Dynamical Systems and Nonlinear Differential Equations.

1. Introduction; Definitions.

1.1 What is a dynamical system (DS)?

It is a "system" whose evolution in time is determined uniquely by its current "state". For the purpose of this course, mathematically it is a continuous map $\varphi: X \times G \to X$, where X is a topological space and $G = \mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_+$, etc.., such that $\varphi(x,0) = x$, $\varphi(x, t+s) = \varphi(\varphi(x,t), s)$.

X = "state space" = "phase space".

G = set of times; continuous or discrete, forwards and possibly backwards.

$\varphi(x,t)$ is the state the system evolves to ~~from~~ from x in time t. $[\varphi(x,t) \equiv \varphi^t(x) \equiv \varphi^t x]$.

Examples in discrete time:

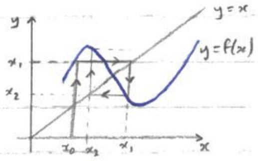
maps: if $f: X \to X$, let $\varphi(x,n) = \overbrace{f \circ \cdots \circ f}^{n}(x) =: f^n(x)$, $n > 0$; $\varphi(x,0) = x$.

If f is invertible, can also define $\varphi(x,n)$ for $n < 0$ by $(f^{-1})^n(x)$. (backward iteration).

Actually, every DS with discrete time is of this form. Simply define $f(x) = \varphi(x,1)$.

Definition: The forward orbit, $\mathcal{O}^+(x)$, of x is the sequence $x, f(x), f^2(x), \ldots$ ie $(f^n(x))_{n \in \mathbb{Z}_+}$.
If f is invertible, can also define the (full) orbit of x, denoted $\mathcal{O}(x)$ to be: $\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots$ [Notation: $f^{-k} = (f^{-1})^k$].

Graphical representation of iteration:



$x_{n+1} = f(x_n)$

Apparent generalisation: recurrence relations of form $x_{n+k} = F(x_{n+k-1}, \ldots, x_n)$ can also be regarded as a DS. Let $X = \mathbb{R}^k$. $f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ F(x_k, \ldots, x_1) \end{pmatrix}$

Examples in continuous time:

- autonomous system of first order ODEs / vector fields, $\dot{x} = V(x)$, $x \in X$ (assume that V is Lipschitz, X compact).

Existence / Uniqueness Theorem for ODEs $\Rightarrow \exists$ unique $\varphi: X \times \mathbb{R} \to X$ such that $\frac{d}{dt}\varphi(x,t) = V(\varphi(x,t))$ with $\varphi(x,0) = x$.

φ is said to be the flow of the vector field V.

Definition: the forward orbit of x (respectively orbit), $\mathcal{O}^+(x)$ (resp. $\mathcal{O}(x)$) is the parametrised curve $\varphi(x,t)$ in X where $t \in \mathbb{R}_+$ (resp. $t \in \mathbb{R}$).
(= integral curve through x - "trajectory").

In this course, we will look at maps of the circle S^1, and the interval $[0,1]$, and also some other topological spaces, and flows on (bounded regions of) $\mathbb{R}$.

1.2. DS viewpoint.

"Traditional" viewpoint on ODEs and recurrence relations is to find explicit formulae for φ. But this is usually impossible for nonlinear systems. DS viewpoint is to study "qualitative features" of φ.

For example, in discrete time: existence of fixed points and periodic orbits / points.

Definition: In discrete time, say x is a fixed point if $f(x)=x$. Say x is a periodic point of (least) period $q>0$ if $x=f^q(x)$ and $f^n(x) \neq x$ for $1 \leq n < q$.

In continuous time, look for: existence of equilibrium / stationary points, ie x such that $V(x)=0$, and for existence of periodic orbits, ie $\mathcal{O}(x)$ such that $\varphi(x,T)=\varphi(x,0)$, some T

More thoroughly, what we mean by a "qualitative feature" is a property of φ which is invariant under changes of coordinates by all homeomorphisms.
(In the continuous-time case, under time-rescaling also)

Recall: a homeomorphism is a continuous map with continuous inverse.

Say $f: X \to X$ and $g: Y \to Y$ are topologically conjugate if $\exists$ homeomorphism $h: X \to Y$ such that $g \circ h = h \circ f$. $\begin{array}{ccc} X & \xrightarrow{f} & X \\ h\downarrow & & \downarrow h \\ Y & \xrightarrow{g} & Y \end{array}$. h is called the conjugacy.

The qualitative features of f and g are identical. For example, if f has a fixed point x, then so does g - $h(x)$.

1.3. Asymptotic Behaviour.

The most interesting qualitative features are those to do with the behaviour as $t \to \pm\infty$. For example, does every orbit tend to a fixed point or periodic orbit? - No.

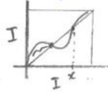
Definition: The ω-limit set of a point $x \in X$ (for a DS φ, t continuous or discrete), denoted $\omega(x)$, is the set $\{ y \in X : \exists (t_n) \to +\infty$ such that $\varphi^{t_n} x \to y \}$.
If G is $\mathbb{R}$ or $\mathbb{Z}$, can also define the α-limit set of x, $\alpha(x)$, by replacing $+\infty$ here by $-\infty$.

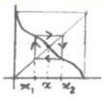
Comments: (i) $\omega(x)$, $\alpha(x)$ depend on φ. So if not clear from context, write $\omega(x,f)$ if DS comes from a map f; write $\omega(x,V)$ if from a vector field V.
(ii) obviously, $\omega(f(x),f) = \omega(x,f)$.

Examples: - if x is a fixed point, then $\omega(x) = x$.
- if x is a periodic point, then $\omega(x) = \mathcal{O}^+(x)$. [N.B. abuse of notation here: $\mathcal{O}(x) = \{ f^n(x) : n \in \mathbb{Z}_+ \}$]

Illustration for the simplest class of DS: homeomorphism of closed interval $I \subset \mathbb{R}$.

(i) f is orientation-preserving (OPH). The only possible $\overset{\alpha-}{\omega-}$ limit sets are fixed points.

<u>Proof</u>: exercise. 

(ii) f is orientation-reversing (ORH). The only possible $\overset{\alpha-}{\omega-}$limit sets are fixed points and period 2 orbits. <u>Proof</u>: exercise. Consider f^2 (an OPH) 

To get more exciting ω-limit sets in one dimension, either consider non-invertible maps of the interval, or maps of the circle.

<u>Definition</u>: A subset $U \subset X$ is <u>invariant</u> for a DS φ if $\varphi^t U = U \ \forall t \in G$.

2. Maps of the Circle.

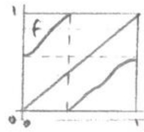
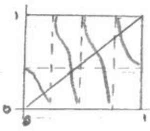
2.0. Generalities.

Consider a continuous map of the circle to itself, $f: S^1 \to S^1$. Two ways of representing the circle S^1:

(a) $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$, or equivalently, $\{z \in \mathbb{C} : |z|=1\} = \{e^{2\pi i x} : x \in \mathbb{R}\}$ (multiplicative notation)

(b) $S^1 = \mathbb{R}/\mathbb{Z}$ (additive notation).

The log map, $e^{2\pi i x} \mapsto x$ establishes an isomorphism between these two representations. Additive representation (b) will be more useful for our present purpose.

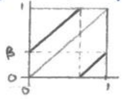
Can draw a graph of f by cutting the circle at one point, say 0, and regarding it as a map of $[0,1)$. For example:  

2.1. Simplest Example.

<u>Rotation</u> (rigid, solid, ...) by angle $2\pi\beta =: r_\beta$.

(a) $r_\beta z = z_0 z$, with $z_0 = e^{2\pi i\beta}$

(b) $r_\beta x = x + \beta \mod 1$, where "mod 1" means that numbers differing by integers are identified.

So: 

There is a crucial distinction between the cases β rational and β irrational.

(i) <u>$\beta \in \mathbb{Q}$</u>: Write $\beta = p/q$, p,q coprime. Then $r_\beta^q x = x \ \forall x \in S^1 \Rightarrow$ all points periodic, period q.

(ii) <u>$\beta \notin \mathbb{Q}$</u>:

<u>Definition</u>: a <u>minimal set</u> for f is a closed, (f-) invariant subset possess no proper closed invariant subset.

Equivalent definition: a closed, invariant subset in which every forward orbit is dense.
(for example, a periodic orbit is a minimal set).

Proposition 1: If $\beta \in \mathbb{R}\setminus\mathbb{Q}$, the orbit of every point under r_β is dense in S^1.

Remarks: $\omega(x) = S^1 \ \forall x \in S^1$. S^1 is a minimal set for r_β (so there are no others).

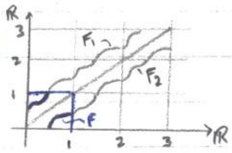
Proof: Given $\beta \in \mathbb{R}\setminus\mathbb{Q}$, $x \in S^1$, $\varepsilon > 0$, all points $\{r_\beta^n x : n \in \mathbb{Z}_+\}$ are distinct. (Else $\beta \in \mathbb{Q}$, for $r_\beta^n x = r_\beta^m x \ (m \neq n) \Rightarrow (n-m)\beta \in \mathbb{Z}$). S^1 has bounded length, thus $\exists\ m \neq n$ such that $0 < d(r_\beta^m x, r_\beta^n x) < \varepsilon$. Let $N = |n-m|$ and $\beta_N = d(r_\beta^m x, r_\beta^n x)$. Now, r_β preserves lengths and orientation in S^1, so that r_β^N rotates by amount β_N. Thus, $r_\beta^{jN} x$ $(j \in \mathbb{N}_0)$ are equally spaced. So, $\{r_\beta^{jN} x : j = 0, 1, .., \lceil \frac{1}{\beta_N} \rceil\}$ come within ε of every point of S^1.

For more general OPHs of the circle, do we have the same property?

2.2. Lift; Degree.

Define projection $\pi : \mathbb{R} \to S^1$ by $\pi(x) = x \bmod 1 = [x]$. Then, given continuous $f : S^1 \to S^1$, $\exists$ (not unique) continuous function $F : \mathbb{R} \to \mathbb{R}$ such that $f\pi(x) = \pi F(x)\ \forall x$.
F is called a lift of the circle map f.
$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \pi \downarrow & & \downarrow \pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$
commutes. (π is a covering map)

Example:



Lemma 2 (properties of lift): (i) if $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ are two lifts of the same continuous map $f : S^1 \to S^1$ then $\exists\ k \in \mathbb{Z}$ such that $F_1(x) = F_2(x) + k \ \ \forall x$.

(ii) given continuous $f : S^1 \to S^1$, $\exists\, d \in \mathbb{Z}$ such that $\forall$ lifts F and $x \in \mathbb{R}$, $F(x+1) = F(x) + d$. d is called the degree of f ($\deg f$).

(iii) if F is a lift of f, then F^k is a lift of f^k $\forall k \in \mathbb{Z}_+$.

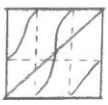
(iv) $\deg(F^k) = (\deg F)^k$

Proof: (i) choose $x \in \mathbb{R}$. $f\pi(x) = \pi F_1(x) = \pi F_2(x)$. So $F_1(x) - F_2(x) \in \mathbb{Z}$. Since F_1, F_2 are continuous, so is $F_1 - F_2$, hence $F_1 - F_2$ is constant.

(ii) $\pi(x+1) = \pi(x)$, so $f\pi(x+1) = f\pi(x) = \pi F(x) = \pi F(x+1)$, so $F(x+1) - F(x) = d \in \mathbb{Z}$ (as for (i))
Let $\bar{F}$ be another lift, then $\bar{F} = F + k$, some k. So $\bar{F}(x+1) - \bar{F}(x) = F(x+1) + k - (F(x)+k) = F(x+1) - F(x) = d$.

(iii), (iv) Exercise – use induction.

Intuitively, $|\deg f|$ measures how many times the circle is wrapped around itself by f.

Examples:  : $|d| = 2$.  : $|d| = 3$

If f is a homeomorphism, then $\deg f = \pm 1$. If f is an $\left\{\begin{matrix}\text{OPH}\\ \text{ORH}\end{matrix}\right\}$, then $\deg f = \left\{\begin{matrix}+1\\ -1\end{matrix}\right\}$

Example: r_β. $\deg r_\beta = +1$. Lift: $R_\beta : x \mapsto x+\beta$, translation by β. Then, $\frac{R_\beta^n(x)-x}{n} = \frac{n\beta}{n} = \beta \quad \forall n \in \mathbb{N}, x \in \mathbb{R}$.

We want to generalise this to degree one homeomorphisms. From now on, restrict attention to degree one maps.

Properties of degree-1 continuous circle maps:

Lemma 2 $\Rightarrow$ $F(x+1) = F(x)+1$ — (P1)

$F^k(x+1) = F^k(x)+1, \; k \in \mathbb{N}$ — (P2)

$F^k(x+m) = F^k(x)+m$, by induction. — (P3)

$F(x)-x$ is periodic, period 1: $F(x+1)-(x+1) = F(x)+1-(x+1) = F(x)-x$ — (P4)

Similarly, $F^k(x)-x$ is 1-periodic. — (P5)

2.3. Rotation Number.

Let $f: S^1 \to S^1$ be a degree 1 continuous map; F a lift of f. Define $\bar{\rho}(F,x) = \lim_{n\to\infty} \frac{F^n(x)-x}{n}$, if this limit exists. $\bar{\rho}(F,x)$ is called the rotation number of x under F. It measures the average rotation rate for an orbit.

Theorem 3: Let $f: S^1 \to S^1$ be an OPH. Then:

(i) $\forall x \in \mathbb{R}$, the limit $\bar{\rho}(F,x)$ and is independent of x, ie. $\bar{\rho} = \bar{\rho}(F)$.

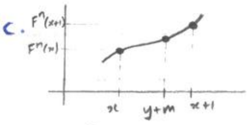
(ii) If we define $\rho(f) = \bar{\rho}(F,x) \bmod 1$, then ρ is independent of F.

(iii) The rotation number of f depends continuously on f.

[$\rho(f)$ is called the rotation number of f.]

Lemma 4: Given $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$, then $F^n(x)-x-1 < F^n(y)-y < F^n(x)-x+1$.

Proof: $\exists\, m \in \mathbb{Z}$ such that $x \leq y+m \leq x+1$. F monotonic $\Rightarrow$ F^n monotonic.



So $F^n(x) \leq F^n(y+m) \leq F^n(x+1)$. $\Rightarrow F^n(x)-x-1 \leq F^n(y+m)-(x+1) \ldots$

$< F^n(y+m)-(y+m) < F^n(x+1)-x = F^n(x)-x+1$, by (P1)

But (P5) $\Rightarrow F^n(y+m)-(y+m) = F^n(y)-y$.

Proof of Theorem 3: (i) Begin by proving that $\bar{\rho}(F,0)$ exists.

For $nk \in \mathbb{N}$, $F^{nk}(0) = \left(F^{nk}(0)-F^{n(k-1)}(0)\right) + \left(F^{n(k-1)}(0)-F^{n(k-2)}(0)\right) + \cdots + \left(F^{2n}(0)-F^n(0)\right) + \left(F^n(0)-0\right)$

Lemma 4, with $x=0$ and $y = F^{n(m-1)}(0)$ for $m=1,2,\ldots,k \Rightarrow k(F^n(0)-1) < F^{nk}(0) < k(F^n(0)+1)$.

Divide by nk: $\frac{F^n(0)-1}{n} < \frac{F^{nk}(0)}{nk} < \frac{F^n(0)+1}{n}$, ie $\left|\frac{F^{nk}(0)}{nk} - \frac{F^n(0)}{n}\right| < \frac{1}{n}$

Switch rôles of k and $n \Rightarrow \left|\frac{F^{nk}(0)}{nk} - \frac{F^k(0)}{k}\right| < \frac{1}{k}$.

Hence, $\left|\frac{F^k(0)}{k} - \frac{F^n(0)}{n}\right| < \frac{1}{k}+\frac{1}{n} \Rightarrow \left(\frac{F^n(0)}{n}\right)_{n\in\mathbb{N}}$ is Cauchy and hence converges.

But, from lemma 4, $\frac{F^n(0)-1}{n} < \frac{F^n(x)-x}{n} < \frac{F^n(0)-1}{n} \quad \forall x \in \mathbb{R} \Rightarrow \frac{F^n(x)-x}{n} \xrightarrow[n\to\infty]{} \bar{\rho}(F,0) = \bar{\rho}(F)$

(ii) Assume F_1 and F_2 are two lifts of f. Thus $\exists\, k \in \mathbb{Z}$ such that $F_2(x) = F_1(x)+k$.

By induction, $F_2^n(x) = F_1^n(x) + nk \quad \forall n \in \mathbb{N}$.

Therefore, $\rho(F_2) = \lim_{n\to\infty} \frac{F_2^n(x)-x}{n} = \lim_{n\to\infty}\left(\frac{F_1^n(x)-x}{n} + \frac{nk}{n}\right) = \rho(F_1)+k$.

So $\rho(F_2) = \rho(F_1) \bmod 1$.

(iii) $\left[\text{Lemma 5: Let } F \text{ be a lift of an OPH } f: S^1 \to S^1 \text{ and } n \in \mathbb{N}. \text{ Then, } \exists\, k(n) \in \mathbb{Z} \text{ such that } k-1 < F^n(x) - x < k+1 \ \forall x \in \mathbb{R}. \text{ See example sheet 1}\right]$

Let F be a lift of f. Lemma 5 $\Rightarrow$ given n, $\exists\, k$ such that $k-1 < F^n(x) - x < k+1$ $\forall x \in \mathbb{R}$. Given $\varepsilon > 0$, choose n such that $\frac{2}{n} < \varepsilon$. For g close enough to f in the C^0-topology [ie, $d(f,g) = \sup_{x \in S^1} d(f(x), g(x))$], a lift G can be chosen so that $k-1 < G^n(x) - x < k+1$ (same n, k).

Now, $F^{nl}(0) - 0 = \sum_{j=0}^{l-1} (F^{n(j+1)}(0) - F^{nj}(0))$ and similarly for G.

Therefore, $l(k-1) < F^{nl}(0) < l(k+1)$, $l(k-1) < G^{nl}(0) < l(k+1)$

Now, $\rho(F) = \lim_{n\to\infty} \frac{F^{nl}(0)}{nl}$ and $\rho(G) = \lim_{n \to \infty} \frac{G^{nl}(0)}{nl}$, so $\frac{k-1}{n} \le \rho(F) \le \frac{k+1}{n}$, $\frac{k-1}{n} \le \rho(G) \le \frac{k+1}{n}$.

Hence, $|\rho(F) - \rho(G)| \le \frac{2}{n} < \varepsilon$.

The rotation number of an OPH of the circle is a topological invariant, ie:

Proposition 6: Suppose $f, g: S^1 \to S^1$ are OPHs and $\exists$ an OPH h such that $h.f = g.h$. Then $\rho(F) = \rho(G)$

Proof: See example sheet 1.

2.4. OPH's of the circle with rational rotation number

Proposition 7: Let f be an OPH of S^1. Then $\rho(F) \in \mathbb{Q} \Leftrightarrow f$ has a periodic point. In fact, $\rho(f) = p/q \Leftrightarrow f$ has a periodic point of period q.

Proof: ($\Leftarrow$) Suppose f has a point x_0 of least period q. Let F be a lift of f. $\exists\, k \in \mathbb{Z}$ such that $F^q(x_0) = x_0 + k$. $F^{nq}(x_0) - x_0 = nk$, so $\frac{F^{nq}(x_0) - x_0}{nq} = k/q$ $\forall n$.

Take limit $n \to \infty$: so $\rho(F) = k/q$ and $\rho(f) = p/q$ where $p \equiv k \bmod q$.

($\Rightarrow$) Assume $\rho(f) = p/q$ in lowest terms. Let $\bar{F}$ be a lift of f. Then $\exists\, k \in \mathbb{Z}$ such that $\rho(\bar{F}) = \frac{p}{q} + k$. Then, $F(x) = \bar{F}(x) - k$ is another lift of f, with rotation number p/q. Also, $\rho(F^q - p) = \rho(F^q) - p = q\rho(F) - p = 0$. Let $G(x) = F^q(x) - p$. Need to show that G has a fixed point on $\mathbb{R}$, so that f has a point of period q on S^1.

Three cases: (i) $G(0) = 0$ - done.

(ii) $G(0) > 0$. G is increasing, so $0 < G(0) < G^2(0) < \cdots < G^n(0) < \cdots$ Either:

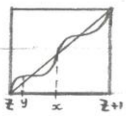
(a) $0 < G^n(0) < 1$ $\forall n$, then an increasing sequence bounded above $\Rightarrow$ converges to some point x_0, and, by continuity, $G(x_0) = x_0$.

(b) $\exists\, k > 0$ such that $G^k(0) > 1$. Then, $G^{2k}(0) = G^k(G^k(0)) > G^k(1) = G^k(0) + 1 > 2$, by monotinicity of G^k. By induction, $G^{jk}(0) > j$, and $\frac{G^{jk}(0)}{jk} > \frac{1}{k}$, and $\rho(G) \geq 1/k$. #.

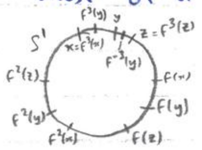
(iii) $G(0) < 0$. Same reasons as in (ii)

Theorem 8: Let $f: S^1 \to S^1$ be an OPH with rotation number $p/q = \rho(f)$, in lowest terms. Then every orbit is either periodic with period q, or is forward asymptotic to a periodic orbit and backward asymptotic to a periodic orbit.

Proof (sketch): $\rho(f) = p/q$, so $\exists$ orbit of period q for f. f^q can be identified with a homeomorphism of $I = [z, z+1]$, by cutting S^1 at a fixed point z of f^q.



Remark: The periodic orbits of an OPH of S^1 are ordered on S^1 like those of a rigid rotation with the same rotation number.



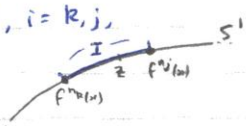
ie, If x is a periodic point and $\rho(f) = p/q$, then ordering of $\{x, f(x), .., f^{q-1}(x)\}$ is the same as $\{0, p/q, 2p/q, .., (q-1)p/q\}$.

2.5. OPH's of the circle with irrational rotation number.

Theorem 9: Assume $f: S^1 \to S^1$ is an OPH, and $\rho(f) \in \mathbb{R}\setminus\mathbb{Q}$. Then, (i) $\omega(x)$ is independent of x,
(ii) $E := \omega(x)$ is the unique minimal set of f.
(iii) E is either S^1 or a Cantor subset of S^1.

Definition: A Cantor subset of $\mathbb{R}^n$ is a compact, totally disconnected set with no isolated points. On S^1, can replace "totally disconnected" by "empty interior."

Proof: (i) Choose $x, y \in S^1$. Want to prove $\omega(x) \subseteq \omega(y)$, and hence, by symmetry, $\omega(x) = \omega(y)$.
Since S^1 is compact, $\omega(x) \neq \phi$, and $\rho(f) \notin \mathbb{Q}$, so all points in $\mathcal{O}(x)$ are distinct.
Let $z \in \omega(x)$, ie $\exists\, n_i \to \infty$ (as $i \to \infty$) such that $|f^{n_i}(x) - z| \to 0$ as $i \to \infty$.
Choose $\varepsilon > 0$. Then we can find $n_k > n_j > 0$ such that $|f^{n_i}(x) - z| < \varepsilon$, $i = k, j$, and $|f^{n_k}(x) - f^{n_j}(x)| < \varepsilon$. Let I be the closed interval of length $< \varepsilon$ with end points $f^{n_k}(x), f^{n_j}(x)$.



Let $N = n_k - n_j$ and note that $f^N(f^{n_j}(x)) = f^{n_k}(x)$, and $f^{-N}(f^{n_k}(x)) = f^{n_j}(x)$.
So $f^{-N}(I) \cap I = \{f^{n_j}(x)\}$. Arguing similarly, $\{f^{-mN}(I)\}_{m \in \mathbb{Z}_+}$ is a sequence of closed intervals joined end to end. Either $f^{-N}(I)$ accumulate to some point p, which must by continuity satisfy $f^{-N}(p) = p$. ※. Or, they cover the whole of S^1: $S^1 = \bigcup_{m \in \mathbb{Z}_+} f^{-mN}(I)$.
Thus, for any point $y \in S^1$, $\exists\, l \in \mathbb{Z}_+$ such that $y \in f^{-lN}(I)$, so that $f^{lN}(y) \in I$, so that $|f^{lN}(y) - z| < \varepsilon$. Hence $z \in \omega(y)$, so $\omega(x) \subseteq \omega(y)$.

(ii) E is closed, and invariant (under the mapping f). Let $A \subset S^1$ be a non-empty, closed, invariant set, and $x \in A$. Then $\mathcal{O}^+(x) \subset A$ since A is invariant, and $E = \omega(x) \subset A$ since A closed. Hence E is the unique invariant set.

(iii) ϕ and E are the only closed invariant subsets of E. The boundary ∂E of E is a closed, invariant subset of E, so $\partial E = \phi$ or $\partial E = E$. If $\partial E = \phi$ then $E = S^1$.
If $\partial E = E$, then E has empty interior. Let $x \in E$. Since $E = \omega(x)$, $\exists$ sequence $(k_n) \to \infty$ such that $\lim_{n\to\infty} f^{k_n}(x) = x$. But f has no periodic point, so $f^{k_n}(x) \neq x$ $\forall n$.
Hence x is an accumulation point of E, since $f^{k_n}(x) \in E$ by invariance of E.

Examples: of maps with $\omega(x) = S^1$: $x \mapsto r_\beta(x) = x + \beta \mod 1$, $\beta \in \mathbb{R}\setminus\mathbb{Q}$.
$\omega(x) \neq S^1$: do they exist?

Examples of maps with $E \neq S^1$ are called Denjoy counterexamples.

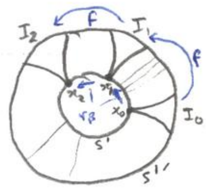
Recall: $f: X \to Y$ is a diffeomorphism if it is a bijection and both f and f^{-1} are differentiable.

Theorem [redacted] 10 (Denjoy): Assume $f: S^1 \to S^1$ is a C^2-OP-diffeomorphism and $\beta = \rho(f)$ is irrational. Then f is topologically conjugate to the rigid rotation r_β.

Proof*: ... So $E = S^1$. In particular, all orbits are dense in S^1.

Proposition 11 (Denjoy): Let β be irrational. Then $\exists$ a C^1-OP diffeomorphism $f: S^1 \to S^1$ such that $\rho(F) = \beta$ and $E \neq S^1$.

Proof (Denjoy counterexample): Want to find a circle map with an orbit that is not dense in S^1, so that $E \neq S^1$.



Idea:
- start from rigid rotation r_β, $\beta \in \mathbb{R} \setminus \mathbb{Q}$, on S^1
  - choose an orbit $(x_n)_{n\in\mathbb{Z}}$ of r_β, and "blow it up" into an orbit of closed intervals $(I_n)_{n\in\mathbb{Z}}$, with lengths l_n such that $\sum_{n\in\mathbb{Z}} l_n < \infty$, to obtain a new circle $S^{1\prime}$ (see handout)
- extend r_β to a map $f: S^1 \to S^1$ by choosing, for each $n \in \mathbb{Z}$, an OPH mapping I_n onto I_{n+1}. Could choose $f: I_n \to I_{n+1}$ affine $\Rightarrow f$ is C^0 but not C^1 (exercise).
- to make f C^1, we need to make $f' = 1$ at the endpoints of each I_n and $\max_{x\in I_n} |f'(x) - 1| \to 0$ as $n \to \infty$ (See handout).

Notes: (i) f has same rotation number as r_β.

(ii) No point in I_n ever returns to I_n under iteration of f (and f^{-1}). [We say the I_n are wandering intervals.] So if $p \in \mathrm{Int}(I_n)$ then $f^m p \notin \mathrm{Int}(I_n)$ $\forall m \neq 0$, so $\mathcal{O}(p)$ is not dense in S^1. So $E \neq S^1$, and by Theorem 9, E is a Cantor set.

In fact, $E = S^1 \setminus \bigcup_{n\in\mathbb{Z}} \mathrm{Int}(I_n)$. The open sets, $\mathrm{Int}(I_n)$, $n \in \mathbb{Z}$, are the gaps of the Cantor set. E is nowhere dense, since $\bigcup_{n\in\mathbb{Z}}$ is dense in $S^{1\prime}$. So E has empty interior.

Exercise: Show that E has no isolated points, and that $\forall x \in S^1$, $\omega(x) = E$.

Definition: An orbit $\mathcal{O}(x)$ of a map is said to be homoclinic to an invariant set $S \subset S^1 \setminus \mathcal{O}(x)$ if $\alpha(x) = \omega(x) = S$.

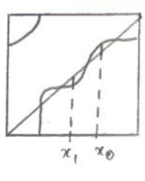
Theorem 12: Let $f: S^1 \to S^1$ be an OPH with irrational rotation number. Then every orbit is either: (i) dense in S^1; (ii) dense in a Cantor set; (iii) homoclinic to a Cantor set.

Remark: All orbits of an OPH $f: S^1 \to S^1$ with $\rho(f) \notin \mathbb{Q}$ are ordered as those of a rigid rotation $r_{\rho(f)}$.

2.6. Families of Circle Maps.

Proposition 13: Let $f: S^1 \to S^1$ be an OPH with lift F such that $\rho(F) = \rho(f) = \frac{p}{q}$, and suppose that the graph $(F^q - p)$ has points on both sides of the diagonal. Then all small enough perturbations $\bar{f}$ of f have rotation number p/q. This phenomenon is called frequency locking.

Proof: $\exists\, x_0, x_1 \in \mathbb{R}$ such that $F^q(x_0) - p > x_0$, $F^q(x_1) - p < x_1$. Then, for any small enough perturbation $\bar{f}$ of f with corresponding lift $\bar{F}$, we have $\bar{F}^q(x_0) - p > x_0$ - (1), $\bar{F}^q(x_1) - p < x_1$ - (2).



(1) $\Rightarrow \rho(\bar{F}) \geq p/q$, (2) $\Rightarrow \rho(\bar{F}) \leq p/q$. Hence $\rho(\bar{F}) = p/q$.

<u>Monotinicity of rotation number:</u> Let $F_1(x)$, $F_2(x)$ be lifts of OPH's $f_1, f_2 : S^1 \to S^1$. If $F_1(x) < F_2(x)$ $\forall x \in \mathbb{R}$, then $\rho(F_1) \leq \rho(F_2)$, from definition of rotation number. At irrational values, the rotation number is strictly increasing.

<u>Proposition 14:</u> Let F_1, F_2 be lifts of OPH's of S^1 with $\rho(F_1) \notin \mathbb{Q}$. If $F_1(x) < F_2(x)$ $\forall x \in \mathbb{R}$, then $\rho(F_1) < \rho(F_2)$

<u>Proof:</u> By continuity and periodicity, $F_2(x) - F_1(x) > \delta$ for some $\delta > 0$ and all $x \in \mathbb{R}$. Take $p/q \in \mathbb{Q}$ such that $\frac{p}{q} - \frac{\delta}{q} < \rho(F_1) < p/q$ (check). Then $\exists\, x_0 \in \mathbb{R}$ such that $F_1^q(x_0) - x_0 > p - \delta$. $\left[\text{Otherwise, } \rho(F_1) = \lim_{n\to\infty} \frac{F^{nq}(x) - x}{nq} \leq \lim_{n\to\infty} \frac{n(p-\delta)}{nq} = \frac{p}{q} - \frac{\delta}{q}\right]$

$F_2^q(x_0) = F_2(F_2^{q-1}(x_0)) > F_1(F_2^{q-1}(x_0)) + \delta > F_1(F_1^{q-1}(x_0)) + \delta = F_1^q(x_0) + \delta > x_0 + p.$

Hence, $\rho(F_2) \geq p/q$.

<u>The Arnold Family.</u>

2-parameter family: $f_{k,\omega} : S^1 \to S^1$; $k, \omega \in \mathbb{R}$, with lift $F_{k,\omega} : x \mapsto x + \omega + \frac{k}{2\pi} \sin 2\pi x$.

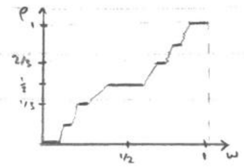
$$\left.\begin{array}{ll} k = 0: & \text{rigid rotation.} \\ 0 \leq k < 1: & \text{diffeomorphism.} \\ k = 1: & \text{homeomorphism.} \\ k > 1: & \text{no longer 1-1.} \end{array}\right\} \text{ so consider } k, \omega \in [0,1].$$

Fix k: If $\omega_1 < \omega_2$, then $F_{k,\omega_1}(x) < F_{k,\omega_2}(x)$ $\forall x \in \mathbb{R}$, so $\rho(F_{k,\omega_1}) \leq \rho(F_{k,\omega_2})$.
Hence, $\rho(F_{k,\omega})$ is a non-decreasing function of ω for each fixed $k \in [0,1]$.
It is also continuous by Theorem 3(iii).

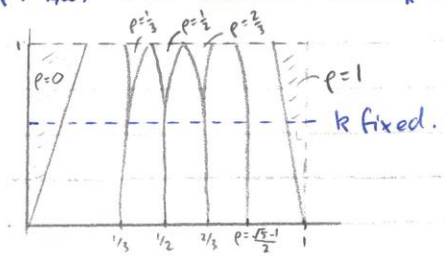
<u>Definition:</u> a monotone continuous function $\varphi : [0,1] \to \mathbb{R}$ is called a <u>Devil's Staircase</u> if $\exists$ a family $\{I_\alpha\}_{\alpha \in A}$ of disjoint closed subintervals of $[0,1]$ with dense union such that φ takes distinct constant values on these subintervals.

<u>Proposition 15:</u> For $k \in [0,1]$, $\varphi : \omega \mapsto \rho(F_{k,\omega})$ is a devil's staircase, and $\varphi^{-1}(p/q)$ is an interval for each rational p/q.

<u>Proof:</u> exercise sheet 1, questions 16, 17.



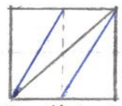
In (k,ω)-parameter space, $\rho(F_{k,\omega}) = 0 \iff \exists\, x$ such that $F_{k,\omega}(x) = x$, that is, $x + \omega + \frac{k}{2\pi} \sin 2\pi x = x$, ie, $\sin 2\pi x = -\frac{2\pi\omega}{k}$. So $\exists$ solutions if $\left|\frac{2\pi\omega}{k}\right| \leq 1$.
$\rho(F_{k,\omega}) = 1 \iff \sin 2\pi x = (1-\omega) \cdot \frac{2\pi}{k}$, with solutions if $\left|\frac{2\pi(1-\omega)}{k}\right| \leq 1$



Regions in parameter space where ρ is rational are called <u>Arnold tongues</u>.

3. Chaos and Maps of the Interval

3.1 Chaos



Start with an example: $f: S^1 \to S^1;\ z \mapsto z^2$, or, $x \mapsto 2x \bmod 1$:
$\deg(f) = 2$.

Write x as a binary expansion: $x = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$, $a_i \in \{0,1\}$.

[NB: dyadic rationals, ie $m/2^n$ have two expansions, eg $0.1000\ldots = 0.0111\ldots$]

Then $2x = a_1 + \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i}$, so $f(x) = 2x \bmod 1 = \sum_{i=1}^{\infty} \frac{a_{i+1}}{2^i}$

Notes: (i) if $x \in \mathbb{Q}$, then x is periodic, or eventually periodic.

[A point x is eventually periodic of period n if x is not periodic, but $\exists\, k \in \mathbb{N}$ such that $f^{n+i}(x) = f^i(x)\ \forall i \geq k$]

(ii) irrationals are neither periodic nor eventually periodic.

Let $P_n(f) :=$ number of periodic points of f with (not necessarily least) period n, ie, the number of fixed points of f^n. Then:

(iii) $P_n(f) = 2^n - 1$ $[z^{2^n} = z$, ie $z^{2^n-1} = 1$ $]$

(iv) $x \neq y \Rightarrow \exists\, n$ such that $|f^n(x) - f^n(y)| \geq 1/4$ (expansive). [if $x-y \geq \frac{1}{4}$, done, else iterate].

(v) for every open interval J, $\exists\, n \in \mathbb{Z}_+$ such that $f^n(J) = S^1$.

(vi) periodic points are dense on S^1.

Remark: some of these properties are particular to this example, but it has two more general properties.

Definition: $f: L \to L$ is topologically transitive (TT) if, for any pair of open sets $U, V \subseteq L$, $\exists\, n > 0$ such that $f^n(U) \cap V \neq \emptyset$.

[Equivalently, $f: L \to L$ is TT if it has a dense (forward) orbit in L]

(v) above $\Rightarrow$ TT. (Property (v) is called transitivity)

Definition: $f: L \to L$ has sensitive dependence on initial conditions (SDIC) if $\exists\, \delta > 0$ such that $\forall x \in L$ and any neighbourhood U of x, $\exists\, y \in U$ and $n > 0$ such that $|f^n(x) - f^n(y)| > \delta$.

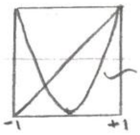
(iv) above $\Rightarrow$ SDIC.

Definition: An invariant set Λ for $f: L \to L$ is a chaotic complement if $f|_\Lambda$ is TT and has SDIC. Also, say f is chaotic on Λ.

Example: $f: S^1 \to S^1;\ x \mapsto 2x \bmod 1$ is chaotic on S^1.

Alternative definitions: f has a horseshoe if $\exists$ closed intervals $J_1, J_2 \subseteq I$ (or S^1) such that
(i) $\operatorname{int} J_1 \cap \operatorname{int} J_2 = \emptyset$, (ii) $J_1 \cup J_2 \subseteq f(J_i)$ for $i = 1, 2$
f is chaotic if $\exists\, n > 0$ such that f^n has a horseshoe.

Example 2: $L := [-1,1]$. $g: L \to L$; $x \mapsto 2x^2 - 1$. Then g is chaotic on L.



Proof: Consider $h: S^1 \to L$; $\theta \mapsto \cos 2\pi\theta$. h is continuous and onto.

$hf(\theta) = \cos 4\pi\theta = 2\cos^2 2\pi\theta - 1 = gh(\theta)$. So $hf = gh$.

Note that h is not injective, but we don't actually need injectivity.

To prove TT: given open $I, J \subset L$, then $h^{-1}(I)$, $h^{-1}(J)$ are open in S^1, so $\exists\, n>0$ such that $f^n(h^{-1}(I)) \cap h^{-1}(J) \neq \phi \Rightarrow g^n(I) \cap J \neq \phi$.

To prove SDIC for g: given $x \in L$, open $U \ni x$, $\exists\, n > 0$ such that $g^n(U) = L$ $-(*)$ (same true for f). Let $y = \begin{cases} 1 & \text{if } g^n(x) \geq 0 \\ -1 & \text{if } g^n(x) < 0 \end{cases}$. By $(*)$, $\exists\, z \in U$ such that $g^n(z) = y$, so $|g^n(x) - g^n(z)| \geq 1$.

Definition: If $f: I \to I$, $g: J \to J$, both continuous, and $h: I \to J$, a continuous surjection such that $hf = gh$, say h semiconjugates f to g.

[Say conjugate if, in addition, h is injective and h^{-1} is continuous].

If so, the dynamics of f are at least as complicated as dynamics of g.

3.2. Symbolic Dynamics; The Shift Map.

The first example introduced the important idea of symbolic dynamics: f was equivalent to the left shift on binary sequences.

Definition: $\Sigma_N := \{ \underset{\sim}{a} = (a_0 a_1 \ldots) : a_i \in \{0,1,\ldots,N-1\}, \text{ i.e. } \mathbb{Z}_+ \}$ = sequence space on n symbols.

Make Σ_N a metric space by defining the distance: $d(\underset{\sim}{a}, \underset{\sim}{b}) = \sum_{n=0}^{\infty} \frac{\gamma(a_n, b_n)}{3^n}$, where $\gamma(i,j) = \begin{cases} 0 & \text{if } i=j \\ 1 & \text{if } i \neq j \end{cases}$.

So two points in Σ_N are close if they agree for a long initial segment.

Suppose $\underset{\sim}{a}, \underset{\sim}{b} \in \Sigma_N$ and $a_i = b_i$, $i < m$, $a_m \neq b_m$. Then $\frac{1}{3^m} \leq d(\underset{\sim}{a}, \underset{\sim}{b}) \leq \sum_{n=m}^{\infty} \frac{1}{3^n} = \frac{1}{3^m} \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{3}{2} \cdot \frac{1}{3^m}$

Properties of Σ_N: Σ_N is a Cantor set - compact, non-empty, totally disconnected, no isolated points.

Definition: $\sigma: \Sigma_N \to \Sigma_N$; $(a_0 a_1 a_2 \ldots) \mapsto (a_1 a_2 \ldots)$ is the (left) shift map.

Proposition 1: (i) σ is continuous, (ii) $P_k = N^k$ (number of fixed points of σ^k),
(iii) Set of periodic points, $\mathrm{Per}(\sigma)$, is dense in Σ_N, (iv) $\exists$ a dense (forward) orbit in Σ_N ($\Rightarrow$ TT).

Proof: (i) $d(\underset{\sim}{a}, \underset{\sim}{b}) = \gamma(a_0, b_0) + \frac{1}{3} d(\sigma(\underset{\sim}{a}), \sigma(\underset{\sim}{b})) \Rightarrow d(\sigma(\underset{\sim}{a}), \sigma(\underset{\sim}{b})) \leq 3 d(\underset{\sim}{a}, \underset{\sim}{b})$. Pick n such that $1/3^n < \varepsilon$ and let $\delta = \frac{1}{3^{n+1}}$

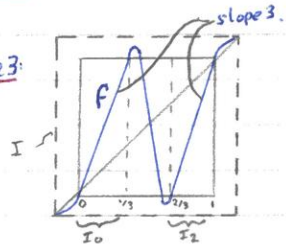
(ii) $\sigma^k \underset{\sim}{a} = \underset{\sim}{a} \iff a_{k+j} = a_j \ \forall j \geq 0$. Given k, there are N^k blocks of length k.

(iii) Given $\underset{\sim}{a} \in \Sigma_N$ and $\varepsilon > 0$, take n such that $\frac{3}{2 \cdot 3^n} < \varepsilon$. Let $\underset{\sim}{b} = (a_0 a_1 \ldots a_{n-1} a_0 \ldots a_{n-1} a_0 \ldots)$. Then $d(\underset{\sim}{a}, \underset{\sim}{b}) < \varepsilon$.

(iv) Let $\underset{\sim}{b}$ be a sequence which lists all blocks of length n for each successive n.
[Eg, in Σ_2, $\underset{\sim}{b} = 0\ 1\ 00\ 01\ 10\ 11\ 000\ 001 \ldots$] Then, given $\underset{\sim}{a} \in \Sigma_N$ and $k \in \mathbb{Z}_+$, $\exists\, n$ such that $\sigma^n(\underset{\sim}{b})$ and $\underset{\sim}{a}$ agree in the first k places. $d(\sigma^n(\underset{\sim}{b}), \underset{\sim}{a}) \leq \frac{1}{2 \cdot 3^{k-1}}$. But $k, \underset{\sim}{a}$ arbitrary $\Rightarrow \mathcal{O}^+(\underset{\sim}{b})$ is dense in Σ_N.

Remark: σ is chaotic on Σ_N.

Example 3:



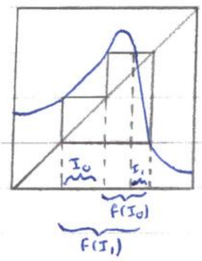
Let $\Lambda = \{x : f^n(x) \in I_0 \cup I_2 \ \forall n \geq 0\}$. Then Λ is the middle third Cantor set.
Ie. $\Lambda = \{x \in [0,1] :$ base 3 expansion has no 1's$\}$.
So $x = \sum_{n=0}^{\infty} \frac{a_n}{3^{n+1}}$, $a_n \in \{0,2\}$.

Proposition: $f|_\Lambda$ is topologically conjugate to σ on Σ_2.
Proof: the conjugacy $h : \{0,1\}^{\mathbb{Z}_+} \to \Lambda$; $(b_0 b_1 b_2 \dots) \mapsto x = \sum_{n=0}^{\infty} \frac{2b_n}{3^{n+1}}$. So f is chaotic on Λ.

3.3. Subshifts of Finite Type.

General setting, $f: I \to I$ (or $S^1 \to S^1$). $(I_i)_{i=0,1,\dots,N-1}$ - disjoint closed intervals in I.
Let $\Lambda = \{x \in I : f^n(x) \in \bigcup_{i=0}^{N-1} I_i \ \forall n \geq 0\}$. Say I_i f-covers I_j (written $I_i \to I_j$ or $i \to j$), if $f(I_i) \supset I_j$.
Let Γ be the directed graph (with N vertices) indicating all the f-covering relations, and let A be the $N \times N$ matrix indicating the allowed transitions, ie $(A)_{ij} = \begin{cases} 1 & \text{if } I_i \to I_j \\ 0 & \text{if } I_i \not\to I_j \end{cases}$.
A is called the transition matrix.

Example 4:



$\Gamma : 0 \rightleftarrows 1 \circlearrowleft$
I_0 f-covers I_1.
I_1 f-covers I_0 and I_1
$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

Let $\Sigma_{N,A} = \{\underline{a} \in \Sigma_N : \forall n \geq 0, \ A_{a_n a_{n+1}} = 1\}$. Note that $\Sigma_{N,A}$ is closed and invariant under σ.

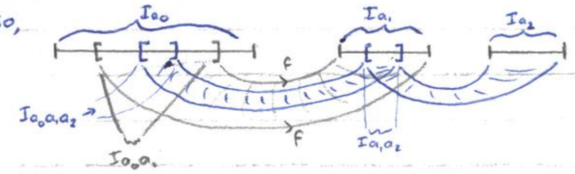
Definition: the restriction of σ to $\Sigma_{N,A}$ is called a subshift of finite type, $=: \sigma_A$.

Proposition 2: $f|_\Lambda$ is semi-conjugate to $\sigma|_{\Sigma_{N,A}}$.
Proof: Define $h : \Lambda \to \Sigma_{N,A}$ by $x \mapsto$ path of $(f^n(x))_{n \in \mathbb{Z}_+}$ in Γ, ie, $\underline{a}$, where $f^n(x) \in I_{a_n} \ \forall n \in \mathbb{Z}_+$.
(i) h is continuous, because given M, $\exists \delta > 0$ such that $x, y \in \Lambda$, $|x-y| \leq \delta \Rightarrow f^n(x), f^n(y) \in I_{a_n}$ for $0 \leq n \leq M$.
(ii) h is surjective, because given an allowed sequence $a_0 a_1 a_2 \dots \in \Sigma_{N,A}$,
claim: $\exists$ nested sequence of closed intervals $I_{a_0 a_1 \dots a_m} \subset I_{a_0 a_1 \dots a_{m-1}} \subset \dots \subset I_{a_0}$, $m \in \mathbb{Z}_+$, such that $x \in I_{a_0 a_1 \dots a_m} \Rightarrow f^n(x) \in I_{a_n}$, $0 \leq n \leq m$.

Lemma 3: If I and J are closed intervals and I f-covers J then $\exists$ a closed subinterval $K \subset I$ such that $f(K) = J$.
Proof: Let $J = [a,b]$. $f^{-1}(a)$ and $f^{-1}(b)$ are closed and non-empty. So can choose $u \in f^{-1}(a)$, $v \in f^{-1}(b)$ such that $(u,v) \cap (f^{-1}(a) \cup f^{-1}(b)) = \emptyset$ (wlog $u < v$).
Set $K = [u,v]$ and use the Intermediate Value Theorem.

Proof of claim: Using lemma 3, $I_{a_0a_1} \subset I_{a_0}$ exists such that $f(I_{a_0a_1}) = I_{a_1}$.

Also,



Also, $I_{a_1a_2} \subset I_{a_1}$ exists such that $f(I_{a_1a_2}) = I_{a_2}$. Now let $I_{a_0a_1a_2} \subset I_{a_0a_1}$ be such that $f(I_{a_0a_1a_2}) = I_{a_1a_2}$. Continue inductively to obtain $I_{a_0a_1 \ldots a_m}$. Finally, $\bigcap_{m \geq 0} I_{a_0a_1 \ldots a_m} \neq \emptyset$, so h_0 is surjective.

(iii) $h \circ f|_\Lambda = \sigma_A \circ h$, by construction.

Properties of σ_A, 1: Periodic Orbits.

A word is a finite string $\underline{w} = w_0 \ldots w_k$. Given a transition matrix A, a word is said to be allowable, or allowed, if the transition from w_i to w_{i+1} is allowed for $0 \leq i < k$. Ie, $A_{w_i w_{i+1}} = 1$ for $i = 0, \ldots, k-1 \iff A_{w_0w_1} \cdot A_{w_1w_2} \cdots A_{w_{k-1}w_k} = 1$.

Lemma 4: Let $N_{ij}^{(n)}$ = number of allowed words $i w_1 w_2 \ldots w_{n-1} j$ of length $n-1$ from i to j. Then, $N_{ij}^{(n)} = (A^n)_{ij}$.

Proof: The product $A_{iw_1} A_{w_1w_2} \cdots A_{w_{n-1}j} = 1$ if $iw_1w_2 \ldots w_{n-1}j$ is an allowed word, and is 0 otherwise. Thus $N_{ij}^{(n)} = \sum_{w_1, \ldots, w_{n-1}} A_{iw_1} A_{w_1w_2} \cdots A_{w_{n-1}j} = (A^n)_{ij}$.

Proposition 5: $P_n(\sigma_A) = \# \mathrm{Fix}(\sigma_A^n) = \mathrm{Tr}(A^n)$

Proof: Fixed points of σ_A^n are in 1-1 correspondance with allowed words of length $n+1$ with same start and end. So $P_n(\sigma_A) = \sum_i N_{ii}^{(n)} = \sum_i (A^n)_{ii} = \mathrm{Tr}(A^n)$.

Remark: Can compute $\mathrm{Tr}\, A^n$ from a recurrence relation using the fact that A satisfies its own characteristic equation (Cayley-Hamilton Theorem). Let $P(\lambda) = (\lambda I - A)$, then $P(A) = 0$. So $A^m P(A) = 0 \ \forall \ m \geq 0$, and $\mathrm{Tr}(A^m P(A)) = 0$.

Example: $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. $P(\lambda) = \lambda^2 - \lambda - 1 = 0$. $\mathrm{Tr}\, A^{2+m} - \mathrm{Tr}\, A^{1+m} - \mathrm{Tr}\, A^m = 0$ (Fibonacci recurrence)
$P_1(\sigma_A) = 1$, $P_2(\sigma_A) = 3$, $P_3(\sigma_A) = 4$, $P_4(\sigma_A) = 7$.

Note: $N_q :=$ number of periodic orbits of least period q. Then $P_n(\sigma_A) = \sum_{q|n} q N_q$.

Properties of σ_A, 2: Chaos.

Definition: A is irreducible if $\forall i, j \ \exists n$ such that $(A^n)_{ij} \neq 0$, ie, an allowed word from i to $j \iff \exists$ a path from i to j in Γ.

Proposition 6: If A is irreducible then σ_A is topologically transitive.

Proof: Need to find a dense orbit. Choose a sequence which contains all allowed words, with proper choice of transition words between them to make the sequence allowable. Such transition words must exist as A is irreducible.

Definition: A is non-trivial if $\exists\ i, j_1 \neq j_2$ such that $i \to j_1, i \to j_2$ are allowed.

Example: This excludes permutation matrices, such as $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ - irreducible but trivial.

Proposition 7: If A is irreducible and non-trivial then σ_A has SDIC.

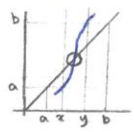
Proof: Given $\underline{a} = a_0 a_1 \dots \in \Sigma_{N,A}$ and $m \in \mathbb{Z}_+$, $\exists$ allowed word $a_m w_{m+1} \dots w_k = i$ (since A is irreducible), where i is followed by j_1 or j_2 (since A is non-trivial).
Either $w_{m+1} \dots w_k = a_{m+1} \dots a_k$, and then choose $\underline{b} = a_0 a_1 \dots a_k b_{k+1} \dots$, where $b_{k+1} \neq a_{k+1}$, and let $n = k$. Or, $w_{m+1} \dots w_k \neq a_{m+1} \dots a_k$, and then choose n to be the last integer such that $w_j = a_j$ and take $\underline{b} = a_0 a_1 \dots a_m \dots a_n b_{n+1}$, where $b_{n+1} \neq a_{n+1}$.
Then, $d(\underline{a}, \underline{b}) \leq \frac{1}{2} \cdot \frac{1}{3^m}$ and $d(\sigma^{n+1}(\underline{a}), \sigma^{n+1}(\underline{b})) \geq 1$. Hence σ_A is chaotic.

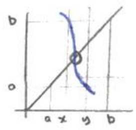
Remarks: (i) $f|_\Lambda$ semiconjugate to an irreducible non-trivial subshift $\not\Rightarrow$ $f|_\Lambda$ is chaotic, but if it is actually conjugate to σ_A, then it is chaotic. It can be proved that if f is expanding on $\bigcup_n I_n$, ie $\exists c > 1$ such that $\forall n, \forall x, y \in I_n$, $|f(x) - f(y)| \geq c|x - y|$, then h is a conjugacy.
(ii) $f|_\Lambda$ is semiconjugate to σ_A is enough to deduce $\exists$ at least as many periodic orbits of $f|_\Lambda$ as of σ_A.

Proposition 8: $\forall$ closed paths $a_0 a_1 \dots a_k = a_0$ in Γ, $\exists$ a periodic orbit for f in Λ, $(x_0, x_1, \dots, x_{k-1})^\infty$ such that $x_n \in I_{a_n}\ \forall n \geq 0$.

Lemma 9: If $I = [a, b]$ f-covers itself, then f has a fixed point in I.

Proof: By lemma 3, $\exists K = [x, y] \subset I$ such that $f(K) = I$.
Then either: (i) $f(x) = a \leq x$, $f(y) = b \geq y$ (i):





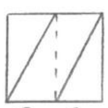
or: (ii) $f(x) = b > x$, $f(y) = a < y$.
In both cases, apply IVT to get $g(x) = f(x) - x = 0$:
(i) $g(x) < 0$ and $g(y) > 0 \Rightarrow \exists c \in [x, y]$ such that $g(c) = 0$, ie $f(c) = c$.

Proof of Proposition 8: As in proof of Theorem 2, $\exists$ subinterval $K_k \subset I_{a_0}$ such that for $n = 1, \dots, k$, $f^n(K_k) \subset I_{a_n}$ and $f^k(K_k) = I_{a_0}$ $-(*)$. In particular, K_k f^k-covers itself. By lemma 9, it has a fixed point x_0, and by $(*)$, $f^n(x_0) \in I_{a_n}\ \forall n$.

Note: If the loop $a_0 a_1 \dots a_k = a_0$ is of least period, then x_0 has least period k.

Remark: (iii) If the I_n's are not disjoint, but their interiors are, then can construct Γ in same way. Cannot deduce semiconjugacy to σ_A but can still get lots of periodic orbits for $f|_\Lambda$ - proposition 8 works just the same, but the only problem is that two different paths might lead to the same orbit (of common points)

Example:



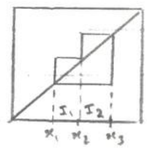
I_0 I_1

$f: S^1 \to S^1$, $x \mapsto 2x \bmod 1$. Γ: $0 \circlearrowleft 0 \rightleftarrows 1 \circlearrowright$
0^∞ and 1^∞ give the same fixed point for f.

3.4. Sharkovskii's Theorem

Proposition 9: Suppose $f: I \to I$, continuous, has a periodic point of period 3. Then f has periodic points of all periods.

Proof: Consider the period 3 orbit and label its points by $\{x_1, x_2, x_3\}$. Let $I_1 = [x_1, x_2]$, $I_2 = [x_2, x_3]$. Suppose $f(x_2) = x_3$. Then $f^2(x_2) = x_1$. So f-covering relations are Γ: $I_1 \rightleftarrows I_2 \circlearrowleft$
(If $f(x_2) = x_1$, then ~~set~~ relabel I_1, I_2. Get the same conclusion).
$\forall n \in \mathbb{N}$, we have a loop $I_1 \to \underbrace{I_2 \to \dots \to I_2}_{n-1} \to I_2 \to I_1$, which by Proposition 8 gives a periodic point of period n, which cannot have smaller period.



This is a special case of Sharkovskii's Theorem, which gives the precise order in which periodic orbits appear, for different periods.

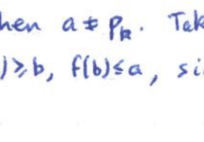
Definition: The Sharkovskii ordering of $\mathbb{N}$ is defined by:
$$1 \lhd 2 \lhd 2^2 \lhd 2^3 \lhd \dots \lhd 2^n \lhd \dots \lhd 2^n.7 \lhd 2^n.5 \lhd 2^n.3 \lhd \dots \lhd 2^{n-1}.5 \lhd 2^{n-1}.3 \lhd \dots \lhd 2^2.3 \lhd \dots \lhd 2.3 \lhd \dots \lhd 5 \lhd 3.$$

Theorem 10 (Sharkovskii's Theorem): Let $I \subseteq \mathbb{R}$ be a closed interval and $f: I \to I$ a continuous map. If f has a periodic point of period k and $n \lhd k$, then f has a periodic point of period n.

Proof: In stages. Start with case $k > 1$, odd:

Lemma 11: Let $f: I \circlearrowleft$, continuous, have a periodic point x of least period $k \geq 3$, odd, and no point of odd period n with $1 < n < k$. Then f has an orbit of least period n for all $n > k$, and all even $n < k$.

Proof: Let $J = [\min \mathcal{O}(x), \max \mathcal{O}(x)]$. Make a partition of J by the elements of $\mathcal{O}(x) = \{p_1 < p_2 < \dots < p_k\}$.
So define intervals I_i, $i = 1, 2, \dots, k-1$ of form $[p_l, p_{l+1}]$ (l not necessarily equal to i).
Aim: show we can choose labelling of the I_i's such that we get the following directed graph for the f-covering relations. (Stefan Graph):
Ie, loop $I_1 \circlearrowleft$, loop $I_1 \to I_2 \to \dots \to I_{k-1} \to I_1$, and directed edges from I_{k-1} to all odd vertices.



Once this is established, just need to note distinct loops of period: $1: I_1 \circlearrowleft$,
$n > k$: $(I_1^{\,n-k+2} \to I_2 \to \dots \to I_{k-1}) \to I_1$, even $< k$: $(I_{k-1} \to I_{2l+1} \to I_{2l+2} \to \dots \to I_{k-2}) \to I_{k-1}$.

Claim 1: $I_1 \to I_1$

Proof: Note $f(p_1) > p_1$ and $f(p_k) < p_k$. Take $a = \max\{y \in \mathcal{O}(x) : f(y) > y\}$. Then $a \neq p_k$. Take $I_1 = [a, b]$, where b is the closest point of $\mathcal{O}(x)$ to the right of a. Then $f(a) \geq b$, $f(b) \leq a$, since $b > a$. Hence $f(I_1) \supset I_1$

Claim 2: $f^{k-2}(I_1) \supset J$. (ie, $\exists$ paths from I_1 to all other vertices)

Proof: $f(I_1) \supset I_1$, with proper inclusion (else $k=2$, #). So $f^{j+1}(I_1) \supset f^j(I_1)$. There are $k-2$ points in $\mathcal{O}(x) \setminus \{a, b\}$, so $p_k \in f^j(I_1)$, some $0 \leq j \leq k-2$, and by nested property, $p_k \in f^{k-2}(I_1)$.
Similarly, $p_1 \in f^{k-2}(I_1)$. Since I_1 is connected, $f^{k-2}(I_1) \supset [p_1, p_k] = J$.

Claim 3: $\exists j \neq 1$ such that $I_j \to I_1$.

Proof: Let $B_l = \{y \in \mathcal{O}(x) : y \leq a\}$, $B_r = \{y \in \mathcal{O}(x) : y \geq b\}$. k odd $\Rightarrow \#B_l \neq \#B_r$. Let B be whichever of B_l, B_r has more elements. Then $y_1, y_2 \in B$, adjacent, with $f(y_1) \in B$ and $f(y_2) \in \mathcal{O}(x) \setminus B$.
Take I_j = interval from y_1 to y_2. Then $I_1 \subset f(I_j)$.

Now label the intervals such that $I_1 \to I_2 \to \dots \to I_l \to I_1$ is the shortest loop containing I_1.

Claim 4: The shortest loop with $l \geq 2$ has $l = k-1$.

Proof: $\exists$ only $k-1$ distinct intervals, so shortest loop has $l \leq k-1$. Assume $l < k-1$. Either l or $l+1$ is odd. Let $q \in \{l, l+1\}$, odd. So $1 < q < k$. Use loop $I_1 \to \dots \to I_l \to I_1$ or $I_1 \to \dots \to I_l \to I_1 \to I_1$ depending whether $q = l$ or $l+1$. By propostion 8, $\exists$ $y \in I_1$ such that $f^{q}(y) = y$. Now $y \notin \partial I_1$, since points on ∂I_1 have period $k > q$. Hence y has period $q < k$ ※

Claim 5: (a) If $f(I_i) \supset I_1$ then either $i = l$ or $i = k-1$.

(b) for $j > i+1$, $I_i \not\to I_j$.

(c) I_1 f-covers only I_1 and I_2.

Proof: Claim 4 $\Rightarrow$ (a). Shortest loop $\Rightarrow$ (b), (c).

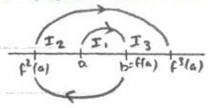
Remains to show $I_{k-1} \to I_j$, j odd.

Claim 6: Orderings (in terms of $\mathbb{R}$) of I_i's and of $\mathcal{O}(x)$ are either:

(i) $I_{k-1} \leq I_{k-3} \leq \dots \leq I_2 \leq I_1 \leq I_3 \leq \dots \leq I_{k-2}$, $f^{k-1}(a) < f^{k-3}(a) < \dots < f^2(a) < a < f(a) < f^3(a) < \dots < f^{k-2}(a)$,

or (ii) both exactly reversed.

Proof: $I_1 = [a,b]$ f-covers only I_1 and I_2, so by connectedness, I_1 and I_2 must be adjacent. Assume $I_2 \leq I_1$ (to get (i); $I_1 \leq I_2$ gives (ii)). Then, must have $f(a) = b$ and $f(b) =$ left point of I_2. $f(a) = b$ and $I_2 \not\to I_1 \Rightarrow f(I_2) \geq a$.



$I_2 \to I_3$, but $I_2 \not\to I_j$ $(j > 3)$ $\Rightarrow$ I_3 adjacent to I_1. Continue inductively.

Claim 7: $I_{k-1} \to I_j$, j odd.

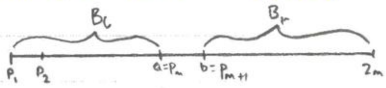
Proof: Note $I_{k-1} = [f^{k-1}(a), f^{k-3}(a)]$. Then, $f(f^{k-1}(a)) = f^k(a) = a$. Also, $f^{k-3}(a) \in I_{k-3}$, so $f(f^{k-3}(a)) = f^{k-2}(a) \in I_{k-2}$. Thus, $f(I_{k-1}) \supset [a, f^{k-2}(a)] = I_1 \cup I_3 \cup \dots \cup I_{k-2}$.

End of proof of lemma. For all (least period) loops in Stefan graph, use Proposition 8 to get a periodic orbit of same least period.

Lemma 12: If f has an orbit of least period $k = 2m$, $m \geq 1$, and no odd period ≥ 3, then f has a fixed point and f^2 has two periodic orbits of period m. These are: $\{p_1, \dots, p_m\}$ and $\{p_{m+1}, \dots, p_{2m}\}$.

Proof: Define a, b and set $I_1 = [a,b]$ as before. Then $I_1 \to I_1$ and $\exists$ a fixed point in I_1.

Note: in lemma 11 we used the fact that k was odd only in Claim 3, to show $\exists$ I_j, $j \neq 1$, such that $I_j \to I_1$.



Here, if such an I_j exists, then get Stefan graph as before, but with k even. So $\exists$ a loop of period $k-1$, ie an odd period. ※. Hence I_1 only f-covers itself. Since $f(a) \geq b$, at least one point changes side wrt I_1, so all points in $\mathcal{O}(x)$ change side. Ie, $f(B_l) = B_r$ and $f(B_r) = B_l$, so $\#B_l = \#B_r$, so I_1 is in the middle. Hence $f^2(B_l) = B_l$, $f^2(B_r) = B_r$, so B_l and B_r are permuted independently by f^2.

Proof of Sharkovskii's Theorem: If k odd, done by lemma 11.

If $k = 2r$, r odd, then f has period k, and f^2 splits $\mathcal{O}(x)$ into two components, by lemma 12, each with period r (for f^2). So f^2 has period r, odd. So

(i) $r = 1 \Rightarrow$ period 2 point for f.

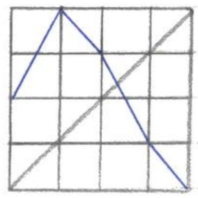
(ii) $r \geq 3 \Rightarrow$ by lemma 11, f^2 has all periods $\geq r$ and all evens $\leq r$, and period 1.

$\Rightarrow$ f has periods $2m$ (all $m \geq r$), $2p$ (all even p), period 2, period 1.

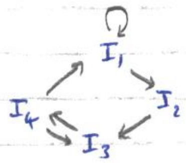
If $k = 2^l r$, r odd, $l > 1$, then just repeat the argument.

Remark (i): There are examples of maps with exactly the periodic orbits implied by the Sharkovskii ordering.

Example:



$k=5$, f - piecewise linear.



Exercise: prove this map has no period 3 orbit.

Remark (ii): $\forall n \geq 3$, $\exists$ permutation of n elements such that a periodic orbit $\{p_1 < \dots < p_n\}$ of least period n that realises the permutation, forces existence of all other periods.

4. Differential Equations In The Plane

4.1. Linearisation and Stationary Points (revision).

Given system: $\dot{x} = f(x)$, $x \in \mathbb{R}^2$, $f: \mathbb{R}^2 \to \mathbb{R}^2$, a point x^* is a stationary point (or fixed, equilibrium) if $f(x^*) = 0$.

Near x^*, write $x = x^* + v$, $|v| \ll 1$, so $\dot{x} = \dot{v} = f(x^* + v) = Df(x^*)v + \mathcal{O}(|v|^2)$, which can be approximated by linear equation: $\dot{v} = Av$ (Linearisation), where A is a constant 2×2 matrix: $A_{ij} = \frac{\partial f_i}{\partial x_j}(x^*)$ - the Jacobian.

Case 1: A has distinct real eigenvalues, λ_1, λ_2, with eigenvectors e_1, e_2.

Set $M = [e_1, e_2]$ - matrix with eigenvectors as columns.

Then, $AM = [\lambda_1 e_1, \lambda_2 e_2] = [e_1, e_2]\begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix} =: M\Lambda$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$.

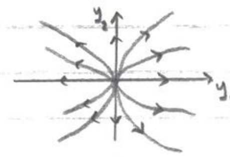
M is invertible as e_1, e_2 independent.

$\dot{x} = Ax$, and change of coordinates: $y = M^{-1}x$, then $\dot{y} = M^{-1}Ax = M^{-1}AMy = \Lambda y$.

So, in coordinates given by basis of eigenvectors, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $\dot{y}_1 = \lambda_1 y_1$, $\dot{y}_2 = \lambda_2 y_2$.

(i) $\lambda_1 > \lambda_2 > 0$:

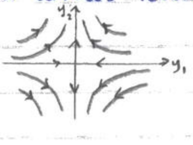
$\frac{dy_1}{dy_2} = \frac{\lambda_1}{\lambda_2} \cdot \frac{y_1}{y_2} \Rightarrow y_1 = c\, y_2^{\lambda_1/\lambda_2}$:



- unstable node. Trajectories tangent to eigenvector corresponding to smallest eigenvalue.

(ii) $\lambda_1 < \lambda_2 < 0$: same picture, but with arrows reversed. - stable node.

(iii) $\lambda_1 < 0 < \lambda_2$: then $y_1 = c\, y_2^{\lambda_1/\lambda_2}$:



- saddle.

Case 2: Complex conjugate eigenvalues: $\lambda = \rho \pm i\omega$, $\omega \neq 0$.

Have complex conjugate eigenvectors, $Az = (\rho + i\omega)z$. Set $z = p + iq$.

So $Ap = \rho p - \omega q$, $Aq = \rho q + \omega p$. Let $M = [q, p]$, then $AM = [\rho q + \omega p, \rho p - \omega q]$ $= [q, p]\begin{pmatrix}\rho & -\omega \\ \omega & \rho\end{pmatrix}$. Hence, in coordinates $y = M^{-1}x$, $\dot{y} = \begin{pmatrix}\rho & -\omega \\ \omega & \rho\end{pmatrix} y$.

Digression: $(x,y) \to (r,\theta)$. $r^2 = x^2+y^2$, so $r\dot r = x\dot x + y\dot y \Rightarrow \dot r = \frac{x\dot x + y\dot y}{r}$

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dot x = \dot r\cos\theta - r\dot\theta\sin\theta \\ \dot y = \dot r\sin\theta + r\dot\theta\cos\theta \end{array}\right\}$ $\sin\theta\,\dot x - \cos\theta\,\dot y = -r\dot\theta \Rightarrow \dot\theta = \frac{x\dot y - y\dot x}{r^2}$.

Here, $\dot x = \rho x - \omega y$, $\dot y = \omega x + \rho y \Rightarrow \dot r = \rho r$, $\dot\theta = \omega$. $\Rightarrow \frac{dr}{d\theta} = \frac{\rho}{\omega} r \Rightarrow r = ce^{(\rho/\omega)\theta}$.

- Logarithmic spiral.

$\omega > 0, \rho > 0$:  - unstable focus.

$\omega > 0, \rho < 0$:  - stable focus.

If $\rho = 0$, $\dot r = 0$: centre: 

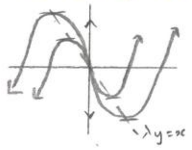
Case 3: Real equal eigenvalues, λ.

The characteristic equation of A is $(s-\lambda)^2 = 0$. Hence $(A-\lambda I)^2 x = 0 \quad \forall x \in \mathbb{R}^2$.

Two cases: (i) $\exists\, e_1 \in \mathbb{R}^2 \setminus \{\binom{0}{0}\}$ such that $(A-\lambda I)e_1 \neq 0$. Set $(A-\lambda I)e_1 =: e_2$.

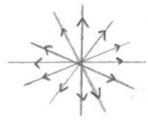
Then, $Ae_1 = e_2 + \lambda e_1$, $Ae_2 = \lambda e_2$. Set $M = [e_1, e_2]$. Then,

$AM = [\lambda e_1 + e_2, \lambda e_2] = [e_1, e_2]\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$. Ie, $\dot x = \lambda x$, $\dot y = x + \lambda y$.



unstable degenerate node. (stable if $\lambda < 0$)

(ii) $(A-\lambda I)x = 0 \quad \forall x \in \mathbb{R}^2$, ie $A = \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Then, $\dot x_1 = \lambda x_1$, $\dot x_2 = \lambda x_2 \Rightarrow \frac{dx_1}{dx_2} = \frac{x_1}{x_2}$.

So $x_1 = Cx_2$: 

$\lambda > 0$: unstable star.

$\lambda < 0$: stable star (reverse the arrows)

Definition: A stationary point x^* is hyperbolic if $\mathrm{Re}(\lambda) \neq 0$ for all eigenvalues λ of $Df(x^*)$.

Statement: Given $\dot x = f(x)$, $x \in \mathbb{R}^2$, $f: \mathbb{R}^2 \to \mathbb{R}^2$, smooth, then the linearised flow near a stationary point x^* is locally a good picture of the nonlinear flow near x^* if x^* is hyperbolic.

Example: $\left\{\begin{array}{l} \dot r = -r^2 \\ \dot\theta = \omega. \end{array}\right.$: gives  Linear version: $\left\{\begin{array}{l} \dot r = 0 \\ \dot\theta = \omega \end{array}\right.$: 

0 is not hyperbolic.

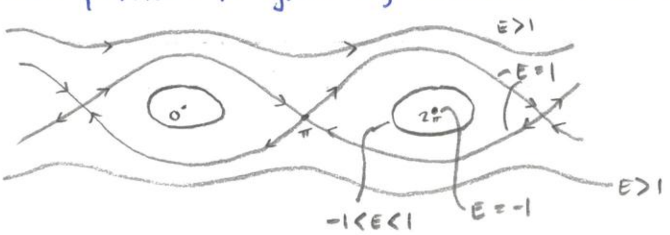
But, there are examples where we get a nonlinear centre.

Consider $\ddot x + \sin x = 0$. $\left\{\begin{array}{l} \dot x = y \\ \dot y = -\sin x. \end{array}\right.$ Stationary points at $(n\pi, 0)$.

At $(0,0)$, Jacobian $= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, with eigenvalues $\pm i$, so linearly a centre.

Note that $\frac{1}{2}y^2 - \cos x =: E$ is constant on trajectories.

Phase portrait is given by contours of constant E:



4.2 Global Phase Portrait.

Methodology: locate stationary points, determine their type, sketch trajectories by patching together local pictures.

Example: model of population dynamics (rabbits and sheep)

Populations of rabbits, r, and sheep, s, on a finite grassy island.

$\dot r = r(3-r-2s)$, $\dot s = s(2-r-s)$, $r,s \geq 0$.

Stationary points: $(r,s) = (0,0), (0,2), (3,0), (1,1)$

Jacobian $= \begin{pmatrix} 3-2r-2s & -2r \\ -s & 2-r-2s \end{pmatrix}$.

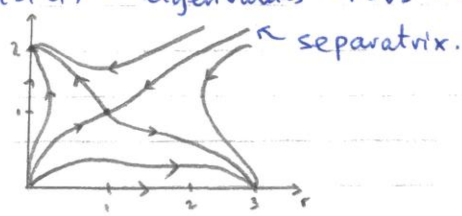
At $(0,0)$, $J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ - unstable node.

At $(0,2)$, $J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$ - eigenvalues $-1, -2$ - stable node.

At $(3,0)$, $J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$ - eigenvalues $-3, -1$ - stable node.

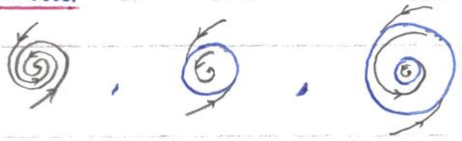
At $(1,1)$, $J = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$ - eigenvalues $-1 \pm \sqrt{2}$ - saddle.

This gives:



4.3. Existence of periodic orbits — negative criteria.

How do we tell the difference between:



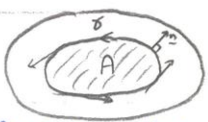
?

In sketches, small r and large r behaviours are the same in all three.

Consider $\begin{cases} \dot x = f(x,y) \\ \dot y = g(x,y) \end{cases}$, $(x,y) \in \mathbb{R}^2$, f, g are C^1 $-(1)$

Theorem 1 (Bendixson Criterion): If, on a simply connected region $\mathcal{D} \subset \mathbb{R}^2$, $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is not identically zero and does not change sign, then (1) has no periodic orbit lying entirely inside $\mathcal{D}$.

Proof: Suppose periodic orbit γ lies entirely in $\mathcal{D}$, enclosing area $A \subset \mathcal{D}$.



Then, $\iint_A \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dx\,dy \neq 0$. But by the divergence theorem,

$\iint_A \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dx\,dy = \oint_\gamma (f,g)\cdot \underline{n}\, dl$, and $(f,g)\cdot\underline{n} = 0$ on γ, since vector field tangent to orbits ※

An extension of this is:

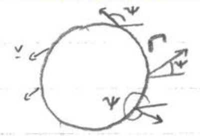
Theorem 2 (Dulac Criterion): If $\exists$ a C^1 function $B: \mathbb{R}^2 \to \mathbb{R}^2$ such that the conditions in Theorem 1 hold for $\frac{\partial (Bf)}{\partial x} + \frac{\partial (Bg)}{\partial y}$, then we have the same result.

Proof: exercise.

Example: Duffing oscillator: $\begin{cases} \dot x = y \\ \dot y = x - x^3 - \delta y, \ \delta > 0 \end{cases}$, $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\delta < 0 \Rightarrow$ no periodic orbit in $\mathbb{R}^2$

Poincaré Index: Given a continuous vector field $\underline{v}(f,g)$ on $\mathbb{R}^2$ and a simple closed curve Γ such that $\underline{v} \neq 0$ on Γ, define the index of Γ, I_Γ, to be the number of times $v(x)$ goes round 0 when $\underline{x}$ goes round Γ, counted anticlockwise. Clearly, $I_\Gamma \in \mathbb{Z}$.

Algebraic representation: $I_\Gamma = \frac{1}{2\pi} \oint_\Gamma d\Psi = \frac{1}{2\pi} \oint_\Gamma \frac{f dg - g df}{f^2+g^2}$



Since $I_\Gamma \in \mathbb{Z}$ and I_Γ is continuous if $f^2+g^2 \neq 0$, then a smooth deformation Γ' which does not pass through a fixed point of $\underline{v}$ has $I_{\Gamma'} = I_\Gamma$. Thus we can define the index of an isolated fixed point p as the index of any curve containing p and no other fixed points.

Properties: Index of : saddle is -1

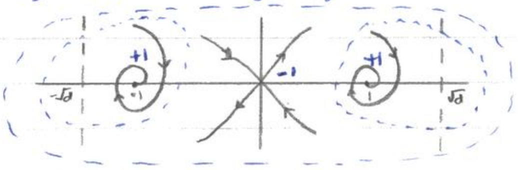
focus, node, centre, periodic orbit is $+1$

closed curve containing no fixed points is 0.

general closed curve is the sum of indices of fixed points within it.

Example: van der Pol - Duffing : $\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 - \delta y + x^2 y, \ \delta > 0. \end{cases}$

$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -\delta + x^2.$

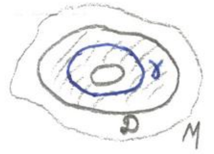


4.4. Positive Criteria for Periodic Orbits - the Poincaré-Bendixson Theorem.

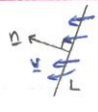
Theorem 3 (Poincaré-Bendixson Theorem): Let M be a simply connected subset of $\mathbb{R}^2$ and $\varphi^t(x)$ a C^1 flow on M. Let $p \in M$ be such that $\mathcal{O}^+(p)$ is bounded and $\omega(p)$ does not contain any fixed points. Then $\omega(p)$ is a periodic orbit. [Same result for $\mathcal{O}^-(p)$ and $\alpha(p)$]

If p is not itself on a closed periodic orbit then the closed orbit is said to be a limit cycle. (So γ = limit cycle $\Leftrightarrow \exists\, p \notin \gamma$ such that $\omega(p) = \gamma$)

Corollary 4 (P-B): Let M, $\varphi^t(x)$ be as above. Let $\mathcal{D} \subset M$ be a positively invariant (ie $\varphi^t \mathcal{D} \subset \mathcal{D}\ \forall\, t>0$) bounded annular region containing no fixed points. Then $\mathcal{D}$ contains a periodic orbit.

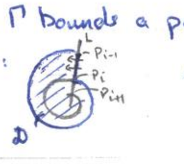


Definition: Let $\underline{v}$ be the vector field, L a C^1 arc in M. Then L is said to be transverse to the flow, or a (local) transversal, if, at each point of L, $\underline{v} \cdot \underline{n} \neq 0$. In particular, $\underline{v}$ has no fixed points on L and is not tangent to L.

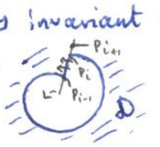


Lemma 5: Let L be a transversal to the flow. Then $\mathcal{O}^+(p)$ intersects L in a monotonic sequence, ie, if p_i is the ith intersection of $\mathcal{O}^+(p)$ with L, then $p_i \in [p_{i-1}, p_{i+1}]$.

Proof: Let Γ be the piece of $\mathcal{O}^+(p)$ from p_{i-1} to p_i along with segment $[p_i, p_{i-1}] \subset L$ (so Γ is a Jordan Curve). Then Γ bounds a positively invariant region $\mathcal{D}$. Hence $\mathcal{O}^+(p_i) \subset \mathcal{D}$, and thus $p_{i+1} \in \mathcal{D}$. So:



, or :



Lemma 6: $\omega(p) \cap L$ is at most one point.

Proof: By contradiction. Suppose $q_1, q_2 \in \omega(p) \cap L$, $q_1 \neq q_2$. Then, by definition of $\omega(p)$, $\exists\ (p_n), (\bar{p}_n) \subset \mathcal{O}^+(p) \cap L$ such that $p_n \to q_1$, $\bar{p}_n \to q_2$ as $n \to \infty$. This contradicts the monotonicity of intersections of $\mathcal{O}^+(p)$ with L.

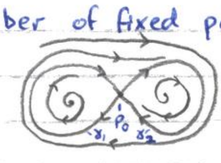
Proof of Theorem 3: Strategy - choose $q \in \omega(p)$. Show $\mathcal{O}(q)$ is periodic and then show $\omega(p) = \mathcal{O}(q)$. Since $\mathcal{O}^+(p)$ is bounded, $\omega(p) \neq \phi$. Let $q \in \omega(p)$. Then $\omega(q) \subset \omega(p)$. Choose $x \in \omega(q)$. Then x is not a fixed point. So can take transversal L at x. Since $x \in \omega(q)$, $\exists\ (t_n) \to \infty$ such that $q_n := \varphi^{t_n}(q) \xrightarrow[n\to\infty]{} x$ and $\{q_n\} \subset L$. Now, $\{q_n\} \subset \omega(p)$ by its invariance under the flow. Since $\omega(p) \cap L$ is at most one point (lemma 6), $q_n = x\ \forall n$. Hence $\varphi^{t_{n+1}-t_n}(q) = q$, so q is a periodic point of period $t_{n+1} - t_n$, and $\mathcal{O}(q)$ is a closed curve. So $\omega(q) = \mathcal{O}(q)$ and $\mathcal{O}(q) \subset \omega(p)$. Hence $\omega(p)$ is a union of periodic orbits. But $\omega(p)$ is connected (exercise). Therefore, $\omega(p)$ is a single periodic orbit, so $\omega(p) = \mathcal{O}(q)$.

Remarks: (i) Theorem 3 needs the existence of a Jordan Curve. So valid on S^2 and $\mathbb{T}^1 \times \mathbb{R}$ (cylinder), but not on $\mathbb{T}^2$ (torus).

(ii) Can generalise the P-B Theorem for positively invariant domain $\mathcal{D}$ including a finite number of fixed points $\{p_i : i = 1, .., n\}$. If $p \in \mathcal{D}$, then either

(a) $\omega(p)$ is a fixed point, (b) $\omega(p)$ is a closed orbit,

(c) $\omega(p)$ consists of a finite number of fixed points and orbits γ with $\alpha(\gamma) = p_i$ and $\omega(\gamma) = p_j$. Eg:



$\omega(p) = \gamma_1 \cup \gamma_2 \cup p_0$
$\omega(\gamma_1) = \alpha(\gamma_1) = p_0$
$\omega(\gamma_2) = \alpha(\gamma_2) = p_0$

Applications of P-B.

Need to find a bounded annular region $\mathcal{D}$ which is positively invariant and contains no fixed points - "P-B domain".

Example 1: normal application.

$\dot{r} > 0$ for $0 < r \leq R_1$, $\dot{r} < 0$ for $r \geq R_2$.

Then, $\mathcal{D} = \{(r, \theta) : R_1 \leq r \leq R_2\}$ is a P-B domain if $\mathcal{D}$ contains no fixed points. $\exists$ at least one periodic orbit in $\mathcal{D}$. If this is unique, this must be $\omega(p)$ for all $p \in \mathcal{D}$ (stable).

Example 2: $\ddot{x} + f(x)\dot{x} + g(x) = 0$. Set $y = \dot{x} + F(x)$ where $F(x) = \int_0^x f(x')\,dx'$.

Then $\begin{cases} \dot{x} = y - F(x) \\ \dot{y} = -g(x) \end{cases}$ (Liénard coordinates). - Liénard oscillators.

Theorem 7: Consider $\ddot{x} + f(x)\dot{x} + g(x) = 0$ and suppose

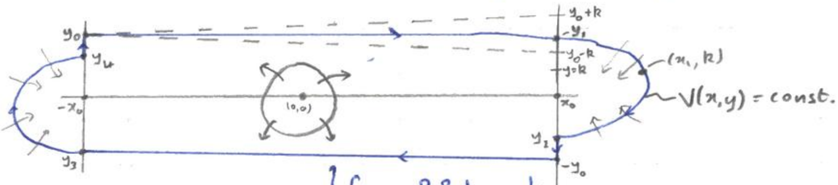
(i) $g(0) = 0$, $xg(x) > 0$ for $x \neq 0$, $f(0) < 0$, $g'(0) > 0$.

(ii) $\exists\ k > 0$ such that $\text{sign}(x)\, F(x) > k$ for $|x|$ sufficiently large.

(iii) $G(x) = \int_0^x g(x')\,dx' \to \infty$ as $|x| \to \infty$.

Then, the system has at least one periodic orbit.

Proof: By (i), $(0,0)$ is the unique fixed point and it is unstable. So $\exists$ a small closed curve around the origin that orbits cross outwards. This gives inner boundary of P-B domain. Now construct the outer boundary in 3 stages.



forms P-B boundary.

(a) $\frac{dy}{dx} = \frac{-g(x)}{y-F(x)}$. Choose x_0 sufficiently large that $\text{sign}(x)F(x) > k > 0$. Choose $y_0 > 2k$ such that $\left|\frac{dy}{dx}\right| < \frac{k}{2x_0}$, $\forall y \in [y_0-k, y_0+k]$, $\forall x \in [-x_0, x_0]$. Then, orbit through $(-x_0, y_0)$ strikes $x = x_0$ at (x_0, y_1) with $k < y_0 - k < y_1 < y_0 + k$.

(b) Define $V(x,y) := \frac{1}{2}(y-k)^2 + G(x)$ and consider $V(x,y) = \text{const.}$, through (x_0, y_1). Since G is an increasing function of $|x|$, $\exists\, x_1 > x_0$ such that $V(x,y) = V(x_0, y_1)$ passes through (x_1, k). But V is symmetric about $y = k$ and hence curve strikes $x = x_0$ at (x_0, y_2), $y_2 = -y_1 + 2k$.

Now, $\dot{V} = \dot{y}(y-k) + \dot{x}\,g(x) = -g(x)(y-k) + [y-F(x)]g(x) = -g(x)(F(x)-k) < 0$ if $x > x_0$.

(c) $y_2 = -y_1 + 2k \leq -y_0 + 3k < k$. Now, $\dot{x} = y - F(x) \leq y - k$ on $x = x_0$, so $\dot{x} < 0$ on $x = x_0$, $-y_0 < y < y_2$.

Finally, repeat (a), (b), (c) as in diagram, but using $V(x,y) = \frac{1}{2}(y+k)^2 + G(x)$. Have a P-B domain.

Uniqueness, Number of Periodic Orbits.

Not an easy problem in general, but uniqueness can be proved in some particular cases, such as the Van der Pol oscillator: $\ddot{x} + (x^2-1)\dot{x} + x = 0$.

There is, however, a criterion for uniqueness: if in a P-B domain $\mathcal{D}$ the vector field $\underline{v}$ satisfies $\underline{\nabla}\cdot\underline{v} < 0$ then there is a unique periodic orbit in $\mathcal{D}$, and it attracts all orbits in $\mathcal{D}$. (See examples sheet 3, question 6).

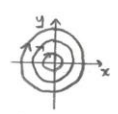
4.5 Pertubation Methods.

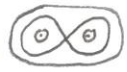
(i) Nearly-Hamiltonian Systems.

A Hamiltonian system satisfies $\dot{x} = \frac{\partial H}{\partial y}$, $\dot{y} = -\frac{\partial H}{\partial x}$, some $H: \mathbb{R}^2 \to \mathbb{R}^2$ (independent of t).

Then $\frac{dH}{dt} = 0$ along orbits.

Every compact connected component of a level set containing no equilibrium points is a periodic orbit. (See sheet 3, question 8).

Example: $H(x,y) = \frac{1}{2}(x^2+y^2)$, harmonic oscillator: 

More generally, plane divides into 2 regions foliated by periodic orbits: 

Suppose we have a (not necessarily Hamiltonian) perturbation of the Hamiltonian system:
$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} + \varepsilon f(x,y) \\ \dot{y} = -\frac{\partial H}{\partial x} + \varepsilon g(x,y). \end{cases}$$

Question: which, if any, of the periodic orbits of the Hamiltonian system persist for small ε?

For a family of periodic orbits of the system labelled by the value E of H, let $M(E) = \oint_{\gamma(E)} g\,dx - f\,dy$, oriented in the direction of time.

Theorem 8: If, for $0 \le \varepsilon < \varepsilon_0$, γ_ε is a periodic orbit of the perturbed system, depending continuously on ε, then the periodic orbit γ_0 has $M(E) = 0$.

Proof: $\oint_{\gamma(E)} dH = 0$, as it is a closed curve. So, $0 = \oint_{\gamma_\varepsilon} \frac{\partial H}{\partial x}dx + \frac{\partial H}{\partial y}dy = \oint_{\gamma_\varepsilon} (-\dot y + \varepsilon g)\,dx + (\dot x - \varepsilon f)\,dy$
$= \varepsilon \oint_{\gamma_\varepsilon} g\,dx - f\,dy$, as $\int -\dot y\,dx + \dot x\,dy = \int (-\dot y \dot x + \dot x \dot y)\,dt = 0$.
True $\forall\ 0 \le \varepsilon < \varepsilon_0 \Rightarrow \oint_{\gamma_0} g\,dx - f\,dy = 0$.

Conversely:

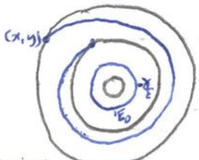
Theorem 9: Every simple zero of M gives rise to a continuous family γ_ε of periodic orbits for sufficiently small ε.

Proof: Not given - implicit function theorem.

Example: Van der Pol oscillator for small ε: $\ddot x + \varepsilon(x^2-1)\dot x + x = 0 \Rightarrow \begin{cases} \dot x = y \\ \dot y = -x - \varepsilon(x^2-1)y. \end{cases}$
When $\varepsilon = 0$, this is Hamiltonian, with $H = \frac{1}{2}(x^2+y^2)$. So, $M(E) = -\oint_{x^2+y^2=2E} (x^2-1)y\,dt$.
(Note orientation). Let $x = \sqrt{2E}\cos\theta$, $y = \sqrt{2E}\sin\theta$.
So, $M(E) = -\int_0^{2\pi} (2E\cos^2\theta - 1)(-\sqrt{2E}\sin\theta)(-\sqrt{2E}\sin\theta)\,d\theta = -\pi E^2 + 2\pi E$.
$M(E) = 0 \iff E = 0$ or $E = 2$. $E = 0$ is not a periodic orbit. So the only periodic orbit which can persist is $x^2+y^2 = 4$.

Stability of the resulting periodic orbit: For small ε, $\begin{cases} \varepsilon M'(E) < 0 \Rightarrow \text{attracting} \\ \varepsilon M'(E) > 0 \Rightarrow \text{repelling}. \end{cases}$

Proof: For the orbit of a general point (x,y) of the perturbed system we can evaluate $\Delta H(x,y) = \int_0^T dH$, where T is the period of (x,y) under the system.



$M(E_0) = 0$.

This equals, as before, $\varepsilon \int g\,dx - f\,dy = \varepsilon M(E) + O(\varepsilon^2)$, where $E = H(x,y)$.
So if $M(E_0)$, $M'(E_0) < 0$, $\varepsilon > 0$, then $\Delta H < 0$ for $E > E_0$. (for small ε)
Similarly, $\Delta H > 0$ for $E < E_0$
So γ_ε is attracting.

Exercise: van der Pol: $M'(E) = -2\pi E + 2\pi = -2\pi$ for $E = 2$. So periodic orbit is attracting for small ε.

(ii) Method of Multiple Scales.

Suppose we know all the solutions of an N^{th} order ODE, $L(x, \dot x, \ddot x, .., t) = 0$. These form an N-parameter family, with arbitrary constants, $\underline{A} = A_1, .., A_N$, $x_{\underline{A}} = \tilde{x}(t; \underline{A})$. Then, if we are interested in the perturbed equation $L(x, \dot x, .., t) = \varepsilon f(x, \dot x, .., t)$ - (1), $0 < \varepsilon \ll 1$, try to represent solutions as $x(t,\varepsilon) = \tilde{x}(t; A(\varepsilon t)) + \varepsilon x_1(t, \varepsilon t) + O(\varepsilon^2)$, for $|t| < \frac{1}{\varepsilon}$, with x_1 bounded.
More specifically, $\sup_{\varepsilon \ge 0} \sup_{0 \le t \le 1/\varepsilon} |x_1(t, \varepsilon t)| < \infty$.

General principle: boundedness of $x_1 \Rightarrow$ equation of evolution for $\underline{A}$ on slow time scale, εt.
More generally, consider solutions as functions of two different time scales: $\tau := t$ ("fast time"), $T := \varepsilon t$ ("slow time"). $x(t,\varepsilon) \sim x_0(\tau, T) + \varepsilon x_1(\tau, T) + \varepsilon^2 x_2(\tau, T) + \dots$ - (2), with $x_n = O(1)$ for T of order 1.
Then, $\frac{d}{dt} = \frac{\partial}{\partial \tau} + \varepsilon \frac{\partial}{\partial T}$ - (3)
Substitute (2) and (3) in (1) and compare coefficients of $\varepsilon^0, \varepsilon^1, \varepsilon^2, \dots$

<u>Example:</u> van der Pol: $\ddot{x} + x = -\varepsilon\dot{x}(x^2-1)$, $0<\varepsilon\ll 1$. Try $x(t,\varepsilon) \sim x_0(\tau,T) + \varepsilon x_1(\tau,T) + \dots$

Then $\dot{x}(t,\varepsilon) \sim x_{0\tau} + \varepsilon(x_{0T} + x_{1\tau}) + \dots$

$\ddot{x}(t,\varepsilon) \sim x_{0\tau\tau} + \varepsilon(2x_{0\tau T} + x_{1\tau\tau}) + \dots$

In equation, $O(\varepsilon^0)$: $x_{0\tau\tau} + x_0 = 0 \Rightarrow x_0 = A(T)e^{i\tau} + A^*(T)e^{-i\tau}$ ($A(T)$ as yet undetermined).

$O(\varepsilon^1)$: $x_{1\tau\tau} + x_1 = -2x_{0\tau T} - x_{0\tau}(x_0^2-1)$

RHS $= -2i\left[A_T e^{i\tau} - A_T^* e^{-i\tau}\right] - \left[A^2 e^{2i\tau} + 2|A|^2 + A^{*2} e^{-2i\tau} - 1\right](iAe^{i\tau} - iA^*e^{-i\tau})$

Have $e^{3i\tau} \to$ PI of form $e^{\pm 3i\tau}$. No problem here.

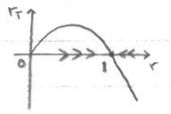
$e^{i\tau} \to$ PI of form $\tau e^{\pm i\tau}$ - not uniformly bounded wrt ε on time interval $|t| \le 1/\varepsilon$ - called <u>resonant</u>, or <u>secular</u> terms.

Condition for bounded solutions is to kill secular terms, ie, set to zero the coefficients of $e^{\pm i\tau}$ in RHS $\Rightarrow$ differential equation for A.

$e^{i\tau}$: $-2iA_T + iA^2A^* - 2i|A|^2A + iA = 0 \iff 2A_T = A(1-|A|^2)$

Let $A(T) = r(T)e^{i\theta(T)}$. Im: $\theta_T = 0 \Rightarrow \theta$ is constant, θ_0.

Re: $2r_T = r - r^3$.



Fixed points: 0 (unstable), 1 (stable).

If $r(0) \neq 0$, then $r \to 1 \Rightarrow$ for non-trivial initial condition,

$x_0(\tau,T) \to e^{i(\tau+\theta_0)} + e^{-i(\tau+\theta_0)} = 2\cos(t+\theta_0)$ - limit cycle of radius 2.

Solving the differential equation for $r(T)$ gives the way solutions approach the periodic orbit, $2\cos(t+\theta_0) + O(\varepsilon)$

<u>Example:</u> Mathieu equation: $\ddot{x} + x = -4\varepsilon x\cos 2t$, $0<\varepsilon\ll 1$.

$O(\varepsilon^0)$: $x_{0\tau\tau} + x_0 = 0$, $x_0 = A(T)e^{i\tau} + A^*(T)e^{-i\tau}$

$O(\varepsilon^1)$: $x_{1\tau\tau} + x_1 = -2x_{0\tau T} - 4x_0\cos 2\tau$.

RHS $= -2i(A_T e^{i\tau} - A_T^* e^{-i\tau}) - 2(Ae^{i\tau} + A^*e^{-i\tau})(e^{2i\tau} + e^{-2i\tau})$

Secular terms: $e^{\pm i\tau}$, set their coefficients to zero. $e^{i\tau}$: $-2iA_T - 2A^* = 0$.

So $A_T = iA^*$ (non-secularity condition). Write $A(T) = u(T) + iv(T)$.

Re: $u_T = v$. Im: $v_T = u \Rightarrow u_{TT} = u \Rightarrow u = Ce^T + De^{-T}$, $v = Ce^T - De^{-T}$.

- Amplitudes of most solutions grow exponentially on slow time scale - resonance phenomenon.

<u>Example: Asynchronous Quenching.</u>

Periodically forced van der Pol: $\ddot{x} + x + a(\omega^2-1)\cos\omega t = -\varepsilon\dot{x}(x^2-1)$, $0<\varepsilon\ll 1$, $a, |\omega-1| \gg \varepsilon$, $\omega \ge 0$.

Usual rigmarole: try $x(t,\varepsilon) = x_0(\tau,T) + \varepsilon x_1(\tau,T) + O(\varepsilon^2)$

$O(\varepsilon^0)$: $x_{0\tau\tau} + x_0 + a(\omega^2-1)\cos\omega\tau = 0$, so $x_0 = A(T)e^{i\tau} + A^*(T)e^{-i\tau} + a\cos\omega\tau$.

$O(\varepsilon^1)$: $x_{1\tau\tau} + x_1 = -2x_{0\tau T} - (x_0^2-1)x_{0\tau}$

RHS $= -2i(A_T e^{i\tau} - A_T^* e^{-i\tau}) - \left\{\left(Ae^{i\tau} + A^*e^{-i\tau} + \tfrac{1}{2}a(e^{i\omega\tau} + e^{-i\omega\tau})\right)^2 - 1\right\}.\left\{iAe^{i\tau} - iA^*e^{-i\tau} + \tfrac{ia\omega}{2}(e^{i\omega\tau} - e^{-i\omega\tau})\right\}$.

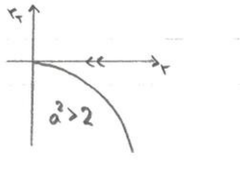
Get terms like: $e^{\pm i\tau}$ (secular), $e^{(2\omega+1)i\tau}$ (okay, $\omega\neq 0$), $e^{i(\omega-2)\tau}$ (okay, $\omega\neq 1,3$), $e^{3i\omega\tau}$ (okay, $\omega\neq 1/3$)

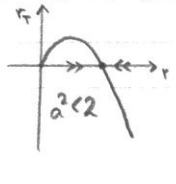
$\omega = 0, 1/3, 1, 3$ are the resonant values of ω. All other terms give no new resonant values of ω.

Provided $\omega \neq 0, \frac{1}{3}, 1, 3$, get non-secularity condition:

$e^{i\tau}$: $0 = -2iA_T - \left\{iA(2AA^* + \tfrac{1}{2}a^2 - 1) - iA^*A^2 + \frac{ia^2\omega A}{2} - \frac{ia^2\omega A}{2}\right\}$. $\iff 2A_T = (1-\tfrac{1}{2}a^2)A - |A|^2A$.

Let $A = re^{i\theta} \Rightarrow \begin{cases} 2r_T = (1-\tfrac{1}{2}a^2)r - r^3 \\ \theta_T = 0 \Rightarrow \theta = \text{constant}. \end{cases}$





If $a^2 > 2$: Then $r_T < 0 \quad \forall r \neq 0 \Rightarrow r(T) \to 0$

$x(t,\varepsilon) \to a\cos\omega t + O(\varepsilon)$ - Asynchronous Quenching.

if $a^2 < 2$: Then $r_T \to \sqrt{1-\frac{1}{2}a^2}$ (if $r(0) \neq 0$).

$x(t,\varepsilon) \to \sqrt{2-a^2}\cos(t+\theta_0) + a\cos\omega t + O(\varepsilon)$, $0 \le |t| < \frac{1}{\varepsilon}$ - Soft Excitation.

Example: Subharmonic Resonance.

$\ddot{x} + x + a(\omega^2-1)\cos\omega t = -\varepsilon\dot{x}(x^2-1)$ -(1)

If $\omega \simeq 3$, set $\omega = 3(1+\delta\varepsilon)$ in (1). So get ε terms in the cosine. Avoid by rescaling time: $s = (1+\delta\varepsilon)t$. So $\frac{d}{dt} = (1+\delta\varepsilon)\frac{d}{ds}$. Let $'$ signify $\frac{d}{ds}$.

$\Rightarrow (1+\delta\varepsilon)^2 x'' + x + a(9(1+\delta\varepsilon)^2 - 1)\cos 3s = -\varepsilon(1+\delta\varepsilon)x'(x^2-1)$

$\Rightarrow x'' + (1-2\delta\varepsilon)x + a(8+2\delta\varepsilon)\cos 3s = \varepsilon x'(1-x^2) + O(\varepsilon^2)$

If we ignore $O(\varepsilon^2)$ terms and the $\varepsilon\cos 3s$ forcing, get: $x'' + (1-2\delta\varepsilon)x + 8a\cos 3s = -\varepsilon x'(x^2-1)$ -(2)

So, $\omega \approx 3$, natural frequency $\Leftrightarrow$ $\omega = 3$, natural frequency $\simeq 1$.

Cf. Question 14, sheet 3. Will get $x_0 = A(T)e^{i\tau} + A^* e^{-i\tau} + a\cos 3\tau$. If $A \to$ non-trivial fixed point (as $T \to \infty$) then response solution has period 2π, three times the period of external forcing, $2\pi/3$.

This is called subharmonic resonance - ie, response frequency is a fraction of the frequency of external forcing.

Example: Frequency Locking.

$\ddot{x} + \varepsilon\dot{x}(x^2-1) + (1+\delta\varepsilon)x = \varepsilon a\cos\omega t$. - weak periodic forcing by an almost resonant term. ($\delta \neq 0$).

$O(\varepsilon^0)$: $x_{0\tau\tau} + x_{0\tau} = 0 \Rightarrow x_0(\tau,T) = A(T)e^{i\tau} + A^*(T)e^{-i\tau}$.

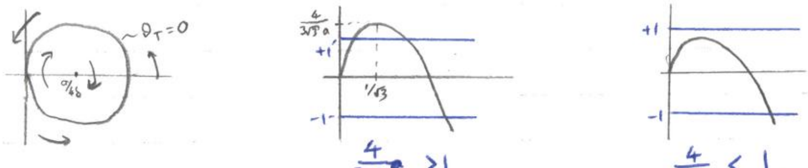
$O(\varepsilon^1)$: $x_{1\tau\tau} + x_1 = -2x_{0\tau T} - x_{0\tau}(x_0^2-1) - \delta x_0 + a\cos\omega\tau$.

Non-secularity condition: $2A_T = (1+i\delta)A - |A|^2 - \frac{ia}{2}$.

Let $A(T) = r(T)e^{i\theta(T)} \Rightarrow \begin{cases} r_T = \frac{1}{2}(r - r^3 - \frac{1}{2}a\sin\theta) \\ r\theta_T = \frac{1}{2}(\delta r - \frac{1}{2}a\cos\theta) \end{cases}$ - Note that $r=0$ is not a fixed point.

$r \neq 0$, fixed points: $\begin{cases} \sin\theta = \frac{2}{a}(r-r^3) & -(1) \\ r = \frac{a}{2\delta}\cos\theta & -(2) \end{cases}$

θ-motion:

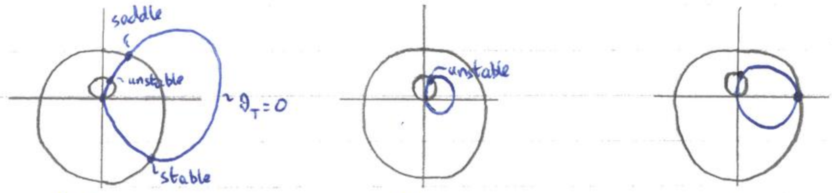


$\frac{4}{3\sqrt{3}a} > 1$ $\quad\quad$ $\frac{4}{3\sqrt{3}a} < 1$. $\quad\quad$ $\Leftarrow$ RHS of (1)

r-motion:

$\frac{4}{3\sqrt{3}a} > 1$ $\quad\quad$ $\frac{4}{3\sqrt{3}a} < 1$

<u>Example:</u> $\frac{4}{3\sqrt{3}a} > 1$.



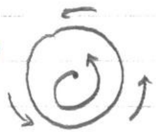
(1): $a/\delta > \mu_c$. (2): $\frac{a}{\delta} < \mu_c$ (3): $a/\delta = \mu_c$.

In (1), all solutions are attracted by $2r_0\cos(t+\theta_0) + O(\varepsilon^2)$

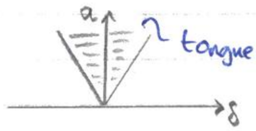
In (2), use Poincaré-Bendixson to get a limit cycle.

In (1):

In (2):



In (3):



tongue of frequency locking.