II B Differentiable manifolds

Section 1 : The Concepts and definitions

1.1 Revision of calculus in $\mathbb{R}^n$

If U is an open subset of $\mathbb{R}^n$ a map $f : U \to \mathbb{R}^m$ is <u>differentiable</u> at $x \in U$ if there is a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$f(x+h) = f(x) + Ah + R(x,h),$$

for all sufficiently small h, where $\|R(x,h)\| / \|h\| \to 0$ as $h \to 0$.

We write $A = Df(x)$.

If x is differentiable for all $x \in U$ then $Df : U \to \mathrm{Hom}(\mathbb{R}^n; \mathbb{R}^m) \cong \mathbb{R}^{nm}$.

The <u>second derivative</u> $D^2f(x)$, if it exists, is then an element of $\mathrm{Hom}(\mathbb{R}^n; \mathrm{Hom}(\mathbb{R}^n; \mathbb{R}^m))$, i.e. a <u>bilinear</u> map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$. And so on.

I shall say f is <u>smooth</u> if it has derivatives of all orders. In that case $D^kf(x)$ is a <u>symmetric</u> multilinear map $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \to \mathbb{R}^m$.

The "chain rule" asserts that if $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, and $U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}^k$ are differentiable, then so is $g \circ f$, and

$$D(g\circ f)(x) = Dg(y) \circ Df(x), \quad \text{where } y = f(x).$$

The <u>mean value theorem</u> asserts that if f is differentiable in U then

$$\|f(y) - f(x)\| \leqslant K\|x-y\|, \quad \text{where } K = \sup_{t\in[0,1]} \|Df((1-t)x+ty)\|.$$

The <u>inverse function theorem</u> asserts that if $f : U \to \mathbb{R}^n$ has a continuous derivative, and $Df(x)$ is invertible for some $x \in U$, then there is a neighbourhood V of $y = f(x)$, and a map $g : V \to U$ with continuous derivative such that $f \circ g = \mathrm{id}_V$ and $(g \circ f) | g(V) = \mathrm{id}_{g(V)}$.

<u>Proof</u> We can assume that $x = y = 0$, and that $Df(0) = 1$. Also that $U \supset D_\varepsilon = \{\xi \in \mathbb{R}^n : \|\xi\| \leqslant \varepsilon\}$, and that $\|Df(\xi_1) - Df(\xi_2)\| < \frac{1}{4}$ for $\xi_1, \xi_2 \in D_\varepsilon$.

Write $f(\xi) = \xi + \varphi(\xi)$. To solve $f(\xi) = \eta$ is to find a fixed point of F_η, where $F_\eta(\xi) = \eta - \varphi(\xi)$.

Suppose that $\eta \in V = \{\eta \in \mathbb{R}^n : \|\eta\| < \frac{1}{2}\varepsilon\}$. Then, by the mean value theorem, as $\|D\varphi\| < \frac{1}{4}$ in D_ε, we see that F_η is a contraction mapping $: D_\varepsilon \to D_\varepsilon$, and so has a unique fixed point, say $g(\eta) \in D_\varepsilon$. To see that $g : V \to U$ is differentiable, with $Dg(\eta) = Df(\xi)^{-1}$ when $\xi = g(\eta)$ we write $g(\eta + h) = g(\eta) + k$.

Then $\eta + h = f(\xi + k)$, so $h = f(\xi+k) - f(\xi) = Ak + R(k)$ ...(*), where $A = Df(\xi)$, and $\|R(k)\| / \|k\| \to 0$ as $k \to 0$. Thus

$$k = A^{-1}h - A^{-1}R(k),$$

and we must show that $\|A^{-1}R(k)\| / \|h\| \to 0$ as $h \to 0$.

But $\|A^{-1}R(k)\| \leqslant \|A^{-1}\| . \dfrac{\|R(k)\|}{\|k\|} . \|k\|$, so it is enough to show that $\|k\|/\|h\|$ is bounded as $h \to 0$. But the MVT applied to the function $v \mapsto f(v) - Av$ between ξ and $\xi + k$ tells us that $\|R(k)\| \leqslant \frac{1}{4}\|k\|$, while $\|1 - A\| < \frac{1}{4}$ implies that $\|Ak\| \geqslant \frac{3}{4}\|k\|$. So $\|h\| \geqslant \frac{3}{4}\|k\| - \frac{1}{4}\|k\| = \frac{1}{2}\|k\|$ from (*), as we want.

Bump functions

For any $\varepsilon > 0$ we can find a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = 1$ if $\|x\| \leqslant \varepsilon$, and $f(x) = 0$ if $\|x\| \geqslant 2\varepsilon$.

In fact we define $f(x) = 1 - g(\|x\|/\varepsilon)$, where $g : \mathbb{R} \to \mathbb{R}$ is smooth and such that
$$g(t) = 1 \text{ for } t \leqslant 1$$
$$= 0 \text{ for } t \geqslant 2.$$

We can take $g(t) = c\int_1^t e^{-1/(s-1)^2 - 1/(2-s)^2}\, ds$ for $t \in [1, 2]$ and suitable $c > 0$.

1.2 The definition of a smooth manifold

A smooth manifold is a set X together with a maximal atlas.

An atlas is a collection of compatible charts whose domains cover X.

A chart is a bijection $\varphi : U \to V$, where V is an open subset of $\mathbb{R}^n$.

Two charts $\varphi_1 : U_1 \to \mathbb{R}^n$ and $\varphi_2 : U_2 \to \mathbb{R}^m$ are compatible if $\varphi_1(U_1 \cap U_2)$ and $\varphi_2(U_1 \cap U_2)$ are open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and

the bijections $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ and
$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(U_1 \cap U_2) \to \varphi_1(U_1 \cap U_2)$
are smooth. (By the chain rule this can happen only if $n = m$ or else $U_1 \cap U_2 = \emptyset$.)

An atlas is <u>maximal</u> if it contains all charts compatible with all of its charts.

<u>Lemma</u> Any atlas is contained in a unique maximal atlas, for if $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}$ is an atlas, and $\varphi : U \to \mathbb{R}^n$ and $\tilde{\varphi} : \tilde{U} \to \mathbb{R}^n$ are charts compatible with all φ_α then they are compatible with each other.

<u>Proof</u> We must show that $\varphi(U \cap \tilde{U})$ and $\tilde{\varphi}(U \cap \tilde{U})$ are open, and that the bijections $\varphi(U \cap \tilde{U}) \rightleftarrows \tilde{\varphi}(U \cap \tilde{U})$ are smooth. But $U \cap \tilde{U} = \bigcup_\alpha (U \cap \tilde{U} \cap U_\alpha)$, so $\varphi(U \cap \tilde{U}) = \bigcup \varphi(U \cap \tilde{U} \cap U_\alpha) = \bigcup \varphi(U \cap U_\alpha) \cap \varphi(\tilde{U} \cap U_\alpha)$, which is open. Similarly $\tilde{\varphi}(U \cap \tilde{U})$ is open. As to the smoothness, it is enough to show that $\varphi(U \cap \tilde{U} \cap U_\alpha) \rightleftarrows \tilde{\varphi}(U \cap \tilde{U} \cap U_\alpha)$ are smooth. But these are composites of the smooth maps

$$\varphi(U \cap \tilde{U} \cap U_\alpha) \rightleftarrows \varphi_\alpha(U \cap \tilde{U} \cap U_\alpha) \rightleftarrows \tilde{\varphi}(U \cap \tilde{U} \cap U_\alpha). \quad //$$

<u>Examples</u>

(i) $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

Let $U_1^+ = \{x \in S^n : x_1 > 0\}$, and define $\varphi_1^+ : U_1^+ \to \mathbb{R}^n$ by $\varphi_1^+(x) = (x_2, \ldots, x_{n+1})$. There are $2(n+1)$ charts $\varphi_i^\pm : U_i^\pm \to \mathbb{R}^n$ like this which cover S^n, and they are compatible.

<u>Stereographic projection</u> is the bijection $\varphi : U \to \mathbb{R}^n$, where $U = S - \{e_{n+1}\}$, given by $\varphi(x) = \left(\frac{x_1}{1 - x_{n+1}}, \ldots \frac{x_n}{1 - x_{n+1}}\right)$.

We have $\varphi^{-1}(y) = \left(\frac{2y_1}{1 + \|y\|^2}, \ldots, \frac{2y_n}{1 + \|y\|^2}, \frac{-1 + \|y\|^2}{1 + \|y\|^2}\right) \in S^n$.

This is a chart compatible with the preceding ones. There is a similar chart $\tilde{\varphi} : \tilde{U} \to \mathbb{R}^n$, where $\tilde{U} = S^n - \{-e_{n+1}\}$. We have $\varphi(U \cap \tilde{U}) = \tilde{\varphi}(U \cap \tilde{U}) = \mathbb{R}^n - \{0\}$, and both transition maps $\varphi(U \cap \tilde{U}) \rightleftarrows \tilde{\varphi}(U \cap \tilde{U})$ are given by <u>inversion</u>, i.e. $y \mapsto y / \|y\|^2$.

(ii) Let X be the torus $X = \{(x,y,z) \in \mathbb{R}^3 : (\rho-2)^2 + z^2 = 1$, where $\rho^2 = x^2+y^2\}$.

We can cover X by two compatible charts: define

$$\varphi_1 : U_1 = \{(x,y,z) \in X : \rho < 3\} \longrightarrow \mathbb{R}^2 \text{ by}$$

$$(x,y,z) \longmapsto \left(\frac{\varphi x}{\rho}, \frac{\varphi y}{\rho}\right), \text{ where } \varphi \in (0, 2\pi) \text{ and } \begin{cases} \cos\varphi = \rho - 2 \\ \sin\varphi = z \end{cases};$$

$$\varphi_2 : U_2 = \{(x,y,z) \in X : \rho > 1\} \longrightarrow \mathbb{R}^2 \text{ by}$$

the same formulae, but with $\varphi \in (\pi, 3\pi)$.

The same smooth manifold X can be more conveniently regarded as the subset $\{(x,y,z,t) \in \mathbb{R}^4 : x^2+y^2 = z^2+t^2 = 1\}$ by the map $(x,y,z) \mapsto \left(\frac{x}{\rho}, \frac{y}{\rho}, z, \rho-2\right)$.

(iii) Let $X = O_n = \{$real $n \times n$ matrices A such that $A^tA = 1\} \subset \mathbb{R}^{n^2}$.

Because A^tA is symmetric for all A the subset X is actually defined by $\frac{1}{2}n(n+1)$ equations, and we expect it to have dimension $\frac{1}{2}n(n-1)$.

We have a bijection $\varphi : U = \{A \in O_n : \det(A+1) \neq 0\} \longrightarrow \{\text{skew matrices}\} = \mathbb{R}^{\frac{1}{2}n(n-1)}$

given by $\qquad A \longmapsto \varphi(A) = (A-1)(A+1)^{-1}$.

The inverse is $S \mapsto (1+S)(1-S)^{-1}$, which is called the <u>Cayley parametrization</u> of O_n. Notice that $\det(1-S) \neq 0$ if S is skew.

The charts $\varphi_g = \varphi \circ g^{-1} : gU \longrightarrow \mathbb{R}^n$, where g runs through O_n, cover O_n and are compatible. (In fact it is enough to let g run through the 2^n diagonal elements of O_n.)

Notice that $\varphi(U \cap gU) = \{$skew S such that $\det(g^{-1}(1+S)(1-S)^{-1}+1)) \neq 0\}$ is open, and the transition map $\varphi(U \cap gU) \longrightarrow \varphi_g(U \cap gU)$ is

$$S \longmapsto \left(g^{-1}(1+S)(1-S)^{-1} - 1\right)\left(g^{-1}(1+S)(1-S)^{-1}+1\right)^{-1}$$

which is smooth.

There is another smooth map $\psi : \{\text{skew matrices}\} \longrightarrow O_n$ given by $\psi(S) = e^S$. It is bijective when restricted to $\{S : \|S\| < \pi\}$, and its inverse is a chart $U \to \{\text{skew matrices}\}$, with the <u>same</u> U as above. It is compatible with the Cayley charts.

(iv) <u>Real projective space</u> $\mathbb{P}^n$ is defined as the set of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$. A point of $\mathbb{P}^n$ is described by "homogeneous coordinates" $(x_0, \ldots, x_n)$, not all 0, i.e. by $x \neq 0$ in $\mathbb{R}^{n+1}$, subject to the equivalence relation $(x_0, \dots, x_n) \sim (x_0', \dots, x_n')$ if $x_i' = \lambda x_i$ for some $\lambda \neq 0$.

Let $U_i = \{x \in \mathbb{P}^n : x_i \neq 0\}$. The sets $U_0, \dots, U_n$ cover $\mathbb{P}^n$, and if $i < j$ then $\varphi_i(U_i \cap U_j) = \{y \in \mathbb{R}^n : y_j \neq 0\}$, and the transition map $\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is

$$(y_1, \dots, y_n) \longmapsto \left(\frac{y_1}{y_j}, \dots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \frac{y_i}{y_j}, \dots, \frac{y_n}{y_j}\right),$$

which is smooth.

So we have an atlas making $\mathbb{P}^n$ a smooth manifold. It is <u>compact</u>.

<u>Miscellaneous vocabulary</u>

If X is a smooth manifold we say a subset U of X is <u>open</u> if $\varphi_\alpha(U \cap U_\alpha)$ is open in $\mathbb{R}^n$ for each chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$. The collection of open sets so defined gives us a <u>topology</u> on X, i.e. the collection is closed under arbitrary unions and finite intersections.

We say X is (i) <u>compact</u>, if every covering of X by open sets has a finite subcovering,

(ii) <u>Hausdorff</u>, if for each $x_1 \neq x_2$ in X there are disjoint open sets U_1, U_2 with $x_i \in U_i$,

(iii) <u>metrizable</u>, if there is a metric on X which defines the topology, i.e. a subset is open in terms of the metric if and only if it is open in terms of the charts,

(iv) <u>connected</u>, if X is not the union of two nonempty disjoint open subsets.

Non-metrizable manifolds are of no interest to us, but the definition we have given does not even imply that X is Hausdorff. We shall prove later that any compact Hausdorff manifold is metrizable.

If X and Y are smooth manifolds, a map $f : X \to Y$ is <u>smooth</u> if for each chart $\varphi : U \to \mathbb{R}^n$ for X and each chart $\tilde{\varphi} : \tilde{U} \to \mathbb{R}^m$ for Y the map $\tilde{\varphi} \circ f \circ \varphi^{-1} : \varphi(U \cap f^{-1}\tilde{U}) \to \mathbb{R}^m$ is smooth.

A smooth map is a <u>diffeomorphism</u> if it is bijective, and its inverse is smooth. (Thus the bijection in Ex. (ii) above between the torus in $\mathbb{R}^3$ and the torus in $\mathbb{R}^4$ is a diffeomorphism.)

If U and V are open sets in $\mathbb{R}^n$, a diffeomorphism $f : U \to V$ is orientation-preserving if $\det(Df(x)) > 0$ for all $x \in U$, and orientation-reversing if $\det(Df(x)) < 0$ for all $x \in U$. If U is connected, one or the other must be true.

A smooth manifold X is orientable if it possesses an atlas for which the transition maps between charts are orientation-preserving — call this an oriented atlas. An orientation of X is a choice of a maximal oriented atlas.

1.3 Submanifolds

If X is an n-dimensional smooth manifold, a subset Y of X is an m-dimensional smooth submanifold if for each $y \in Y$ there is a chart $\varphi : U \to \mathbb{R}^n$ for X, with $y \in U$, such that $\varphi(Y \cap U) = \varphi(U) \cap (\mathbb{R}^m \oplus 0)$.

The charts $Y \cap U \to \mathbb{R}^m$ obtained in this situation are clearly compatible, and define an atlas making Y a smooth manifold.

We are usually interested in submanifolds Y which are either open or closed subsets of X, but the definition does not require that. Thus the set $Y = \{(x, \sin\frac{1}{x}) : 0 < x < 1\}$ is a submanifold of $\mathbb{R}^2$. But the definition does not allow, say, the image of the smooth injective map $\mathbb{R} \to \mathbb{R}^2$ which looks like:

It also does not allow the "dense winding" on the torus $X = \{(x,y,z,t) \in \mathbb{R}^4 : x^2+y^2 = z^2+t^2 = 1\}$, which is the image of $f : \mathbb{R} \to X$ given by $f(t) = (\cos t, \sin t, \cos at, \sin at)$, where a is irrational.

The most important way of defining submanifolds of $\mathbb{R}^n$ is by the Implicit function theorem: If U is open in $\mathbb{R}^n$, and $f : U \to \mathbb{R}^k$ is a smooth map such that $Df(x) : \mathbb{R}^n \to \mathbb{R}^k$ has rank k whenever $f(x) = 0$, then $Y = f^{-1}(0)$ is a smooth submanifold of U of dimension $m = n-k$.

Proof We shall find a suitable chart in a neighbourhood of $x \in Y$.

First choose a linear map $p : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$Df(x) \oplus p : \mathbb{R}^n \to \mathbb{R}^k \oplus \mathbb{R}^m = \mathbb{R}^n$$

is invertible. (One can take p to be the orthogonal projection on to the vector subspace spanned by the standard basis vectors $e_{i_1}, \ldots, e_{i_m}$ for some $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\}$.)

By the inverse-function theorem there is a neighbourhood V of $0 \in \mathbb{R}^n$ and a smooth map $g : V \to U$ such that g is the inverse of $f \times p : g(V) \to V$. Then $f \times p : g(V) \to \mathbb{R}^k \oplus \mathbb{R}^n$ is a chart for U, and $p : g(V) \cap Y \to \mathbb{R}^m$ is the induced chart for Y. //

<u>Example</u> Let $U = \{$ real $n \times n$ matrices $A\}$ such that $\det(A) \neq 0\}$, and define $f : U \to \{$symmetric matrices$\} \cong \mathbb{R}^{\frac{1}{2}n(n+1)}$ by $f(A) = A^t A - 1$.

Then $Df(A).h = A^t h + h^t A$, and $Df(A)$ is surjective, for $A^t h + h^t A = S$ has the solution $h = \frac{1}{2}(A^t)^{-1} S$. So $f^{-1}(0) = O_n$ is a smooth submanifold of $\mathbb{R}^{n^2}$.

1.4 <u>Tangent vectors</u>

Let X be a smooth manifold, and $x \in X$.

Let $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}_{\alpha \in A}$ be the collection of all charts for X with $x \in U_\alpha$, and write $x_\alpha = \varphi_\alpha(x) \in \mathbb{R}^n$.

<u>Definition</u> $T_x X$ is the set of all functions $\xi : A \to \mathbb{R}^n$, written $\alpha \mapsto \xi_\alpha$, such that

$$D(\varphi_\beta \circ \varphi_\alpha^{-1})(x_\alpha)\, \xi_\alpha = \xi_\beta \quad \text{for all } \alpha, \beta \in A.$$

Clearly $T_x X$ is a vector space.

<u>Proposition</u> (i) For any α, the map $\xi \mapsto \xi_\alpha$ is an isomorphism $T_x X \to \mathbb{R}^n$.

(ii) If X is a vector space, then $T_x X$ is canonically isomorphic to X.

(iii) If X' is an open subset of X, then $T_x X'$ is canon. isomorphic to $T_x X$.

<u>Proof</u> (i) Given ξ_α we define $\xi_\beta = D(\varphi_\beta \varphi_\alpha^{-1})(x_\alpha)\xi_\alpha$ for all other β, and then the chain rule shows we have an element of $T_x X$.

(ii) If X is a vector space then $D(\varphi_\alpha^{-1})(x_\alpha).\xi_\alpha \in X$ is independent of α.

(iii) is obvious. //

If $f : X \to Y$ is a smooth map we define $Df(x) : T_x X \to T_{f(x)} Y$

by $(Df(x)\xi)_\beta = D(\tilde{\varphi}_\beta \circ f \circ \varphi_\alpha^{-1})(x_\alpha)\,\xi_\alpha$, where $\{\tilde{\varphi}_\beta\}$ are the charts for Y whose domains contain $f(x)$. One checks that this is well-defined, and that one has the

<u>Chain Rule</u> If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps, then

$$D(g \circ f)(x) = Dg(y) \circ Df(x), \quad \text{where} \quad y = f(x).$$

There are many other ways of defining $T_x X$, of which I shall mention two.

(i) Consider smooth curves $\gamma : (-\varepsilon, \varepsilon) \to X$ such that $\gamma(0) = x$, for variable $\varepsilon > 0$. Introduce the equivalence relation $\gamma \sim \tilde{\gamma}$ if γ and $\tilde{\gamma}$ <u>touch</u> at x, i.e. if for one (and hence every) chart $\varphi : U \to \mathbb{R}^n$ with $x \in U$ we have $D(\varphi \circ \gamma)(0) = D(\varphi \circ \tilde{\gamma})(0)$. Let $\mathcal{C}_x X$ denote the set of equivalence classes. It is almost trivial to see that $\mathcal{C}_x \mathbb{R}^n \cong \mathbb{R}^n$. On the other hand, a chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ for X clearly induces an isomorphism $\mathcal{C}_x X \to \mathcal{C}_{x_\alpha} \mathbb{R}^n = \mathbb{R}^n$. So we get an isomorphism $\mathcal{C}_x X \to T_x X$.

(ii) Let $\mathcal{D}_x X$ be the vector space of <u>derivations</u> at x of the ring $C^\infty(X)$ of all smooth functions $X \to \mathbb{R}$. A derivation at x means a linear map $\theta : C^\infty(X) \to \mathbb{R}$ such that

$$\theta(fg) = f(x)\theta(g) + \theta(f)g(x). \quad \text{(The 'Leibniz property'.)}$$

Then there is a canonical isomorphism $\mathcal{D}_x X \to T_x X$.

To prove this we first consider the case $X = \mathbb{R}^n$, and $x = 0$. Then any $f \in C^\infty(\mathbb{R}^n)$ can be expressed $f = a + \sum x_i g_i$, where a is constant, x_i is the ith coordinate function, and $g_i \in C^\infty(\mathbb{R}^n)$ is such that $g_i(0) = D_i f(0)$, the ith partial derivative. (To prove that we write

$$f(x) = f(0) + \int_0^1 \frac{d}{dt} f(tx)\,dt = \sum x_i \int_0^1 D_i f(tx)\,dt = \sum x_i g_i(x), \text{ say.})$$

Now $\theta(1) = 0$, for $\theta(1) = \theta(1^2) = 2\theta(1)$. So $\theta(a) = 0$, and so

$$\theta(f) = \sum D_i f(0)\, \theta(x_i).$$

Thus proves that $\theta \mapsto (\theta(x_1), \ldots, \theta(x_n))$ is an isomorphism $\mathcal{D}_0(\mathbb{R}^n) \to \mathbb{R}^n$.

~~Now, for the general case, choose a chart $\varphi : U \to \mathbb{R}^n$ with $x \in U$.~~

~~We have a ring homomorphism $C^\infty(X) \to C^\infty(V)$, where $V = \varphi(U)$, given~~

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To treat the general case, we shall have to assume that X is Hausdorff. First observe that a smooth map $f : X \to Y$ induces a ring homomorphism $C^\infty(Y) \to C^\infty(X)$ by $F \mapsto F \circ f$, and hence a linear map $\mathcal{D}_x f : \mathcal{D}_x X \to \mathcal{D}_{f(x)} Y$. Clearly there is a "chain rule" $\mathcal{D}_x (g \circ f) = (\mathcal{D}_{f(x)} g) \circ (\mathcal{D}_x f)$. Furthermore, the preceding argument for $\mathbb{R}^n$ shows that $\mathcal{D}_x f = Df(x)$ when $X = Y = \mathbb{R}^n$.

To prove that $\mathcal{D}_x X \cong T_x X$ in general, it is therefore enough to show that when U is an open subset of X we have $\mathcal{D}_x U \xrightarrow{\cong} \mathcal{D}_x X$. For then each chart $\varphi_\alpha : U_\alpha \xrightarrow{\cong} V_\alpha \subset \mathbb{R}^n$ gives us an isomorphism

$$\mathcal{D}_x X \cong \mathcal{D}_x U_\alpha \xrightarrow{\mathcal{D}_x \varphi} \mathcal{D}_{x_\alpha} V_\alpha \cong \mathcal{D}_{x_\alpha} \mathbb{R}^n \cong \mathbb{R}^n,$$

and so $\mathcal{D}_x X \xrightarrow{\cong} T_x X$.

To prove that $\mathcal{D}_x U \cong \mathcal{D}_x X$ we need the

Bump function lemma. If $x \in U$, where U is an open subset of a smooth manifold X, and X is Hausdorff, then there is a smooth function $h : X \to \mathbb{R}$ such that $h = 1$ in a neighbourhood of x, and $\mathrm{supp}(h) = \overline{\{x \in X : h(x) \neq 0\}}$ is a compact subset of U.

Granting this, we have the

Corollary In the same situation, $\mathcal{D}_x U \xrightarrow{\cong} \mathcal{D}_x X$.

Proof of Corollary Injectivity: Suppose $\theta \in \mathcal{D}_x U$ and $\theta(f) = 0$ whenever f is the restriction of $\tilde{f} \in C^\infty(X)$. Choose h as in the lemma. Then $\theta(h) = 0$, for we can find another bump function h_1 such that $h h_1 = h_1$, and then $\theta(h_1) = \theta(h h_1) = \theta(h) + \theta(h_1)$. But for any $f \in C^\infty(U)$ the function fh extends by zero to X, so

$\theta(f) = \theta(fh) = 0$.

Surjectivity: Given $\theta \in \mathcal{D}_x X$, define $\hat{\theta} \in \mathcal{D}_x U$ by $\hat{\theta}(f) = \theta(fh)$, where h is a bump function. The previous argument with h_1 shows that $\hat{\theta}(f)$ does not depend on the choice of h, and that $\hat{\theta}(f) = \theta(f)$ if f extends to X. Finally, $\hat{\theta}$ is a derivation, because

$\hat{\theta}(fg) = \theta(fgh) = \theta(fgh^2) = \theta((fh)(gh)) = f(x)\,\hat{\theta}(g) + \hat{\theta}(f)\,g(x)$. //

<u>Proof of the lemma</u> We can assume that U is the domain of a chart $\varphi : U \to \mathbb{R}^n$. If $V = \varphi(U) \subset \mathbb{R}^n$ we can find a smooth function $k : \mathbb{R}^n \to \mathbb{R}$ with compact support inside V which $= 1$ in a neighbourhood of $\varphi(x)$. Define $h(\xi) = k(\varphi(\xi))$ if $\xi \in U$, and $h(\xi) = 0$ otherwise. Clearly $h = 1$ near x, and $\mathrm{supp}(h)$ is the compact set $\varphi^{-1}(\mathrm{supp}(k)) \subset U$. To prove that h is smooth, let $\tilde{\varphi} : \tilde{U} \to \mathbb{R}^n$ be an arbitrary chart for X. We must show that $h\tilde{\varphi}^{-1}$ is smooth in $\tilde{\varphi}(\tilde{U})$. But in $\tilde{\varphi}(U \cap \tilde{U})$ we have $h\tilde{\varphi}^{-1} = k \circ (\varphi\tilde{\varphi}^{-1})$, which is smooth. Finally, if $\eta = \tilde{\varphi}(\xi)$ with $\xi \notin U$ then, because X is Hausdorff and $\mathrm{supp}(h)$ is compact, ξ is contained in an open subset of X which does not meet $\mathrm{supp}(h)$, and so $h\tilde{\varphi}^{-1}$ is zero in a neighbourhood of η. //

The fact that $T_x X$ can be constructed algebraically from the ring $C^\infty(X)$ is part of a larger and more interesting story. Indeed, when X is compact and Hausdorff, the smooth manifold X with its maximal atlas can be completely reconstructed purely algebraically from the ring $C^\infty(X)$ alone, and smooth maps $X \to Y$ between smooth manifolds correspond 1-1 to ring homomorphisms $C^\infty(Y) \to C^\infty(X)$. I shall not prove this, but only a weaker result.

<u>Proposition</u> If X is compact and Hausdorff there is a 1-1 correspondence between points $x \in X$ and unital ring homomorphisms $A : C^\infty(X) \to \mathbb{R}$, given by $x \mapsto A_x$, where $A_x(f) = f(x)$.

<u>Note</u> A homomorphism A is <u>unital</u> if $A(1) = 1$. In that case $A(\lambda) = \lambda$ if λ is a constant function, for the only unital ring homomorphism $\mathbb{R} \to \mathbb{R}$ is the identity.

<u>Proof</u> If $x_1 \neq x_2$ then $A_{x_1} \neq A_{x_2}$, for we can find a bump function $h \in C^\infty(X)$ such that $h(x_1) = 1$ and $h(x_2) = 0$.

Given $A : C^\infty(X) \to \mathbb{R}$ we must show that $A = A_x$ for some x. It is enough to show

$$A(f) = 0 \implies f(x) = 0 \qquad (*)$$

for some x, for then if $A(f) = \lambda$ we have $A(f - \lambda) = 0$, so that $f(x) - \lambda = 0$ and $A(f) = f(x)$.

But if there is no x satisfying $(*)$ we can find $f_x \in C^\infty(X)$ for each $x \in X$ such that $A(f_x) = 0$ and $f_x(x) \neq 0$. By the compactness of X we can find a finite set $x_1, \ldots, x_k$ such that $g = \sum f_{x_i}^2$ is > 0 everywhere in X. Then $A(g) = \sum A(f_{x_i})^2 = 0$, so $A(1) = A(g \cdot g^{-1}) = A(g)\,A(g^{-1}) = 0$, a contradiction. //

The tangent spaces of a submanifold of $\mathbb{R}^N$

If X is a submanifold of $\mathbb{R}^N$ the inclusion $i : X \hookrightarrow \mathbb{R}^N$ induces a linear inclusion $Di(x) : T_xX \to T_x\mathbb{R}^N = \mathbb{R}^N$ for any $x \in X$. (We know that $Di(x)$ is injective because in a suitable chart for $\mathbb{R}^N$ the submanifold X looks like $\mathbb{R}^n \oplus 0 \subset \mathbb{R}^N$.) In this situation we usually think of T_xX as a subspace of $\mathbb{R}^N$: this corresponds to the usual geometric notion of the "tangent space".

The most common case is when $X = f^{-1}(0) \subset U$, where U is open in $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ is a smooth map such that $Df(x)$ has rank k for all $x \in X$.

<u>Proposition</u> In this situation, the image of T_xX in $\mathbb{R}^N$ is the kernel of $Df(x) : \mathbb{R}^N \to \mathbb{R}^k$.

<u>Proof</u> We have $f \circ i = 0$, so $Df(x) \circ Di(x) = 0$, so $T_xX \subset \ker Df(x)$. But both of these vector spaces have the same dimension.

<u>Examples</u> (i) If $x \in S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ then

$$T_xS^n = \{y \in \mathbb{R}^{n+1} : \langle x, y\rangle = 0\}.$$

(ii) If $A \in O_n \subset \{n \times n \text{ matrices}\}$ then $T_AO_n = A.\Sigma = \Sigma.A$, where $\Sigma = \{\text{all skew matrices}\}$. For $O_n = f^{-1}(0)$, where

$f : \{\text{matrices}\} \to \{\text{symmetric matrices}\}$ is $A \mapsto A^t A - 1$. Then

$\ker Df(A) = \{h : A^t h + h^t A = 0\} = \{h : A^t h \text{ is skew}\} = A.\Sigma$.

The tangent bundle

The tangent spaces $T_x X$ at the different points x of a smooth manifold X are by definition disjoint: a tangent vector should best be regarded as a pair (x, ξ), with $x \in X$ and $\xi \in T_x X$. The union

$$TX = \bigcup_{x \in X} T_x X$$

is called the tangent bundle of X. It is naturally a smooth manifold of dimension $2n$, where $n = \dim X$, for if $\{\varphi_\alpha : U_\alpha \to \mathbb{R}^n\}$ is the atlas of X then TX is covered by the sets TU_α, and we have bijections $D\varphi_\alpha : TU_\alpha \to U_\alpha \times \mathbb{R}^n \subset \mathbb{R}^n \oplus \mathbb{R}^n$ which take (x, ξ) to (x_α, ξ_α). The transition map $(D\varphi_\beta) \circ (D\varphi_\alpha)^{-1}$ is

$$(x_\alpha, \xi_\alpha) \longmapsto (x_\beta, D(\varphi_\beta \circ \varphi_\alpha^{-1})(x_\alpha)\, \xi_\alpha),$$

which is smooth.

If X is a submanifold of $\mathbb{R}^N$ then TX is a submanifold of $\mathbb{R}^{2N}$. For example, if $X = S^n \subset \mathbb{R}^{n+1}$ then

$$TX = \{(x, \xi) \in \mathbb{R}^{2(n+1)} : \|x\| = 1 \text{ and } \langle x, \xi \rangle = 0\}.$$

A smooth tangent vector field on X is a smooth map $s : X \to TX$ such that $s(x) \in T_x X$ for each $x \in X$.

1.4 Embedding manifolds in Euclidean space

The Whitney embedding theorem asserts that if a smooth manifold of dimension n is Hausdorff then it is diffeomorphic to a submanifold of $\mathbb{R}^{2n}$ — for brevity, one usually says "it can be embedded in $\mathbb{R}^{2n}$". Here I shall prove a weaker result.

<u>Theorem</u> A compact Hausdorff manifold is diffeomorphic to a submanifold of $\mathbb{R}^N$ for some N.

<u>Corollary</u> Such a manifold is metrizable.

The main tool needed to prove the theorem is the existence of "partitions of unity".

<u>Proposition</u> A finite covering $\{U_\alpha\}$ of a compact Hausdorff manifold by open subsets has a subordinate "partition of unity", i.e. there is a collection of smooth functions $f_\alpha : X \to \mathbb{R}$ such that

(i) $f_\alpha \geqslant 0$ everywhere,

(ii) $\mathrm{Supp}\,(f_\alpha) = \overline{\{x : f_\alpha(x) \neq 0\}} \subset U_\alpha$, and

(iii) $\Sigma f_\alpha = 1$.

<u>Proof</u> We have seen that for each $x \in X$ we can find a smooth $h_x : X \to \mathbb{R}$ such that $h_x \geqslant 0$, $\mathrm{supp}\,(h_x) \subset$ some U_α, and $h_x(x) \neq 0$. Choose a finite set $x_1, \ldots, x_k$ so that $h = \Sigma h_{x_i} > 0$ everywhere, and choose α_i so that $\mathrm{supp}\,(h_{x_i}) \subset U_{\alpha_i}$. Define $f_\alpha = \frac{1}{h} \sum_{\alpha_i = \alpha} h_{x_i}$. Then $\mathrm{supp}\,(f_\alpha) \subset U_\alpha$, and $\Sigma f_\alpha = 1$. //

<u>Proof of the embedding theorem</u>

Choose a finite collection of charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ (for $\alpha = 1, \ldots, k$) which cover X, and a partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$.

Define

$$F : X \longrightarrow \mathbb{R}^k \oplus \underbrace{\mathbb{R}^n \oplus \cdots \oplus \mathbb{R}^n}_{k} = \mathbb{R}^{k(n+1)}$$

$$x \longmapsto (f_1(x), \ldots, f_k(x) \,;\, f_1(x)\varphi_1(x), \ldots, f_k(x)\varphi_k(x)),$$

where $f_i(x)\varphi_i(x)$ is taken to mean 0 if $x \notin U_i$.

Let $V_i = \{x \in X : f_i(x) > 0\}$. To prove that F is injective it is enough to prove that $F|V_i$ is injective for each i, for if $x \in V_i$ and $x' \notin V_i$ then $f_i(x) \neq f_i(x')$, so $F(x) \neq F(x')$.

Let $W_i = \{(\lambda_1, \dots, \lambda_k ; v_1, \dots, v_k) \in \mathbb{R}^k \oplus (\mathbb{R}^n)^k : \lambda_i \neq 0\}$.
Define $p_i : W_i \to \mathbb{R}^n$ by $(\lambda_1, \dots, \lambda_k ; v_1, \dots, v_k) \longmapsto \lambda_i^{-1} v_i$.
Then $F(V_i) \subset W_i$, and $p_i \circ F = \varphi_i | V_i$, which is injective.

Furthermore, $DF(x) : \mathbb{R}^n \to \mathbb{R}^{k(n+1)}$ is injective for all x, for if $x \in V_i$ then

$$Dp_i \circ DF(x) = D(p_i \circ F)(x) = D\varphi_i(x) : \mathbb{R}^n \to \mathbb{R}^n,$$

which is an isomorphism. We have now completed the proof, because of the

<u>Criterion</u> If X is a compact smooth n dimensional manifold, and $F : X \to \mathbb{R}^N$ is a smooth injective map such that $DF(x)$ has rank n for all $x \in X$, then F is a diffeomorphism on to a submanifold of $\mathbb{R}^N$.

<u>Proof of the criterion</u> For any $x \in X$, let $\varphi : U \to \mathbb{R}^n$ be a chart for X with $x \in U$ and $\varphi(x) = \tilde{x}$. We shall find a chart for $\mathbb{R}^N$ near $F(x)$ which maps $F(\varphi^{-1}(\xi))$ to $(\xi ; 0) \in \mathbb{R}^n \oplus \mathbb{R}^{N-n}$.

Choose a neighbourhood V of $F(x)$ in $\mathbb{R}^N$ such that $F^{-1}(V) \subset U$: this can be done because $F(X - U)$ is a compact subset of $\mathbb{R}^N$ which does not contain $F(x)$. Choose a linear map $A : \mathbb{R}^{N-n} \to \mathbb{R}^N$ such that

$$D(F \circ \varphi^{-1})(\tilde{x}) \oplus A : \mathbb{R}^n \oplus \mathbb{R}^{N-n} \to \mathbb{R}^N$$

is invertible. This map is the derivative at $(\tilde{x}, 0)$ of

$$(\xi, \eta) \longmapsto F(\varphi^{-1}(\xi)) + A\eta : \varphi(U) \times \mathbb{R}^{N-n} \to \mathbb{R}^N, \qquad (*)$$

so, after making V smaller if necessary, we can use the inverse-function theorem to choose a smooth map $g : V \to \mathbb{R}^N$ inverse to $(*)$. Then $g : V \to \mathbb{R}^N$ is a chart for $\mathbb{R}^N$, and if $y \in V \cap F(X)$ then $y = F(\varphi^{-1}(\xi))$ for some $\xi \in \varphi(U)$, and

$$g(y) = g(F(\varphi^{-1}(\xi))) = (\xi, 0) \in \varphi(U) \times \mathbb{R}^{N-n}. \quad //$$

Having proved that X can be embedded in <u>some</u> $\mathbb{R}^N$ it is fairly easy to prove that it can be embedded in $\mathbb{R}^{2n+1}$.

To get the dimension down to $\mathbb{R}^{2n}$ is quite a lot harder.

The argument for $\mathbb{R}^{2n+1}$ can be sketched as follows. Given an embedding $F : X \hookrightarrow \mathbb{R}^N$ we should like to prove that if $N > 2n+1$ we can find a unit vector $u \in \mathbb{R}^N$ such that the composition $\pi \circ F : X \longrightarrow u^\perp \cong \mathbb{R}^{N-1}$ is still an embedding, where π is the orthogonal projection $\mathbb{R}^N \to u^\perp$.

But $\pi \circ F$ will be injective providing there is no pair of distinct points x_1, x_2 of X such that $F(x_1) - F(x_2)$ is a multiple of u, i.e. providing u is not in the image of the map

$$X \times X - (\text{diagonal}) \longrightarrow S^{N-1}$$

given by $(x_1, x_2) \longmapsto (F(x_1) - F(x_2))/\|x_1 - x_2\|$. But this can always be achieved if $N-1 > 2n$, for the domain has dimension $2n$, and the map is smooth (and so cannot be like a Peano curve).

To ensure that $\pi \circ F$ also satisfies the condition that $D(\pi \circ F)(x)$ is injective we must also avoid vectors u which are parallel to tangent vectors to X; but that excludes a subspace which has at most dimension $2n-1$.

Part II Complex manifolds

2.1 Holomorphic maps

If U is open in $\mathbb{C}^n$ then $f : U \to \mathbb{C}^m$ is differentiable at $z \in U$ in the complex sense if there is a $\mathbb{C}$-linear map $Df(z) : \mathbb{C}^n \to \mathbb{C}^m$ such that

$$f(z+h) = f(z) + Df(z).h + R(z;h),$$

where $\|R(z;h)\| / \|h\| \to 0$ as $h \to 0$. Thus the definition is the same as in the real case, except that $Df(z)$ is required additionally to commute with multiplication by i.

The condition $Df(z) \circ i = i \circ Df(z)$ is called the Cauchy-Riemann equation, for if $n = m = 1$ and we write $z = x + iy$, $f = u + iv$ then $Df(z) = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix}$, and $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and we get the usual C-R equations.

We say $f : U \to \mathbb{C}^m$ is <u>analytic</u> if it is differentiable at each $z \in U$, and $Df(z)$ is a continuous function of z. Analyticity is a <u>very</u> much stronger condition than differentiability in the real sense. For if f is analytic then

(i) it has derivatives of <u>all</u> orders, and

(ii) it can be expanded in a convergent Taylor series in the neighbourhood of each point.

From (ii) it follows that if U is connected and $f_1, f_2 : U \to \mathbb{C}^m$ are analytic functions which coincide in an open subset of U then $f_1 = f_2$.

The following properties are also worth mentioning. (Here U is always open in $\mathbb{C}^n$.)

(iii) <u>Hartogs's theorem</u> If $n > 1$ a holomorphic function cannot have an isolated singularity, i.e. if $f : U - \{z\} \to \mathbb{C}^m$ is analytic then f extends analytically to U. (If $n = 1$ one needs to assume in addition that f is bounded.)

(iv) If $f : U \to \mathbb{C}$ is analytic and not constant then it is <u>open</u>, i.e. $f(V)$ is open whenever V is open.

(v) If $f : U \to \mathbb{C}^n$ is analytic and 1–1 then $Df(z)$ is invertible for all $z \in U$. I.e. if $\det Df(z) = 0$ then f cannot be 1–1 in a neighbourhood of z. (This is well-known when $n = 1$, and the general case is harder to prove, and we shall not need it in this course.)

(vi) If $f : U \to \mathbb{C}^n$ is analytic and 1–1 then it is orientation-preserving as a smooth map $U \to \mathbb{R}^{2n}$. For if A is a complex $n \times n$ matrix, and B is the same transformation written as a $2n \times 2n$ real matrix, then $\det(B) = |\det(A)|^2 \geq 0$.

2.2 Complex manifolds

The definition of a complex manifold X is exactly the same as that of a real smooth manifold, except that the charts are maps $\varphi : U \to \mathbb{C}^n$, and the transition maps $\varphi_2 \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ are required to be analytic. Thus an n-dimensional complex manifold IS a $2n$-dimensional smooth manifold. – with some additional structure.

Analytic maps between complex manifolds are defined just as in the real case.

A one-dimensional Hausdorff complex manifold is called a Riemann surface.

Examples

(i) The Riemann sphere $S = \mathbb{C} \cup \{\infty\}$ is covered by two charts $\varphi_1 : \mathbb{C} \to \mathbb{C}$ and $\varphi_2 : (\mathbb{C} - \{0\}) \cup \{\infty\} \to \mathbb{C}$, where φ_1 is the identity, and $\varphi_2(z) = z^{-1}$ (and $\varphi_2(\infty) = 0$).

If U is open in $\mathbb{C}$, and $f : U - \{a\} \to \mathbb{C}$ is analytic with a pole at a we can extend f to an analytic map $f : U \to S$ by defining $f(a) = \infty$. For f has a pole at a iff $1/f(z) \to 0$ as $z \to a$. Similarly, if $V = \{z \in \mathbb{C} : |z| > R\}$, and $f : V \to \mathbb{C}$ is analytic, then f extends to an analytic map $f : V \cup \{\infty\} \to S$ iff $|f(z)| \leq K |z|^n$ as $z \to \infty$ for some K and n.

In particular, a rational function $f = p/q$, where p and q

are non-zero polynomials, defines an analytic map $f : S \to S$, and it is easy to prove the converse, that every analytic map $f : S \to S$ is given by a rational function.

(ii) Complex projective space $\mathbb{P}^n_{\mathbb{C}}$ is defined just like real projective space.

(iii) The real spheres S^{2n} cannot be made into complex manifolds except for the Riemann sphere S^2, and conceivably also S^6. (It is a famous unsolved problem to prove that S^6 cannot be complex.)

(iv) Tori Let L be a lattice in $\mathbb{C}$, i.e. an additive subgroup of the form $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\{\omega_1, \omega_2\}$ is a basis of $\mathbb{C}$ as a vector space over $\mathbb{R}$. (I.e. $\omega_1/\omega_2 \notin \mathbb{R}$.)

Let $X = \mathbb{C}/L$ be the quotient group. There is a natural map $\pi : \mathbb{C} \to X$. To define an atlas for X, choose $\varepsilon > 0$ such that $|\lambda| > 2\varepsilon$ for all $\lambda \in L - \{0\}$. Let $\tilde{U} = \{z \in \mathbb{C} : |z| < \varepsilon\}$, and $U = \pi(\tilde{U}) \subset X$. Then $\pi^{-1} : U \to \tilde{U} \subset \mathbb{C}$ is a chart.

For $\zeta \in X$, let $U_\zeta = \zeta + U \subset X$, and define $\varphi_\zeta : U_\zeta \to \mathbb{C}$ by $\varphi_\zeta(\xi) = \varphi(\xi - \zeta)$. Then $\{\varphi_\zeta : U_\zeta \to \mathbb{C}\}$ is an atlas for X, which compact and Hausdorff. (It is compact because the map $(s,t) \longmapsto s\omega_1 + t\omega_2$ from $[0,1] \times [0,1]$ to X is continuous and surjective.)

Whereas the manifolds $\mathbb{C}/L_1$ and $\mathbb{C}/L_2$ for different lattices L_1, L_2 are always diffeomorphic as real smooth manifolds — for one can find an $\mathbb{R}$-linear isomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(L_1) = L_2$ — we shall see that $\mathbb{C}/L_1$ and $\mathbb{C}/L_2$ are not isomorphic as complex manifolds unless $L_1 = \alpha L_2$ for some $\alpha \in \mathbb{C}$.

Lemma For any lattice L we have $L = \alpha.(\mathbb{Z} + \tau\mathbb{Z})$ for some τ in $UHP = \{\tau \in \mathbb{C} : \operatorname{Im}(\tau) > 0\}$. And if $\tau, \tau' \in UHP$ then $\mathbb{Z} + \tau\mathbb{Z} = \alpha(\mathbb{Z} + \tau'\mathbb{Z})$ for some $\alpha \in \mathbb{C}$ iff $\tau' = g\tau$ for some $g \in SL_2(\mathbb{Z})$. (I.e. g is a Möbius transformation $\tau \longmapsto (a\tau + b)/(c\tau + d)$ with

$a, b, c, d \in \mathbb{Z}$, and $ad - bc = 1$.

(v) A cubic curve. Let p be a monic cubic polynomial in $\mathbb{C}[x]$, with <u>distinct</u> roots $\alpha, \beta, \gamma \in \mathbb{C}$. Let

$$Y = \{(x,y) \in \mathbb{C}^2 : y^2 = p(x)\}.$$

One can use the implicit function theorem (which is true in the complex-analytic context) to see that Y is a complex manifold. Alternatively, one can cover Y by seven charts, as follows.

Let $V_+ = \mathbb{C} - \{\text{negative real axis}\}$, $V_- = \mathbb{C} - \{\text{positive real axis}\}$. Define $\sqrt{z}$ for $z \in V_+$ so that $\mathrm{Re}(\sqrt{z}) > 0$, and so that $\mathrm{Im}(\sqrt{z}) > 0$ for $z \in V_-$.

Let
$$U_{++} = \{(x,y) \in Y : x \in V_+, \ y = +\sqrt{p(x)}\},$$
$$U_{+-} = \{(x,y) \in Y : x \in V_+, \ y = -\sqrt{p(x)}\},$$
and similarly U_{-+}, U_{--}. There are charts $\varphi_{\pm\pm} : U_{\pm\pm} \to \mathbb{C}$ given by $(x,y) \mapsto x$.

Now choose some small $\varepsilon > 0$, and let $U_\alpha = \{(x,y) \in Y : |x - \alpha| < \varepsilon\}$. We have a chart $\varphi_\alpha : U_\alpha \to \mathbb{C}$ given by $(x,y) \mapsto y$.

Define U_β, U_γ similarly. It is easy to check that we have an atlas.

It is natural to "complete" Y by adding a "point at infinity". We can do this most simply by defining $\hat{Y} = Y \cup \{\infty\}$, and adding one chart $\varphi_\infty : U_\infty \to \mathbb{C}$ to the atlas. We take $U_\infty = \{(x,y) \in Y : |x| > R\} \cup \{\infty\}$ and $\varphi_\infty(\infty) = 0$, $\varphi_\infty(x,y) = x/y \in \mathbb{C}$. Providing R is chosen large enough, φ_∞ is a chart, and is compatible with the earlier charts.

A more illuminating way to define $\hat{Y}$ is as the subset of $\mathbb{P}^2_{\mathbb{C}}$ given, in terms of homogeneous coordinates $(t, x, y) \in \mathbb{C}^3 - \{0\}$ by

$$ty^2 = t^3 p(x/t) = x^3 + a_1 t x^2 + a_2 t^2 x + a_3 t^3$$

if $p(x) = x^3 + a_1 x^2 + a_2 x + a_3$. If we identify $\mathbb{C}^2$ with the points $(1, x, y) \in \mathbb{P}^2_{\mathbb{C}}$ then $Y = \mathbb{C}^2 \cap \hat{Y}$. And $\hat{Y}$ has just one point $(0, 0, 1)$ on the line $t = 0$ at infinity in $\mathbb{P}^2_{\mathbb{C}}$. In the open subset of $\mathbb{P}^2_{\mathbb{C}}$ consisting of points $(t, x, 1)$ the curve $\hat{Y}$ is given by $t = (\text{homogeneous cubic in } (x,t))$, and the chart φ_∞ is $(t, x, 1) \mapsto x$;

which is chosen because we can solve for t as a function of x near $(x,t) = (0,0)$. Notice that $t = x^3 + O(x^4)$, so that $\hat{Y}$ touches the line at infinity $t = 0$ in a <u>point of inflexion</u>.

The compact manifold $\hat{Y}$ we have constructed will turn out to be isomorphic to the torus $\mathbb{C}/L$ for a suitable lattice L. But in the constructing it we could equally well have been carried out for an arbitrary non-singular curve given by a homogeneous polynomial equation $F(t,x,y)=0$ in $\mathbb{P}^2_{\mathbb{C}}$: the only condition needed (to ensure non-singularity) is that the four numbers $F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial t}$ do not vanish simultaneously, except at $(0,0,0)$. A good example to think about is the "Fermat" curve $x^n + y^n = t^n$.

The degree of an analytic map

If $f : X \to Y$ is a non-constant map of connected compact Riemann surfaces the inverse-image $f^{-1}(y)$ of each $y \in Y$ has no points of accumulation (as the zeros of an analytic function are isolated), and so is finite. By the same argument the derivative $Df(x) : T_x X \to T_y Y$ vanishes at most at a finite number of points $x \in X$, say $x_1, \ldots, x_k$. (The tangent spaces of a complex manifold are defined just as for a real smooth manifold: in terms of a pair of charts, f will be represented by an analytic function $\tilde{f}$ between open sets in $\mathbb{C}$, and $Df(x)$ by the complex number $\tilde{f}'(x)$.) Let $y_i = f(x_i)$. The inverse-function theorem tells us that the number of points in $f^{-1}(y)$ is a locally constant, and hence <u>constant</u>, function of y for $y \in Y - \{y_1, \ldots, y_k\}$. This number is called the <u>degree</u> of f. The points y_i are called the <u>ramification points</u> of f.

<u>Example</u> If $X = Y = \mathbb{C} \cup \{\infty\}$ then f is a rational function $p(z)/q(z)$, and the degree of f is whichever is greater of $\deg(p), \deg(q)$, providing p and q have no common factor.

In fact it is easy (see Problem Sheet) to define the <u>multiplicity</u> of f at the points x_i, and if one counts the points of $f^{-1}(y)$ with multiplicities then the number is the same for <u>all</u> $y \in Y$.

<u>Proposition</u> An analytic map $f : X \to Y$ of degree 1 between compact Riemann surfaces is an isomorphism.

<u>Proof</u> The derivative cannot vanish anywhere, for if, in local charts, f is represented by $\tilde{f}$, and $\tilde{f}'$ vanishes to order k at z_0, then $\tilde{f}$ is $(k+1)$-to-1 in the neighbourhood of z_0. Thus f must be a bijection, and its inverse is analytic by the inverse-function theorem.

2.3 <u>The Weierstrass elliptic function</u>

Let $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ be a lattice in $\mathbb{C}$.

A function f is <u>L-periodic</u> if $f(z+\lambda) = f(z)$ for all $\lambda \in L$. Such a function cannot be analytic in all of $\mathbb{C}$, unless it is constant. For it would be bounded on the compact set $\{\lambda\omega_1 + \mu\omega_2 : 0 \le \lambda, \mu \le 1\}$, and hence bounded everywhere, and hence constant by Liouville's theorem.

Furthermore it cannot have just one simple pole in each "cell" $C_{A,B} = \{a\omega_1 + b\omega_2 : A \le a < A+1,\ B \le b < B+1\}$. For if it had a pole in the interior of $C_{A,B}$ then the integral $\int f(z)\,dz$ around the boundary of $C_{A,B}$ would be zero by periodicity, but non-zero by the residue theorem.

Suppose we try to construct f to be L-periodic with just one double pole at each lattice point. The Laurent expansion of $f(z)$ near $z = 0$ can then be taken to be $f(z) = z^{-2} + (\text{analytic})$, for the coefficient of z^{-1} must vanish by the preceding residue argument. If the series $\sum_{\lambda \in L} \frac{1}{(z-\lambda)^2}$ converged, we could conclude that $f(z) - \sum \frac{1}{(z-\lambda)^2}$ was constant. In fact that series does not converge absolutely, so we define

$$\wp_L(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

This series <u>does</u> converge absolutely for $z \notin L$, and it even converges uniformly in any compact region providing one omits the finite number of terms which have poles in that region. For if $|z| \le R$ then

$$\left| \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right| \le \frac{K}{|\lambda|^3}$$

when $|\lambda| > 2R$ (for some K), and the number of terms with $r \le |\lambda| \le r+1$ is $\le K'r$ for some K'.

Thus $\wp_L$ is meromorphic throughout $\mathbb{C}$, with double poles at the lattice points. Furthermore, $\wp_L$ is L-periodic, for after a little rearrangement we find, for $\mu \in L$,

$$\wp_L(z \mp \mu) - \wp_L(z) = \sum_{\substack{\lambda \in L \\ \lambda \neq 0, \mu}} u_\lambda \,, \quad \text{where} \quad u_\lambda = \frac{1}{\lambda^2} - \frac{1}{(\lambda-\mu)^2} .$$

The last series is absolutely ~~uniformly~~ convergent, and $u_\lambda = -u_{\mu-\lambda}$, so its sum is zero.

Because of the L-periodicity, $\wp_L$ defines an analytic map $\wp_L : X = \mathbb{C}/L \longrightarrow \mathbb{C} \cup \{\infty\} = S$. The inverse-image of ∞ is the single point $0 \in X$, at which $1/\wp_L$ has a 2^{nd}-order zero. So $\wp_L : X \to S$ is degree 2.

<u>Proposition</u> $\wp_L$ satisfies the differential equation

$$\wp_L'(z)^2 = 4\wp(z)^3 - a\wp(z) - b, \qquad (\text{where } \wp = \wp_L)$$

where $a = 60 \sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \lambda^{-4}$ and $b = 140 \sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \lambda^{-6}$.

<u>Proof</u> We have $\wp(0) = 0$, and $\wp(-z) = \wp(z)$, and one readily finds that the Laurent expansion of $\wp$ at $z = 0$ is

$$\wp(z) = \frac{1}{z^2} + \frac{a}{20} z^2 + O(z^4).$$

So
$$\wp'(z) = \frac{-2}{z^3} + \frac{a}{10} z + O(z^3).$$

Calculating the expansion of $\wp'(z)^2 - 4\wp(z)^3 + a\wp(z)$ we find this function has no pole at $z = 0$, so is analytic everywhere, and doubly periodic, hence constant. To find the constant term we must consider the next term in the Laurent expansion of $\wp(z)$, which is $\frac{b}{28} z^4$. //

<u>Corollary</u> The map $f : \mathbb{C}/L \longrightarrow \mathbb{P}^2_{\mathbb{C}}$ given by

$$\begin{aligned} f(z) &= (1, \wp(z), \wp'(z)) && \text{if } z \neq 0 \\ &= (0, 0, 1) && \text{if } z = 0 \end{aligned}$$

defines an analytic map $f : \mathbb{C}/L \to \hat{Y}$, where $\hat{Y}$ has the equation $ty^2 = 4x^3 - axt^2 - bt^3$.

The analytic map $\hat{Y} \to S$ given by $(x,y) \mapsto x$ and $\infty \mapsto \infty$ clearly has degree 2, and we have seen that the composite $\mathbb{C}/L \to \hat{Y} \to S$, which is $z \mapsto \wp(z)$, $0 \mapsto \infty$, also has degree 2. It follows that $f : \mathbb{C}/L \to \hat{Y}$ has degree 1, giving us the

<u>Corollary</u> $f : \mathbb{C}/L \to \hat{Y}$ is an isomorphism of complex manifolds.

The analytic map $\wp_L : \mathbb{C}/L \to S$ of degree 2 is ramified at the three roots of $4x^3 - ax - b = 0$, say α, β, γ, for then $\wp_L'(z) = 0$. It is also ramified at $\infty = \wp_L(0)$, where $1/\wp_L$ vanishes to second order. (As a map of complex manifolds $\wp_L$ has derivative 0 at 0.) The four inverse-image points in $\mathbb{C}/L$ are the four solutions of $2z = 0$ in the group $\mathbb{C}/L$, i.e. $z = 0, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}(\omega_1+\omega_2)$. For $\wp_L(z) = \wp_L(\lambda - z)$ for any $\lambda \in L$, and so $\wp_L'(z) = -\wp_L'(\lambda - z)$, whence $\wp_L'(\lambda/2) = 0$.

If we take for L the lattice $L = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau \in \mathrm{UHP}$ there is a canonical way to order the four ramification points of $\wp_L$, namely $(\infty, \wp(\frac{1}{2}), \wp(\frac{1}{2}\tau), \wp(\frac{1}{2}+\frac{1}{2}\tau))$. Let us write $k(\tau) \in \mathbb{C} - \{0,1\}$ for the cross-ratio of these points. Thus

$$k : \mathrm{UHP} \longrightarrow \mathbb{C} - \{0,1\}$$

is an analytic map, to which we shall return.

2.4 <u>Covering spaces</u>

A smooth map $p : X \to Y$ between smooth manifolds is a <u>covering map</u> if each $y \in Y$ has a neighbourhood V such that $p^{-1}(V)$ is the <u>disjoint</u> union of sets U_α each of which is mapped to V by a diffeomorphism $p : U_\alpha \to V$. (I shall refer to such a neighbourhood V as a "good" neighbourhood.) One says that X is a <u>covering space</u> of Y.

<u>Examples</u> (i) The map $z \mapsto e^z$ is a covering map $\mathbb{C} \to \mathbb{C} - \{0\}$.

(ii) The obvious map $\mathbb{C} \to \mathbb{C}/L$ (for a lattice L) is a covering map.

(iii) If Y is the cubic curve in $\mathbb{C}^2$ described above then the map $Y' \to \mathbb{C} - \{\alpha, \beta, \gamma\}$, where $Y' = \{(x,y) \in Y : y \neq 0\}$, given by

$(x,y) \mapsto x$ is a covering map.

(iv) More generally, if $p: X \to Y$ is any analytic map between compact Riemann surfaces, and $\{y_1, \ldots, y_k\} \subset Y$ is the set of ramification points, then $p|p^{-1}(Y'): p^{-1}(Y') \to Y'$ is a covering map, where $Y' = Y - \{y_1, \ldots, y_k\}$.

(v) We shall see that the map $k: UHP \to \mathbb{C} - \{0,1\}$ mentioned above is a covering map.

(vi) (A non-complex example). Let $S^{n-1} = \{\xi \in \mathbb{R}^n : \|\xi\| = 1\}$, and let $p: S^{n-1} \to \mathbb{P}^{n-1}$ be the obvious 2-to-1 map. This is a covering map.

When one has a covering map $p: X \to Y$ one is usually interested in "lifting" things — especially paths — from Y to X. If $f: Z \to Y$ is any map, a <u>lift</u> of f is a map $\tilde{f}: Z \to X$ such that $p \circ \tilde{f} = f$.

Let us recall some definitions.

(a) Two paths $\gamma_0, \gamma_1 : [a,b] \to X$ are <u>homotopic</u> if there is a continuous map $\gamma : [a,b] \times [0,1] \to X$ such that $\gamma(t,0) = \gamma_0(t)$ for $t \in [a,b]$ and $\gamma(t,1) = \gamma_1(t)$ for $t \in [a,b]$.

The paths are <u>homotopic relative to their end-points</u> ("rel. ends" for short) if the ends stay fixed during the homotopy, i.e. $\gamma(a,s) = \gamma(a,0)$ and $\gamma(b,s) = \gamma(b,0)$ for all $s \in [0,1]$. Homotopy, and homotopy rel. ends, are equivalence relations.

(b) A space Z is <u>simply connected</u> if it is path-connected, and any two paths from z_0 to z_1 are homotopic rel. ends, for all $z_0, z_1 \in Z$.

In proving that homotopy is an equivalence relation, and also in proving the results below, we constantly use the almost trivial

<u>Lemma</u> If a space Z is the union of two <u>closed</u> subsets Z_1 and Z_2, and $f_1: Z_1 \to X$, $f_2: Z_2 \to X$ are continuous maps which agree on $Z_1 \cap Z_2$, then f_1 and f_2 together define a continuous map $f: Z \to X$.

Now we return to lifting. In the following, $p: X \to Y$ is always a covering map.

<u>Proposition</u> If Z is a connected space, and $\tilde{f}_0, \tilde{f}_1 : Z \to X$ are lifts

of the same map $f : Z \to Y$, and $\tilde{f}_0(z) = \tilde{f}_1(z)$ for one point $z \in Z$, then $\tilde{f}_0 = \tilde{f}_1$.

<u>Proof</u> We show that the set of points $\zeta \in Z$ such that $\tilde{f}_0(\zeta) = \tilde{f}_1(\zeta)$ is open and that its complement is open. Suppose $\tilde{f}_0(\zeta) = \tilde{f}_1(\zeta)$. Choose a "good" neighbourhood V of $f(\zeta)$ in Y, so that $p^{-1}(V) = \coprod U_\alpha$. Suppose $\tilde{f}_0(\zeta) \in U_\alpha$. Then $\tilde{f}_0(\zeta')$ and $\tilde{f}_1(\zeta')$ both belong to U_α for all ζ' in a neighbourhood of ζ. But $p|U_\alpha$ is bijective, so $\tilde{f}_0(\zeta') = \tilde{f}_1(\zeta')$ for all ζ' near ζ. Similarly, if $\tilde{f}_0(\zeta) \neq \tilde{f}_1(\zeta)$ then these points belong to disjoint neighbourhoods U_α, U_β, and so $\tilde{f}_0(\zeta') \neq \tilde{f}_1(\zeta')$ for ζ' near ζ.

<u>Proposition</u> (i) A path $\gamma : [a,b] \to Y$ can be lifted to $\tilde{\gamma} : [a,b] \to X$, with any given starting point in $p^{-1}(\gamma(a))$, uniquely if its starting point $\tilde{\gamma}(a)$ is prescribed.

(ii) A homotopy $\gamma : [a,b] \times [0,1] \to Y$ can be lifted; and if γ is a homotopy rel. ends then so is its lift $\tilde{\gamma}$.

(iii) If Z is a simply connected smooth (or complex) manifold, then any smooth (or analytic) map $f : Z \to Y$ can be lifted to a smooth (or analytic) map $\tilde{f} : Z \to X$.

<u>Proof</u> (i) Choose an open covering $\mathcal{U}$ of Y by good open sets V. By repeated bisection, as in the proof of the Heine-Borel theorem, one can subdivide $[a,b]$ into 2^k consecutive closed subintervals $I_1, \ldots, I_{2^k}$, each of length $(b-a)/2^k$, so that each $\gamma(I_i) \subset$ some V in $\mathcal{U}$. Clearly we can lift $\gamma|I_i$. Choose the lifts successively so that $\tilde{\gamma}(a_i) = \tilde{\gamma}(b_{i-1})$, where $I_i = [a_i, b_i]$. This gives a lift $\tilde{\gamma} : [a,b] \to X$ by the lemma above.

(ii) By repeated quadrisection, subdivide $[a,b] \times [0,1]$ into 2^{2k} subrectangles I_{ij} so that $\gamma(I_{ij}) \subset$ some V in $\mathcal{U}$. Taking the I_{ij} in order lexicographically, lift $\gamma|I_{ij}$ so that $\tilde{\gamma}|I_{ij}$ agrees with $\tilde{\gamma}|J_{ij}$ at the lower left corner of I_{ij}, where J_{ij} = (union of earlier rectangles). Then $\tilde{\gamma}$ is well-defined on $I_{ij} \cap J_{ij}$ by uniqueness of path-lifting, so is continuous on $I_{ij} \cup J_{ij}$. And so on.

The assertion about homotopy rel. ends follows from the uniqueness of path-lifting.

(iii) Choose $z_0 \in Z$ and $x_0 \in X$ such that $p(x_0) = f(z_0)$. Define $\tilde{f}(z)$, for any $z \in Z$, as the end-point of the lift of $f \circ \gamma$, where γ is any path in Z from z_0 to z. By the result of (ii), the map $\tilde{f}: Z \to X$ is well-defined, but we do not yet know it is continuous.

Given $z \in Z$, let W be an open neighbourhood of z which is diffeomorphic to an open ball in $\mathbb{R}^n$ and is such that $f(W) \subset$ some V in $\mathcal{V}$. If $\tilde{f}(z) \in U_\alpha$, we find that $\tilde{f}|W$ is the composite of $f: W \to V$ with the inverse of $p: U_\alpha \xrightarrow{\cong} V$, for we can define $\tilde{f}(z')$, for $z' \in W$, by lifting a path which goes from z_0 to z and then from z to z' inside W. This shows that $\tilde{f}|W$ is smooth (or analytic).

<u>Corollaries</u>

(i) If X is connected and Y is simply connected, then any covering map $p: X \to Y$ is an isomorphism.

(ii) If L is a lattice in $\mathbb{C}$ then any analytic map $f: \mathbb{C} \cup \{\infty\} \to \mathbb{C}/L$ is constant.

(iii) If L and L' are lattices the complex manifolds $\mathbb{C}/L$ and $\mathbb{C}/L'$ are isomorphic if and only if $L' = \alpha L$ for some $\alpha \in \mathbb{C}$.

<u>Proof</u> (i) is obvious, for we can lift the identity map $Y \to Y$, and the resulting smooth map must be surjective because X is path-connected, and any path is the lift of its image in Y.

(ii) holds because $\mathbb{C} \cup \{\infty\}$ is simply connected, so that f can be lifted to an analytic map $\tilde{f}: \mathbb{C} \cup \{\infty\} \to \mathbb{C}$, bounded because $\mathbb{C} \cup \{\infty\}$ is compact, and hence constant by Liouville's theorem.

(iii) If $f: \mathbb{C}/L \to \mathbb{C}/L'$ is an analytic isomorphism, we can assume $f(0) = 0$ (as, otherwise, consider $z \mapsto f(z) - f(0)$). Lift the composite $\mathbb{C} \to \mathbb{C}/L \to \mathbb{C}/L'$ to get an analytic map $F: \mathbb{C} \to \mathbb{C}$ such that

$$\begin{tikzcd} \mathbb{C} \arrow[r, "F"] \arrow[d] & \mathbb{C} \arrow[d] \\ \mathbb{C}/L \arrow[r, "f"] & \mathbb{C}/L' \end{tikzcd}$$

commutes. We can assume $F(0) = 0$. If $\lambda \in L$ then $F(z+\lambda)$ and $F(z)$ have the same image in $\mathbb{C}/L'$; and so $z \longmapsto F(z+\lambda) - F(z)$ is a map

$\mathbb{C} \to L'$, necessarily constant. Thus $F(z+\lambda) = F(z) + F(\lambda)$ when $z \in \mathbb{C}$, $\lambda \in L$. In particular, F induces a group homomorphism $F: L \to L'$, and so $|F(\lambda)| \leq K|\lambda|$ for some K, when $\lambda \in L$. We deduce that $|F(z)| \leq K'|z|$ for all large z, as any $z \in \mathbb{C}$ is within bounded distance of a point of L. So $z \mapsto F(z)/z$ (which is analytic at $z = 0$ because $F(0) = 0$) is a bounded analytic map $\mathbb{C} \to \mathbb{C}$, and hence constant. So $F(z) = \alpha z$ for some $\alpha \in \mathbb{C}$.

Two more exciting corollaries are

(iv) <u>Picard's theorem</u> A non-constant analytic map $f: \mathbb{C} \to \mathbb{C}$ can omit at most <u>one</u> value (e.g. $z \mapsto e^z$ omits $0 \in \mathbb{C}$). That is, an analytic map $f: \mathbb{C} \to \mathbb{C} - \{0,1\}$ is constant.

<u>Proof</u> Any analytic map $f: \mathbb{C} \to UHP$ is constant by Liouville, as $UHP \cong$ (unit disc). But the map $k: UHP \to \mathbb{C} - \{0,1\}$ defined by elliptic functions is a covering map, as will be explained below.

(v) A cubic curve $\hat{Y}$ in $\mathbb{P}^2_{\mathbb{C}}$ has a group-law coming from the isomorphism $\hat{Y} \cong \mathbb{C}/L$. (We shall explain below that <u>any</u> cubic curve is of this form.) We can choose the isomorphism so that any point of $\hat{Y}$ is the identity element, but there is no further freedom, by (iii) above.

<u>Proposition</u> If p_1, p_2, p_3 are three collinear points on a cubic curve $\hat{Y}$ then $p_1 + p_2 + p_3 =$ constant (independent of the line), and
$p_1 + p_2 + p_3 = 0$
if we choose the identity element of $\hat{Y}$ to be a point of inflection.

<u>Proof</u> (sketch only) Choose any point $p_0 \in \mathbb{P}^2_{\mathbb{C}}$, and a line ℓ_0 in $\mathbb{P}^2_{\mathbb{C}}$ not containing p_0. Then $\ell_0 \cong \mathbb{C} \cup \{\infty\}$. We define an analytic map $\ell_0 \to \hat{Y} \cong \mathbb{C}/L$ by $q \longmapsto p_1 + p_2 + p_3$, where p_1, p_2, p_3 are the three points of intersection of the line $p_0 q$ with $\hat{Y}$. One must check that this <u>is</u> analytic. It is then constant by (ii) above. So $p_1 + p_2 + p_3$ is the same for all lines passing through p_0, and hence the same for <u>all</u> lines in $\mathbb{P}^2_{\mathbb{C}}$.

Analytic Continuation

When $f : V \to \mathbb{C}$ is an analytic function defined in an open subset V of the complex plane it may happen that f extends to an analytic function $\tilde{f} : U \to \mathbb{C}$ defined in a larger open set U. If U is connected then $\tilde{f}$ is completely determined by f.

Sometimes there is a biggest open set U to which f extends. Sometimes, however, there is not.

Examples

(i) Let $f(z) = \prod_{n\geq 1} (1 - z^n)$ for $z \in V = \{z \in \mathbb{C} : |z| < 1\}$. Then f cannot be extended to any larger open subset of $\mathbb{C}$, for $f(re^{2\pi i p/q}) \to 0$ as $r \to 1$, so any extension of f would have to vanish on the dense subset $\{e^{2\pi i p/q} : p/q \text{ rational}\}$ of the unit circle. (Notice that $f(z)^{-1} = \sum p_n z^n$, where p_n is the number of partitions of n.)

(ii) The Riemann ζ-function is defined by $\zeta(z) = \sum_{n\geq 1} n^{-z}$ for $z \in V = \{z \in \mathbb{C} : \mathrm{Re}(z) > 1\}$. Then ζ extends to an analytic function in $\mathbb{C} - \{1\}$, and has a simple pole at $z = 1$. (See problem sheet.)

(iii) Similarly, the Γ-function, defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,dt$ for $\mathrm{Re}(z) > 0$ extends to a meromorphic function in all of $\mathbb{C}$, with simple poles at $z = 0, -1, -2, \ldots$

(iv) The function $f(z) = z^{1/2}$ can be defined by $f(z) = (1 - (1-z))^{1/2} = \sum \binom{2k}{k}\left(\frac{1-z}{4}\right)^k$ when $z \in V = \{z \in \mathbb{C} : |z-1| < 1\}$. There is a unique extension of f to $\tilde{f}_1 : U_1 \to \mathbb{C}$, where $U_1 = \mathbb{C} - \{z \in \mathbb{R} : z \leq 0\}$. There is also a unique extension to $\tilde{f}_2 : U_2 \to \mathbb{C}$, where $U_2 = \mathbb{C} - \{z \in i\mathbb{R} : \mathrm{Im}(z) \leq$
But although $\tilde{f}_1 | V = \tilde{f}_2 | V$ we find $\tilde{f}_1 | U_1 \cap U_2 \neq \tilde{f}_2 | U_1 \cap U_2$, for $\tilde{f}_1(z) = -\tilde{f}_2(z)$ when $\mathrm{Re}(z) < 0$ and $\mathrm{Im}(z) < 0$.

(v) If $p(z) = a_0 z^3 + a_1 z^2 + a_2 z + a_3$ is a cubic polynomial with roots $\alpha_1, \alpha_2, \alpha_3$, we can define

$$f(z) = \int_\infty^z \frac{d\zeta}{\sqrt{p(\zeta)}} = z^{1/2} \sum_{n\geq 1} \frac{2b_n}{1-2n} z^{-n} \quad \text{for } \begin{cases} |z| > \sup|\alpha_i|, \text{ where} \\ \mathrm{Re}(z) > 0 \end{cases}$$

$\sum b_n \zeta^{-n-\frac{1}{2}} = p(\zeta)^{-1/2} = \zeta^{-3/2}(a_0 + a_1\zeta^{-1} + a_2\zeta^{-2} + a_3\zeta^{-3})^{-1/2}$. This is an inverse to the Weierstrass elliptic function which satisfies $\wp'(z)^2 = p(\wp(z))$, for if $z = \wp(w)$ then $\frac{dw}{dz} = \frac{1}{\wp'(w)} = \frac{1}{\sqrt{p(z)}}$.

We can extend the function f to a <u>multivalued</u> function on $\mathbb{C} - \{\alpha_1, \alpha_2, \alpha_3\}$ by defining

$$f(z) = \int_{\gamma_z} \frac{d\zeta}{\sqrt{p(\zeta)}},$$

where γ_z is a path in $\mathbb{C} - \{\alpha_1, \alpha_2, \alpha_3\}$ from ∞ to z. The value of $f(z)$ depends on the choice of the homotopy class of the path γ_z. But it is not hard to see that for given z the possible values of $f(z)$ differ by elements of the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, where $\omega_i = \int_{\pi_i} d\xi/\sqrt{p(\xi)}$, for π_1 a closed path enclosing α_1 and α_2 but not α_3, and π_2 a closed path enclosing α_2 and α_3 but not α_1.

(v) A function whose analytic continuation is multivalued in a more complicated way is the <u>dilogarithm</u>, defined for $|z| < 1$ by

$$f(z) = \sum_{n \geq 1} \frac{z^n}{n^2} = -\int_0^z \log(1-\zeta)\frac{d\zeta}{\zeta}.$$

Again we can define $f(z)$ for $z \in \mathbb{C} - \{0, 1\}$ by choosing a path from 0 to z in $\mathbb{C} - \{0, 1\}$, but the effect of changing the homotopy class of the path is more complicated to describe.

It hardly needs emphasizing how inevitably one is led to "multivalued" analytic functions in complex analysis. It is also natural to feel that, say, the two branches of $z^{1/2}$ defined in Ex. (iv) above are both part of the "same" analytic function. One can make sense of this by the following mathematical technology.

<u>Definition</u> If $f_1 : U_1 \to Y$, $f_2 : U_2 \to Y$ are functions, where U_1 and U_2 are open subsets of some topological space X, and $z \in U_1 \cap U_2$, one says that f_1 and f_2 <u>have the same germ</u> at z if $f_1|V = f_2|V$ for

some neighbourhood V of z.

This defines an equivalence relation on the functions defined near z. An equivalence class is called a <u>germ</u> (of a map $X \to Y$ at z).

Now let us define a huge Riemann surface Σ. Let $\{f_\alpha : V_\alpha \to \mathbb{C}\}_{\alpha \in A}$ be the collection of all pairs (V_α, f_α) where V_α is an open subset of $\mathbb{C}$ and f_α is analytic. Consider the set of all pairs $\{(z, \alpha) \in \mathbb{C} \times A : z \in V_\alpha\}$. Introduce an equivalence relation:

$$(z, \alpha) \sim (z', \alpha') \quad \Longleftrightarrow \quad z = z' \text{ and } f_\alpha \text{ and } f_{\alpha'} \text{ have the same germ at } z.$$

Let Σ be the set of equivalence classes. Thus Σ is the set of <u>all</u> germs of analytic maps $\mathbb{C} \to \mathbb{C}$ at all points of $\mathbb{C}$. There is an obvious map $p : \Sigma \to \mathbb{C}$ defined by $p(z, \alpha) = z$, and for each α there is an injective map $i_\alpha : V_\alpha \to \Sigma$ taking z to (z, α).

The sets $U_\alpha = i_\alpha(V_\alpha)$ cover Σ, and $\varphi_\alpha = i_\alpha^{-1} : U_\alpha \xrightarrow{\cong} V_\alpha \subset \mathbb{C}$ is a chart. We have

$$\varphi_\alpha(U_\alpha \cap U_\beta) = \varphi_\beta(U_\alpha \cap U_\beta) = V_{\alpha\beta} = \{z \in V_\alpha \cap V_\beta : f_\alpha \text{ and } f_\beta \text{ have the same germ at } z\}.$$

Clearly $V_{\alpha\beta}$ is open in $\mathbb{C}$.

Thus Σ is a 1-dimensional complex manifold. It is Hausdorff, because two analytic functions with different germs at z have different germs at all points in a neighbourhood of z. (This is <u>not</u> true for continuous or smooth functions — cf. x and $|x|$ near $x = 0$ — and this is the <u>only</u>, albeit crucial, point where we are using analyticity.)

There is a tautological analytic map $f : \Sigma \to \mathbb{C}$ defined by $f(z, \alpha) = f_\alpha(z)$. If $\{\Sigma_i\}$ are the connected components of Σ it is natural to say that Σ_i (together with the tautological map $f : \Sigma_i \to \mathbb{C}$) is a "complete multivalued analytic function": for it consists of all germs which can be obtained by moving continuously from one given germ. If one wants to think of the function as a multivalued function defined on the open subset $p(\Sigma_i) \subset \mathbb{C}$, then it has one value at z for each α such that $z \in U_\alpha \subset \Sigma_i$.

<u>Final remarks about the Weierstrass elliptic function</u>

Given a cubic polynomial $p(z)$ with distinct roots $\alpha_1, \alpha_2, \alpha_3$ we have defined an analytic function $f : \mathbb{C} - \{\alpha_1, \alpha_2, \alpha_3\} \longrightarrow \mathbb{C}/L$ inverse to $\wp_L : \mathbb{C}/L \longrightarrow \mathbb{C} \cup \{\infty\}$, where L is a lattice associated to the polynomial p. In fact L consists of all integrals $\int_C dz/\sqrt{p(z)}$ around all closed contours C in $\mathbb{C} - \{\alpha_1, \alpha_2, \alpha_3\}$ on which $\sqrt{p(z)}$ is continuous (and takes the same value at beginning and end).

The basis elements ω_1 and ω_2 of the lattice L (associated to the contours π_1 and π_2 on page 14) are analytic functions of the roots $\alpha_1, \alpha_2, \alpha_3$, at least locally. (They are multivalued, because to define ω_1 we must <u>choose</u> π_1.) Multiplying the roots α_i by c^2 replaces L by $c^{-1}L$, and replacing $\{\alpha_1, \alpha_2, \alpha_3\}$ by $\{0, \alpha_2 - \alpha_1, \alpha_3 - \alpha_1\}$ does not change L at all.

Observing that the cross-ratio of $(\infty, 0, 1, \kappa)$ is κ, we see that for each κ we can find c^2 uniquely so that the lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ corresponding to the cubic polynomial $p(z) = z(z - c^2)(z - \kappa c^2)$, is of the form $L = \mathbb{Z} + \tau\mathbb{Z}$ with $\operatorname{Im}(\tau) > 0$. This defines τ locally as an analytic function of κ, and implies that the map $k : UHP \rightarrow \mathbb{C} - \{0, 1\}$ of page 12 is a covering map.

Part 3   Differential forms

If U is an open set in $\mathbb{R}^3$ one has

$$\Omega^0(U) \xrightarrow[\text{grad}]{} \Omega^1(U) \xrightarrow[\text{curl}]{} \Omega^2(U) \xrightarrow[\text{div}]{} \Omega^3(U),$$

where $\Omega^0(U) = \Omega^3(U) = \{\text{smooth maps } U \to \mathbb{R}\}$
and $\Omega^1(U) = \Omega^2(U) = \{\text{smooth maps } U \to \mathbb{R}^3\}$

We have $\text{curl} \circ \text{grad} = 0$ and $\text{div} \circ \text{curl} = 0$, and locally the converses are also true, i.e. a vector field is a gradient if its curl vanishes, and it is a curl if its divergence vanishes. The failure of this to be true globally measures the topology of U.

Stokes's Theorem asserts that if $v \in \Omega^1(U)$ and Σ is a piece of surface in U with boundary curve $\partial\Sigma$, then $\int_\Sigma (\text{curl}\, v).dS = \int_{\partial\Sigma} v.ds$.
Similarly, if $v \in \Omega^2(U)$ and R is a region in U with boundary surface ∂R then

$$\int_R \text{div}\, v \, dV = \int_{\partial R} v.dS,$$

and if $f \in \Omega^0(U)$ and γ is a curve in U with ends $\partial\gamma = P_1 - P_0$ then

$$\int_\gamma \text{grad} f . ds = \int_{\partial\gamma} f = f(P_1) - f(P_0).$$

Finally, notice that although $\Omega^0(U) = \Omega^3(U)$, elements of $\Omega^3(U)$ should be thought of as densities, i.e. quantities "per unit volume", while . difference between $\Omega^1(U)$ and $\Omega^2(U)$ is that the values in the latter case are "axial" vectors, characterized by the fact that to define their direction one needs to use the orientation of $\mathbb{R}^3$ — i.e. the right-handed rule.

Concerning the topology of U, we have

(a) ker (grad) is the vector space whose basis is the set of connected components of U;

(b) an element $v \in$ ker (curl) defines an invariant $I_v(\gamma) = \int_\gamma v.ds$ for closed curves γ in U, and $I_v(\gamma) = I_v(\tilde{\gamma})$ if γ and $\tilde{\gamma}$ together form the boundary of a piece of surface in U; Furthermore $I_{v_1}(\gamma) = I_{v_2}(\gamma)$ for all γ if and only if v_1 and v_2 differ by grad f for some $f \in \Omega^0(U)$;

(c) similarly, an element $v \in$ ker (div) defines an invariant $J_v(\Sigma) = \int_\Sigma v.dS$

for closed surfaces Σ in U, with analogous properties to those in (b).

The object of the following part of the course is to generalise the preceding discussion, first to open subsets of $\mathbb{R}^n$, then to general smooth manifolds.

<u>Algebra</u>

Let V be an n-dimensional real vector space.
Let $\mathrm{Alt}^k(V)$ denote the alternating multilinear maps $\alpha : \underbrace{V \times \dots \times V}_{k} \to \mathbb{R}$.

<u>Multilinear</u> means that $v_i \longmapsto \alpha(v_1, \dots, v_k)$ is a linear map $V \to \mathbb{R}$ when $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$ are held fixed.

<u>Alternating</u> means that $\alpha(v_{\pi(1)}, \dots, v_{\pi(k)}) = (-1)^\pi \alpha(v_1, \dots, v_k)$ for any permutation $\pi : \{1, \dots, k\} \to \{1, \dots, k\}$, where $(-1)^\pi$ is the sign of π.

$\mathrm{Alt}^k(V)$ is a real vector space.

If $\{e_1, \dots, e_n\}$ is any basis of V then an element $\alpha \in \mathrm{Alt}^k(V)$ is clearly determined completely by giving the numbers $\alpha(e_{i_1}, \dots, e_{i_k}) = \alpha_{i_1 \dots i_k}$ when $i_1 < i_2 < \dots < i_k$. Clearly also these numbers can be prescribed arbitrarily. Thus $\dim(\mathrm{Alt}^k(V)) = \binom{n}{k}$.

We define multiplication $\mathrm{Alt}^k(V) \times \mathrm{Alt}^m(V) \longrightarrow \mathrm{Alt}^{k+m}(V)$ by $(\alpha, \beta) \longmapsto \alpha \wedge \beta$, where

$$(\alpha \wedge \beta)(v_1, \dots, v_{k+m}) = \sum_{\pi \in Sh_{k,m}} (-1)^\pi \alpha(v_{\pi(1)}, \dots, v_{\pi(k)}) \beta(v_{\pi(k+1)}, \dots, v_{\pi(k+m)})$$
$$= \frac{1}{k!\,m!} \sum (-1)^\pi (\text{same}).$$

Here $Sh_{k,m}$ is the set of <u>(k,m)-shuffles</u>, i.e. of permutations π of $\{1, \dots, k+m\}$ such that $\pi(1) < \pi(2) < \dots < \pi(k)$ and $\pi(k+1) < \pi(k+2) < \dots < \pi(k+m)$.
In the second expression for $\alpha \wedge \beta$ we sum over <u>all</u> permutations of $\{1, \dots, k+m\}$.
From the second expression we see that $\alpha \wedge \beta$ really is alternating.

The product is <u>bilinear</u>, <u>associative</u>, and <u>anticommutative</u> in the sense that

$$\beta \wedge \alpha = (-1)^{km} \alpha \wedge \beta.$$

The proof of associativity amounts to the fact that $Sh_{k,m} \times Sh_{k+m,l}$ and

$Sh_{m,l} \times Sh_{k,m+l}$ can both be identified with the set of all (k,m,l)-shuffles, in an obvious sense. For the anticommutativity, the obvious bijection $Sh_{k,m} \to Sh_{m,k}$ takes π to $\tilde{\pi}$, where $(-1)^{\tilde{\pi}} = (-1)^{km}(-1)^{\pi}$.

Examples

(i) If $\alpha, \beta \in Alt^1(V)$ then $(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1)$.

(ii) If $\alpha_1, \ldots, \alpha_k \in Alt^1(V)$ then

$$(\alpha_1 \wedge \ldots \wedge \alpha_k)(v_1, \ldots, v_k) = \det(\alpha_i(v_j)).$$

(iii) If $\{e_1, \ldots, e_n\}$ is a basis of V, and $\{\alpha_1, \ldots, \alpha_n\}$ is the dual basis of $V^* = Alt^1(V)$ (i.e. $\alpha_i(e_j) = \delta_{ij}$) then $\{\alpha_{i_1} \wedge \ldots \wedge \alpha_{i_k}\}$ for $i_1 < \ldots < i_k$ is the obvious basis of $Alt^k(V)$.

Differential forms in $\mathbb{R}^n$

If U is open in $\mathbb{R}^n$ we define $\Omega^k(U) = \{$smooth maps $U \to Alt^k(\mathbb{R}^n)\}$.
We define $d : \Omega^0(U) \longrightarrow \Omega^1(U)$ by $(df)(x; v) = Df(x)v \in \mathbb{R}$.
If $x^1, \ldots, x^n \in \Omega^0(U)$ are the coordinate functions, then $dx^1, \ldots, dx^n$ are constant elements of $Alt^1(\mathbb{R}^n)$ — in fact they are the natural basis of $(\mathbb{R}^n)^*$.

For any $f \in \Omega^0(U)$ we have $df = \sum D_i f . dx^i$, where $D_i = \frac{\partial}{\partial x_i}$ is the partial derivative. Thus $d : \Omega^0 \to \Omega^1$ is the usual gradient.

If $I = (i_1, \ldots, i_k)$ is any sequence, write $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k} \in \Omega^k(U)$.
efine $d : \Omega^k(U) \longrightarrow \Omega^{k+1}(U)$ by $d(f\,dx^I) = df \wedge dx^I$ for $f \in \Omega^0(U)$ and $i_1 < \ldots < i_k$. This is a valid definition, as any element of $\Omega^k(U)$ is uniquely a sum of terms of the form $f\,dx^I$.

It is immediate that $d(f\,dx^I) = df \wedge dx^I$ for any sequence $I = (i_1, \ldots, i_k)$.
Further, d is an antiderivation, i.e.

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta \quad \text{if } \alpha \in \Omega^k.$$

Proposition $d \circ d = 0$

Proof

$$\begin{aligned} d(d(f\,dx^I)) &= d(df \wedge dx^I) = d^2 f \wedge dx^I - df \wedge d(dx^I) \\ &= d^2 f \wedge dx^I. \end{aligned}$$

So it is enough to show that $d^2 f = 0$ when $f \in \Omega^0(U)$.
But $d^2 f = d(\sum D_i f . dx^i) = \sum D_j D_i f . dx^j \wedge dx^i = 0$, because $D_j D_i f = D_i D_j f$, while $dx^j \wedge dx^i = -dx^i \wedge dx^j$.

The operation $d : \Omega^1(U) \longrightarrow \Omega^2(U)$ takes $\sum f_i \, dx^i$ to $\sum_{i<j} (D_i f_j - D_j f_i) \, dx^i \wedge dx^j$, so it can be thought of as "curl".

The operation $d : \Omega^{n-1}(U) \longrightarrow \Omega^n(U)$ takes $\sum (-1)^i g_i \, dx^{I_i}$, where $I_i = (1, \ldots, i-1, i+1, \ldots, n)$ to $(\sum D_i g_i) \, dx^1 \wedge \ldots \wedge dx^n$, and can be regarded as "div".

Maxwell's equations If we write (t, x^1, x^2, x^3) for the coordinates in $\mathbb{R}^4$, an arbitrary element F of $\Omega^2(\mathbb{R}^4)$ can be written

$$F = dt \wedge E + H,$$

with $E = \sum E_i \, dx^i$ and $H = H_1 \, dx^2 \wedge dx^3 + H_2 \, dx^3 \wedge dx^1 + H_3 \, dx^1 \wedge dx^2$.
If $\mathrm{curl}\, E = (D_2 E_3 - D_3 E_2) \, dx^2 \wedge dx^3$ + cyclic permutations, then

$$dF = - dt \wedge \mathrm{curl}\, E + dt \wedge \dot{H} + (\mathrm{div}\, H) \, dx^1 \wedge dx^2 \wedge dx^3,$$

where $\dot{H} = \frac{\partial}{\partial t} H$. Thus

$$dF = 0 \iff \dot{H} = \mathrm{curl}\, E \quad \text{and} \quad \mathrm{div}\, H = 0 .$$

These are half of Maxwell's equations for the electromagnetic field. The other half can be written

$$d(* F) = J,$$

where $J \in \Omega^3(U)$ is $\rho \, dx^1 \wedge dx^2 \wedge dx^3 + dt \wedge J_1 \, dx^2 \wedge dx^3 + J_2 \, dx^3 \wedge dx^1 + J_3 \, dx^1 \wedge dx^2$ (here ρ = charge density, and $J_1 \, dx^2 \wedge dx^3 + \ldots$ = current density), and $* : \Omega^2(U) \to \Omega^2(U)$ is the linear map defined by $*(dt \wedge dx^i) = dx^j \wedge dx^k$ when $*$ is an cyclic permutation of $(1, 2, 3)$, and $*^2 = 1$.

The electromagnetic potential $\alpha \in \Omega^1(\mathbb{R}^4)$ is $\alpha = \varphi \, dt + \sum A_i \, dx^i$, and satisfies $d\alpha = F$, i.e. $E = \dot{A} - \mathrm{grad}\, \varphi$, $H = \mathrm{curl}\, A$.

Pulling back forms in Euclidean space

If $\varphi : U \to V$ is a smooth map, with U open in $\mathbb{R}^n$ and V open in $\mathbb{R}^m$, we define $\varphi^* : \Omega^k(V) \longrightarrow \Omega^k(U)$ by

$$(\varphi^* \alpha)(x ; v_1, \ldots, v_k) = \alpha(\varphi(x) ; D\varphi(x) v_1, \ldots, D\varphi(x) v_k).$$

The operation φ^* is clearly a ring-homomorphism (i.e. $\varphi^*(\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta$), and satisfies $\varphi^* \circ d = d \circ \varphi^*$.

Proof Because φ^* is a ring-homomorphism and $(\varphi^* \circ d - d \circ \varphi^*)$ commutes

with d it is enough to prove that $\varphi^* df = d\varphi^* f$ for $f \in \Omega^0(U)$.
But $(\varphi^* df)(x; v) = df(\varphi(x); D\varphi(x) v) = Df(\varphi(x)) D\varphi(x) v = D(f \circ \varphi)(x) v$
$= d(\varphi^* f)(x; v)$.

<u>Remark</u> If $\varphi : U \to V$ is the inclusion of a submanifold, then $\varphi^* \alpha$ is just the <u>restriction</u> $\alpha | U$.

<u>The Poincaré lemma</u>

<u>Proposition</u> If U is an open ball in $\mathbb{R}^n$, and $\alpha \in \Omega^k(U)$ is closed, (i.e. $d\alpha = 0$), for some $k > 0$, then $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(U)$.

This follows from a stronger result. Let us consider the open set $\mathbb{R} \times U$ in $\mathbb{R}^{n+1}$. Write $(t, x^1, \ldots, x^n)$ for the coordinates on $\mathbb{R}^{n+1}$. For each $t \in \mathbb{R}$ we have the inclusion $i_t : U \to \mathbb{R} \times U$ taking x to (t, x). Write $\alpha \mapsto \alpha_t = i_t^* \alpha$ for the corresponding restriction $\Omega^k(\mathbb{R} \times U) \to \Omega^k(U)$.

<u>Proposition</u> If $\alpha \in \Omega^k(\mathbb{R} \times U)$ is closed then $\alpha_1 - \alpha_0 = d\eta$ for some $\eta \in \Omega^{k-1}(U)$.

To deduce the preceding result from this, choose a smooth function $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(t) = 0$ when $t \le 0$ and $\theta(t) = 1$ when $t \ge 1$. Define $f : \mathbb{R} \times U \to U$ by $f(t, x) = \theta(t) x$. (I am assuming $U = \{x \in \mathbb{R}^n : \|x\| < R\}$.) Then, if $\alpha \in \Omega^k(U)$ is closed, apply the second result to $f^* \alpha$, noting that $d(f^* \alpha) = f^* d\alpha = 0$. The map $f \circ i_1$ is the identity, and $f \circ i_0$ is constant. So $(f^* \alpha)_1 = \alpha$, and $(f^* \alpha)_0 = 0$, and we have $\alpha = d\eta$ for some η.

To prove the second proposition, notice that any form on $\mathbb{R} \times U$ can be written uniquely $\alpha = \beta + dt \wedge \gamma$, where β and γ involve the dx^i but not dt. We can think of β as a family of elements of $\Omega^k(U)$ depending on t as a parameter: i.e. $\beta = \{\beta_t\}_{t \in \mathbb{R}}$, where

$\beta_t = i_t^* \beta$. Similarly $\gamma = \{\gamma_t\}$, with $\gamma_t \in \Omega^{k-1}(U)$.

We have $d\alpha = d(\beta + dt \wedge \gamma) = dt \wedge \left(\frac{\partial}{\partial t}\beta - d_U\gamma\right) + d_U\beta$, where $d_U\beta$ and $d_U\gamma$ are the parts of $d\beta$ and $d\gamma$ which do not involve dt. Thus $(d_U\gamma)_t = d(\gamma_t) \in \Omega^k(U)$.

If $d\alpha = 0$, we find $\frac{\partial}{\partial t}\beta_t = d\gamma_t$ in $\Omega^k(U)$, and so $\beta_1 - \beta_0 = d\eta$, where $\eta = \int_0^1 \gamma_t\, dt$. (The notation means that if $\gamma_t = \sum \gamma_{t,I}\, dx^I$ then $\eta = \sum \eta_I\, dx^I$, where $\eta_I = \int_0^1 \gamma_{t,I}\, dt$.) This is what we want, as $\alpha_t = \beta_t$ (because $i_t^*(dt) = 0$). //

Differential forms on a smooth manifold

Let X be a smooth n-dimensional manifold. (All manifolds will be assumed to be Hausdorff.)

An element of $\Omega^k(X)$ is a function α which associates to each point $x \in X$ an element $\alpha(x) \in Alt^k(T_xX)$. This is not quite an adequate definition, however; for we want α to be a smooth function in a sense that must be explained. There are very many ways of doing this. I shall adopt the most down-to-each, which is

Definition Let $\{\varphi_i : U_i \to V_i \subset \mathbb{R}^n\}_{i\in A}$ be the atlas of X. Then an element $\alpha \in \Omega^k(X)$ is a family of elements $\alpha_i \in \Omega^k(V_i)$ which are compatible in the sense that

$$\alpha_i | V_{ij} = (\varphi_j \circ \varphi_i^{-1})^* \, (\alpha_j | V_{ji})$$

for all $i, j \in A$, where $V_{ij} = V_i \cap \varphi_i(U_j)$.

We define multiplication of differential forms by $(\alpha\wedge\beta)_i = \alpha_i \wedge \beta_i$, and define $d : \Omega^k(X) \to \Omega^{k+1}(X)$ by $(d\alpha)_i = d(\alpha_i)$. Then the vector spaces $\Omega^k(X)$ form an anticommutative graded ring, and d is an antiderivation satisfying $d \circ d = 0$.

It is trivial (if cumbersome) to see that one does not need to give the

α_i for all charts, but just for a family which covers X. In the light of this we can define the pull-back operation $f^* : \Omega^k(Y) \to \Omega^k(X)$ when $f : X \to Y$ is a smooth map. If $\beta \in \Omega^k(Y)$ is represented by $\{\beta_j\}$ with respect to an atlas $\{\tilde{\varphi}_j : \tilde{U}_j \to \mathbb{R}^m\}$ for Y, we can find an atlas $\{\varphi_i : U_i \to \mathbb{R}^n\}$ for X such that $f(U_i) \subset \tilde{U}_j$ for some $j = j(i)$. Then we define
$$(f^*\beta)_i = (\tilde{\varphi}_j \circ f \circ \varphi_i^{-1})^* \beta_j .$$

As before, f^* is a ring-homomorphism, and $d \circ f^* = f^* \circ d$.

Manifolds with boundary

The definition of a smooth manifold with boundary is exactly analogous to that of a smooth manifold. It consists of a set X together with an atlas $\{\varphi_i : U_i \to V_i\}_{i \in A}$. The only new point is that φ_i is a bijection between U_i and an open subset of $\mathbb{R}^n_- = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}$.

The transition functions $\varphi_j \circ \varphi_i^{-1}$ are diffeomorphisms as before, a diffeomorphism between open subsets of $\mathbb{R}^n_-$ having, by definition, one-sided derivatives of all orders at boundary points.

The boundary ∂X of X is the subset of all points which by some chart (and hence by all charts) are mapped to boundary points of $\mathbb{R}^n_-$. (A diffeomorphism between open subsets of $\mathbb{R}^n_-$ takes boundary points to boundary points, for if a smooth real-valued function f has a local maximum at an interior point y then $Df(y) = 0$; but x^1 has a local maximum at every boundary point, but Dx^1 does not vanish.)

Thus ∂X is a smooth manifold (without boundary) of dimension $n-1$.

Recall that a manifold is oriented if an oriented atlas is given, i.e. an atlas such that, for all transition maps $\varphi_j \circ \varphi_i^{-1}$ we have $\det D(\varphi_j \circ \varphi_i^{-1}) > 0$ everywhere. This makes sense for manifolds with boundary, and furthermore an orientation of X induces an orientation of ∂X.

The definition of differential forms, and their standard properties, apply without change for manifolds with boundary.

Orientation and differential forms

Proposition A smooth manifold X is orientable if and only if there is a nowhere-vanishing element $\omega \in \Omega^n(X)$, where $n = \dim(X)$.

Proof (i) If ω is given we take a chart $\varphi_i : U_i \to V_i$ to be part of a maximal oriented atlas if the representative ω_i of ω in $\Omega^n(V_i)$ is of the form $\omega_i = f\,dx^1 \wedge \ldots \wedge dx^n$ with $f > 0$ throughout V_i.

(ii) If $\{\varphi_i : U_i \to V_i\}$ is an oriented atlas, choose a partition of unity $\{f_i\}$ subordinate to $\{U_i\}$, where f_i has compact support inside U_i. Let $\tilde{\omega}_i$ be the element of $\Omega^n(U_i)$ which is represented by $dx^1 \wedge \ldots \wedge dx^n$ in $\Omega^n(V_i)$. Then $f_i \tilde{\omega}_i$ is a well-defined element of $\Omega^n(X)$, and we can take $\omega = \Sigma f_i \tilde{\omega}_i$.

Example Real projective space $\mathbb{P}^n$ is not orientable if n is even. For consider the 2-to-1 map $\pi : S^n \to \mathbb{P}^n$. Let $\omega_0 \in \Omega^n(S^n)$ be the nowhere-vanishing volume form. If $i : S^n \to S^n$ is $x \mapsto -x$ we have $i^*\omega_0 = (-1)^{n+1}\omega_0$. If $\omega \in \Omega^n(\mathbb{P}^n)$ we can write $\pi^*\omega = f\omega_0$ for some $f \in \Omega^0(S^n)$. If ω never vanishes, neither does f. But $\pi \circ i = \pi$, so

$$f\omega_0 = \pi^*\omega = i^*\pi^*\omega = (i^*f)(i^*\omega_0) = (-1)^{n+1}(i^*f)\,\omega_0 .$$

So $i^*f = (-1)^{n+1} f$, i.e. $f(-x) = (-1)^{n+1} f(x)$. If $f(x)$ is never zero this contradicts the intermediate value theorem when n is even.

We can also see that $\mathbb{P}^n$ *is* orientable when n is odd. For each chart for $\mathbb{P}^n$ gives two charts for S^n, related by $i : S^n \to S^n$. The representatives of ω_0 in both charts for S^n are the *same* if n is odd, so they can be taken as representatives of a nowhere-vanishing element $\omega_0 \in \Omega^n(\mathbb{P}^n)$.

Integration of differential forms

If U is an open subset of $\mathbb{R}^n$, and $\alpha = f\,dx^1 \wedge \ldots \wedge dx^n$ is an element of $\Omega^n(U)$ with compact support inside U then we define $\int_U \alpha$ in the obvious way.

If $\varphi : V \to U$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, recall that the standard change-of-variable formula is

$$\int_V f(\varphi(x))\,|\det D\varphi(x)|\,dx^1\wedge\ldots\wedge dx^n = \int_U f(x)\,dx^1\wedge\ldots\wedge dx^n.$$

Now $\varphi^*(dx^i) = d\varphi^i$, where $\varphi^i = x^i\circ\varphi = \varphi^*(x^i)$. So $\varphi^*(dx^i) = \sum_j (D_j\varphi^i)\,dx^j$,

and $$\varphi^*(dx^1\wedge\ldots\wedge dx^n) = \det(D\varphi(x))\,dx^1\wedge\ldots\wedge dx^n$$

i.e. $$(\varphi^*\alpha)(x) = f(\varphi(x))\,\det(D\varphi(x))\,dx^1\wedge\ldots\wedge dx^n.$$

Thus

<u>Proposition</u> $\int_V \varphi^*\alpha = \pm\int_U \alpha$, with the sign $\pm$ according as φ preserves orientation or not.

<u>Definition</u> If X is a smooth <u>oriented</u> manifold with boundary, and $\alpha\in\Omega^n(X)$ has compact support, we define

$$\int_X \alpha = \sum_i \int_{V_i} (f_i\alpha)_i\,,$$

where the sum is over the charts of an oriented atlas $\{\varphi_i : U_i \to V_i \subset \mathbb{R}^n\}$, and $\{f_i : X\to\mathbb{R}\}$ is a partition of unity subordinate to $\{U_i\}$.

The definition does not depend on the atlas or choice of $\{f_i\}$, for if $\{\tilde\varphi_j : \tilde U_j \to \tilde V_j\}$ and $\{\tilde f_j\}$ are another choice, then

$$\sum_i \int_{V_i} (f_i\alpha)_i = \sum_{i,j} \int_{V_i} (f_i\tilde f_j\alpha)_i\,,$$

and $\int_{V_i} (f_i\tilde f_j\alpha)_i = \int_{\tilde V_j} (f_i\tilde f_j\alpha)_j$ by the proposition above because the charts $U_i\to V_i$ and $\tilde U_j \to \tilde V_j$ are compatible.

<u>Stokes's theorem</u>

<u>Proposition</u> If X is an oriented n-dimensional manifold with boundary, and $\beta\in\Omega^{n-1}(X)$ has compact support, then

$$\int_X d\beta = \int_{\partial X} (\beta|\partial X).$$

<u>Proof</u> It is enough to prove this when $X = \mathbb{R}^n_-$. In that case we can write $\beta = \sum(-1)^{i-1} g_i\,dx^{I_i}$, with $I_i = (1,2,\ldots,i-1,i+1,\ldots n)$.

Then $d\beta = (\sum D_i g_i)\, dx^1 \wedge \dots \wedge dx^n$, and $\beta | \partial X = g_1(0, x^2, \dots, x^n)\, dx^2 \wedge \dots \wedge dx^n$, and the proposition holds because

$$\int_{\mathbb{R}} (D_i g_i)\, dx^i = 0 \quad \text{if } i > 1, \quad \text{while} \quad \int_{-\infty}^{0} (D_1 g_1)\, dx^1 = g_1(0, x^2, \dots, x^n).$$

Corollary If X is a smooth manifold, $\beta \in \Omega^{k-1}(X)$, and Y is a k-dimensional compact oriented manifold with boundary, and $f : Y \to X$ is smooth, then

$$\int_Y f^*(d\beta) = \int_{\partial Y} f^*\beta .$$

Cohomology

If X is a smooth manifold, we say $\alpha \in \Omega^k(X)$ is closed if $d\alpha = 0$, and is exact if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(X)$. The quotient

$$H^k(X) = \{\text{closed } k\text{-forms on } X\} / \{\text{exact } k\text{-forms on } X\}$$

is called the k^{th} (de Rham) cohomology group of X. (Actually it is a vector space.)

We have seen that $H^0(X)$ is the space of locally constant functions on X, so that $\dim(H^0(X)) = $ (number of connected components of X).

An element of $H^k(X)$ represented by a closed form α defines an · riant $I_\alpha(f) = \int_Y f^*\alpha$ for k-dimensional cycles $f : Y \to X$ in X, where a k-cycle means a smooth map f from a compact oriented k-dimensional manifold Y without boundary. By Stokes's theorem the invariant depends only on the class of α in $H^k(X)$, for $I_{\alpha + d\beta}(f) = I_\alpha(f) + I_{d\beta}(f)$, and $I_{d\beta}(f) = \int_Y f^*(d\beta) = \int_{\partial Y} f^*\beta = 0$, as $\partial Y = \emptyset$.

Furthermore, $I_\alpha(f)$ depends only on the homology class of the cycle: we say that two k-cycles $f_0 : Y_0 \to X$ and $f_1 : Y_1 \to X$ are homologous if f_0 and f_1 are the restrictions of $f : Y \to X$, where Y is a $(k+1)$-dimensional compact oriented manifold with boundary, whose boundary ∂Y is the disjoint union $\tilde{Y}_0 \amalg Y_1$, where $\tilde{Y}_0$ denotes Y_0 with reversed orientation.

The Poincaré lemma showed that $H^k(X) = 0$ when $k > 0$ if X is a ball in $\mathbb{R}^n$ (or if $X = \mathbb{R}^n$).

A smooth map $\varphi : X \to X'$ induces a homomorphism $\varphi^* : H^k(X') \to H^k(X)$ by pulling back forms ($\alpha \mapsto \varphi^*\alpha$). The proof of the result after Poincaré's lemma (on page) was written so that it applies without change to prove the following "homotopy invariance" property.

<u>Proposition</u> If $\varphi : \mathbb{R} \times X \to X'$ is smooth, and $\varphi_t(x) = \varphi(t,x)$, then
$$\varphi_0^* = \varphi_1^* : H^k(X') \to H^k(X).$$

A typical application of this is

<u>Proposition</u> If n is odd there is no nowhere-vanishing smooth tangent vector field on the sphere S^{n-1}.

<u>Proof</u> Regard S^{n-1} as $\{x \in \mathbb{R}^n : \|x\| = 1\}$. A tangent vector field is a smooth map $v : S^{n-1} \to \mathbb{R}^n$ such that $\langle x, v(x)\rangle = 0$ for all x. If v never vanishes, we can assume $\|v(x)\| = 1$ for all x. Then define $\varphi_t : S^{n-1} \to S^{n-1}$ by $\varphi_t(x) = (\cos t\pi)\, x + (\sin t\pi)\, v(x) \in \mathbb{R}^n$. Thus $\varphi_0 =$ identity, and φ_1 is the antipodal map $x \mapsto -x$. It is easy to check that φ_1^* is multiplication by $(-1)^n$ (see next section), so we have a contradiction if n is odd.

<u>Cohomology and the degree of maps</u>

<u>Proposition 1</u> If X is a compact connected oriented n-dimensional manifold, then $H^n(X) \cong \mathbb{R}$ by the map $\alpha \mapsto \int_X \alpha$.

To prove this it is helpful to introduce the notion of <u>cohomology with compact supports</u>. For any manifold X, let $\Omega^k_{cpt}(X)$ denote the elements of $\Omega^k(X)$ which have compact support. We define
$$H^k_{cpt}(X) = \{\alpha \in \Omega^k_{cpt}(X) : d\alpha = 0\} / \{\alpha : \alpha = d\beta \text{ for some } \beta \in \Omega^{k-1}_{cpt}(X)\}.$$

Proposition 1 can be generalized to

<u>Proposition 2</u> If X is a connected oriented manifold which is the union of a finite number of open subsets each diffeomorphic to an open ball in $\mathbb{R}^n$, then $H^n_{cpt}(X) \cong \mathbb{R}$ by $\alpha \mapsto \int_X \alpha$.

This is proved by induction on the number of balls covering X, starting with

<u>Proposition 3</u> $H^n_{cpt}(\mathbb{R}^n) \cong \mathbb{R}$ by $\alpha \mapsto \int_{\mathbb{R}^n} \alpha$.

<u>Proof</u> This, in turn, is by induction on n. We must show that if $\int \alpha = 0$ then $\alpha = d\beta$ for some $\beta \in \Omega^{n-1}_{cpt}(\mathbb{R}^n)$.

If $n=1$, and $\alpha \in \Omega^1_{cpt}(\mathbb{R})$, we define $f \in \Omega^0(X)$ by $f(x) = \int_{-\infty}^{x} \alpha(t)\,dt$. Then $df = \alpha$, and f has compact support iff $\int_{\mathbb{R}} \alpha = 0$.

If $n > 1$, and $\alpha = f\,dx^1 \wedge \ldots \wedge dx^n \in \Omega^n_{cpt}(\mathbb{R}^n)$, define $\beta \in \Omega^{n-1}_{cpt}(\mathbb{R}^n)$ by $\beta = g\,dx^1 \wedge \ldots \wedge dx^{n-1}$, where

$$g(x^1, \ldots, x^{n-1}) = \int_{\mathbb{R}} f(x^1, \ldots, x^n)\,dx^n.$$

If $\beta = 0$ then $\alpha = d\gamma$, where $\gamma = h\,dx^1 \wedge \ldots \wedge dx^{n-1}$, and

$$h(x^1, \ldots, x^n) = \int_{-\infty}^{x^n} f(x^1, \ldots, x^{n-1}, t)\,dt.$$

(Note that γ has compact support.) But we can make $\beta = 0$ by replacing α by $\alpha - \beta \wedge \eta$, where $\eta = \varphi(x^n)\,dx^n$ for any $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support such that $\int_{\mathbb{R}} \varphi(t)\,dt = 1$. Thus $\alpha = \beta \wedge \eta + d\gamma$, with $\gamma \in \Omega^{n-1}_{cpt}(\mathbb{R}^n)$.

Now $\int_{\mathbb{R}^n} \alpha = \int_{\mathbb{R}^{n-1}} \beta$, so if $\int \alpha = 0$ we have $\int \beta = 0$, and so, by induction, $\beta = d\xi$ with $\xi \in \Omega^{n-2}_{cpt}(\mathbb{R}^{n-1})$. Then

$$\alpha = d\xi \wedge \eta + d\gamma = d(\xi \wedge \eta + \gamma),$$

as we want.

<u>Proof of Proposition 2</u> We can suppose that $X = X_0 \cup X_1$, where the result is known for X_0, and X_1 is diffeomorphic to $\mathbb{R}^n$. We can also

assume that $X_0 \cap X_1 \neq \emptyset$, as X is connected. Given $\alpha \in \Omega^n_{cpt}(X)$ with $\int \alpha = 0$ we can use a partition of unity for $X_0 \cup X_1$ to write $\alpha = \alpha_0 + \alpha_1$ with $\alpha_i \in \Omega^n_{cpt}(X_i)$. Replacing this by $\alpha = (\alpha_0+\gamma) + (\alpha_1 - \gamma)$ where γ is a suitable "bump" form with compact support in a small ball contained in $X_0 \cap X_1$, we can assume that $\int_{X_0} \alpha_0 = \int_{X_1} \alpha_1 = 0$. But then $\alpha_0 = d\beta_0$ and $\alpha_1 = d\beta_1$, with $\beta_i \in \Omega^{n-1}_{cpt}(X_i)$, and $\alpha = d(\beta_0 + \beta_1)$. //

Now consider a smooth map $f : X \to Y$, where X and Y are compact connected oriented manifolds of dimension n. A point $y \in Y$ is called a <u>regular value</u> of f if $Df(x) : T_x X \to T_y Y$ is invertible for all $x \in f^{-1}(y)$. <u>Sard's theorem</u> (see below) tells us that almost all points of Y are regular. If $y \in Y$ is regular, the inverse-function theorems tells us that $f^{-1}(y)$ is finite, and that y is contained in an open set V such that

$$f^{-1}(V) = \coprod_{x \in f^{-1}(y)} U_x \quad , \text{ where each } U_x \text{ is open in } X \text{ and is mapped}$$

diffeomorphically to V by f (and $x \in U_x$).

If $Df(x)$ is invertible, define $\varepsilon_x = \pm 1$ according as f is orientation-preserving or -reversing at x, i.e. according as $Df(x)$ has positive or negative determinant when represented in oriented charts.

<u>Theorem</u> In the preceding situation

$$\int_X f^*\beta = N \int_Y \beta$$

for all $\beta \in \Omega^n(Y)$, where $N = \sum_{x \in f^{-1}(y)} \varepsilon_x$ for every regular value y of f.

<u>Corollary</u> The number $\sum_{x \in f^{-1}(y)} \varepsilon_x$ is the same for all regular values y.

The integer N in the statement is called the <u>degree</u> of $f : X \to Y$.

<u>Proof</u> By Proposition 1 we can replace β by any other $\tilde\beta \in \Omega^n(Y)$ such that $\int \tilde\beta = \int \beta$: in particular we can assume β has support in any

convenient small open set. Choose a regular value y, and V such that $f^{-1}(V) = \coprod U_x$ as above. Suppose $\operatorname{supp}(\beta) \subset V$. Then $f^*\beta$ is the sum of a finite number of forms with support in the U_x, and the theorems follows from the first proposition on page 9. //

Returning to Sard's theorem, first observe that it does make sense to speak of a set of measure zero on any smooth manifold. It is then enough to show that if $f : U \to \mathbb{R}^n$ is smooth, where U is the unit ball in $\mathbb{R}^n$, then $f(S)$ has measure zero, where $S = \{x \in U : \det(Df(x)) = 0\}$. But, for any $\varepsilon > 0$, each $x \in S$ is contained in a neighbourhood N_x such that ~~vol(N_x)~~

$$\operatorname{vol} f(N_x) < \varepsilon \operatorname{vol}(N_x),$$

and standard measure-theoretical arguments then show that $f(S)$ is contained in an open set of volume $< \varepsilon . \operatorname{vol}(U)$.

Appendix The definition of differential forms

Once one has understood the role they play as tools, one should put "charts" at the very back of one's mind — if not out of it entirely — when thinking of smooth manifolds. So one should certainly think of an element $\alpha \in \Omega^k(X)$ as a smooth map $x \longmapsto \alpha(x)$ with $\alpha(x) \in \operatorname{Alt}^k(T_x X)$. To make sense of this one must think of the disjoint union $\operatorname{Alt}^k(TX) = \coprod_{x \in X} \operatorname{Alt}^k(T_x X)$ as a smooth manifold, equipped with a smooth map $\pi : \operatorname{Alt}^k(TX) \to X$ taking $\operatorname{Alt}^k(T_x X)$ to x. To do this we associate to each chart $\varphi : U \to \mathbb{R}^n$ for X a chart $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^n \times \operatorname{Alt}^k(\mathbb{R}^n)$ taking $\alpha \in \operatorname{Alt}^k(T_x X)$ to $(x, \tilde{\alpha})$, where $\tilde{\alpha} \in \operatorname{Alt}^k(\mathbb{R}^n)$ is the φ-representative of α. Spelling out the definition of a smooth map $X \to \operatorname{Alt}^k(TX)$ then gives back precisely the definition of forms which I used. It is conceivably instructive to check that the charts $\tilde{\varphi}$ are compatible, but it is undoubtedly uninstructive to read someone else's verification.

What <u>is</u> important, however, is to recognize $\operatorname{Alt}^k(TX)$ as a typical example of a <u>vector bundle</u>: a smooth manifold consisting of a family of vector spaces $\operatorname{Alt}^k(T_x X)$ parametrized by another smooth manifold X.