MATHEMATICAL TRIPOS **Michaelmas 1995**
PART II: Alternative (B) **Professor P V Landshoff**

Foundations of Quantum Mechanics - Examples I

1. The normalised ground state position-space wave function of the harmonic oscillator has the form $A\exp(-\alpha^2x^2)$. Determine A and α in terms of m, $\omega, \hbar$. Find the momentum-space wave function by Fourier transformation. Check the result by using the similarity between the Schrödinger equations in position and momentum space.

2. Obtain the Schrödinger equation for the hydrogen atom in the momentum-space representation.

$$\left[Hint : 4\pi(\mathbf{k}^2)^{-1} \;=\; \int d^3\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, |\mathbf{x}|^{-1}\right]$$

3. If A and B are operators which each commute with their commutator, prove that $[A, B^n] = nB^{n-1}[A, B]$; $[A^n, B] = nA^{n-1}[A, B]$. Hence show that, if $f(x)$ can be expanded in a power series, then

$$[p_x, f(x)] = -i\hbar\partial f/\partial x\ .$$

If $f(\lambda) = e^{\lambda A}e^{\lambda B}e^{-\lambda(A+B)}$ defines an operator function of the real variable λ, show that $\frac{df}{d\lambda} = \lambda[A, B]f$ and hence that $e^Ae^B = e^{A+B+\frac{1}{2}[A,B]}$.
Show also that $e^Ae^B = e^Be^Ae^{[A,B]}$.

4. If $H = p^2/2m + V(x)$, consider the expectation value of the double commutator $[[x, H], x]$, and obtain the sum rule

$$\sum_n (E_n - E_m)|\langle n|x|m\rangle|^2 = \hbar^2/(2m),$$

where $|n\rangle$ is the state of energy E_n.

5. If $a^\dagger$ is the hermitian conjugate of a, and $[a, a^\dagger] = 1$, calculate $[a, (a^\dagger)^n]$. If $|0\rangle$ is a state such that $a|0\rangle = 0$ calculate the norm of $(a^\dagger)^n|0\rangle$. Show that $|z\rangle = e^{za^\dagger}|0\rangle$ is an eignvector of a if z is any complex number. Calculate directly $\langle z_1|z_2\rangle$. Use the last result of question 3 to confirm the answer.

6. The operators a and $a^\dagger$ satisfy the relations $aa^\dagger + a^\dagger a = 1, \quad a^2 = 0$. Show that the eigenvalues of $N_a = a^\dagger a$ can only be 0 and 1. If $|0\rangle$ and $|1\rangle$ are the corresponding normalised eigenvectors of N_a, express $|1\rangle$ in terms of $|0\rangle$, and $|0\rangle$ in terms of $|1\rangle$. Obtain the eigenvalues of H

$$H = \alpha a^\dagger a + \beta b^\dagger b + \gamma(a^\dagger b + ab^\dagger),$$

where α, β and γ are real numbers, and

$$bb^\dagger + b^\dagger b = 1, \quad b^2 = 0, \quad [a, b] = 0 = [a, b^\dagger].$$

7. Let $\{|j\rangle,\ j = 1, 2, \cdots N,\}$ be an orthonormal set of eigenvectors of an observable A with non-degenerate eigenvalues $\{a_j\}$. Show that $1 = \sum_i |i\rangle\langle i|$, and hence that $A = \sum_i a_i|i\rangle\langle i|$.
Show that

$$P_i(A) = \prod_{j\neq i} \frac{A - a_j}{a_i - a_j}$$

is a projection operator onto the one-dimensional subspace spanned by $|i\rangle$, i.e., if $|\ \rangle = \sum_i c_i|i\rangle$, show that $P_i(A)|\ \rangle = c_i|i\rangle$.
Show, by considering actions on $|\ \rangle$, that

$$P_i(A)P_j(A) = \delta_{ij}P_j(A), \quad \sum_i P_i(A) = 1.$$

8. Verify that, in **both** the Heisenberg picture **and** the Schrödinger picture, the relation

$$i\hbar\frac{d}{dt}\langle u|A|v\rangle = \langle u|[A, H]|v\rangle$$

is true.

9. Show that, in the nth eigenstate of a linear harmonic oscillator, $\Delta x\Delta p = (n + \frac{1}{2})\hbar$.

10. Express x and p in terms of the ladder operators of a harmonic oscillator. Show that in the state $|z\rangle = e^{za^\dagger}|0\rangle$ the expectation values are

$$\langle x\rangle = \surd(\frac{2\hbar}{m\omega})\ \mathrm{Re}\, z, \quad \langle p\rangle = \surd(2\hbar m\omega)\ \mathrm{Im}\, z\ .$$

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Foundations of Quantum Mechanics - Examples II

1. Construct ladder operators a_j , $a_j^\dagger$ for the three dimensional harmonic oscillator with Hamiltonian

$$H = \sum_{i=1}^{3} \left[\frac{1}{2m}\, p_i^2 + \frac{1}{2}\, m\omega^2 x_i^2\right]$$

Use these to find the energy levels. Show that the degeneracy of the n^{th} excited state is $\frac{1}{2}(n+1)(n+2)$.

2. Show that in coordinate space the annihilation operator for the simple harmonic oscillator has the representation $a = \frac{1}{\sqrt{2}}\left[\frac{d}{dy} + y\right]$ where $y = \left[\frac{m\omega}{\hbar}\right]^{\frac{1}{2}} x$. Use the relation $a|0>= 0$ to calculate the wave function $\psi_o(x)$ of the ground state. Use the creation operator to show that the wave function of the n^{th} excited state is given by

$$\psi_n(y) = h_n(y)\ \psi_0(y)$$

where $h_n(y) = (2^n n!)^{-\frac{1}{2}}\ e^{y^2}\ \frac{d^n}{dy^n}\ e^{-y^2}(-)^n$.

3. S and S' are two independent quantum systems. The state vectors for S and S' are the non-zero vectors of vector spaces V and V'. An orthonormal basis for V is $|a\rangle$, $|b\rangle$, and one for V' is $|a'\rangle$, $|b'\rangle$. An operator A for the system S is defined by $A|a\rangle = |b\rangle$, $A|b\rangle = -|a\rangle$, and an operator A' for the system S' is defined by $A'|a'\rangle = |b'\rangle$, $A'|b'\rangle = -|a'\rangle$. The unit operators for S and S' are I and I'.

Consider the composite system SS'. Find the eigenvalues of the operator

$$W = I\ I' \ +\ A\ A' .$$

[*Hint:* One way to do this is to consider actions of W on the four basis vectors of the vector space $|V\rangle|V'\rangle$.]

4. A harmonic oscillator of charge e is perturbed by a constant electric field of strength E. Calculate the change in energy levels to order E^2 and compare with the exact result.

5. Two harmonic oscillators of frequency ω and mass m interact through a potential $2m\omega^2\lambda x_1 x_2$ where x_1 and x_2 are the displacements of the oscillators. Use perturbation theory to calculate, to second order in powers of λ, the ground state energy of the system. Verify your results by using the change of variables

$$\sqrt{2}\, q_1 = x_1 + x_2, \quad \sqrt{2}\, q_2 = x_1 - x_2,$$

to obtain an exact expression for the energy eigenvalues.

6. The interaction between two harmonic oscillators is described by the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + x_1^2) + \frac{1}{2}(p_2^2 + x_2^2) + f x_1 x_2$$

What are the degeneracies of the three lowest energy levels of H when $f = 0$? For $f \neq 0$, compute to lowest non-trivial order in perturbation theory, the energies of the corresponding states.
[It may be assumed that $|n\rangle = (n!)^{-\frac{1}{2}}(a^\dagger)^n|0\rangle$, where $a = \frac{1}{\sqrt{2\hbar}}(x + ip)$, obeys

$$1/2(p^2 + x^2)|n\rangle = (n + 1/2)\hbar|n\rangle. \ ]$$

7. The quantum-mechanical observable Q has just three eigenstates, $|1\rangle$, $|2\rangle$, $|3\rangle$ which are orthogonal and correspond to eigenvalues 2, 1, 1 respectively. The operator Q' is defined by $Q' = Q + \epsilon T$, where

$$\langle j|T|i\rangle = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & \sqrt{6} \\ 0 & \sqrt{6} & 1 \end{pmatrix} .$$

Deriving the necessary formulae from first principles, apply perturbation theory to find two of the eigenvalues of Q' correct to order ϵ and the third correct to order ϵ^2 .

8. A first excited level of the hydrogen atom has energy E_1 and is four-fold degenerate. The four wave functions can be taken to be $\psi_o = C_o\left[1 - \frac{r}{2a}\right]\ e^{-r/2a}$, $\psi_i = C_i x_i e^{-r/2a}$ where x_i are the Cartesian coordinates of the electron and a, C_o, C_i are constants. The atom is perturbed by a weak electric field ϵ parallel to the z-axis. Calculate to first order in ϵ the new energy levels corresponding to E_1.

9. Repeat question 1 using spherical polar coordinates and writing ∇^2 in terms of $\mathbf{L}^2$ as in the analysis of the hydrogen atom.

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Foundations of Quantum Mechanics - Examples III

1. If **A** and **B** are commuting operators (or simply vectors which are not operators) show that

$$\boldsymbol{\sigma}\cdot\mathbf{A}\,\boldsymbol{\sigma}\cdot\mathbf{B}=\mathbf{A}\cdot\mathbf{B}+i\boldsymbol{\sigma}\cdot\mathbf{A}\times\mathbf{B}$$

Show also that, if **n** is a unit vector,

$$(\boldsymbol{\sigma}\cdot\mathbf{n})\,\boldsymbol{\sigma}\,(\boldsymbol{\sigma}\cdot\mathbf{n})=2\mathbf{n}\,\mathbf{n}\cdot\boldsymbol{\sigma}-\boldsymbol{\sigma},$$

$$\exp(\,i\omega\mathbf{n}\cdot\boldsymbol{\sigma})=\cos\omega+i\mathbf{n}\cdot\boldsymbol{\sigma}\sin\omega$$

2. A quantum particle is in a state with orbital angular momentum $l=1$ and with $L_z=m$. Calculate the expectation values of a component of **L** in a direction making an angle θ with the z-axis, and also of its square. Calculate the probabilities of obtaining the values 1, 0 and -1 in a measurement of this component.

3. A particle of spin $\frac{1}{2}$ and magnetic moment $\frac{1}{2}\hbar\mu\boldsymbol{\sigma}$ is at rest in a magnetic field B, parallel to the z-axis. A magnetic field B' is switched on parallel to the x-axis. Use perturbation theory to calculate the change in the energy to order B'^2 and compare with the exact answer.

4. A particle of spin $\frac{1}{2}$ and spin operator $\mathbf{S}=\frac{1}{2}\boldsymbol{\sigma}$, which is at rest, interacts with a uniform homogeneous magnetic field $B\mathbf{n}$ through its magnetic moment $\boldsymbol{\mu}=\gamma\mathbf{s}$, the Hamiltonian being

$$H=-\tfrac{1}{2}\gamma B\,\mathbf{n}\cdot\boldsymbol{\sigma}\;.$$

Write down the state-vector at time t in terms of the state-vector at $t=0$. The polarization of the particle is defined to be $\langle\mathbf{s}\rangle$ and at time $t=0$ it is $\mathbf{P}_o$. Show that at time t the polarization is given by

$$\mathbf{P}=\mathbf{P}_o\;\cos\omega t+\mathbf{P}_o\times\mathbf{n}\sin\omega t+\mathbf{n}(\mathbf{n}\cdot\mathbf{P}_o)\,(1-\cos\omega t)\;,$$

where $\hbar\omega=\gamma B$. (You will probably find the results of question 1 useful.)

5. A particle of spin $\frac{1}{2}$ is subjected to a magnetic field in the direction of the unit vector **n**. Given that $\boldsymbol{\sigma}$ are the Pauli matrices, show that $P_n\equiv\frac{1}{2}(1+\mathbf{n}\cdot\boldsymbol{\sigma})$ is hermitian , satisfies $P_n^2=P_n$, and has eigenvalues 0 and 1. Given that $|1\rangle$ is the normalised eigenstate of P_n with eigenvalue 1, verify that $P_n=|1\rangle\langle 1|$. Hence, or otherwise, deduce that the expectation value of $\boldsymbol{\sigma}$ in the state $|1\rangle$ is $tr\,(P_n\boldsymbol{\sigma})$, where tr indicates the matrix trace.
The particle is initially in the state $|1\rangle$ and the magnetic field is suddenly rotated to the direction of the unit vector **m**. Show that the probability that a measurement of the spin will find it aligned (rather than anti-aligned) with the field is $\frac{1}{2}(1+\mathbf{n}\cdot\mathbf{m})$.

6. Determine the Clebsch-Gordon coefficients for the addition of angular momentum $\frac{1}{2}$ to angular momentum 1.

7. Two particles, each of spin one, are at rest. It is known that the y-component of the spin of each is zero. Calculate the probability that the total angular momentum of the system is zero.

8. Two non-interacting particles of equal mass are in a rigid box with sides $a<b<c$. Give the two lowest energy levels, their degeneracy and the corresponding wave functions if the particles are (i) spin 0 but not identical (ii) spin 0 and identical (iii) spin $\frac{1}{2}$ and identical.

9. Show that a one-dimensional square well potential always has at least one bound state. Use the variation principle to deduce that this remains true for any shape of potential, provided it is purely attractive and vanishes outside a finite range. Is it true in three dimensions?

10. A particle of mass m moves in one dimension under the influence of a potential $V(x)=\sqrt{\pi}\,A|x|$. Show that the best estimate of the ground state energy with the trial wave function $\psi(x)=N\,\exp(-\frac{1}{2}\,\lambda x^2)$, where λ is a parameter, is

$$E=\frac{3}{2}\left(\frac{\hbar^2A^2}{2m}\right)^{\frac{1}{3}}\;.$$

State carefully any theorem on which you base your calculation.
How would you modify the trial wave function in order to obtain an estimate for the energy of the first excited state? (Do not actually bother to do a calculation of the estimate.)

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PART II: Alternative (B)

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Foundations of Quantum Mechanics - Examples IV

1. It is known that π^- mesons are absorbed from S-state orbits about deuterium nuclei to give two neutrons: $\pi^- + D \to N + N$. Show that this implies that the pion has negative parity. (The spins are π, 0; D, 1; N, $\frac{1}{2}$).

2. Show that a particle of spin 1 cannot decay into two identical particles of spin 0. The ρ-meson has spin 1 and can decay into two pions of different charges. What is its intrinsic parity?

3. The particle K has spin zero. A particle A is observed to undergo the two parity-conserving decays $A \to \rho^+ + \pi^-$, $A \to K + K$. What is the lowest spin value for A that is consistent with this, and what is the corresponding intrinsic parity of A?

4. (II83330) Show that, in the expression for the total spin states $|SM\rangle$, in terms of the product states $|s_1 = 1\, m_1; s_2 = 1\, m_2\rangle$, the states with $S = 0$ and $S = 2$ are symmetric under the interchange of m_1 and m_2, whereas those with $S = 1$ are antisymmetric.
A boson C in its rest frame decays via interactions that conserve total angular momentum and parity into two identical bosons of intrinsic spin 1. If S is the total spin and L is the relative orbital angular momentum in the final state, show that $L + S$ must be even.
Deduce the values of L and S when C has spin 1 and negative intrinsic parity.
[The result $J_-|jm\rangle = c_{jm}\, |j\, m-1\rangle$, where $c_{jm} = [(j+m)(j-m+1)]^{\frac{1}{2}}$, may be assumed.]

5. (a) Show that the operator $U(b) = \exp(-ip\, b/\hbar)$, which describes translations in one dimension, satisfies $U(b)xU(b)^\dagger = x - b$ and $U(b)H_0U(b)^\dagger = H_b$, where

$$H_b = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2(x-b)^2.$$

(b) If

$$a = \frac{m\omega x + ip}{\sqrt{(2m\hbar\omega)}},$$

use Question 3 of Problem Set I to prove

$$U(b) = e^{-\frac{1}{2}C^2}\, e^{Ca^\dagger}\, e^{-Ca}, \quad C = b\Big(\frac{m\omega}{2\hbar}\Big)^{\frac{1}{2}}.$$

(c) Hence show that the ground state of H_b is given, in terms of the standard eigenstates of H_0, by

$$e^{-\frac{1}{2}C^2}\sum_{n=0}^{\infty}(n!)^{-\frac{1}{2}}C^n|n\rangle.$$

6. The unitary matrix $V = \exp(-i\theta S_z)$ should produce a rotation of the spin coordinate system about the z-axis, through angle θ. For the case of spin $\frac{1}{2}$, show that

$$V = \begin{pmatrix} e^{-\frac{1}{2}i\theta} & 0 \\ 0 & e^{\frac{1}{2}i\theta} \end{pmatrix}.$$

Verify that $V^{-1}\sigma V$ can be interpreted as σ in a rotated coordinate system. Show that for a rotation about the axis $\mathbf{n}$,

$$V = \cos\tfrac{1}{2}\theta - i\sigma.\mathbf{n}\sin\tfrac{1}{2}\theta$$

7. (II90130) Let $\mathbf{J} = (J_x, J_y, J_z)$ and $|jm\rangle$ denote the standard angular momentum operators and state vectors, so that, using units in which $\hbar = 1$,

$$\mathbf{J}^2\,|jm\rangle = j(j+1)|jm\rangle, \quad J_z|jm\rangle = m|jm\rangle.$$

If $U(\theta) = \exp(-iJ_y\theta)$, show that $U(\theta)$ is unitary, and that

$$U(\theta)\,J_x\,U(\theta)^{-1} = J_x\cos\theta - J_z\sin\theta,$$

$$U(\theta)\,J_z U(\theta)^{-1} = J_x\sin\theta + J_z\cos\theta.$$

Show that the state vectors $U(\frac{\pi}{2})\,|jm\rangle$ are eigenvectors of J_x. If J_x is measured for a system in the state $|jm\rangle$, calculate the probability that the result of the measurement is m'. Take $j = m = \frac{1}{2}$, and evaluate the probability in the cases $m' = \frac{1}{2}$ and $m' = -\frac{1}{2}$.

[Note that the above result for the rotation J_x, J_z to J'_x, J'_z, where $J'_k = U(\theta)\,J_k\,U(\theta)^\dagger$, $k = x, z$, is best proved by consideration of $\frac{\partial J'_k}{\partial\theta}$.]

8. (II84130) Define the Heisenberg picture of quantum mechanics in relation to the Schrödinger picture. Explain how the two pictures provide equivalent descriptions of observable results, and derive the equations of motion for a Heisenberg picture operator when the corresponding Schrödinger picture operator is independent of time. The behaviour of a particle of spin $\mathbf{S} = (S_1, S_2, S_3)$, located at a fixed point of space, in a uniform time-independent magnetic field $\mathbf{B} = (0,0,B)$, is governed by the Hamiltonian $H = \mu\mathbf{S}.\mathbf{B}$, where μ is a constant. Here H and $\mathbf{S}$ are Schrödinger picture operators. Let $K(t)$ and $\mathbf{P}(t)$ denote the corresponding Heisenberg picture oprators.
How is $K(t)$ related to a) H and b) $\mathbf{P}(t)$? By consideration of $\frac{\partial P_1}{\partial t}$ and $\frac{\partial P_2}{\partial t}$, show that $(P_1(t), P_2(t))$ is related to (S_1, S_2) by a formula which corresponds to a rotation of angle ωt, where $\omega = \mu B$.

9. Let

$$\mathbf{f}(\theta) = \exp(-i\theta\mathbf{n}.\mathbf{J}/\hbar)\mathbf{p}\exp(i\theta\mathbf{n}.\mathbf{J}/\hbar)$$

describe the rotation of the momentum operator $\mathbf{p}$ through angle θ about the direction of the constant unit vector $\mathbf{n}$. Using $[J_i, p_j] = i\hbar\epsilon_{ijk}p_k$, show that $\frac{\partial\mathbf{f}}{\partial\theta} = \mathbf{f}\wedge\mathbf{n}$, and hence that $\mathbf{n}.\mathbf{f}$ is independent of θ and

$$\frac{\partial^2\mathbf{f}}{\partial\theta^2} + \mathbf{f} = \mathbf{n}.\mathbf{p}\,\mathbf{n} \quad .$$

Show that $\mathbf{n}.\mathbf{p}\,\mathbf{n}(1-\cos\theta)$ is a 'particular integral', and hence derive the familiar result

$$\mathbf{f}(\theta) = \mathbf{p}\cos\theta + \mathbf{n}.\mathbf{p}\,\mathbf{n}(1-\cos\theta) + \mathbf{p}\wedge\mathbf{n}\sin\theta.$$