Differentiable manifolds   Lent Term 1996   Problem Sheet 1

1. Mercator's projection maps a place with longitude $\theta \in (-\pi, \pi)$ and latitude $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ to $(\theta, f(\varphi)) \in \mathbb{R}^2$, where f is the unique function such that $f(0) = 0$ and that when a point moves on the earth in a fixed compass direction α its image moves along a straight line in $\mathbb{R}^2$ at an angle α to the φ-axis. Find the function f.

What is the map $(-\pi, \pi) \times \mathbb{R} \longrightarrow \mathbb{C}$ which relates Mercator's chart to the chart given by stereographic projection from the north pole. Check that this map is conformal.

2. Prove that a connected smooth manifold is path-connected.

3. Prove that a smooth manifold is compact if and only if it is the union of a finite number of sets each contained in the domain of a chart and mapped by the chart to a closed ball in $\mathbb{R}^n$.

4. Describe real projective space $\mathbb{P}^n$ as a smooth manifold. Prove that it is compact. Prove that there is a smooth surjective map $S^n \longrightarrow \mathbb{P}^n$.

5. Prove that SO_3 is diffeomorphic to $\mathbb{P}^3$.

6. Prove that the real symmetric matrices A of trace 1 such that $A^2 = A$ form a smooth submanifold of $\mathbb{R}^{n^2}$. Prove that it is diffeomorphic to $\mathbb{P}^{n-1}$. Prove that all real symmetric matrice A such that $A^2 = A$ are also a submanifold of $\mathbb{R}^{n^2}$. [Perhaps best leave this till after Question 9.]

7. Prove that the map $(x, y, z) \longmapsto (x^2 - y^2, yz, zx, xy)$ from $\mathbb{R}^3$ to $\mathbb{R}^4$ induces an embedding of $\mathbb{P}^2$ as a submanifold of $\mathbb{R}^4$.

8. Prove that a smooth manifold is orientable if it can be covered by charts between which the transition maps are orientation-preserving.

9. Let $X = Gr_k(\mathbb{R}^n)$ denote the set of all k-dimensional vector subspaces of $\mathbb{R}^n$. If $V \in X$, let $U_V = \{W \in X : V \cap W^\perp = 0\}$. Prove that the map $\psi_V : \mathrm{Hom}(V; V^\perp) \longrightarrow U_V$ given by $\psi_V(T) = \mathrm{graph}(T) \subset V \oplus V^\perp = \mathbb{R}^n$ is a bijection. Show that $\psi_W^{-1} \circ \psi_V$ is given by

$$T \longmapsto (c+dT)(a+bT)^{-1},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the identity map expressed as a map $V \oplus V^{\perp} \to W \oplus W^{\perp}$. Deduce that the charts (U_V, ψ_V^{-1}) define an atlas for X. Prove that the tangent space to X at V is canonically isomorphic to $\mathrm{Hom}(V; V^{\perp})$.

10. Prove that the set of straight lines in $\mathbb{R}^2$ is naturally a smooth manifold. Prove that it is diffeomorphic to the complement of a point in $\mathbb{P}^2$. Prove that it is a Möbius band.

11. Prove that $\{A \in O_n : A^2 = 1\}$ is a smooth submanifold of O_n with $n+1$ connected components, diffeomorphic to $\coprod_{0 \le k \le n} Gr_k(\mathbb{R}^n)$.

12. Prove that $S^n \times S^m$ can be embedded as a smooth submanifold of S^{n+m+1}, and deduce that it can be embedded as a smooth submanifold of $\mathbb{R}^{n+m+1}$.

13. Prove that $\mathbb{P}^n \times \mathbb{P}^m$ can be embedded in $\mathbb{P}^{nm+n+m}$.

14. A group G acts on a smooth manifold X by smooth maps. Suppose that each $x \in X$ has a neighbourhood U such that the subsets gU, for all $g \in G$, are disjoint. Describe an atlas for the orbit space X/G making it a smooth manifold. If $X = S^{n-1} \subset \mathbb{R}^n$, and $G = \{\pm 1\}$, acting on X by multiplication, prove that X/G is diffeomorphic to $\mathbb{P}^{n-1}$.

Describe a group G which acts on $\mathbb{R}^2$ so that $\mathbb{R}^2/G$ is a Klein bottle.

Part II B Differentiable Manifolds Lent 1996 Problem Sheet 2

1. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation, and let $B : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be A regarded as an $\mathbb{R}$-linear map. Prove that $\det(B) = |\det(A)|^2$.

2. Let U be a neighbourhood of 0 in $\mathbb{C}^2$, and let $f : U - \{0\} \to \mathbb{C}$ be analytic. Prove Hartogs's theorem, that f extends to an analytic map $\hat{f} : U \to \mathbb{C}$.
[Hint: If $|x| < \varepsilon$, define $\hat{f}(x,y) = \frac{1}{2\pi i} \int_{|\zeta| = \varepsilon} \frac{f(\zeta, y)}{\zeta - x} d\zeta$.]

3. Let $f : U \to \mathbb{C}$ be analytic, and not constant, with U open in $\mathbb{C}$. If $f(a) = 0$, the multiplicity of a as a zero of f is k if $f(z) = (z-a)^k g(z)$, with g analytic and $g(a) \neq 0$. If $f : X \to Y$ is a non-constant analytic map of Riemann surfaces, invent a definition of the multiplicity $\nu_a(f)$ of f at $a \in X$ by using charts and checking chart-independence. If X is compact, prove that $\sum_{a \in f^{-1}(b)} \nu_a(f)$ is independent of $b \in Y$.

4. Find an explicit atlas for the complex manifold $Y = \{(x,y) \in \mathbb{C}^2 : x^n + y^n = 1\}$. Prove that one can adjoin n "points at infinity" to Y so as to obtain a compact Riemann surface $\hat{Y}$. Describe a division of $\hat{Y}$ into cells with Euler number $n(3-n)$.
[There is a cell-decomposition such that the map $(x,y) \mapsto y$ from $\hat{Y}$ to $\mathbb{C} \cup \infty$ takes each 2-cell either to $\{y : |y| \leq 1\}$ or $\{y : |y| \geq 1\}$.]

5. Let $X = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^2 + z^2 = 1\}$. Prove that X is a complex manifold which is diffeomorphic to TS^2 as a real smooth manifold. Prove that X is not isomorphic to TS^2 as a complex manifold. [Observe that X can contain no compact complex submanifold, while the complex tangent bundle TS^2 has many such.]

6. The elements ρ of the symmetric group S_3 act on the Riemann sphere S by Möbius transformations g_ρ such that $CR(z_{\rho(1)}, z_{\rho(2)}, z_{\rho(3)}, z_4) = g_\rho(CR(z_1, z_2, z_3, z_4))$ for any $\{z_1, \ldots, z_4\}$, where CR = cross-ratio. Describe the g_ρ explicitly. Prove that there are exactly two orbits which do not have six elements. Find an analytic map $f : S \to S$ of degree 6 such that $f(z) = f(z') \Leftrightarrow z' = g_\rho(z)$ for some $\rho \in S_3$, and which takes the exceptional orbits to 0 and ∞ in S. Is f unique?
[If you know how to classify the finite subgroups of SO_3 — ~~two~~ cyclic, dihedral, plus three Platonic groups, it is worth doing this exercise for the general finite group of Möbius transformations.]

7. Prove that the series $f(x) = \sum (\tanh(n+x) - \tanh n)$ is absolutely and uniformly convergent for $x \in \mathbb{R}$, but that f is not periodic. What is $f(x+1) - f(x)$?

8. Let $X = \mathbb{C}/L$, where L is a lattice. Prove that any analytic map $f : X \to S = \mathbb{C} \cup \infty$ of degree two has exactly four ramification points (i.e. points $w \in S$ such that $f^{-1}(w)$ consists of just one point). Prove that $f = \varphi \circ \wp_L \circ \theta$, where $\wp_L$ is the Weierstrass elliptic function, φ is a Möbius transformation, and $\theta(z) = z + a$ for some a.

9. Using the results of Q6 and Q8, show how to assign a number $j(L) \in \mathbb{C}$ to each lattice $L \subset \mathbb{C}$ so that $j(L) = j(L') \iff L' = \lambda L$ for some $\lambda \in \mathbb{C}$. Prove that j defines a surjective analytic map $j : U \to \mathbb{C}$, where $U = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$, such that $j(z) = j(z') \iff z' = g(z)$ for some $g \in SL_2(\mathbb{Z})$.

10. (a) The gamma-function $\Gamma(s)$ is defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ when $Re(s) > 0$. If γ is the path  from ∞ to ∞ in $\mathbb{C}$, prove that $f(s) = \int_\gamma e^{-t} t^{s-1} dt$ defines an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f(s) = (e^{2\pi i s} - 1)\Gamma(s)$ when $Re(s) > 0$. Deduce that Γ extends to a meromorphic function in all of $\mathbb{C}$.

(b) Do the same for the ζ-function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, defined for $Re(s) > 1$, by considering $g(s) = \int_\gamma \frac{t^{-s} dt}{e^{2\pi t} - 1}$, where γ is as before.

Differentiable Manifolds Lent 1996 Problem Sheet 3

1. If V is a finite-dimensional real vector space, and $\alpha_1, \ldots, \alpha_k \in \mathrm{Alt}^1(V) = V^*$, prove that $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k = 0$ if and only if $\alpha_1, \ldots, \alpha_k$ are linearly dependent.

2. (i) If $\dim(V) = 3$, prove that any $\omega \in \mathrm{Alt}^2(V)$ can be expressed $\omega = \alpha \wedge \beta$ for some $\alpha, \beta \in \mathrm{Alt}^1(V)$.

(ii) If $\dim(V) = 4$, prove that any $\omega \in \mathrm{Alt}^2(V)$ can be expressed $\omega = \alpha \wedge \beta + \gamma \wedge \delta$, and that $\omega \wedge \omega = 0$ if and only if $\omega = \alpha \wedge \beta$ for some $\alpha, \beta \in \mathrm{Alt}^1(V)$.

3. Let $\alpha = \sum_{k=1}^{n} (-1)^k x^k dx^1 \wedge \ldots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \ldots \wedge dx^n \in \Omega^{n-1}(\mathbb{R}^n)$.
Calculate $\varphi^* \alpha$, when $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. Prove that the restriction of α to $S^{n-1} \subset \mathbb{R}^n$ does not vanish at any point.

4. If $A = (a_{ij})$ is a $2n \times 2n$ real skew matrix, let $\alpha = \sum_{i<j} a_{ij}\, dx^i \wedge dx^j \in \Omega^2(\mathbb{R}^{2n})$. If $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a linear transformation, find a formula for the matrix (b_{ij}) in terms of A and the matrix of φ. The Pfaffian $\mathrm{Pf}(A)$ is the real number defined by

$$\underbrace{\alpha \wedge \ldots \wedge \alpha}_{n} = \frac{1}{n!} \mathrm{Pf}(A)\, dx^1 \wedge \ldots \wedge dx^{2n}.$$

Prove that $\mathrm{Pf}(A)$ is a polynomial of degree n in the a_{ij} with integer coefficients, and that $\mathrm{Pf}(A)^2 = \det(A)$. (Hint: Prove first that one can find $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that

$$\varphi^* \alpha = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + \ldots + dx^{2m-1} \wedge dx^{2m} \quad \text{for some } m \leq n.)$$

5. If V is a finite-dimensional real vector space, and $v \in V$, prove that there is a unique antiderivation $i_v : \mathrm{Alt}^k(V) \to \mathrm{Alt}^{k-1}(V)$ such that $i_v(\alpha) = \alpha(v)$ when $\alpha \in \mathrm{Alt}^1(V) = V^*$, and that

$$i_v(\alpha)(v_1, \ldots, v_{k-1}) = \alpha(v, v_1, \ldots, v_{k-1}).$$

6. If U is an open subset of $\mathbb{R}^n$, and $v \in \mathbb{R}^n$, use the preceding result to define an antiderivation $i_v : \Omega^k(U) \to \Omega^{k-1}(U)$. Prove that $L_v = i_v \circ d + d \circ i_v : \Omega^k(U) \to \Omega^k(U)$ is the unique derivation such that

(i) $L_v \circ d = d \circ L_v$, and

(ii) $(L_v f)(x) = Df(x)\, v$ when $f \in \Omega^0(U)$.

(I.e. the operation $L_v : \Omega^0(U) \to \Omega^0(U)$ is the directional derivative corresponding to the vector v.)

7. If X is a smooth manifold and v is a tangent vector field on X define operations $i_v : \Omega^k(X) \to \Omega^{k-1}(X)$ and $L_v : \Omega^k(X) \to \Omega^k(X)$ with the properties of the preceding question. (L_v is called the Lie derivative along the vector field v.)

8. If an element $\alpha \in \Omega^k(X)$ is defined as a compatible collection $\{\alpha_j \in \Omega^k(V_j)\}$ for every chart $\varphi_j : U_j \to V_j \subset \mathbb{R}^n$ for X, prove that it is enough to give α_j for a collection of charts which cover X.

9. A vector bundle on a smooth manifold X is a smooth manifold E together with a smooth map $\pi : E \to X$ and a real vector space structure on each "fibre" $E_x = \pi^{-1}(x)$ for each $x \in X$. It is required to be locally trivial in the sense that each $x \in X$ has an open neighbourhood U such that $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{R}^N$ for some N, by a diffeomorphism which is a vector-space isomorphism $E_x \cong \{x\} \times \mathbb{R}^N$ for each x.

Give a complete proof that $\mathrm{Alt}^k(TX) = \bigcup_{x \in X} \mathrm{Alt}^k(T_x X)$ is a vector bundle on X.

10. If $X = \mathbb{P}^{n-1}$ and $E = \{(x, \xi) \in \mathbb{P}^{n-1} \times \mathbb{R}^n : \xi \in x\}$ prove that E is a vector bundle on X. More generally, do the same when $X = Gr_k(\mathbb{R}^n)$ and $E = \{(x, \xi) \in X \times \mathbb{R}^n : \xi \in x\}$.

11. (retrospective) Review the proof that the tangent space $T_x X$ of a smooth manifold X is isomorphic to the space $\mathcal{D}_x$ of derivations of $C^\infty(X)$ at x, and observe that it factorizes into a very easy proof that $T_x X$ is isomorphic to the space $\check{\mathcal{D}}_x$ of derivations at x of the ring of germs of smooth functions defined in a neighbourhood of x, and a separate proof, involving bump functions, that $\check{\mathcal{D}}_x \cong \mathcal{D}_x$.

The tangent space to a complex manifold at any point is defined in exact analogy to the real case, using charts. There are no bump functions, but you should check that the tangent space at x is isomorphic to the space of derivations at x of the ring $\mathcal{O}_x$ of germs of analytic functions defined near x.

12. If X is a simply connected smooth manifold, prove that $H^1(X) \cong \mathbb{D}$.