with respect to this fundamental system).

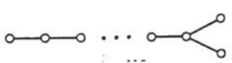
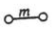
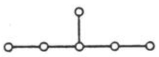
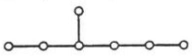
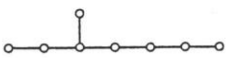
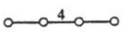
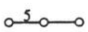
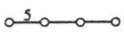
6. Formulate and prove a question corresponding to question 5 for G of type D_n.

7. Let $w \in W$, with $w = s_1 s_2 \cdots s_v$ a reduced expression for w in terms of the fundamental reflections $s_i = s_{\alpha_i}$. Writing $\Pi(w)$ for the set $\Phi^+ \cap w^{-1}\Phi^-$, prove that

$$\Pi(w) = \{\alpha_v, s_v\alpha_{v-1}, s_v s_{v-1}\alpha_{v-2}, \ldots, s_v s_{v-1} \ldots s_2\alpha_1\}.$$

8. Let $w \in W$, with $w = s_1 s_2 \ldots s_v$ an expression in terms of the fundamental reflections. If $l(ws) < l(w)$ for some fundamental reflection s, show that there exists an index i such that $ws = s_1 \cdots \hat{s}_i \cdots s_v$. Deduce that w has a reduced expression ending in s iff $l(ws) < l(w)$.

Classification Theorem The positive definite connected Coxeter graphs are precisely as in the table:

Type	Graph	$\lvert W\rvert$	$\lvert\Phi\rvert$	$\det 2A$
$A_n,\ n \geq 1$		$(n+1)!$	$n(n+1)$	$n+1$
$B_n,\ n \geq 2$		$2^n n!$	$2n^2$	2
$D_n,\ n \geq 4$		$2^{n-1}n!$	$2n(n-1)$	4
$I_2(m),\ m \geq 5$		$2m$	$2m$	$4\sin^2\frac{\pi}{m}$
E_6		$2^7 3^4 5$	72	3
E_7		$2^{10}3^4 5\ 7$	126	2
E_8		$2^{14}3^5 5^2 7$	240	1
F_4		$2^7 3^2$	48	1
H_3		120	30	$3-\sqrt{5}$
H_4		14400	120	$\frac{1}{2}(7-3\sqrt{5})$

9. Let A be the matrix corresponding to the graph Γ (so $a_{ij} = -\cos\frac{\pi}{m_{ij}}$). Verify the formula for $\det 2A$ in the table (at least for the first four lines).

Hint: If the vertex n is joined precisely to one other vertex (say $n-1$) with label $m = m_{n-1,n} \in \{3,4\}$, then $\det 2A = 2\det 2B - (m-2)\det 2C$, where B is the matrix corresponding to the subgraph $\Gamma \setminus \{n\}$ and C is the matrix corresponding to the subgraph $\Gamma \setminus \{n, n-1\}$.

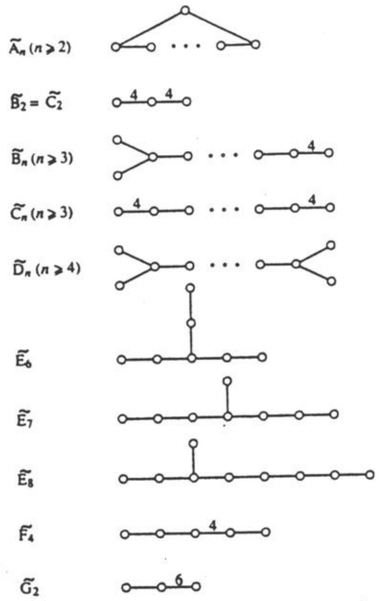
10. Prove that there are no positive definite connected Coxeter graphs other than those listed in the table.

You may use without proof the following facts:

A. Principal Lemma: A non-empty labelled subgraph Γ' of a positive definite graph Γ is positive definite (here Γ' is obtained from Γ by deleting some vertices and/or lowering some labels).

B. The graphs $\circ - \circ \overset{5}{-} \circ - \circ$ and $\circ \overset{5}{-} \circ - \circ - \circ - \circ$ are not positive definite.

C. Each of the following graphs (though positive semidefinite) has determinant of the corresponding matrix equal to 0:



11. Let Φ be a root system in V with fundamental system Δ, let $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$. The fundamental chamber is

$$C = \{v \in V \mid (v, \alpha) > 0 \text{ for all } \alpha \in \Delta\}.$$

The fundamental domain is

$$D = \{v \in V \mid (v, \alpha) \geq 0 \text{ for all } \alpha \in \Delta\}.$$

The chambers associated with Φ are $w(C)$ for various $w \in W$ (these are the connected components of $V \setminus \bigcup H_\alpha$).

a. Prove that any vector $v \in V$ is W-conjugate to some $u \in D$.

b. Prove that if $u, v \in C$ are W-conjugate then they are equal.

c. The number of chambers equals $|W|$.

12. Let W be a reflection group on V, with a fixed fundamental system Δ of roots in its root system Φ.

(a) If $v \in V$ with $(v, \alpha) \geq 0$ for all $\alpha \in \Delta$, then the stabilizer $W_v = \{w \in W \mid w(v) = v\}$ is generated by the fundamental reflections fixing v.

(b) If $v \in V$ then W_v is generated by those reflections s_α $(\alpha \in \Phi)$ it contains.

(c) If U is any subset of V, the pointwise stabilizer $W_{(U)}$ of U is generated by those reflections it contains.

13. Let W be a reflection group, with a fundamental system Δ of roots. For $J \subseteq \Delta$, write $W_J = \langle s_\alpha \mid \alpha \in J \rangle$ - this is the *parabolic* subgroup corresponding to J. For J, $K \subseteq \Delta$, prove that

a. $W_J = W_K$ iff $J = K$,

b. $W_{J \cup K} = \langle W_J, W_K \rangle$,

c. $W_{J \cap K} = W_J \cap W_K$.

14. Let Φ be a root system, let $W = \langle s_\alpha \mid \alpha \in \Phi \rangle$ be the corresponding reflection group. Prove that any reflection in W is s_α for some $\alpha \in \Phi$.

Groups and Representation Theory 3

Unless otherwise stated, all vector spaces are finite dimensional over a field F of characteristic zero.

1. Let $\theta : G \to GL(F)$ be a 1-dimensional representation of the group G and $\rho : G \to GL(V)$ another representation. Show that both

$$\theta' : x \mapsto \theta(x^{-1}) \quad \text{and} \quad \theta.\rho : x \mapsto \theta(x)\rho(x)$$

are representations and that $\theta.\rho$ is irreducible if and only if ρ is irreducible. Show that the set of all 1-dimensional representations of G over F forms an Abelian group under this multiplication. (It is called the dual group of G.)

2. Let $\rho : G \to GL(V)$ be a representation. Show that $\rho^* : G \to GL(V^*)$, given by $\rho^* : \quad x \mapsto \rho(x^{-1})^*$ is also a representation on the dual space V^* of V.
Prove that ρ is irreducible if and only if ρ^* is irreducible.

3. Let $\langle \ , \ \rangle$ be an inner product on a complex vector space V. Let $\rho : G \to GL(V)$ be a representation which maps into the unitary group $U(V)$; then we will say that ρ is an *unitary* representation of G. Prove the following results for any group G, finite or infinite.
   (a) $\rho(x^{-1}) = \rho(x)^*$ for $x \in G$.
   (b) If W is a subrepresentation space of V then its orthogonal complement $W^\perp$ is also a subrepresentation space and $V = W \oplus W^\perp$.
   (c) Any finite dimensional unitary representation is the direct sum of irreducible representations.

4. Let $\rho : G \to GL(V)$ be a complex representation of the <u>finite</u> group G, and let $\langle \ , \ \rangle$ be an inner product on V. Show that

$$\langle\langle v, w \rangle\rangle = \sum_{x \in G} \langle \rho(x)v, \rho(x)w \rangle$$

is also an inner product and, if we give V this inner product, the representation ρ is unitary.
Use this, and the previous question to give an alternative proof of Maschke's theorem for complex (or real) representations.

5. Suppose that a representation space V is the sum of irreducible subrepresentation spaces $V = V_1 + V_2 + \ldots + V_n$. Show that V is the direct sum of some of the V_i.
If W is any subrepresentation of V then $V = W \oplus U$ with U a direct sum of some of the V_i.

6. Let x be an element of a finite group G. Show that x is conjugate to its inverse in G if and only if $\chi(x)$ is real for every character χ of G over the complex field.
Show that the quaternion group Q_8 is an example of a finite group in which every element is conjugate to its inverse but not every complex representation is equivalent to a real one.

7. Let χ be a character of a finite group G and let $g \in G$ have order 2. Show that $\chi(g)$ is an integer congruent to $\chi(e)$ modulo 2. Show further that $\chi(g)$ is congruent to $\chi(e)$ modulo 4 unless G has a subgroup of index 2. [Hint for the last part: consider determinants.]

8. Find the character tables of $C_N, D_{2N}, S_4, A_5, S_5$.

9. (1975/3/4) A group of order 720 has 11 conjugacy classes. Two representations of this group are known and have corresponding characters α and β, The table below gives the sizes of the conjugacy classes and the values which α and β take on them.

	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

Prove that the group has an irreducible representation of degree 16 and write down the corresponding character on the conjugacy classes.

10. (1987/3/3) A finite group has 7 conjugacy classes $C_1 = \{e\}, C_2, \ldots, C_7$ and the values of 5 of its irreducible characters are given in the following table.

1	20	24	15	10	20	30
C_1	C_2	C_3	C_4	C_5	C_6	C_7
1	1	1	1	1	1	1
1	1	1	1	-1	-1	-1
4	1	-1	0	2	-1	0
4	1	-1	0	-2	1	0
5	-1	0	1	1	1	-1
5	-1	0	1	-1	-1	1
6	0	1	2	0	0	0.

Calculate the number of elements in the various conjugacy classes and the remaining ireducible characters.

11. (1987/1/2) State and prove Schur's Lemma. Show that a finite group with a faithful irreducible complex representation must have a cyclic centre.

Let G be a group of order 18 containing an elementary abelian group P of order 9 and an element t of order 2 with $txt = x^{-1}$ for each x in P. By considering the action of P on an irreducible $\mathbf{C}G$-module (or otherwise) prove that G has no faithful irreducible complex representation.

12. Let V be a real representation space for a group G. Define $V_{\mathbf{C}}$ to be the $V \oplus V$ and show that this becomes a complex vector space if we define scalar multiplication by $x + iy$ by

$$(x + iy).(v_1, v_2) = (xv_1 - yv_2, xv_2 + yv_1).$$

Show that the action of G on the direct sum $V \oplus V$ is $\mathbf{C}$-linear so $V_{\mathbf{C}}$ is a complex representation space for G. Prove that if $V_{\mathbf{C}}$ is an irreducible complex representation space then V is an irreducible real representation space but that the converse is false. (It is more elegant to define $V_{\mathbf{C}}$ to be the tensor product $\mathbf{C} \otimes_{\mathbf{R}} V$.)

Find all the irreducible real representations of C_3.

Groups and Representation Theory 4

Unless otherwise stated, all vector spaces are finite dimensional over an algebraically closed field F of characteristic zero.

1. Prove the transitivity of induction: given $H < K < G$, with ϕ a character of H, show that $(\phi^K)^G = \phi^G$.

2. Find all the characters of S_5 induced by the irreducible representations of S_4. Hence find the character table of S_5.

3. Let H be a subgroup of the group G. Show that for every irreducible representation space V for G there is an irreducible representation space W for H with V a component of the induced representation W^G.

Prove that, if A is a commutative subgroup of G, then every irreducible representation for G has degree at most $|G/A|$.

4. a) Prove that if G is a finite group acting on a finite set X with permutation character π and on a set Y with permutation character τ then $\langle \pi, \tau \rangle$ equals the number of orbits of G on the set $X \times Y$.

b) The symmetric group S_N permutes the set $X^{(r)}$ of r element subsets of $\{1, 2, \ldots, N\}$. Let $\chi^{(r)}$ denote the corresponding permutation character. Show that

$$\langle \chi^{(r)}, \chi^{(s)} \rangle = r + 1$$

if $0 \leq r \leq s \leq N - r$. Deduce that there are inequivalent irreducible characters $\psi^{(r)}$ for $0 \leq r \leq N/2$ with $\chi^{(r)}$ being the sum of the $\psi^{(i)}$ with $0 \leq i \leq r$.

c) The linear group $GL_n(q)$ acting naturally on the n-dimensional vector space $V = V_n(q)$ permutes the set $V^{(r)}$ of r-dimensional subspaces of V. Let $\chi^{(r)}$ denote the corresponding permutation character. Show that

$$\langle \chi^{(r)}, \chi^{(s)} \rangle = r + 1$$

if $0 \leq r \leq s \leq N - r$. Deduce that there are inequivalent irreducible characters $\psi^{(r)}$ for $0 \leq r \leq N/2$ with $\chi^{(r)}$ being the sum of the $\psi^{(i)}$ with $0 \leq i \leq r$.

5. Show that the complex character table of a finite group G is invertible when viewed as a matrix.

Prove that the number of irreducible characters of G which take only real values is equal to the number of self-inverse conjugacy classes. [Consider the permutation action induced by complex conjugation on rows and on columns.]

6. Let a finite group G act on itself by conjugation and find the character of the corresponding permutation representation. Prove that the sum of the elements in any row of the character table for G is a non-negative integer.

7. (87/3/3) State and prove the Frobenius Reciprocity Theorem. Illustrate its use by finding the irreducible characters of the dihedral group D_{2n} of order $2n$, where n is even.

8. (88/4/3) Define the character ψ^G of a finite group G which is induced by a character ψ of a subgroup H of G. Prove the Frobenius reciprocity formula

$$\langle \psi^G, \chi \rangle_G = \langle \psi, \chi_H \rangle_H,$$

where χ is any character of G and χ_H is the restriction of χ to H.

Now let H be a subgroup of index 2 in G. An irreducible character ψ of H and an irreducible character χ of G are "related" if

$$\langle \psi^G, \chi \rangle_G = \langle \psi, \chi_H \rangle_H > 0.$$

By considering $\langle \psi^G, \psi^G \rangle_G$ or otherwise show that each ψ of degree n is either "monogamous" in the sense that it is related to one χ (of degree $2n$), or "bigamous" in the sense that it is related to precisely two distinct characters χ_1, χ_2 (of degree n). Show that each χ is either related to one bigamous ψ, or to two monogamous characters ψ_1, ψ_2 (of the same degree).

Write down the degrees of the complex irreducible characters of A_5. Find the degrees of the irreducible characters of a group G containing A_5 as a subgroup of index 2, distinguishing two possible cases.

9. (89/4/3) Let x be an element of order n in a finite group G. Say, without detailed proof, why

(a) if χ is a character of G, then $\chi(x)$ is a sum of n-th roots of unity;

(b) $\tau(x)$ is real for every character τ of G if and only if x is conjugate to x^{-1};

(c) x and x^{-1} have the same number of conjugates in G.

State the orthogonality relations that hold between the rows and columns of the character table of G.

A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have corresponding characters α, β and γ. The table below gives the sizes of the conjugacy classes and the values α, β and γ take on them.

	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

Construct the character table of the group.

[*You may assume, if needed, the fact that* $\sqrt{7}$ *is not in the field* $\mathbf{Q}(\zeta)$, *where* ζ *is a primitive 7th root of unity.*]

10. Construct the character table of the symmetric group S_6 of degree 6.

11. If V is an irreducible complex representation space for G with character χ, find the characters of the representation spaces $V \otimes V$, $Sym^2(V)$ and $\Lambda^2(V)$.

Deduce that:

$$\frac{1}{|G|}\sum_{x\in G}\chi(x^2) = \begin{cases} 0, & \text{if } \chi \text{ is not real-valued;} \\ +1, & \text{if } V \text{ is equivalent to a real representation;} \\ -1, & \text{if } \chi \text{ is real-valued but } V \text{ is not} \\ & \text{equivalent to a real representation.} \end{cases}$$

12**. (Brauer) If θ is a faithful character of the group G, which takes r distinct values on G, prove that each irreducible character of G is a constituent of θ to power i for some $i < r$.

[*The Vandermonde $r \times r$ matrix involving the column of the distinct values $a_1, \ldots, a_r$ of θ is nonsingular.*]

13**. (Burnside) If θ is an irreducible character of degree > 1 of the group G then $\theta(g) = 0$ for some g in G.