Denjoy Counterexample

1. Blowup

wlog blowup $\mathcal{O}(0)$.

- choose l_n, $n \in \mathbb{Z}$ s.t. $\sum_{n\in\mathbb{Z}} l_n < \infty$
- choose $k \in \mathbb{R}_+$, e.g. $k=1$ or $k=0$.

Let $L = k + \sum_{n\in\mathbb{Z}} l_n$

We will define a continuous order-preserving mapping h from a circle $S^{1\prime}$ of length L onto a circle S^1 of length 1 such that $h^{-1}(r_\beta^n(0))$ is a closed interval I_n of length l_n and $\forall x \notin \mathcal{O}(0)$ $h^{-1}(x)$ is a single point. This is most easily done by defining $h^{-1}(x)$ for each $x \in S^1$.

Think of S^1 as $[0,1)$ with ends identified

$S^{1\prime}$ as $[0,L]$ — " — .

Let $h^{-1}(0)$ be the interval $[0, l_0]$.

For $x_n = r_\beta^n(0)$, $h^{-1}(x_n)$ is the interval $[a_n, b_n]$ with $a_n = k x_n + \sum_{i:\, i\beta (\mathrm{mod}\, 1) \in [0,x_n)} l_i$ and $b_n = k x_n + \sum_{i:\, i\beta (\mathrm{mod}\, 1) \in [0,x_n]} l_i$.

For $x \notin \mathcal{O}(0)$, $h^{-1}(x)$ is the point $y = kx + \sum_{i:\, i\beta(\mathrm{mod}\, 1) \in [0,x)} l_i$.

- Comment: $\dfrac{\mathrm{mes}(E)}{L} = \dfrac{k}{k + \sum_{n\in\mathbb{Z}} l_n}$ can be made anything in $[0,1)$.

2. Choice of f on $\bigcup_{n\in\mathbb{Z}} I_n$ for a C^1 Denjoy map

Let $(I_n)_{n\in\mathbb{Z}}$ be the sequence of intervals arising from the blow up of an orbit of r_β, with lengths $l_n > 0$.

1. Choose the lengths l_n such that

(i) $\lim_{|n|\to\infty} \frac{l_{|n+1|}}{l_{|n|}} = 1$ (ii) $\sum_{n\in\mathbb{Z}} l_n < \infty$

(iii) $l_{|n|} > l_{|n+1|}$ (iv) $3l_{n+1} - l_n > 0$ for $n \geq 0$

e.g. $l_n = \frac{c}{(|n|+2)(|n|+3)}$, some $c \in \mathbb{R}$

2. Define f on $\bigcup_{n\in\mathbb{Z}} I_n$:

Write $I_n = [a_n, b_n]$, $l_n = b_n - a_n$

- $\int_{a_n}^{b_n} (b_n - t)(t - a_n)\,dt = \frac{l_n^3}{6}$ so $\frac{6(l_{n+1}-l_n)}{l_n^3}\int_{a_n}^{b_n}(b_n-t)(t-a_n)\,dt = l_{n+1} - l_n$.

Therefore if we define f for $x \in I_n$ by

$$f(x) = a_{n+1} + \int_{a_n}^{x} 1 + \frac{6(l_{n+1}-l_n)}{l_n^3}(b_n - t)(t - a_n)\,dt$$

then $f(b_n) = a_{n+1} + l_n + (l_{n+1} - l_n) = b_{n+1}$.

Also, f is continuously differentiable on I_n with

$$f'(x) = 1 + 6\,\frac{(l_{n+1}-l_n)}{l_n^3}(b_n - x)(x - a_n)$$

Thus $f'(a_n) = f'(b_n) = 1$.

Notice that – for $n < 0$, $l_{n+1} - l_n > 0$ so that $f'(x) \geq 1$, all $x \in I_n$,
and $f'(x) \leq 1 + 6\left(\frac{l_{n+1}-l_n}{l_n^3}\right)\left(\frac{l_n}{2}\right)^2 = \frac{3l_{n+1} - l_n}{2l_n}$

– for $n \geq 0$ and $x \in I_n$, $1 \geq f'(x) \geq \frac{3l_{n+1}-l_n}{2l_n} > 0$ (by (iii)),

and $\lim_{|n|\to\infty} \frac{3l_{n+1}-l_n}{2l_n} = 1$ (by (i)).

So $\max_{x\in I_n} |f'(x) - 1| \xrightarrow[|n|\to\infty]{} 0$. (*)

By construction f is C^1 on $\mathrm{Int}(I_n)$. Now show that for $y \in S^1 \setminus \bigcup_{n\in\mathbb{Z}} \mathrm{Int}(I_n)$

(i) $f'(y)$ exists and $= 1$ [i.e. prove $\frac{f(y') - f(y)}{y' - y} \to 1$ as $y' \to y$]

(ii) f' is continuous at y. (easy given (i) & (*))

- $f''(x) = \frac{6(l_{n+1}-l_n)}{l_n^3}\big((b_n - x) - (x - a_n)\big)$. So $f''\left(\frac{a_n+b_n}{2}\right) = 0$ but

$\lim_{x\to a_n} f''(x) = 6\,\frac{(l_{n+1}-l_n)\,l_n}{l_n^3} = 6\frac{(l_{n+1}-l_n)}{l_n}\cdot l_n^{-1}$: unbounded as $|n| \to \infty$.

Thus f is <u>not</u> C^2.