Exercise Sheet 1

C. Baesens, Lent 1996

1. (Homeomorphisms of the interval) Let f be a homeomorphism of a closed interval $I \subset \mathbb{R}$ onto itself.

(i) If f is orientation-preserving, prove that the set P of fixed points of f is a closed subset of I containing the end points. If (a,b) is a complementary interval of P in I prove that

either $f(x) > x \ \forall x \in (a,b)$ and $\omega(x) = b$, $\alpha(x) = a$,

or $f(x) < x \ \forall x \in (a,b)$ and $\omega(x) = a$, $\alpha(x) = b$.

(ii) If f is orientation-reversing prove that the ω(resp. α)-limit set of every point is either a fixed point or a period 2 orbit. [Hint: consider f^2]

2. (Lift, degree, topological conjugacy) Each of the following complex functions defines a map of the circle, $\{ z \in \mathbb{C} \mid |z| = 1\}$.
In each case find lifts of the circle map and determine the degree of the map:

$\bar{z}$, $-z$, $-\bar{z}$, z^m $(m \in \mathbb{Z})$, $\bar{z}^m$ $(m \in \mathbb{Z})$.

3. Let $f : S^1 \to S^1$ be a continuous map, and n a positive integer. If F is a lift of f prove that

(i) F^n is a lift of f^n;

(ii) $\deg(F^n) = [\deg(F)]^n$.

4. Show that the maps $x \mapsto 2x$ and $x \mapsto 3x$ are topologically conjugate as maps of $\mathbb{R}$ onto itself. Are they differentiably conjugate?
[Hint: the function $\operatorname{sign}(x)\,|x|^d$ might be helpful.]
Are the circle maps $x \mapsto 2x$ and $x \mapsto 3x$ topologically conjugate ?

5. Suppose f and g are continuous maps of the circle and f is topologically conjugate to g by a degree one homeomorphism h. Prove that

(i) $\deg(f)=\deg(g)$;

(ii) $h \circ f^n = g^n \circ h$, for $n \in \mathbb{N}$;

(iii) if $f^n(x) = x$ then $g^n(h(x)) = h(x)$.

If, in addition, f and g are degree one homeomorphisms, and F, G and H are lifts of f, g and h resp. prove

(iv) there exists a lift F_k of f such that F_k is topologically conjugate to G by H.

6. For $m \geq 1$ let $f_m : S^1 \to S^1$ be continuous maps of the circle with lifts

$$F_m(x) = x + \frac{1}{2\pi m}\sin(2\pi m x)$$

Prove (i) f_m is a degree one homeomorphism of S^1 onto itself;

(ii) $\rho(f_m) = 0$ for all $n \geq 1$;

(iii) f_m has precisely 2m fixed points;

(iv) if $m \neq n$ then f_m is not topologically conjugate to f_n.

7. Show that an orientation-reversing homeomorphism of the circle onto itself has two fixed points.

8. (Rotation number) Let f and g be orientation-preserving homeomorphisms of the circle. Prove that $\rho(f \circ g) = \rho(g \circ f)$ and hence deduce that if f is topologically conjugate to g then $\rho(f) = \rho(g)$.
If F and G commute prove that $\rho(F \circ G) = \rho(F) + \rho(G)$.

9. Let F be a lift of an orientation-preserving homeomorphism $f : S^1 \to S^1$ and $n \in \mathbb{N}$. Prove that there exists $k \in \mathbb{Z}$ (depending on n) such that $k - 1 < F(x) - x < k + 1$ for all $x \in \mathbb{R}$.

10. Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with rotation number p/q, p and q coprime. Prove that *all* periodic orbits have period q.

11. Suppose p, q and r are positive integers, and p and q are coprime. For what values of μ is $F(x) = x + p/q + \mu \sin 2\pi qrx$ the lift of a diffeomorphism of the circle, f ? Show that f has 2r periodic orbits of period q and has rotation number p/q.

12. Let F be a lift of an orientation-preserving homeomorphism $f : S^1 \to S^1$ with $\rho(F) = \beta \in \mathbb{R}\setminus\mathbb{Q}$. Then for $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ and $x \in \mathbb{R}$

$$n_1\beta + m_1 < n_2\beta + m_2 \iff F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2 .$$

Hence deduce that the orbits of f are ordered on S^1 like those of the rigid rotation $r_\beta : x \mapsto x + \beta$.

13. (Invariant set, ω-limit set, minimal set)
Let X be a complete metric space and $f : X \to X$ a homeomorphism.

(a) Prove that for any $x \in X$ the ω-limit set $\omega(x)$ is closed and invariant under f.

(b) Let $S \subset X$ be a closed invariant set. Prove that the boundary ∂S of S is a closed invariant subset of S.

Recall that a set S is a *minimal set* for f provided (i) S is a closed, non-empty, invariant set, and (ii) if A is a closed, non-empty invariant subset of S then A=S.

(c) Let $S \subset X$ be a non-empty compact subset. Prove

S is a minimal set $\iff \omega(x) = S$ for all $x \in S$

$\iff$ for every $x \in S$ the orbit $\mathcal{O}(x)$ is dense in S.

14. (Denjoy counterexample) Construct explicitly an orientation-preserving homeomorphism $f : S^1 \to S^1$ with irrational rotation number, for which the minimal set E is a Cantor set and the complementary intervals I_n, $-\infty<n<\infty$, could be indexed so that $f(I_n)=I_{n+1}$ and such that $f : I_n \to I_{n+1}$ is an affine map. Show that f is not C^1.

15. Consider a modification of the Denjoy counterexample in which two distinct orbits are blown up. Is this possible? What is (are) the minimal set(s) of the resulting map?

16. (Families of circle maps) If F is a lift of an orientation-preserving circle homeomorphism with $\rho(F) = p/q$ and $F^q(x) > x + p$ for some $x \in \mathbb{R}$, $q \in \mathbb{N}$, $p \in \mathbb{Z}$, prove that $\rho(\bar{F}) \geq p/q$ for all near enough $\bar{F}$. Deduce that $\rho(\bar{F}) = p/q$ for all near enough $\bar{F} < F$.

17. (Arnold family - devil's staircase) Take the lift $F_\omega(x) = x + \omega + (k/2\pi) \sin 2\pi x$ of the Arnold map. For $k \in (0,1]$, $\omega \in \mathbb{R}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$ prove that $F_\omega{}^q - p \neq \mathrm{Id}$. (optional part - hard!)
[Hint: consider the analytic continuation of F to $\mathbb{C}$ and use the fact that F' has a zero in $\mathbb{C}$.]
Hence deduce that for every rational p/q there is an interval of ω values such that $\rho(F_\omega) = p/q$.

18. (Arnold tongues) Show that the Arnold circle map, with lift

$$F_{k,\omega}(x) = x + \omega + \frac{k}{2\pi} \sin 2\pi x$$

is a diffeomorphism if $|k| < 1$. Suppose $0 < k \ll 1$ and set $\omega = \frac{1}{2} + \varepsilon$, $|\varepsilon| \ll 1$. Show that the region in (ω,k) parameter space for which $\rho(F_{k,\omega}) = \frac{1}{2}$ is given by

$$-\frac{1}{8\pi}k^2 + O(k^3) \leq \varepsilon \leq \frac{1}{8\pi}k^2 + O(k^3).$$

Exercise Sheet 2

C. Baesens, Lent 1996

Let $\Sigma_N = \{\mathbf{a} = (a_0,a_1,...) \mid a_i \in \{0,1,...,N-1\}\}$ with metric $d(\mathbf{a},\mathbf{b}) = \sum_{n=0}^{\infty} \frac{\delta(a_n,b_n)}{3^n}$ where

$\delta(i,j) = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \neq j \end{cases}$, and define the shift map by $\sigma(a_0,a_1,...) = (a_1,a_2,...)$.

1. Prove that for $\mathbf{a} \neq \mathbf{b}$,
   (a) $d(\sigma\mathbf{a},\sigma\mathbf{b}) = 3\, d(\mathbf{a},\mathbf{b})$ if and only if $a_0 = b_0$;
   (b) $d(\sigma\mathbf{a},\sigma\mathbf{b}) = d(\mathbf{a},\mathbf{b})$ if and only if $a_n \neq b_n$ for all $n \geq 0$;
   (c) for any $\mathbf{a} \in \Sigma_N$

$$\inf\{ d(\sigma^m\mathbf{a},\sigma^n\mathbf{a}), m \neq n\} = 0.$$

2. Suppose that for some $\mathbf{x}$ and $\mathbf{y}$ in Σ_2, $\sigma^n(\mathbf{x}) \to \mathbf{y}$ as $n \to \infty$. Show that $\mathbf{y}$ is one of the sequences (0,0,0,...) or (1,1,1,...). Show further that the set $\{\mathbf{x} \in \Sigma_2 \mid \sigma^n(\mathbf{x}) \to (0,0,0,...)\}$ is dense in Σ_2 but that there is no open set in Σ_2 such that all points in the set tend to (0,0,0,...).

3. Let $$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

(i) Show that
$$A^{n+2} = A^{n+1} + A^n.$$
(ii) Find all the periodic orbits with least period ≤ 5 of the subshift defined by A.
(iii) Prove that $\Sigma_A = \{ a_0a_1... \in \Sigma_2 \mid A_{a_i,a_{i+1}} = 1\ \forall\, i \geq 0\}$ is a Cantor set.

4. Suppose that $$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
How many periodic orbits of length 3 does the corresponding subshift has?

5. Let $X = \{\mathbf{x} \in \Sigma_2 \mid \mathbf{x} = x_0x_1x_2...,\ x_ix_{i+1}x_{i+2} \neq 011 \text{ for all } i \geq 0\}$.

Let $$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$ and let $\Sigma_A = \{ a_0a_1... \in \Sigma_4 \mid A_{a_i,a_{i+1}} = 1\ \forall\, i \geq 0\}$.

Show that the map $h : X \to \Sigma_A$ defined by
$$h(x_0x_1x_2...) = (y_0y_1y_2...),$$
where $y_i = 0$ if $x_ix_{i+1} = 00$, $y_i = 1$ if $x_ix_{i+1} = 01$, $y_i = 2$ if $x_ix_{i+1} = 10$, and $y_i = 3$ if $x_ix_{i+1} = 11$, is a topological conjugacy between the shift map, σ, on X, and the shift map on Σ_A.
Is either of these maps topologically transitive? Justify your answer.

6. Show that if p is prime then either p divides n or $n^{p-1} = 1 \pmod p$.
[Hint: take the space of all sequences of n symbols and consider the number of cycles of least period p.]
Let A be an $n \times n$ matrix with $A_{ij} \in \{0,1\}$ and suppose that p is prime. Prove that

$Tr(A^p) - Tr(A) = kp$ for some natural number, k.

7. Let $T_s(x) = \begin{cases} sx & \text{if } x \in [0,1/2] \\ s(1-x) & \text{if } x \in (1/2,1] \end{cases}$

with $\sqrt{2} < s \leq 2$.

(a) Show that the interval $I_s = [T_s^2(1/2), T_s(1/2)]$ is mapped into intself by T_s.

(b) For any open interval $J \subset I_s$ show that there exists $m \geq 0$ such that $1/2 \in T_s^m(J)$.

(c) Let $K \subset I_s$ with $1/2 \in K$. Show that $|T_s(K)| \geq s/2\, |K|$ where $|K|$ denotes the length of the interval K.

(d) Hence show that there exists $n > 0$ such that $1/2 \in T_s^n(J)$ and $1/2 \in T_s^{n+1}(J)$.

(e) Show that T_s is transitive on I_s and that periodic orbits are dense in I_s.

8. A continuous map of the interval has points $p_1 < p_2 < p_3 < p_4$ on an orbit of least period four, which defines a permutation on $\{1,2,3,4\}$, π, such that $f(p_i) = p_{\pi(i)}$. Show that there is a permutation that implies that the map has a periodic orbit of every period.

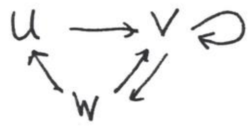
9. Let $p_1 < p_2 < p_3 < \ldots < p_6$ be points on a periodic orbit of period six for a continuous map of the interval, f. Suppose that

$$f(p_1) = p_6, \quad f(p_2) = p_5, \quad f(p_3) = p_4,$$
$$f(p_4) = p_2, \quad f(p_5) = p_1, \quad f(p_6) = p_3\,.$$

Show that f has periodic points of every even period.

10. * Let f be a degree one continuous map of the circle with lift F. Suppose that there is a point x with $F(x) = x$, and points y_i $(i=1,2)$ with $y_1 < x < y_2$ and $F(y_1) = y_2$, $F(y_2) = y_1 + 1$.

(i) Let U, V, W be the projections to the circle of the intervals $[y_1,x]$, $[x,y_2]$, $[y_2,y_1+1]$. Prove the f-cover relations



(ii) For every rational $\omega \in [0,\frac{1}{2}]$ prove that there exists $x \in S^1$ such that $\rho(F,x) = \omega$.

(iii) Let a_n satisfy the difference equation

$$a_n - a_{n-1} - a_{n-2} - a_{n-3} = 0$$

with $a_1 = 1$, $a_2 = 3$ and $a_3 = 7$. If $p \geq 5$ is prime prove that $\dfrac{a_p - 1}{p} \in \mathbb{N}$.

11. * If $f : I \circlearrowleft$, continuous, has a period 3 orbit let I_1, I_2 be the intervals between its points and suppose $f(I_1) = I_2$, $f(I_2) = I_1 \cup I_2$. Prove that f restricted to $I_1 \cup I_2$ is <u>not</u> semiconjugate to the shift on Σ_A with $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. [Hint : the image of a connected set by a continuous map is connected.]

Find two symbol sequences $a_0a_1\ldots$, $b_0b_1\ldots$ in Σ_A such that $\bigcap_{M\geq 0} I_{a_0\ldots a_M} = \bigcap_{M\geq 0} I_{b_0\ldots b_M}$ (using the notation of the proof of proposition 2).

Exercise Sheet 3 C. Baesens, Lent 1996

Some of the first questions will be familiar/revision for those who took the O course on Nonlinear Dynamical Systems last year. Feel free to ignore them if you are happy with this material.

1. A periodic orbit encloses precisely M fixed points, each of which is isolated and hyperbolic. If N of these fixed points are either nodes or foci and all the remainder are saddles, find a relationship between M and N and hence show that M is odd.

2. Prove Dulac's criterion for non-existence of periodic orbits.

3. Let $\phi^t(.)$ be a flow generated by a vector field on $\mathbb{R}^2$ and let D be a positively invariant compact set for this flow. Prove that for $p \in D$
   (i) $\omega(p) \neq \emptyset$;
   (ii) $\omega(p)$ is closed;
   (iii) $\omega(p)$ is invariant under the flow, i.e., $\omega(p)$ is a union of orbits;
   (iv) $\omega(p)$ is connected;
   (v) if $q \in \omega(p)$ then $\omega(q) \subset \omega(p)$.

4. Consider the differential equation
$$\dot{x} = x - y - (x^2 + \tfrac{3}{2}y^2)x, \quad \dot{y} = x + y - (x^2 + \tfrac{1}{2}y^2)y.$$
Use the Poincaré–Bendixson Theorem to show that there is at least one periodic orbit and determine the smallest annular region in which these orbits can be shown to lie.

5. Consider the differential equation
$$\dot{x} = y + cx(1 - 2b - r^2), \quad \dot{y} = -x + cy(1 - r^2),$$
where $r^2 = x^2 + y^2$ and the constants b and c satisfy $0 \le b \le \frac{1}{2}$ and $0 < c \le 1$. Prove that there are at least one periodic orbit and that if there are several then they all have the same period, $P(b,c)$. If $b = 0$ prove that there is one and only one periodic orbit.

6. Suppose $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ and that $D \subset \mathbb{R}^2$ is an annular Poincaré–Bendixson domain. If $\nabla . f < 0$ in D, prove that there is a unique periodic orbit in D.

   If $\dot{r} = \mu r - r^3(1 + R(r,\theta))$, $\dot{\theta} = \omega + Q(r,\theta)r^3$, where $\omega > 0$ and both R and Q are continuously differentiable with $R(0,\theta) = 0$, prove that there is a neighbourhood of the origin such that for sufficiently small $|\mu|$, if $\mu < 0$ then all trajectories in this neighbourhood tend to the origin whilst if $\mu > 0$ then there is a unique periodic orbit in this neighbourhood of the origin.

7. Consider $\dot{x} = Ax + O(|x|^2)$, where A is a constant 2×2 real matrix. Prove that there is a closed curve C enclosing the origin which trajectories cross outwards if all eigenvalues of A have strictly positive real parts.

8. Consider $\ddot{x} + g(x) = 0$. If $g(0) = 0$, $xg(x) > 0$ if $x \neq 0$, and $\int_0^\infty g(u)\,du \to \infty$ as $|x| \to \infty$, prove that all non-stationary trajectories are periodic.

By considering $g(x) = xe^{-x^2}$, show that if the integral condition does not hold then not all trajectories need be periodic.

9. If $a > 0$, show that $\ddot{v} + \dot{v}^3 - a\dot{v} + v = 0$ has at least one periodic orbit.

10. Consider the oscillator

$$\ddot{x} + \varepsilon(\dot{x}^2 - \alpha + 2E(x))\dot{x} + \frac{dE}{dx} = 0$$

where $E(x) = \beta x^2 + x^4$. Discuss the nature of the fixed points in the case $\varepsilon = 0$. If $0 < \varepsilon \ll 1$, find the number of periodic orbits as a function of α and β. What can you deduce if ε is not small?

11. Find the amplitude (to leading order) of the periodic solution(s) to

$$\ddot{x} + \varepsilon(x^2\dot{x}^2 - 2)\dot{x} + 4x = 0, \quad 0 < \varepsilon \ll 1.$$

12. Find the resonant values of ω ($\omega > 0$) for solutions of

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + (1 + \varepsilon\delta)x = 2\varepsilon\dot{x}^2\cos\omega t,$$

where δ is a positive real parameter and $0 < \varepsilon \ll 1$. Describe the dynamics if $\omega = 3$ for the two cases (a) $0 < \delta \ll 1$; and (b) $\delta \gg 1$.

13. Consider the equation

$$\ddot{x} + x = -\varepsilon x(\delta + 4\cos 2t), \quad 0 < \varepsilon \ll 1.$$

Show that solutions grow (on time scales of order ε^{-1}) if $\delta^2 < 4$.

14. Consider the equation

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + (1 + \delta\varepsilon)x = -8a\cos 3t, \quad 0 < \varepsilon \ll 1.$$

Use the method of multiple scales to derive a slow time amplitude equation at first order. If $\delta = 0$, describe the different behaviours of the system as a function of a^2.

All corrections to: c.baesens@damtp, *please.*