Differentiable Manifolds

Assume smooth = differentiable = indefinitely often differentiable.

Example 1: Consider \( S^2 = \{ x \in \mathbb{R}^3 : \| x \| = 1 \} \)

Conformal maps preserve angles.

Mercator's projection:

\[ \psi : \mathbb{S}^1 \ni \theta \mapsto (x, y) = (\cos \theta, \sin \theta) \in \mathbb{R}^2 \]

Stereographic projection:

\[ \phi : \mathbb{S}^2 \setminus \{ N \} \ni (x, y, z) \mapsto (x, y) \in \mathbb{R}^2 \]

Definition: A smooth manifold is a set \( X \) together with a maximal atlas.

An atlas is a compatible collection of charts which cover \( X \).

A chart is a pair \((U, \phi)\) consisting of a subset \( U \) of \( X \) and a map \( \phi : U \rightarrow \mathbb{R}^n \) which is injective, whose image is an open subset of \( \mathbb{R}^n \).

Two charts \( \phi_i : U_i \rightarrow \mathbb{R}^n \), \( i = 1, 2 \), are compatible if:

1. \( \phi_i(U \cap U_j) \) is an open subset \( V_{ij} \) of \( \mathbb{R}^n \), \( \phi_j(U \cap U_j) \) is an open subset \( V_{ij} \) of \( \mathbb{R}^n \).
2. The bijections \( \phi_i^{-1} : V_{ij} \rightarrow V_{ji} \) and \( \phi_j^{-1} : V_{ij} \rightarrow V_{ji} \) are smooth (i.e., \( C^\infty \)).

"Charts cover \( X \)" means the sets \( U \) cover \( X \).

An atlas is maximal means that any chart which is compatible with all the charts in the atlas is in the atlas.

Example 2: Torus \( \mathbb{T}^2 \) is \( \mathbb{R}^3 \), \( \mathbb{S}^1 \times \mathbb{S}^1 \), or \( \{(x, y, z) \in \mathbb{R}^3 : (x - a)^2 + (y - b)^2 + (z - c)^2 = 1 \} \).

Surface of revolution: \( (\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1 \). Let \( x = \frac{x}{\sqrt{x^2 + y^2}}, y = \frac{y}{\sqrt{x^2 + y^2}} \).

So, Circle: \( (2 + \cos \theta, \sin \theta) \), Surface: \( (2 + \cos \theta \cos \phi, 2 + \cos \theta \sin \phi, \sin \phi) \).

Torus \( \leftrightarrow \mathbb{S}^1 \times \mathbb{S}^1 \), \( (\theta, \phi) \leftrightarrow ((\cos \theta, \sin \theta), (\cos \phi, \sin \phi)) \).

Torus is naturally 1-1 correspondence with the subset of \( \mathbb{R}^4 \) consisting of all \((x, y, z, t)\) with \( x^2 + y^2 = 1, t^2 + t^2 = 1 \).

\[ X = \text{Torus}\subset \mathbb{R}^3, X = \{ (x, y, z) \in \mathbb{R}^3 : (x-2)^2 + (y-2)^2 = 1 \} \text{ where } p = \sqrt{x^2 + y^2} \}

Find charts for \( X \).

Define \( \theta : X \rightarrow [0, 2\pi] \) by \( \cos \theta = \frac{x}{p}, \sin \theta = \frac{y}{p} \), \( \phi : X \rightarrow \mathbb{R}^2 \) by \( \cos \phi = p - 2, \sin \phi = z \).

One possible chart: take \( U = \text{all points of } X \text{ with } p > 0 \) and define \( \phi : U \rightarrow \mathbb{R}^2 \)

by \( \phi(x, y, z) = (q \cos \phi, q \sin \phi) \), where \( q \in [0, 2\pi] \).

\( \psi \) is a 1-1 correspondence, \( \psi \leftrightarrow \{(x, y, z) \in \mathbb{R}^3 : 0 < x^2 + y^2 < (2\pi)^2\} \).

Another chart: take \( \tilde{U} = \text{all points of } X \text{ except those with } p = 1 \).

Define \( \tilde{\psi} : \tilde{U} \rightarrow \mathbb{R}^2 \) by \( \tilde{\psi}(x, y, z) = (q \cos \theta, q \sin \theta) \), where \( q \in (\pi, 3\pi) \).

This is a 1-1 correspondence, \( \tilde{U} \leftrightarrow \{(x, y, z) \in \mathbb{R}^3 : \pi < x^2 + y^2 < (3\pi)^2\} \).

\( \tilde{\psi}(U) = \tilde{\psi}(U \cap \tilde{U}) \) is punctured disk - circle radius \( \pi \), \( \tilde{\psi}(U) = \tilde{\psi}(U \cap \tilde{U}) \) is annulus - circle radius \( \pi \).

We now have a 1-1 correspondence: \( \psi(\tilde{U} \cap \tilde{U}) \Rightarrow \tilde{\psi}(\tilde{U} \cap \tilde{U}) \) given by:

\( (x, y) \mapsto (x, y) \), if \( \pi < x^2 + y^2 < (2\pi)^2 \), \( (x, y) \mapsto (\lambda x, \lambda y) \) if \( 0 < x^2 + y^2 < \pi^2 \), \( \lambda = 2 + \sqrt{x^2 + y^2} \).

This is a smooth map. So is its inverse.
Remark: A smooth manifold has been described completely when we give a collection of compatible charts which cover it, because of:

Lemma: If \( \{ U_\alpha, \phi_\alpha \} \) is a collection of compatible charts such that the \( U_\alpha \) cover \( X \), and \( (U', \phi') \), \( (U'', \phi'') \) are charts which are compatible with all \( (U_\alpha, \phi_\alpha) \), then \( (U', \phi'), (U'', \phi'') \) are compatible with each other.

Example 3: Orthogonal group, \( O_n \) is the \( n \times n \) real matrices \( A : A^T A = I \). We are especially interested in the case \( n = 3 \), and in \( SO_n = \{ A \in O_n : \det A = 1 \} \).

\( O_3 \) is \( 3 \times 3 \) real matrices = \( IR^3 \). \( O_3 \) is solution of 6 different equations.

We expect \( O_3 \) then to be 3-dimensional. Expect \( O_n \) to be of dimension \( n^2 - \frac{1}{2} n(n+1) = \frac{1}{2} n(n-1) \).

Elements of \( SO_3 \) are rotations about axes in \( IR^2 \) through angle \( \theta \), with \( 0 \leq \theta < \pi \). But, if \( \theta = \pi \), the axis orientation is undefined.

Points of \( SO_3 \) ⇔ closed ball in \( IR^3 \) of radius \( \pi \), with antipodal points on the boundary sphere representing the same element of \( SO_3 \).

First chart for \( SO_3 \): Take \( U = \) all rotations through \( \pi \), all \( A \) with \( Tr(A) \neq -1 \).

Define \( \Psi : U \to IR^3 \) \( \Psi(A) \) = vector of length \( \theta \) along the axis of rotation. 

Recall: \( 2 \cos \theta \) = \( Tr(A) \), so \( A = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \) is some basis, and trace is invariant under change of basis.

\( \Psi(U) = \) open ball of radius \( \pi \) in \( IR^3 \).

Another chart for \( O_n \): "Cayley parametrisation".

Let \( \hat{U} = \{ A \in O_n : \det(A + I) \neq 0 \} \). Then, \( (A - I)(A + I)^{-1} = S \) is skew, since \( S^T = (A^T + I)(A^{-1} - I) = (A^{-1} + I)(A^T - I) = (A^T - I)(I + A^T) = (I + A^T)(I - A) = -S \).

Conversely, if \( S \) is skew, then \( (S - I)(S + I)^{-1} \) is orthogonal.

Note that \( \det(S + I) \neq 0 \), so the eigenvalues of \( S \) are pure imaginary, as \( IS \) is Hermitian.

\( \Phi : \hat{U} \to \{ n \times n \) real skew matrices \( \} \) is \( \frac{1}{2} n(n-1) \). \( A \to (A - I)(A + I)^{-1} \) is a chart.

If we identify \( \{ skew \ 3 \times 3 \} \leftrightarrow IR^5 \) by \( \Phi \), then the second chart is defined in the same region as the first, and \( A \to \) vector along axis of rotation, with length \( \tan \theta \).

So they are compatible.

Third chart for \( O_n \): "Exponential map".

If \( S \) is \( n \times n \) real skew, then \( e^S = I + S + \frac{1}{2!} S^2 + \ldots \) is orthogonal.

For, \( (e^S)^T = e^S = e^{-S} = (e^S)^{-1} \). Easy to check that \( S \to e^S \) is a bijection of skew matrices \( S \) such that \( ||S|| < \pi \), and orthogonal matrices \( A \) such that \( ||A - I|| < 2 \).

Chart for \( SO_3 \): given by "Euler angles":

(\( \phi, \theta \)) = longitude, latitude of \( N \).

Let \( A_\theta \in SO_3 \) be rotation through \( \theta \) about \( OZ \).

Let \( A_\phi \in SO_3 \) be rotation through \( \phi \) about \( OY \).
An element $A$ of $SO_3$ has Euler angles $(\theta, \phi, \psi)$ if $A = A_{\phi} B_{\theta} A_{\psi}$.
This gives a 1-1 correspondence between a subset of $[0, 2\pi] \times [0, \pi] \times [0, 2\pi]$ and a subset of $SO_3$. It defines a chart of $SO_3$

$O_n$ is a group. Cayley parameterisation is a bijection $U \mapsto \{\text{skew matrices}\}$, where $U = \{A \in O_3 : \det(A + I) \neq 0\}$. Let $U' = \{A \in O_3 : \det(A - I) \neq 0\}$.
We have a bijection $\Phi: U \mapsto \{\text{skew matrices}\}$, by $\Phi(A) = (A + I)(A - I)^{-1}$.
$\Phi(U')$ is the set of invertible skew matrices, open subset of all skew matrices.
Similarly, $\Phi(U)$ is the same.
The bijection $\Phi': U \mapsto U'$ is smooth, with smooth inverse.
So these two charts are compatible.

Other charts: choose any $g \in O_3$, and define $\Phi_g: U_g \mapsto \{\text{skew matrices}\}$, where $U_g = gU$ and $\Phi_g(A) = \Phi(g^{-1}A)$. Because $g \in U_g$, the sets $\{U_g\}_{g \in O_3}$ cover $O_3$ and are compatible, because $\Phi_g(U_g \cap U_{g'}) = U_g \cap U_{g'} = \Phi_{g'}(U_g \cap U_{g'})$, and $\Phi_g(U_g \cap U_{g'}) = \{A : (g^{-1}A + I) \text{ and } (g^{-1}A - I) \text{ are invertible}\}$ is open in $\Phi_g(U_g) = \{\text{skew matrices}\}$.
$\Phi_g(U_g \cap U_{g'}) = \{S : S^T(A - I)\}$ and $\Phi_g^{-1}(S^T(A - I)) = \Phi_g^{-1}(S^T(A + I)) = I \mapsto (I - I)^{-1}$. So these charts are compatible.

(Real) Projective Space.

The real projective plane $\mathbb{RP}^2$ is the set of 1-dimensional subspaces of $\mathbb{R}^3$.
$\mathbb{RP}^2$ is a smooth manifold. Let $U \subset \mathbb{RP}^2$ be all lines which meet the affine plane $x=1$, ie, all lines which contain a vector of the form $(1, y, z)$. We have a 1-1 correspondence $\Phi: U \mapsto \mathbb{R}^3$, $\Phi(\text{line through } (1, y, z)) \mapsto (y, z)$.
$\mathbb{RP}^2 \setminus U = \{\text{points at } 0\}$ from point of view of the plane $x=1$.
Define $U$ all lines which contain a point of the form $(x, 1, z)$. We have a 1-1 correspondence $\Phi': U \mapsto \mathbb{R}^3$, $\Phi'(\text{line through } (x, 1, z)) \mapsto (x, y, z)$. Similarly, define $U''$ with $(x, y, 1)$ and $\Phi'': U'' \mapsto \mathbb{R}^3$.
Clearly, $U, U', U''$ cover $\mathbb{RP}^2$. The charts are compatible.
$\Phi(U) = \{(y, z) \in \mathbb{R}^2 : y \neq 0\}$, $\Phi'(U') = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$, both open in $\mathbb{R}^2$.
The 1-1 correspondence $\Phi(U) \cup U'' \rightarrow U' \rightarrow \Phi'(U')$ takes $\{(y, z) \mapsto \text{line through } (1, y, z) \mapsto \text{line through } (y', y+z) \mapsto (y',y,z)\}$. This is smooth.

Let $X$ be a smooth manifold, with charts $\Phi_k: U_k \mapsto \mathbb{R}^n$. A subset $W$ of $X$ is called open if $\Phi_k(W \cap U_k)$ is open in $\mathbb{R}^n$ for each chart. $W$ is closed if $\Phi_k(W \cap U_k)$ is a closed subset of $\Phi_k(U_k)$ for each $k$. These "open" subsets give $X$ the structure of a topological space, ie. (i) union of any family of open sets is open, (ii) intersection of any finite number of open sets is open, (iii) $\emptyset$ and $X$ are open. We can now say "$X$ is compact" or "$X$ is connected."
Two properties $X$ may not have are:

(i) **Hausdorffness**: two points $x, x_1$ are contained in two disjoint sets $U_1, U_2$. 
   Example: $X = IR$ together with another point $v$. Take two charts $(U_1, \phi_1) \to IR$, and $U_2 = \{w : \phi_2(1 < \phi_2) \to IR\}$, where $\phi_2(w) = 0, \phi_2(1 \leq \phi_2) = id$. 

(ii) **Metrizability**: a metric on the set $X$ such that a subset is open in the above sense (i.e., for the atlas) iff it is open for the atlas.

If $X, Y$ are two manifolds, then we say a map $f : X \to Y$ is 
1) **smooth** if for every chart $\phi : U \to IR^n$ of $X$ and $\psi : V \to IR^m$ of $Y$, the map $\psi \circ f \circ \phi^{-1} : IR^n \to IR^m$ is 
2) **continuous**.

We can thus speak about continuous or smooth $IR$-valued functions on $X$ and continuous or smooth paths in $X$.

**Exercise**: A manifold is connected iff it is path-connected.

**Proof of Compatibility Lemma**: We must show (i) $\phi'(U \cap U')$ is open in $\phi'(U)$, and $\phi''(U \cap U')$ is open in $\phi''(U)$.

Now, $U \cap U'$ is the union of $U \cap U' = \cup_{x \in U} U_x$, for all $x$, so enough to show $\phi'(w) = \phi'(U \cap U' \cap U_x)$ is open in $\phi'(w)$, $\phi''(w) = \phi''(U \cap U' \cap U_x)$ is open in $\phi''(w)$, and that the bijections $\phi'(w) \to \phi''(w)$ are smooth.

Compatibility of $U$ and $U_x$ and of $U'$ and $U_x$ says that $\phi_x(U \cap U_x)$ is open in $\phi_x(U)$ and $\phi_x(U' \cap U_x)$ is open in $\phi_x(U)$.

So, $\phi_x(U) = \phi(U \cap U_x)$ is open in $\phi(U)$ (intersection of these is open in $\phi(U)$).

But, we have smooth bijections $\phi_x(U \cap U_x) \to \phi''(U \cap U_x)$. These take open sets to open sets, so $\phi'(w)$ is open. Similarly, $\phi''(w)$ is open, and we have smooth bijections $\phi'(w) \to \phi''(w)$, as we want.

Suppose $X$ is a smooth manifold and $Y \subset X$. Say $Y$ is a submanifold of dimension $m$ if for every point $y$ of $Y$, there is a chart $\phi : U \to IR^n$ for $X$ with $x \in U$ such that $\phi(U \cap Y) = IR^m \oplus \{0\}$, $\phi(U) \cap IR^m = y$.

**Example**: Let $Y = S^2 \subset IR^3$, $Y = \{y \in IR^3 : \|y\|^2 = 1\}$. Consider $\left( \frac{y}{\|y\|} \right) \in Y$. Take the chart for $IR^3$ given by $\phi' : U \to IR^n$ defined in $U$, where

$$U = \{x : x, x^2 + x^2 + x^2 + \cdots + x^2 \leq 1\}.$$

It takes $U \cap S^{n-1}$ to $IR^n \cap 0 \cap \phi(U)$.

**Example**: To show $O_m$ is a submanifold of $IR^n$, is a mess if done directly.

Later, we shall prove a criterion for submanifolds.

Usually, we want only submanifolds which are either open or closed.
Example: We do not want the graph of $x \mapsto \sin \frac{1}{x}$ on $0 < x < 1$, although the
definition allows it. This is not closed.

$O_r$ - This depicts a $1$-smooth map $F : \mathbb{R} \to \mathbb{R}^3$ whose image is closed and
whose derivative never vanishes. But the image is not a submanifold.

Example: Dense winding on the torus, $S^1 \times S^1 = X$. Let $X = \{ (u, v) \in \mathbb{C}^2 : |u| = 1, |v| = 1 \}$
Define $F : \mathbb{R} \to X$ by $F(t) = (e^{2\pi i ut}, e^{2\pi i vt})$; if $a \in \mathbb{Q}$, say $a = \frac{p}{q}$, in lowest terms,
then $F(\mathbb{R})$ is compact; in fact, a closed curve $X \cap \{ u^q = v^p \}$, and is homeomorphic
to a circle. If we embed $X$ in $\mathbb{R}^3$ in the usual way, it is a "torus
knot of type $(p, q)$". For example, $(2, 3)$ gives the trefoil knot.
Then, $F(\mathbb{R})$ is a closed submanifold of $X$.
But if $a \not\in \mathbb{Q}$, then $F$ is $1$-1 and its image is dense in $X$.

Suppose $F : U \to \mathbb{R}^m$ is a continuously differentiable map, where $U$ is open in $\mathbb{R}^n$.
This means that for each $x \in U$, there is a linear map $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$, such that
$f(x+h) = f(x) + Df(x)h + R(x, h)$, where $\lim_{h \to 0} \frac{\|R(x, h)\|}{\|h\|} = 0$.
"Continuously
differentiable" means that, in addition, $x \mapsto Df(x)$ is a continuous map $\mathbb{R}^n \to \text{Hom}(\mathbb{R}^m, \mathbb{R}^n)$.

Chain Rule: If $U \subseteq V \subseteq \mathbb{R}^n$, $U$ open in $\mathbb{R}^m$, $V$ open in $\mathbb{R}^n$, $f, g$ continuously
differentiable, then so is $g \circ f$, and $D(g \circ f)(x) = Dg(f(x)) \cdot Df(x)$, where $y = f(x)$.

In particular, if $U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^m$ is $1$-1 with differentiable inverse, then $Df(x)$ is
invertible for each $x \in U$ and so $n = m$.

Suppose $U$ open in $\mathbb{R}^n$, $F : U \to \mathbb{R}^m$ continuously differentiable. $Df(x) : \mathbb{R}^n \to \mathbb{R}^m$, linear.
\textbf{Inverse function theorem:} Let $f : U \to \mathbb{R}^m$ be continuously differentiable, $U$ open in $\mathbb{R}^n$.
Suppose $f(y) = y$, and $Df(y) : \mathbb{R}^n \to \mathbb{R}^m$ is invertible. Then, $f$ neighbourhood $V$ of
$y$ in $\mathbb{R}^n$ and $g : V \to U$, continuously differentiable, with $f \circ g = \text{id.}$; $V \subseteq U$.
and $g \circ f = \text{id.}$ in $f^{-1}(V)$.

\textbf{Proof:} See handout I.

\textbf{Implicit function theorem:} Suppose $f : U \to \mathbb{R}^n$ ($U$ open in $\mathbb{R}^n$) is continuously differentiable, and
$Df(y)$ has rank $m$, i.e., is surjective, $y \in U$. Then, $f^{-1}(y)$ is a smooth
submanifold of $U$ of dimension $n - m$.

\textbf{Proof:} We shall choose a linear map $p : \mathbb{R}^m \to \mathbb{R}^m$ such that $f^{-1}(y) = p^{-1}(f(y))$.\[p^{-1}(f(y))\]has
invertible $Df(y) = Df(x) \circ p$. Choose $g : V \to \mathbb{R}^m$, where $V$ is a neighbourhood of
$y(0)$ in $\mathbb{R}^m \oplus \mathbb{R}^m$, such that $g \circ F = \text{id.}$
Then, $g^{-1} : V \to \mathbb{R}^m$ is a chart for $\mathbb{R}^m$ near $x$, and takes $g(V) \subseteq \mathbb{R}^m$
bijectively to $V \times \{ y \} \times \mathbb{R}^m$. This is an open subset of $y \times \mathbb{R}^m$.
So it is a chart for $f^{-1}(y)$. \[\]$
Example: Take $U = \text{all } n \times n$ invertible real matrices, and $f: \{\text{all symmetric matrices}\}$, with $f(A) = A^T A$. Then $f$ is continuously differentiable.

$$f(A + h) - f(A) = (A^T h + A h^T) + (h^T h).$$

Clearly, $h^T h \to 0$ as $\|h\| \to 0$, and $h \to A^T h + A h^T$ is linear. So $DF(0,h) = A^T h + A h^T \in \{\text{symmetric matrices}\}$.

This is surjective, for if $S$ is symmetric, take $h = \frac{1}{2} [A^T]^T S$.

We have $DF(0,h) = S$. Hence $Df_0$ is a submanifold of $\mathbb{R}^n$.

Lemma: If $A: \mathbb{R}^n \to \mathbb{R}^m$ is linear and of rank $m$, then $A$ is linear $p: \mathbb{R}^n \to \mathbb{R}^{n-m}$ such that $A @ p: \mathbb{R}^n \to \mathbb{R}^n$.

Proof: We can choose $p$ to be projection onto a coordinate subspace, ie $p(Y) = \begin{pmatrix} 0 \\ Y \end{pmatrix}$ where $p_1(Y) = Y_j$ for some $j(i)$.

Have: $\begin{pmatrix} A \\ p \end{pmatrix}$ Either, regard $A$ as an $m \times n$ matrix with column rank $m$, hence row rank $m$, and add $n-m$ linearly independent rows.

Or, choose the linear maps $p_1, \ldots, p_{n-m}: \mathbb{R}^n \to \mathbb{R}$ successively so that $p_i$ does not vanish identically on $\text{ker}(A @ p_1, \ldots, p_{i-1})$.

Inverse function theorem: If $f: U \to V$ is a continuously differentiable bijection between open sets of $\mathbb{R}^n$, and $Df(u)$ is invertible $V \times U$, then $f^{-1}$ is continuously differentiable.

Definition: A diffeomorphism is a smooth map which is bijective with smooth inverse.

(defined first for open subsets of $\mathbb{R}^n$, then for maps between smooth manifolds)

Theorem: $f \in C^\infty$, a polynomial, $f: \mathbb{C} \to \mathbb{C}$, $f$ is surjective, say $f(z) = w$.

Proof 1: Consider the winding number of $f(z)$ around $w$, when $z$ traverses a large circle $|z| = 1$. Algebraic topology proof.

Proof 2: By inverse function theorem. Consider $f$ as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ and look at $Df(z): \mathbb{R}^2 \to \mathbb{R}^2$ (multiplication by $f'(z)$). $Df(z) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$

$\det Df(z) = |f'(z)|^2$.

At most $n-1$ points $\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix}$, where $Df$ vanishes.

Inverse function theorem: $\text{number of points in } f^{-1}(w)$ is locally constant as a function of $w$ in the open set $\{w: w \neq f(z) \text{ with } f'(z) = 0\}$. But, $\mathbb{C} \setminus \{\text{finite number of points}\}$ is connected, so the number of points in $f^{-1}(w)$ is independent of $w$, so always $\neq 0$.

Orientability: Suppose $f: U \to V$ is a diffeomorphism where $U, V$ are connected open subsets of $\mathbb{R}^n$. Then, $\det Df(w) \neq 0$ for all $w \in U$. So it is either $>0$ everywhere — say $f$ is orientation preserving, or $<0$ everywhere — say $f$ is orientation reversing.

More generally, say $f: U \to V$ is orientation preserving if $\det Df(w) > 0$ everywhere. If $X$ is a smooth manifold, say $X$ is orientable if the set of all charts $\phi: U \to \mathbb{R}^n$ can be divided into subsets $C_1$ and $C_2$ such that if $\phi_1: U \to \mathbb{R}^n$, $\phi_2: V \to \mathbb{R}^n$ both belong to $C_1$ or both to $C_2$, then $\phi_2 \circ \phi_1: \phi(U \cap V) \to \phi(U \cap V)$ is orientation preserving.
Definition: An orientation of \( X \) is a choice of one of the classes \( \tau_1, \tau_2 \).

Example: Möbius band. A choice of chart: "what you see" / edge lines.
- another chart.

Tangent Vectors.

Definition: Let \( X \) be a smooth manifold. The tangent space \( T_x X \) to \( X \) at \( x \in X \) is the set of all functions which assign to each chart \( (U, \varphi_u) \) with \( x \in U \) a vector \( \mathbf{v}_u \in \mathbb{R}^n \) such that \( \mathbf{v}_u = D(\varphi_u \circ \varphi^{-1}_u)(y) \mathbf{v}_u \), where \( y = \varphi_u(x) \), whenever \( (U, \varphi_u) \) and \( (V, \varphi_v) \) are two charts containing \( x \).

Clearly, (i) \( T_x X \) is a vector space. \( \text{dim}(T_x X) = n = \text{dim} X \).

(ii) an element is completely determined by giving \( \mathbf{v}_u \) for one chart \( U \) containing \( x \).

(iii) if \( f: X \to Y \) is a smooth map between smooth manifolds, then \( f \) induces a linear map \( T_x X \to T_{f(x)} Y \) \( \forall x \in X \). This is called \( Df(x) \).

\[ Df(x) \] \( \mathbf{v}_u \) takes the family \( \{ \mathbf{x}_u \} \) to the family \( \{ \mathbf{y}_u \} \), where \( \mathbf{y}_u = D(\varphi_u \circ f)(\varphi_u^{-1})(\mathbf{x}_u) \).

where \( \mathbf{v}_u = D(\varphi_u \circ f)(\mathbf{v}_u)(\varphi_u^{-1})(\mathbf{y}_u) \), where \( w = \varphi_u(x) \).

Alternatively: \( X \) a smooth manifold. \( \{ \varphi_u: U_u \to \mathbb{R}^n \}_{x \in U} \) = collection of all charts for \( X \).

For \( x \in X \), write \( x = \varphi_u(x) \in \mathbb{R}^n \) if \( x \in U_u \). To define \( T_x X \): an element of \( T_x X \) is a function \( \mathbf{v}_u \) \( \{ x \to \mathbf{v}_u \} \in \mathbb{R}^n \) or a family \( \{ \mathbf{v}_u \}_{x \in U, x \in U} \), with the property that \( \mathbf{v}_u = D(\varphi_u \circ f)(\mathbf{v}_u)(\varphi_u^{-1})(\mathbf{y}_u) \), where \( w = \varphi_u(x) \).

(i) \( T_x X \) is a vector space (because \( D(\varphi_u \circ f)(\mathbf{v}_u)(\varphi_u^{-1})(\mathbf{y}_u) \) is a linear map).

(ii) For any \( x \) with \( x \in U_u \), the map \( \mathbf{v}_u \mapsto \mathbf{y}_u \) is an isomorphism \( T_x X \to \mathbb{R}^n \).

\[ \text{For if we have } \mathbf{v}_u \text{ we can define } \mathbf{y}_u \text{ for any } \beta \text{ with } x \in U_\beta \text{ by } \mathcal{O}, \text{ and then } \mathbf{y}_u \text{ holds for all pairs } \beta, \gamma \text{ with } x \in U_\beta \cap U_\gamma, \text{ by chain rule.} \]

\[ \frac{D}{D(x_1)} (\varphi_\beta \varphi_\beta^{-1})(x_1) \mathbf{v}_u = D(\varphi_\beta \varphi_\beta^{-1})(x_1) \frac{D}{D(x_1)} (\varphi_\beta \varphi_\beta^{-1})(x_1) \mathbf{y}_u = D(\varphi_\beta \varphi_\beta^{-1})(x_1) \mathbf{y}_u \]

\[ \text{Thus this doesn't depend on } x, \text{ because } \frac{D}{D(x_1)} (\varphi_\beta \varphi_\beta^{-1})(x_1) = \varphi_\beta \text{ if } \varphi_\beta \text{ is linear.} \]

(iii) If \( X \) is a vector space, then \( T_x X \) is canonically isomorphic to \( X \), by the map \( \mathbf{v} \mapsto \varphi_\beta^{\mathbf{v}} \varphi_\beta^{-1} \in X \), where \( \varphi_\beta: x \mapsto \mathbb{R}^n \) is any linear isomorphism.

\[ \text{This doesn't depend on } x, \text{ because } \frac{D}{D(x_1)} (\varphi_\beta \varphi_\beta^{-1})(x_1) = \varphi_\beta \text{ if } \varphi_\beta \text{ is linear.} \]

(iv) if \( f: X \to Y \) is any smooth map, then \( f \) induces a linear map \( Df: T_x X \to T_{f(x)} Y \), for any \( x \in X \).

Then define \( Df(x) \mathbf{v}_u = D(\varphi_\beta \varphi_\beta^{-1})(x_1) \mathbf{y}_u \). The chain rule shows this is a well-defined tangent vector in \( T_{f(x)} Y \).

(v) \( X \) \( \ni x \to g(x) \text{ smooth maps, then } Dg(x) \circ Df(x) = Dg(x) \circ Df(x) \subseteq T_x X \to T_{g(x)} Y \), with \( y = g(x) \) \( z = g(f(x)) \). "Chain rule" = follows from chain rule for open subsets of \( \mathbb{R}^n, \mathbb{R}^m \).

Corollary: If we have \( X \) a smooth manifold \( x \) a set \( \mathcal{C}_x \), and to each smooth \( f: X \to Y \) a map \( \mathcal{C}(f)(x) = \mathbb{R}^n \mathcal{C}(f)(x) \) when \( x \to y \), we have \( \mathcal{C}(f)(x) \subseteq \mathcal{C}(f)(x) \subseteq \mathbb{R}^n \mathcal{C}(f)(x) \), such that:

(i) when \( x \) is a vector space, \( T_x X \cong X \), canonically.

(ii) \( \mathcal{C}(y \circ f)(x) = \mathcal{C}(y)(y \circ f)(x) \) when \( x \to y \), \( y \to y \).

(iii) if \( x \) is an open submanifold of \( X \), then \( D(x): T_x X \to T_x X \), where \( i: X \to X \) is inclusion. Then, \( T_x X \cong T_x X \), canonically.
Example: Suppose \( X \) is an \( n \)-dimensional submanifold of \( \mathbb{R}^n \). Then, \( T_x X \) can be identified with an \( n \)-dimensional vector subspace of \( \mathbb{R}^n \), as follows: Choose a chart \( \varphi_x : U_x \to \mathbb{R}^n \) with \( x \in U_x \). Let \( \varphi_x : U_x \to X \to \mathbb{R}^n \) be \( \varphi_x \) regarded as a map \( \varphi_x : U_x \to \mathbb{R}^n \). Then, image \( D\varphi_x(x) \) is an \( n \)-dimensional subspace of \( \mathbb{R}^n \) which does not depend on choice of chart. If \( X \subset \mathbb{R}^n \) is "defined by equations", i.e., we have \( F : U \to \mathbb{R}^{N-n} \), where \( U \) is open in \( \mathbb{R}^n \), \( F \) is smooth, and \( D\varphi(x) \) has rank \( N-n \) for all \( x \in U \), and \( x = F(y) \), some \( y \in \mathbb{R}^{N-n} \), then, \( T_x X = \text{Null} \text{ of } D\varphi(x) : \mathbb{R}^n \to \mathbb{R}^{N-n} \). More precisely, \( \text{Ker}(DF(x)) = \text{image } D\varphi(x) \), with \( \varphi_x \) as before.

For, \( F \circ \varphi : U_x \to \mathbb{R}^n \) is constant, so \( D(F \circ \varphi)(x) = 0 \), so \( \text{im}(D\varphi(x)) \subset \text{Ker}(DF(x)) \), and they have the same dimension, so are equal.

Example: \( X = O_n \subset \mathbb{R}^{n^2} \), \( \{ F = f^{-1}(1) \} \), where \( f : \mathbb{R}^n \to \mathbb{R}^n \); \( f(A) = A^t A \). \( Df(A) = A^t + A \).

So \( \text{Ker}(DF(x)) = \text{skew matrices} \) of all \( n \times n \) matrices.

Alternatively, consider the parametrisation, \( [\text{skew matrices}] \to O_n \), \( S \mapsto (1 + S)^{-1} \).

\( f(S) = \frac{(S + IS) - (S - IS)}{2} = 1 + 2S + R(S) \), where \( R(S) \to 0 \) as \( S \to 0 \).

So, \( T_x O_n = \text{image } Df(x) = \text{all skew matrices} \).

\( T_x X \) can be defined also:

(1) by means of curves through \( x \).

Consider all smooth curves \( \gamma : (-\varepsilon, \varepsilon) \to X \) such that \( \gamma(0) = x \) (for any \( \varepsilon > 0 \)). Say two such curves are equivalent if they touch at \( x \), i.e., \( \gamma_1 \sim \gamma_2 \) if for any chart \( \varphi : U \to \mathbb{R}^n \) in a neighbourhood of \( x \), we have \( \gamma_1(0) = \gamma_2(0) \).

Let \( T_x X \) be the set of equivalence classes.

The characterisation of \( T_x X \) shows that \( T_x X = \varphi_{x*} X \), canonically.

(2) in terms of derivatives.

Let \( C^\infty(X) \) be all smooth maps \( X \to \mathbb{R}^n \). This is a ring, and also a real vector space over the subring \( \mathbb{R} \) of constant functions \( X \to \mathbb{R} \). We can get the group structure back from the ring \( C^\infty(X) \), because \( X \subset \mathbb{R}^n \) is a set of all ring homomorphisms \( C^\infty(X) \to \mathbb{R}^n \) via \( \phi \mapsto \xi \phi \). \( x \mapsto \xi_x \). (Shall prove later that \( \text{Hom}(C^\infty(X), \mathbb{R}^n) = \mathbb{R}^n \) for some \( \xi \).)

Given \( x \in X \), let \( \mathcal{D}_x X \) be the set of all linear maps \( \theta : C^\infty(X) \to \mathbb{R} \) which have the "Leibnitz property at \( x \)." i.e., \( \theta(fg) = \theta(f)g(x) + f(x)\theta(g) \). (Such a \( \theta \) is called a derivation.)

Notice that this is the same as saying that \( f \to f(x) + \theta(f) \) is a ring homomorphism from \( C^\infty(X) \to \mathbb{R}[\mathbb{E}] / (\mathbb{E}^2) \).

Clearly, \( \mathcal{D}_x X \) is a vector space.

Theorem: \( \mathcal{D}_x X \) is canonically isomorphic to \( T_x X \).

Proof: First consider derivatives of \( C^\infty \) at \( 0 \in \mathbb{R} \). Shall prove that \( \theta(f) = Df(0) f \) for some \( f \in \mathbb{R}^n \).

First note that \( \theta(1) \), for \( 0(1) = \theta(1) = 1 \).

In particular, \( \theta(g(x)) = \theta(g) \) is constant. By linearity, any \( f \in C^\infty(\mathbb{R}^n) \) can be expressed as \( f = c + \sum \delta \xi_i g_i \), where \( \xi_i \in C^\infty(\mathbb{R}^n) \) is the \( i \)-th coordinate function,
and $g \in C^0(\mathbb{R}^n)$ is such that $g(1) = D(0)$. (For $F(t) = F(0) + \int_0^t \frac{d}{dt} F(t) \, dt = \int_0^t D(t) \, dt$, set $g(t) = \int_0^t g(s) \, ds$. So take $g(x) = \int_0^x g(t) \, dt$. $g(1) = \int_0^1 D(t) \, dt = D(0)$.)

Now, $\Theta(1) = \Theta(0) + \int D(1,0) \, dt = 0.$

Now let us prove that $D_x X$ is an $n$-dimensional vector space for any $n$-dimensional smooth manifold $X$.

**Lemma:** If smooth $f: \mathbb{R}^n \to \mathbb{R}$ with $f(0) = \{ 1 \}$ if $|x| > e$ for any $e > 0$.

**Proof:** First observe that $e^{-|x|^2}$ on $[0,0]$ is $C^\infty$ with all derivatives zero at 0.

So let $f(t) = e^{-|t|^2} - 1_{|t| > 2}$ if $x \in (0,2)$, and 0 otherwise. $f: \mathbb{R} \to \mathbb{R}$ is $C^0$.

Let $X(x) = \int_0^x f(0) \, dt$, if $x > 0$, 0 otherwise.

Then, $X$ is $C^\infty$ and constant, say $c$, for $x > 0$.

Finally, define $F(x) = 1 - \frac{e}{c} X \bigg( \frac{|x| - 1}{2} \bigg)$.

**Corollary:** If $X$ is any smooth manifold and $x \in X$ and $U$ is a neighbourhood of $x$, we can find $f: X \to \mathbb{R}$ such that $\supp f \subseteq U$, and $f = 1$ in a neighbourhood of $x$. (Recall $\supp f = \{ y : f(y) > 0 \}$)

Now suppose that $\Theta: C^0(\mathbb{R}^n) \to \mathbb{R}$ is a derivation at $x$. Choose $f$ such that $f = 0$ away from $x$. Then $\Theta(\bar{f}) = \bar{f}$, because we can find $\Phi$ such that $\bar{f} \Phi = \bar{f}$, and then $\Theta(\bar{f}) = \Theta(\bar{f}) \Phi(x) + \Theta(\bar{f}) \Phi(x) = \Theta(\bar{f}) + \Theta(\bar{f})$. Choose $\Phi$ to be 1 near $x$, and with $\supp(\Phi) = \text{region where } f = 1$. So, for any $f \in C^0(\mathbb{R}^n)$ we have $\Theta(f) = \Theta(f)$.

This tells us that if $f: U \to \mathbb{R}^n$ is a chart near $x$, then $D_x f \subseteq D_{\bar{x}} U$.

**Main point:** If $f: X \to Y$ is smooth, then $f$ induces a ring homomorphism $f^*: C^0(Y) \to C^0(X)$, so we have $D_x X \to D_{f(x)} Y$.

The disjoint union $\bigcup_x T_x X$ is called the tangent bundle, written $TX$, of $X$.

*It is a smooth manifold of dimension $2n$, where $n = \dim X$.*

To give charts: Let $f: U \to \mathbb{R}^n$ be a chart for $X$, then let $f' := f \circ f$, and define $\bar{f}: U \to \mathbb{R}^n \oplus \mathbb{R}^n$ by $\bar{f}(x) = (f(x), \tilde{f}_x)$, where $\tilde{f}_x$ is the representation of $f$ in the chart. We have $\bar{f}(TU) = f(U) \times \mathbb{R}^n$, which is open in $\mathbb{R}^{2n}$.

The transition between two charts corresponding to $g_1: U_1 \to \mathbb{R}^n$ and $g_2: U_2 \to \mathbb{R}^n$ is $(g_2 \circ g_1^{-1}) \times D(g_2 \circ g_1^{-1})$, i.e., $(y, z) \mapsto (g_2(y), D(g_2)(y)(z))$, which is smooth.

*A tangent vector field on $X$ is a smooth map $\sigma: X \to TX$ such that $\sigma(x) \in T_x X$ for all $x$.***
Let \( x \in X, \ x \in \text{open } U \subset X \). If \( X \) is a compact smooth manifold, and \( \{ U_\alpha \} \) is a finite open covering of \( X \) by the domains of charts \( \varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n \), then we can find smooth functions \( f_\alpha : X \rightarrow \mathbb{R} \) such that:
1. \( f_\alpha (x) \geq 0 \) everywhere,
2. \( \text{supp}(f_\alpha) \subset U_\alpha \),
3. \( \sum \alpha f_\alpha (x) = 1 \forall x \).

The set \( \{ f_\alpha \} \) is called a partition of unity, subordinate to the covering \( \{ U_\alpha \} \).

**Proof:**
For each \( x \), choose \( g_\alpha : X \rightarrow \mathbb{R} \) smooth and \( \geq 0 \) such that \( g_\alpha (x) = 1 \) near \( x \), and \( \text{supp}(g_\alpha) \subset U_\alpha \). By compactness, \( \exists x_1, \ldots, x_n \) such that for every \( x \), \( g_\alpha (x) \neq 0 \) for some \( \alpha \). For each \( x_\alpha \), choose \( \alpha \) such that \( \text{supp}(g_\alpha) \subset U_\alpha \).

Let \( h_\alpha = \sum x_\alpha g_\alpha \). Then, \( h = \sum h_\alpha \) is smooth and \( \geq 0 \) everywhere. Take \( f_\alpha = \frac{h_\alpha}{h} \).

**Proposition:** \( X \) is set of non-zero ring homomorphisms \( \Theta : C^0 (X) \rightarrow \mathbb{R} \).

**Proof:** See handout.

**Theorem:** Any smooth manifold of dimension \( n \) is diffeomorphic to a submanifold of \( \mathbb{R}^{2n+1} \) - Whitney embedding theorem.

**Proof:** (Sketch for when \( X \) is compact)

- These are 1-d, but cannot embed in \( \mathbb{R}^2 \), but they are not manifolds.

First prove \( X \) is submanifold of \( \mathbb{R}^n \) for some \( N \).

Cover \( X \) by finitely many charts, \( \varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n \), \( \alpha = 1, \ldots, k \). Then choose a partition of unity \( f_\alpha : X \rightarrow \mathbb{R}^n \) with \( \text{supp}(f_\alpha) \subset U_\alpha \). Define \( F : X \rightarrow \mathbb{R}^{n+k} \) by \( F(x) = (f_1(x), \varphi_1(x), \ldots, f_k(x), \varphi_k(x)) \), where we define \( f_\alpha (x), \varphi_\alpha (x) = 0 \) if \( x \not\in U_\alpha \).

Then, \( F \) is smooth, and \( F|_{\text{region where } f_\alpha \neq 0} \) is obviously 1-1.

But, points \( x \) for which \( \{ \alpha : f_\alpha (x) \neq 0 \} \) is different obviously have different images. So \( F \) is 1-1.

We now need: Criteria: if \( F : X \rightarrow \mathbb{R}^n \) is smooth, 1-1, and \( X \) is compact, and \( DF(x) \) has rank \( n \) for each \( x \in X \), then \( F \) is a diffeomorphism between \( X \) and a submanifold of \( \mathbb{R}^n \).

In our case, the criterion is satisfied, because \( DF(x) = (\partial F(x), \partial \varphi_1(x), \ldots) \), and at each point \( DF(x) \) has rank \( n \) and at least one \( DF(x) \) is \( \neq 0 \).

Consider the criterion above. To get a good chart for \( \mathbb{R}^n \) near \( f_\alpha \), choose linear \( h : \mathbb{R}^{n-k} \rightarrow \mathbb{R}^n \) such that \( DF(x) \otimes h : T_xX \otimes \mathbb{R}^{n-k} \rightarrow \mathbb{R}^n \). Then apply inverse function theorem to \( (F \otimes h) \), where \( (T_xX \otimes h) : X \rightarrow \mathbb{R}^n \) is the inverse of a chart for \( X \).

**Consider:** \( X \otimes h : \mathbb{R}^n \rightarrow \mathbb{R}^n \) orthogonal projection, \( \text{if } N > 2n+1 \)

The space of all pairs of points of \( X \) joined by a line: \( \dim \mathbb{R}^{2n+1} \)

(Avoiding lines tangent to \( X \)).
**Complex Manifolds (mostly one-dimensional)**

Suppose \( U \) is an open subset in \( \mathbb{C}^n \), and \( f: U \to \mathbb{C}^m \) is a continuous map. Say \( f \) is analytic at \( z \in U \) if \( f \) is \( \mathbb{C} \)-linear map \( A: \mathbb{C}^n \to \mathbb{C}^m \) such that \( f(z + h) = f(z) + Ah + R(z, h) \), where \( |R(z, h)| / |h| \to 0 \) as \( h \to 0 \). In, \( f \) is differentiable in the real sense, and the derivative, \( Df(z) \), is \( \mathbb{R} \)-linear. I.e, \( Df(z) \) satisfies the Cauchy-Riemann equations.

**Example:** \( f = u + iv \). \( Df = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}) : \mathbb{R}^2 \to \mathbb{R}^2 \), \( i = (1, 0, 0, 1) \). Need \( Df \cdot i = i \cdot Df \).

Analytic maps are very tightly constrained. For example:

(i) **Once continuously differentiable \( \Rightarrow \) derivatives of all orders exist.**

(ii) **Two analytic functions \( f, g: U \to \mathbb{C}^m \) agree in any open subset of \( U \), they agree everywhere.**

(iii) **If \( m = n = 1 \), then either \( f \) = constant or \( f(\text{open}) = \text{open} \).**

**Definition:** \( X \) is a **complex manifold of dimension \( n \)** if it has a maximal atlas of compatible charts \( \Phi: U \to \mathbb{C}^n \), where \( \Phi(U) \) is open in \( \mathbb{C}^n \), and "compatible" means that the transition maps \( \Phi_1 \circ \Phi_2^{-1}: \Phi_2(U \cap U_1) \to \Phi_1(U \cap U_1) \) are analytic.

A complex manifold \( X \) is **canonically oriented**, for if \( \Phi: U \to V \) is an analytic bijection, with \( U, V \) open in \( \mathbb{C}^n \), then \( X \) is automatically orientation preserving when regarded as a smooth map \( U \to V \) between subsets of \( \mathbb{R}^{2n} \).

**Example:** \( \det \left( \frac{\partial \Phi_2}{\partial x}, \frac{\partial \Phi_2}{\partial y} \right) = (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 = f(z)^2 > 0 \) (if Cauchy-Riemann equations hold).

In general, when \( Df(z) \), which is an \( \mathbb{R}^{2n} \times \mathbb{R}^{2n} \) real matrix \( T \), have \( \det T = \det Df(z) \).

**Examples (i):** **Riemann sphere, \( X = \{ z \in \mathbb{C}, z \neq 0 \} \).** Charts: \( \Phi: U = \{ z \in \mathbb{C}, \Phi(z) = \overline{z} \}, \overline{\Phi}: \Phi(U) = \{ z \in \mathbb{C}, \Phi(z) = \overline{z} \} \), and the transition map is \( z \to \overline{z} \).

\( U \) is open in \( \mathbb{C} \). Suppose \( z \in U \) and \( f: U \to \mathbb{C} \) is analytic. When can we extend \( f \) to an analytic map \( f: U \to S \)?

**Suppose \( f \) has at most a pole at \( z \), i.e., no essential singularity. Because \( f \) has a pole at \( z \), if \( w \to \frac{1}{f(z-w)} \) has a removable singularity at \( w = z \), but then, if we define \( f(z) = 0 \) at \( z \),** then using the chart \( S \to \mathbb{C}, S \to \mathbb{C}, s \to \mathbb{C}, s \to \mathbb{C} \), we see \( f \) is analytic.

Of course, if \( f \) has a removable singularity, we can extend \( f \) to an analytic map \( f: U \to \mathbb{C} \). Notice that \( z \to e^{1/z} \) cannot be extended to \( \mathbb{C} \).

**Suppose \( f \) is analytic, \( U = \{ z \in \mathbb{C}, |z| > R \} \). When does \( f \) extend to \( f: U \cup \{ 0 \} \to \mathbb{C} \)?**

**Answer:** when \( |f(z)| < K|z|^k \), some \( K, k \) as \( z \to 0 \).

We consider the chart \( \Phi \) given by \( z \to \frac{1}{z} \). Then consider \( \zeta \to f(1/\zeta) : \{ \zeta \in \mathbb{C}, 0 < |\zeta| < 1 \} \to \mathbb{C} \). When does this extend to \( z = 0 \)?

**Precisely when \( \zeta \to f(1/\zeta) \) has at most a pole at \( \zeta = 0 \), i.e., when \( |f(1/\zeta)| < C|\zeta|^\delta \) for some \( C, \delta \).**
Theorem: If $f: S \to S$ is analytic, then it is of the form $f(z) = \frac{p(z)}{q(z)}$, with $p, q$ polynomials, i.e., $f$ is a "rational function."

Proof: First observe that $f^{-1}(a)$ consists of isolated points, for the zeroes of $1/f$ must be isolated. So, $f$ has at most a finite number of poles, say at $z = a_1, ..., a_k$. Near $z = a_i$, it can be expanded: $f(z) = \frac{g_i(z)}{(z-a_i) \cdot h_i(z)} + g(z)$, where $g_i$ is analytic near $a_i$. Doing this at each $a_i$, we find $f = r + g$, where $r$ is rational and $\to 0$ as $z \to a_i$, and $g$ is analytic $\forall z \in S$. But $|r(z)| \leq K|z|^R$ and $|g(z)| \leq K|z|^R$ as $z \to \infty$. So $g$ is a polynomial.

Examples (ii): Let $L \leq \mathbb{C}$ be a lattice, i.e., a subgroup under addition, of the form $\mathbb{Z} w_1 + \mathbb{Z} w_2$ where $w_1, w_2 \in \mathbb{C}$ and $\mathbb{Z} w_2 \in \mathbb{R}$. Example:

Let $X = \{ x \in \mathbb{C} \mid x \leq 5 \}$. Let $U = \pi(V) \subset X$, where $\pi: \mathbb{C} \to X$ is the obvious projection. Let $\pi': U \to V$ be $\pi^{-1}$. By construction, this is 1-1, so is our first chart for $X$. For any $w \in X$, let $U_w = w + U \subset X$ (i.e., a group).

Define $\pi_w: U_w \to \mathbb{C}$ by $\pi_w(z) = \pi(z - w)$.

Clearly $\pi_w(U)$ and $U_w \cap U_{w'}$ are open in $V \subset X$, and the transition map is just $\pi_k \to \pi_n$, where $w \in w'$ such that $w \equiv w - w_2 \mod L$.

An atlas, so $X$ is a complex manifold. $X$ is compact because there is a continuous map $K \to X$ which is surjective, where $K = \{ 0, 1 \} \cup \mathbb{Z}$ (i.e., closed parallelogram), $K$ is compact. $X$ is also Hausdorff (check).

Consider $\mathbb{R}^2/\mathbb{Z}^2 \cong \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$, $t \mapsto (\cos \frac{2\pi t}{\Lambda}, \sin \frac{2\pi t}{\Lambda})$.

Over $\mathbb{R}$, 3 real roots:

\[ \sqrt{\Lambda} \quad -\sqrt{\Lambda} \quad 0 \]

Consider $\{ (w, z) \in \mathbb{C}^2 : w^2 = f(z) \}$, with $f(z) = z^2 + a_1 z + b_1$, $a, b \in \mathbb{R}$.

Consider $f(z) \in \mathbb{C} - \{ 0 \}$ (negative real axis).

Consider $F(w, z) = w^2 - f(z)$. $Y = \text{Curve: } F^{-1}(0)$. $F: \mathbb{C} \to \mathbb{C}$.

$DF(w, z) = 2w F'(z)$ for $(w, z) \in F^{-1}(0)$, as long as $f$ has distinct roots.

So, implicit function theorem $\Rightarrow F^{-1}(0)$ is a submanifold of $\mathbb{C}^2$.

Atlas for $Y$: Let $V = \mathbb{C}$, cut where $f(z) \in \mathbb{R}_+: \{ x \in \mathbb{R} : x \geq 0 \}$.

Then, $\{ (w, z) \in \mathbb{C}^2 : f(z) = 0 \} = U_1 \cup U_2$, and $\pi_2: U_1 \to V \subset \mathbb{C}$ -- two disjoint charts.

Note: A complex manifold of dimension 1 is called a Riemann surface.
Let \( W = \mathbb{C} - \{z : f(z) \in \mathbb{R}^+\} \). Then, \( \{w, z : z \in W\} = \tilde{U} \cup \tilde{U} \), where \( \tilde{U} = \{w, z : w = \sqrt[3]{f(z)}\} \), \( \tilde{U} = \{w, z : w = -\sqrt[3]{f(z)}\} \). Again, two disjoint charts: \( \tilde{U}_x : \tilde{U}_x \rightarrow \mathbb{C} \).

\[ \tilde{U}_x \cap \tilde{U}_y = \mathbb{C} - \mathbb{R}^+ \to \mathbb{C} - \mathbb{R}^+ \]

We need three more charts to cover \( Y \). Let \( z_1, z_2, z_3 \) be the zeroes of \( f(z) \).
We want a chart in the neighborhood of \( (z_1, z_2, z_3) \). Let \( U \) be a small neighborhood of \( (0, 0, 0) \) in \( Y \). Define \( \tilde{U} : U \rightarrow \mathbb{C} \) by \( (w, z) \rightarrow w \). This is injective if we can solve \( f(z) = w^3 \) uniquely as a function of \( w^2 \) for \( z \) near \( z_1 \). Similarly, for \( z = z_2, z_3 \).

**Example:** Fermat curve, \( x^n + y^n = 1 \), with \( (x, y) \in \mathbb{A}^2 \). \( y = \sqrt[n]{1-x^n} \).

Euler's formula: \( V - E + F = 2 \) for polyhedra.

Euler number: \( 2 \) for sphere, \( 2 - 2g \) for a manifold of genus \( g \).

For the Fermat curve: \( n \) edges, \( n \) faces, \( 0 = n(3-n) \), so \( g = \frac{1}{2}(n-1)(n-2) \).

Let \( \tilde{Y} = \tilde{Y} \cup \{0\} \). Chart near \( 0 \) is given by: \( \tilde{Y} \tilde{U} \rightarrow \mathbb{A}^2 \).

**Example:** \( \mathbb{P}^n = \text{complex projective space of dimension } n \), \( \mathbb{C}^n \). \( \mathbb{P}^n = \underbrace{\mathbb{C}^n \cup \mathbb{C}^n \cup \cdots \cup \mathbb{C}^n}_{n \text{ times}} \). \( \mathbb{P}^n \) is a complex manifold.

\( \mathbb{P}^2 \) : points are \( \{w, z : w^2 - z^2 \} / \mathbb{R}^+, \) where \( (w, z) \sim (Aw, A^2z) \) if \( A \neq 0 \).

\( \mathbb{C}^2 = \mathbb{P}^2 \), \( (w, z) \sim (1, w, z) \). "Line at infinity" is \( \mathbb{P}^2 \).

In \( \mathbb{P}^2 \), we have the curve \( \tilde{Y} : \tilde{Y} = \tilde{Y} / \mathbb{A}^2 \), \( \tilde{Y} = \tilde{Y} / \mathbb{A}^2 \). \( \tilde{Y} \cap \mathbb{A}^2 = \mathbb{C}^2 \).

\( \tilde{Y} \) (line at \( 0 \)) = points where \( 4z^2 = 0, \) i.e., \( z = 0 \), i.e., \( (w, z) = (0, 0) \).

We shall now define an isomorphism of complex manifolds, \( X = \mathbb{C}^2 / \mathbb{Z} \), \( X \rightarrow \mathbb{C} \), using the Weierstrass elliptic function \( \wp : \mathbb{C} \rightarrow \mathbb{C} \), defined by:

\[ \wp(z) = \sum_{\lambda \in \Lambda} \left( \frac{1}{(w - \lambda)^2} - \frac{1}{(w + \lambda)^2} \right) + \frac{1}{z^2} = \frac{1}{z^2} + A z^4 + B z^6 + \cdots \]

Weierstrass \( \wp \) function: associated to a lattice \( \Lambda \subseteq \mathbb{C} \), \( \wp : \mathbb{C} \rightarrow \mathbb{C} \). This is a meromorphic function \( \wp : \mathbb{C} \rightarrow \mathbb{C} \), with poles at the points of \( \Lambda \). It is \( \Lambda \)-periodic.

First notice that \( \wp : \mathbb{C} \rightarrow \mathbb{C} \), analytic everywhere, which is \( \Lambda \)-periodic \( \{ \Lambda \text{-constant}\} \)

This would contradict Liouville's Theorem, for \( f \) would be bounded on any open closed parallelogram (continuous on compact set), hence bounded everywhere, so constant.

We also cannot have an \( \Lambda \)-periodic function \( f \) with one simple pole in each parallelogram \( \{0, w, w, w, w, w\} \), as this would contradict Cauchy's integral formula.

For if we had, consider \( \int_{\delta} f(z) dz = 0 \) by periodicity.

But, \( \int_{\delta} f(z) dz = 2 \pi i \text{(residue of } f) = 2 \pi i A \), if \( f(z) = \frac{A}{z^2} + B + \cdots \)

So \( A = 0 \), hence \( f \), analytic, so constant.

Also, if the pole is on the lattice, we may translate to put it in the interior.

But, we can have, say, two simple poles or one double pole.
$g_L$ is characterised by the fact that it has a double pole at $z=0$ with principal part $\frac{1}{2z^2}$, i.e. $f(z) = \frac{1}{2} + g(z)$ near $z=0$, with $g$ analytic in a neighbourhood of $z=0$. Consider $\frac{1}{\chi} \sum_{\lambda \in \Lambda} \frac{1}{|z-\lambda|^2}$. For given $\epsilon \in C$, number of points of the form $z-\lambda$ ($\lambda \in \Lambda$) with $R \leq |z-\lambda| \leq R+\epsilon$ is $\leq KR$ for some constant $K$.

For such $z-\lambda$ we have $|\frac{1}{|z-\lambda|^2}| \leq \frac{1}{R^2}$. Now, $\sum_{\lambda \in \Lambda} \frac{1}{|z-\lambda|^2}$ does not converge. Compare with $\sum_{n \geq 1} \frac{1}{n^2} = \pi \cot \pi z = \frac{1}{z} + \text{(analytic near } 0)$.

This sum equals $\frac{1}{z} + \sum_{n \not= 0} \left( \frac{1}{z+n} - \frac{1}{n} \right)$. Note: $\frac{1}{z+n} - \frac{1}{n} = \frac{2n}{z^2 - n^2} = O(1/n^2)$

Define $g_L(z) = \frac{1}{2} + \sum_{\lambda \in \Lambda} \left( \frac{1}{z-\lambda} - \frac{1}{\lambda} \right)$. This does converge absolutely, because $|\frac{1}{z-\lambda} - \frac{1}{\lambda}| \leq \frac{1}{18z^2}$ if $|z| \geq 18$, and $\sum_{n \geq 1} \frac{1}{n^2} < 1$.

The same argument shows that the series converges absolutely in any disc $12 < |z| \leq 13$ which contains no lattice points. If the disc contains one lattice point $\lambda$, and we omit the term $\frac{1}{z-\lambda} - \frac{1}{\lambda}$ from the series, it will converge absolutely and uniformly in the disc. Hence, $g_L(z)$ converges to an analytic function in the disc. Hence, $g_L(z)$ is meromorphic, with second order poles at $z=\lambda$.

Clearly, $g_L(z)$ depends only on $z$ modulo $L$, so it is really an analytic function $g_L: C/L \to C \cup \{\infty\}$, i.e. torus $\to$ Riemann sphere.

All points on the Riemann sphere have two distinct points in their pre-image except $\infty$ and 3 other points. See:

**Theorem:** $g_L(z)$ satisfies the equation: $g_L(z)^2 = 4B(g(z))^2 - 60A g(z) + 140B$, where

\[A = \sum_{\lambda \in \Lambda} \frac{1}{\lambda^2}, \quad B = \sum_{\lambda \in \Lambda} \frac{1}{\lambda} \]

Let $x = (g(z), g(\bar{z})) \in \text{curve} Y = \{(x,y): x^2 + Ay^2 + Bx^2 = 0\}$

We shall show that $g_L$ induces an isomorphism from $X = C/L$ to $Y = C \cup \{\infty\}$.

**Proof:** Near $z=0$ we have $g(z) = \frac{1}{2} + g(z)$ with $g(z)$ analytic. But, $g(z) = g(-z)$, so $L$ is a lattice, a group. So, $g(1) = g(-1)$. And, $g(0) = 0$. So, $g(z) = \frac{1}{2z} + \frac{1}{2}z^2 + \beta z^2 + \ldots$. So $g'(z) = -\frac{1}{2z^2} + 2z + 4\beta z^3 \ldots$. So, $g''(z) = -\frac{1}{2z^4} + \text{(analytic terms)}$.

So, $g''(z) - 4g'(z)^2 = -\frac{20\beta}{z^2} + \text{(analytic terms)} = -20\alpha g''(z) + \text{(analytic terms)}$.

So, $g''(z) - 4g'(z)^2 + 20\alpha g''(z)$ is analytic near $0$, hence analytic everywhere, hence constant.

We have $X = C/L \to \tilde{Y} = \{(x,y) \in C^2 : x^2 + Ay^2 + Bx^2 = 0\}$; $z \to (g'(z), g(\bar{z})) \in \tilde{Y}$ if $z=0$.

We want to show that $g(z+\alpha) = g(z)$ for all $L$.

$g(z + \alpha) - g(z) = \sum_{\lambda \in \Lambda} \left( \frac{1}{|z+\alpha-\lambda|^2} - \frac{1}{|z-\lambda|^2} \right) = \sum_{\lambda \in \Lambda} \frac{1}{|z+\alpha-\lambda|^2} - \sum_{\lambda \in \Lambda} \frac{1}{|z-\lambda|^2} = \frac{1}{\lambda} - \frac{1}{\lambda} = 0$. 


Theorem 1: $\mathbb{C}/\mathbb{Z} \to \hat{\mathbb{C}}$ is an isomorphism of complex manifolds.

(i) It is an analytic map, $z \mapsto (\mathbb{g}(z), \mathbb{g}(z)) \mapsto \mathbb{g}(z) \in \hat{\mathbb{C}}$, under two kinds of chart. To see what happens near $z = 0$ in $\mathbb{C}/\mathbb{Z}$, consider $\mathbb{g} \in \mathbb{C}^\ast$. $\mathbb{g} \to \mathbb{g} \mapsto (\mathbb{g}(z), \mathbb{g}(z))$ if $z \mapsto 0$, $0 \mapsto (0, 0)$. Consider the chart for $\mathbb{C}$ given by $(p, q, r) \mapsto (\mathbb{g}(p), \mathbb{g}(q)) \in \mathbb{C}^\ast$, when $q \neq 0$. In this chart, our map is $z \mapsto (\mathbb{g}(z), \mathbb{g}(z)) \in \mathbb{C}^\ast$. in the chart for $\mathbb{C}$, we have $z \mapsto \mathbb{g}(z) = \epsilon z$.

(ii) It is 1-1, onto, and its inverse is analytic.

(iii) Follows from:

Theorem 2: Suppose $X$ and $Y$ are two Riemann surfaces, both compact and Hausdorff, and suppose that $F: X \to Y$ is an analytic map. Then $V \neq Y$, the number of points in $F^{-1}(Y)$ is finite, and this number of points is the same, except for finitely many points ye $Y$. This number is called the degree of $F$. If the degree is 1, the map is an isomorphism.

Proof that Theorem 2 $\Rightarrow$ Theorem 1: $X$ and $Y$ are compact, Hausdorff, and $X$ is connected.

Enough to prove $\exists$ infinitely many points with $F^{-1}(Y)$ having just one point. Clearly, only one point $p \in X$. By considering the chart near $p_0 \in \mathbb{C}^\ast$, we have $z \mapsto \mathbb{g}(z) = \epsilon z$. This shows that $DF(z) = 0$ $DF(z) = \epsilon z$ in this chart.

So $F$ is 1-1 by the inverse function theorem in a neighborhood of $p_0$.

Proof of Theorem 2: Given $F: X \to Y$, suppose $F(z) = w$. Pick charts near $p, q$, say $\mathbb{C} \to \mathbb{C}$. Then $\mathbb{C} \to \mathbb{C}$ is an analytic function between open subsets of $\mathbb{C}$, and takes $\mathbb{C}$ to $\mathbb{C}$. As zeroes of analytic functions are isolated $\mathbb{C}$ does not take any point near $\mathbb{C}$ to $\mathbb{C}$. So, the points in $F^{-1}(w)$ are isolated in $X$. But, $X$ is compact and Hausdorff, so $F^{-1}(w)$ is finite. Now observe that for the same reason $DF(z) = 0$ except for finitely many $z$. (Look in terms of charts: $DF(z) = 0 \Leftrightarrow DF(\mathbb{C} \to \mathbb{C}) = 0$, and $DF(\mathbb{C} \to \mathbb{C})$ has isolated zeroes.) So, let $z_1, \ldots, z_n$ be the points where $DF(z) = 0$. Let $w_1 = F(z_1)$, and $Y = Y - w_1, w_n$. We prove that the number of points in $F^{-1}(y)$ is locally constant as a function of $y$ for $y \neq Y$. Hence it is constant in $y$ because removing $< \infty$ points cannot make a surface disconnected. Number of points in $F^{-1}(y)$ is locally constant, by inverse function theorem, for if $y \neq Y$ and $F(z) = y$, then $DF(z) = 0$, so $F$ gives a bijection $U_z \to U_y$ for some neighborhoods $U_z \subset X, U_y \subset Y$. Notice that because $X$ is Hausdorff, we can choose disjoint open neighborhoods $U_z$ of the points $z$ in $F^{-1}(y)$, and $F(U_z) = F(\text{compact}) = \text{compact, closed, and degree 1}$. If any point $y$ near $y$ has $F^{-1}(y) \subset U_z$. Finally, suppose the degree is 1. Then, 1 unique $F^{-1}: Y \to X$, and the inverse function theorem $\Rightarrow F^{-1}$ is analytic. But, in fact $Y = Y$, because if $DF(z) = 0$ for some $z$, then in terms of charts, $\mathbb{C} \to \mathbb{C}$ for $\mathbb{C}$ is not locally 1-1.
Covering Spaces

Let $X, Y$ be smooth manifolds of dimension $n$. Note: from now on, all manifolds are Hausdorff.

A smooth map $p : X \to Y$ is a **covering map** if every $y \in Y$ has a neighbourhood $V$ such that $p^{-1}(V) = \bigcup U_x$, where $F$ maps each $U_x$ by a diffeomorphism onto $V$.

**Examples:**
1. The inclusion of an open subset of $Y$ in $Y$ is not a covering map.
2. The map $p : \mathbb{C} \to \mathbb{C}$, $p(z) = z^2$, is covering map because $\text{if } |z| < 1$, then $z \to \log z$ is an inverse map $V = \{ z \in \mathbb{C} : |z| < 1 \} \to C$, and $p^{-1}(V) = \{ \log(z) + 2\pi i \alpha : \alpha \in \mathbb{Z} \}$, and the $U_x$ are disjoint. This deals with the property near $1 \in \mathbb{C}$.

   For other points outside $C$, consider $V_y = p^{-1}(V)$.
3. $p : \mathbb{C} \to \mathbb{C}/L$, natural map, where $L$ is a lattice.
4. $X = \mathbb{R}^2$, $Y = \mathbb{R}^2 \setminus \{ 0 \}$, $p(x) = \frac{x}{\|x\|}$. Then, $p : X \to Y$, $p(x) = x$ is a covering map.
5. $X = \mathbb{R}^2$, $Y = \mathbb{RP}^2$. Define $p : X \to Y$ by $p(x) = [x, y]$ for all pairs $x, y$. Then $p$ is a covering map.

**Lemma:** If $F : A \to B$ is a map of topological spaces and $A = A_1 \cup A_2$ where $A_1$ and $A_2$ are closed in $A$, and $F_{|A_1}$, $F_{|A_2}$ are continuous, then $F$ is continuous.

**Theorem:** Suppose $p : X \to Y$ is a covering map and $Y : [0, 1] \to Y$ is a path. Suppose $p(x_0) = y_0 = Y(a)$. Then, there is a unique lift $\tilde{X}$ of $Y$ starting at $x_0$, i.e., unique $\tilde{X} : [0, 1] \to X$ such that $p \tilde{X} = \tilde{Y}$ and $\tilde{X}(0) = x_0$.

**Proof:** Supposing $\tilde{Y}$ and $\tilde{X}$ are two such lifts, suppose $\tilde{Y}(t) = \tilde{X}(t)$ for some $t \in [0, 1]$. Choose a neighborhood $V$ of $Y(t)$ as in definition of covering. Then $\tilde{X}(s)$ and $\tilde{Y}(s)$ both belong to the same $U_x \subset F^{-1}(V)$. Take the interval $I$ in turn from the left. Suppose $\tilde{X}$ has been defined on $I_0 \cup I_1$. Clearly, if $\tilde{X}(I_0) \subset V$ and $p^{-1}(V) = \bigcup U_x$, then $\tilde{X}$ can be defined on $I_0 \cup I_1$ with $\tilde{X}(I_0) \subset U_x$.

**Existence:** Because $[0, 1]$ is compact we can successively bisect $I_1$ until it is the union of $2^n$ closed subintervals $I_i$ such that each $Y(I_i) \subset$ some $U_x$, with $p^{-1}(V) = \bigcup U_x$. Take the intervals $I$ in turn from the left. Suppose $\tilde{X}$ has been defined on $I_0 \cup I_1$. Clearly, if $\tilde{X}(I_0) \subset V$ and $p^{-1}(V) = \bigcup U_x$, then $\tilde{X}$ can be defined on $I_0 \cup I_1$ with $\tilde{X}(I_0) \subset U_x$.

**Definition:** Paths $\gamma, \gamma' : [0, 1] \to Y$ are homotopic if there is a continuous map $F : [0, 1] \times [0, 1] \to Y$ such that $F(0, t) = \gamma(t)$, $F(1, t) = \gamma'(t)$ for all $t \in [0, 1]$. Say $\gamma, \gamma'$ are homotopic rel end $x$ if $F(a, t) = \gamma(a) = \gamma'(a)$, $F(1, t) = \gamma(1) = \gamma'(1)$.

**Definition:** $Y$ is simply connected if it is path-connected and for any $y_0, y_1 \in Y$, any two paths from $y_0$ to $y_1$ are homotopic rel end $x$. 


If \( p: X \to Y \) is a covering map and \( F: [a,b] \times [0,1] \to Y \) and \( \pi(x) = y_0 = F(a,0) \), then \( F \) has a unique lift \( \tilde{F} \) such that \( \tilde{F}(0) = \tilde{F}(1) = x_0 \). In particular, if \( X, Y \) are paths in \( Y \) which are homotopic rel ends, and \( \tilde{x}, \tilde{x}' \) are lifts both starting at \( x_0 \), then \( \tilde{x}, \tilde{x}' \) are homotopic rel ends.

**Sketch of proof:** Uniqueness and "in particular" follow from the earlier lifting theorem.

- **Uniqueness:**
- **Existence:**

**Theorem:** \( p: X \to Y \) as before. Suppose \( Z \) is a simply connected manifold and \( F: Z \to Y \) is a smooth map. If, for some \( z_0 \in Z \), we have \( \pi(z_0) = F(z_0) = y_0 \in Y \), then \( \exists \) unique smooth \( \tilde{F}: Z \to X \) such that \( p \circ \tilde{F} = F \) and \( \tilde{F}(z_0) = x_0 \).

To show \( \tilde{F} \) is analytic near \( z = z_0 \), choose a neighborhood \( V \) of \( F(z_0) \) in \( Y \) such that \( V \) is open in \( Y \) and \( p^{-1}(V) = U \cup V \) as in definition of a covering map.

Clearly \( \exists \) a unique analytic map \( \Phi: V \to X \) such that \( p \circ \Phi = \tilde{F} \mid V \).

So \( \Phi \circ F \) and \( \tilde{F} \) are two lifts of \( F \mid V \) which agree at \( F(z_0) \), so they agree in all of \( p^{-1}(V) \). So \( \tilde{F} = \Phi \circ F \) in \( p^{-1}(V) \), and this is analytic.

**Corollary:** If \( p: X \to Y \) is a covering map, \( X \) connected, \( Y \) simply connected, then \( p \) is an isomorphism.

**Proof:** Take \( y = z_0 \) in preceding, and \( F: Y \to Y \) the identity. Get \( \tilde{F}: Y \to X \) such that \( p \circ \tilde{F} = \text{id} \).

And, \( \tilde{F} \) is surjective, because any two points \( x_0, x \) can be joined by a path which is the unique lift of a path from \( \pi(y_0) \) to \( \pi(y) \).

**Theorem:** Let \( X = \mathbb{C}/L, X' = \mathbb{C}/L' \). Then, \( X \cong X' \) as complex manifolds iff \( L' = \alpha L \) for some \( \alpha \in \mathbb{C} \).

**Proof:** We have a covering map \( p: C \to X \), \( Z = C \) is simply connected. Suppose we have an isomorphism \( \Psi: X \to X' \). Then consider: \( \begin{array}{c} C \xrightarrow{\Psi} X \xrightarrow{p} Z \xrightarrow{z \mapsto \gamma(z)} \mathbb{C} \end{array} \).

Consider the map \( z \mapsto \Psi(z + \lambda) - \Psi(z) \) from \( C \to \mathbb{C} \), for some \( \lambda \in \mathbb{C} \). The images of \( \Psi(z + \lambda) \) and \( \Psi(z) \) in \( \mathbb{C}/L' \) are the same, i.e., \( p(\Psi(z + \lambda)) = p(\Psi(z)) \).

So, \( \Psi(z + \lambda) - \Psi(z) \) is independent of \( z \). We can assume \( \Psi(0) = 0 \), as we have a group law on \( X \) and \( X' \). So we can assume that \( \Psi(\lambda) = \lambda \).

So, \( \Psi(z + \lambda) = \Psi(z) + \lambda \) if \( \lambda = \lambda \in \mathbb{C} \). In particular, \( \Psi \) is a homomorphism \( \mathbb{C} \to \mathbb{C}/L' \).

So \( \Psi \) is the restriction of an \( \mathbb{R} \)-linear map, \( \mathbb{R}^2 \to \mathbb{R}^2 \). So \( \Psi(\lambda) \in \mathbb{K} \mathbb{N} \), some \( \mathbb{K} \in \mathbb{R} \).

But if \( L = L_1 + L_2 \), let \( M = \{ 2w : \exists \lambda \in \mathbb{R} \} \). Clearly, \( \Psi(\lambda) \in \mathbb{K} \mathbb{N} \), some \( \mathbb{K} \in \mathbb{R} \).

But \( \tilde{F} \) is analytic \( \mathbb{C} \to \mathbb{C} \), so \( \tilde{F}(z) = az + b \), for some \( a, b \in \mathbb{C} \).

So \( \tilde{F}(0) = 0 \), so \( \tilde{F}(z) = az \), so \( aL = L' \).

But, \( \phi: \mathbb{C}/L \to \mathbb{C}/L' \) is bijective, so \( \phi(L) = L' \), so \( aL = L' \).

Conversely, if \( L' = aL \), then \( z \mapsto az \) induces an isomorphism \( \mathbb{C}/L \to \mathbb{C}/L' \).
How do we classify lattices \( L \) up to the equivalence relation \( L \sim \alpha L \)?

Clearly we may assume \( L = \mathbb{Z} + \mathbb{Z} \tau \) (as \( L = \mathbb{Z} \omega + \mathbb{Z} \omega = \mathfrak{n}([Z + 2\tau]/\omega) \), with \( \mathfrak{n} \tau > 0 \).

When is \( [Z + \mathbb{Z} \tau = a[Z + \mathbb{Z}] \) for some \( \tau, \alpha \in \mathbb{Z} \) and some \( a \)?

If so, \( \tau = a(\alpha + b \tau) \), some \( \alpha, b \in \mathbb{Z} \) and \( \lambda = a(\alpha + b \tau) \), \( \alpha, b \in \mathbb{Z} \). So \( \tau = \frac{a \omega + b \tau}{a \omega + b \tau} \). Similarly, \( \tau = \frac{a \omega + b \tau}{a \omega + b \tau} \), with \( \{a \omega + b \tau \} \in M_2(\mathbb{Z}) \). So, \( \det \{a \omega + b \tau \} = 1 \).

But, have \( \det = 4 \), else \( \tau \) takes \( \mathfrak{n} \tau > 0 \) to \( \mathfrak{n} \tau < 0 \). \( \left[ \mathfrak{n} \tau \right] = \frac{(\alpha - b \tau, \alpha - b \tau)}{a \omega + b \tau} \).

So the condition is \( \tau = \eta \tau \) for some Möbius transformation \( \{a \omega + b \tau \} \in SL_2(\mathbb{Z}) \).

Recall that given a lattice \( \mathfrak{g} \), we defined \( \mathfrak{g}(\mathfrak{g}) \to \mathfrak{g} \), where \( \mathfrak{g}(\mathfrak{g}) \) is the cubic curve with equation \( y^2 = 4x^3 - ax - b \) over \( \mathfrak{g} \).

Let \( \mathfrak{g} \) be \( \mathbb{Z} + \tau \mathbb{Z} \).

Say \( y^2 = \mathfrak{g}(x) \). The roots of \( \mathfrak{g} \) are the points \( \mathfrak{g}(1), \mathfrak{g}(2), \mathfrak{g}(3), \) because \( \mathfrak{g}(a) = \mathfrak{g}(a + \tau) \) for any \( a \in \mathfrak{g} \), \( \mathfrak{g}(a) = \mathfrak{g}(a + \tau) \), \( \mathfrak{g}(a) = \mathfrak{g}(a + \tau) \). So, \( \mathfrak{g}(1) = \mathfrak{g}(2) = \mathfrak{g}(3) = \mathfrak{g}(1) - 2 \). So, \( \mathfrak{g}(1) = 0 \) for any \( \mathfrak{g} \) such that \( \frac{1}{\mathfrak{g}} \).

Consider the cross-ratio of \( \{ \mathfrak{g}(1), \mathfrak{g}(2), \mathfrak{g}(1 + \tau), \mathfrak{g}(1) \} \).

Call it \( \mathfrak{g}(1) \). We have a holomorphic map \( \{ \mathfrak{g}(1), \mathfrak{g}(2), \mathfrak{g}(1 + \tau), \mathfrak{g}(1) \} \to \mathfrak{g} \) by \( \mathfrak{g}(1) \).

This map \( \mathfrak{g}(1) \) is a covering map.

**Corollary (Picard's Theorem):** If \( f: \mathfrak{g} \to \mathfrak{g} \) is analytic, then \( f \) is constant.

Suppose we had \( C/L \to C/\mathfrak{g}(\mathfrak{g}) \) degree. This would be an isomorphism, which is impossible, as \( C/L \to C/\mathfrak{g}(\mathfrak{g}) \) is a Riemann surface, which are topologically different. Any analytic map \( C/\mathfrak{g}(\mathfrak{g}) \to C/L \) is constant.

The \( \mathfrak{g}(\omega) \) group, cubic curve \( \mathfrak{g}(\omega) \) parametrised by \( z \to (\mathfrak{g}(z), \mathfrak{g}(z)) \).

"Addition formula for elliptic functions" \( z + z + z + z + l \to \det \left( \begin{array}{cc} \mathfrak{g}(1) & \mathfrak{g}(2) \\ \mathfrak{g}(3) & \mathfrak{g}(4) \end{array} \right) = \mathfrak{g}(1) \mathfrak{g}(2) - \mathfrak{g}(3) \mathfrak{g}(4) = 0. \)

Take \( P \) a curve, \( L \) a line in \( P \).

Define \( F: \mathfrak{g} \to C/L \) by \( F(1) = \mathfrak{g}(1), \mathfrak{g}(2), \mathfrak{g}(3), \mathfrak{g}(4) \), where \( \mathfrak{g}(1), \mathfrak{g}(2), \mathfrak{g}(3) \) are the parameters of the three points where the line \( P \mathfrak{g} \) meets the curve \( \mathfrak{g}(\omega) \).

This is an analytic map, so is constant. A pencil is the set of lines through a point. Any two pencils have a line in common.

Take \( L \in C, \mathfrak{g}: C/L \to C/\mathfrak{g}(\mathfrak{g}) \) and consider cross-ratio of 4 points. \( L \to j(L) \in C \) such that \( j(L) = j(L) \) \( L \to \alpha L \), some \( \alpha \in C \). \( j(\tau) = j(\tau) \) \( L \to \tau \), some \( \tau \in SL_2(\mathbb{Z}) \). \( j: \{ \mathfrak{n} \mathfrak{g}: \mathfrak{n} \mathfrak{g} > 0 \} \to C \) analytic, surjective.

- Classical modular function \( j(\tau + 1) = j(\tau), j(l \tau) = \sum_{\alpha \in 1 + \mathfrak{n}} q^\alpha \), \( q = e^{2\pi i \tau} \), \( \tau = \frac{1}{2} + a_0 + a_1 \).

**Modular form of weight \( k \):** \( f(L) \) such that \( f(L) = 2^{-k} h_f(L) \).

**Example:** \( f(L) = \sum_{\alpha \in 1 + \mathfrak{n}} \frac{1}{\alpha} \) - Eisenstein Series.

Suppose \( X \) is a topological space, \( Y \) a set. Two maps, \( F: U \to Y \), \( g: V \to Y \), where \( U, V \) are neighbourhoods of some \( x \in X \), have the same germ at \( x \) if \( f_x = g_x \) for some neighbourhood \( W \) of \( x \) in \( U \).
Analytic Continuation.

Examples: 1) \( s \in \mathbb{C}, \quad \text{Re}(s) > 1, \quad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) - Analytic. But, \( s \) extends to an analytic function, \( \zeta : \mathbb{C} - \{ 2 \} \to \mathbb{C}, \) with simple poles at some negative integers.
   2) \( f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} (1 - q^n) \zeta(z) \), where \( p_n \) = number of partitions of \( n; \) analytic for \( |q| < 1. \)

   a) \(-\log(1-z) = \sum \frac{z^n}{n} \) becomes multivalued.
   b) \( \frac{d}{dz} \log(1-z) = \sum_{n=1}^{\infty} \frac{1}{n} (1-z)^{n-1} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{dz}{dz} \) becomes multivalued.

Construct a Riemann surface as follows: Take all pairs \( (U, \overline{U}), \) \( U \) open \( \mathbb{C}, \) \( f : U \to \mathbb{C}, \)
   analytic. Introduce an equivalence relation: \( (z \in U, f) \sim (z \in \overline{U}, f') \) iff germ of \( f \) at \( z \) = germ of \( f' \) at \( \overline{z}. \)

   Set of equivalence classes, \( X \) = all germs of maps \( \mathbb{C} \to \mathbb{C}. \) This is a manifold.
   It has obvious charts, \( \Phi : (U, f) \to U \times \mathbb{C}. \) Write \( U \) for \( (U, f). \) What is \( \Phi : (U \times \mathbb{C}) \to (U, f)? \)
   It is \( \psi \in U \cap V \) where \( \phi \) and \( \psi \) have the same germ at \( z, \) an open subset of \( U. \)
   \( X \) is a Riemann 1-dimensional complex manifold. \( f \) function \( F : X \to \mathbb{C}, \) \( F|_U = f. \)

Definition: A complete multivalued analytic function is a connected component of \( X. \)

Note: Inverse function to \( g : x \to \exp(x). \) \( g(x) = y. \)
   So \( g^{-1} \) is the inverse function to \( g. \)

Simple-connectedness:

\[ f \text{ is continuous, locally constant, when is } f \text{ globally constant?} \]

Consider \( U = \mathbb{R}^3 \) (2-axis). \( f : U \to \mathbb{R}, \)
   \( f \left( \frac{x}{y} \right) = \left( \begin{array}{c} x \\ y \\ 
\end{array} \right) \)

For \( U \to \mathbb{R}, \) \( \Phi : [0, 1] \to U, \) suppose we can write \( \int \Phi(w) dw \)
   \( = \int \Phi(1) \Phi(0) = \int \Phi(1) \Phi(0) = 0. \)

Locally, the converse is true: \( \text{curl} f = 0 \Rightarrow f = \text{grad } \Phi \) in some neighborhood of any point.

In the example, \( f = \text{grad } \Phi, \) where \( \Phi : \cos \left( \frac{\sqrt{x^2+y^2}}{2} \right) = r \) in cylindrical polar coordinates.

Whenever we have a vector-valued function \( f : U \to \mathbb{R}^3 \) such that \( \text{curl } f = 0, \) we can define an invariant \( \delta \) for closed curves \( \gamma \) in \( U \) by \( \delta(f) = \oint \Phi(w) dw. \) If \( f_1, f_2 \) are two such functions, then \( \delta(f_1) = \delta(f_2) \) for all \( f \) iff \( f_1 - f_2 = \text{grad } \Phi \) for some \( \Phi : U \to \mathbb{R}. \)
We are led to introduce a group $H^1(U)$, the first de Rham cohomology of $U$, by:

$$H^1(U) = \{ \text{smooth functions } f: U \to \mathbb{R}^3 \text{ such that } \text{curl}(f) = 0 \} / \{ \text{smooth } F: U \to \mathbb{R}^3 \text{ such that } F = d\varphi \text{ for some } \varphi: U \to \mathbb{R} \}.$$ 

Note that $\text{curl,grad} = 0$, so $\{ f: f = \text{grad} \varphi \} \subset \{ f: \text{curl}(f) = 0 \}.$

If $U = \mathbb{R}^3 \setminus \{ \text{2-axis} \}$, then $H^1(U) = \mathbb{R}$, corresponding to the fact that the homotopy type of a closed curve is completely determined by just one invariant, $\delta_f = \text{"winding number"}$ for the particular $f$ described.

$$H^1(U) = \mathbb{R}, \text{ generated by } \left( \frac{2\pi}{2\pi} \right) = f \mapsto \int_{\gamma} f \text{ d}v \Rightarrow \text{ If } f \text{ satisfies } \text{curl}(f) = 0, \text{ then } f = \lambda f + \text{grad} \varphi, \text{ some } \lambda \in \mathbb{R}, \text{ some } \varphi: U \to \mathbb{R}.$$ 

Sketch proof: Choose $\gamma$ which winds once around the $z$-axis. Then, $\frac{2\pi}{2\pi} \int_{\gamma} f \text{ d}v = 2\pi.$ Let $\lambda = \frac{1}{2\pi} \int_{\gamma} f \text{ d}v$. Then, $\frac{2\pi}{2\pi} \int_{\gamma} (f - \lambda f) \text{ d}v = 0$. Now define $\varphi(p) = \frac{1}{2\pi} \int_{\gamma_p} (f - \lambda f) \text{ d}v$, where $\gamma_p$ is a path from some $p_0$ to $p$.

Similarly, $H^0(U) = \{ \text{functions } \varphi: U \to \mathbb{R} \text{ such that } \text{grad} \varphi = 0 \} / \{ \text{locally constant functions } \varphi: U \to \mathbb{R} \}.$

$\Rightarrow$ dim $H^0(U) =$ number of connected components of $U$.

$H^1(U) = \{ \text{functions } g: U \to \mathbb{R} \text{ such that } \text{div}(g) = 0 \} / \{ \text{functions } h: U \to \mathbb{R} \text{ such that } h = \text{curl } f \text{ for some } f: U \to \mathbb{R} \}.$

Notice that $\text{div, curl} = 0$, so $\{ g: g = \text{curl } f \} \subset \{ g: \text{div } g = 0 \}$. Locally, the converse is true. I.e., if $\text{div } g = 0$ then $g = \text{curl } f$ for some $f$ defined in a neighbourhood.

If one has $g$ such that $\text{div } g = 0$, we can define an invariant $\Phi_g$ for closed surfaces $\Sigma$ in $U$ by $\Phi_g(\Sigma) = \frac{1}{4\pi} \int_{\Sigma} \text{curl } g \text{ dvol} =$ "flux of $g$ through $\Sigma".$

Then, $\frac{1}{4\pi} \int_{\Sigma} \text{curl } g \text{ dvol} = \frac{1}{4\pi} \int_{\Sigma} \text{div } g \text{ dvol} - \frac{1}{4\pi} \int_{\Sigma} \text{curl } (\text{div } g) \text{ dvol} =$ "Green's Theorem" = 0 if $\text{div} g = 0$.

Notice that $\Phi_g(\Sigma) = \Phi_{g_2}(\Sigma)$ for all closed $\Sigma$ if $g_1 - g_2 = \text{curl } f$, because $\frac{1}{4\pi} \int_{\Sigma} (g_1 - g_2) \text{ dvol} = \frac{1}{4\pi} \int_{\Sigma} \text{curl } f \text{ dvol} = 0 \text{ if } \Sigma \text{ closed}.$

Example: $U = \mathbb{R}^3 \setminus \{ 0 \}$, $g: U \to \mathbb{R}^3$, field of point charge at $0$. $g\left( \frac{x}{|x|} \right) = \left( \frac{x_1}{|x|}, \frac{x_2}{|x|}, \frac{x_3}{|x|} \right).$

Then, $\text{div } g = 0$, but $g \neq \text{curl } f$ for any $f: U \to \mathbb{R}^3$, because $\frac{1}{4\pi} \int_{\Sigma} g \text{ dvol} = 4\pi \times \text{(number of times $\Sigma$ encloses the origin)}.$

This all leads to: $\mathcal{N}^0(U) \xrightarrow{\text{grad}} \mathcal{N}^1(U) \xrightarrow{\text{curl}} \mathcal{N}^2(U) \xrightarrow{\text{div}} \mathcal{N}^3(U)$ where $\mathcal{N}^0(U) = \mathcal{N}^3(U) = \{ \text{smooth } F: U \to \mathbb{R}^3 \}$ and $\mathcal{N}^1(U) = \mathcal{N}^2(U) = \{ \text{smooth } F: U \to \mathbb{R}^3 \}$.

Example: $\int_{\Sigma} \text{grad } \varphi \text{ d}S = \varphi(\text{end}) - \varphi(\text{end}) = \int_{\text{boundary}} \varphi \text{ d}S = \int_{\text{surface}} \varphi \text{ d}S = \int_{\Sigma} \varphi \text{ d}S$.

$\int_{\text{volume}} \varphi \text{ d}vol = \int_{\text{volume}} \varphi \text{ d}vol.$

"Stokes' Theorem."
U open in \( \mathbb{R}^n \). Define \( \mathcal{L}^k \) for \( k = 0, \ldots, n \) by:

\[ \mathcal{L}^k(U) = \{ \text{smooth maps } U \to \mathbb{R} \} \]

\[ \mathcal{L}^k(U) = \{ \text{smooth maps } U \to \mathbb{R}^n \} \]

\( \alpha \) is graded if some \( \frac{\partial \alpha}{\partial x_i} = 0 \), \( \mathcal{L}^2(U) = \{ \text{smooth maps } U \to \mathbb{R}^2 \} \), \( \mathcal{L}^n(U) = \{ \text{smooth maps } U \to \mathbb{R}^n \} \).

Have \( \mathcal{L}^2 \xrightarrow{\text{grad}} \mathcal{L}^1 \xrightarrow{\text{div}} \mathcal{L}^0 \xrightarrow{\text{rot}} \mathcal{L}^n \). In general, call each map \( d \) the exterior derivative.

Let \( V \) be a real vector space of dimension. Let \( \text{Alt}^k(U) = \text{alternating multilinear maps} \)
\( \alpha : V \times \ldots \times V \to \mathbb{R} \). Multilinear: linear in each variable separately, for example,

\[ \alpha(v_1, \ldots, v_k) = \alpha(\lambda v_1, v_2, \ldots, v_k) + \sum_{i=1}^k \lambda \cdot \alpha(v_1, \ldots, v_i, \ldots, v_k) \]

Alternating: if \( \pi : [1, \ldots, k] \to [1, \ldots, k] \)

is a permutation, then \( \alpha(\pi(v_1), \ldots, v_k) = (-1)^\pi \alpha(v_1, \ldots, v_k) \), where \( (-1)^\pi = \text{sign } \pi \).

By convention, \( \text{Alt}^0(U) = \mathbb{R} \), \( \text{Alt}^k(U) = V^k \), dual of \( V = \text{linear maps } U \to \mathbb{R} \).

\( \dim(\text{Alt}^k(U)) = \binom{n}{k} \), because if \( e_i, \ldots, e_n \) is a basis for \( U \) then any \( \alpha \in \text{Alt}^k(U) \) is completely determined by giving \( \alpha(e_{i_1}, \ldots, e_{i_k}) \) for all \( k \)-tuples \( (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \), with \( 1 < \ldots < i_k \).

Consider, if we give \( \binom{n}{k} \) numbers \( \alpha(i_1, \ldots, i_k) \) and define \( \alpha(i_1, \ldots, i_k) = 0 \) if not all \( i_1, \ldots, i_k \) are distinct, and \( \alpha(i_1, \ldots, i_k) \) is the permutation such that \( \pi(i_1) < \cdots < \pi(k) \), then we can define an element \( \alpha \in \text{Alt}^k(U) \) by \( (e_{i_1}, \ldots, e_{i_k}) \) and extend linearly.

There is a multiplication \( \text{Alt}^k(U) \times \text{Alt}^m(U) \to \text{Alt}^{k+m}(U) \); \( (\psi, \phi) \mapsto \psi \phi \), defined by

\[ \psi(\phi(u_1, \ldots, u_k)) = \sum_{\pi} (-1)^\pi \psi(\pi(u_1), \ldots, \pi(u_k)) \phi(\pi(u_{k+1}, \ldots, u_{k+m})) \]

A \( (k, m) \)-shuffle is a permutation \( \pi : [1, \ldots, k+m] \to [1, \ldots, k+m] \) such that \( \pi(1) < \cdots < \pi(k) \)

and \( \pi(k+1) < \cdots < \pi(k+m) \). There are \( \binom{n}{k+m} \) such shuffles.

This multiplication is bilinear, associative, and anticommutative. \( \psi, \phi \mapsto \psi \phi \equiv \alpha \beta \)

We have an anticommutative graded ring: \( \text{Alt}^*(U), \text{Alt}^*(U), \text{Alt}^*(U), \ldots \)

(\text{Associativity: } \alpha(\beta \psi) \psi = \alpha(\beta \psi) \psi = \alpha \beta(\psi) \psi = (-1)^{\text{sign } \beta} \alpha(\phi) \beta(\phi) \psi(\phi(u_{k+m+1}, \ldots, u_{k+m+n})) \]

Examples: Suppose \( \psi, \phi \in \text{Alt}^k(U) = V^k \). Then:

\[ \psi(\phi(u_1, \ldots, u_k)) = \psi(\phi) \phi(u_1, \ldots, u_k) = \det(\phi)(\psi(u_1, \ldots, u_k)) \]

If \( \alpha, \beta \in \text{Alt}^k(U) \), then \( \alpha(\beta(u_1, \ldots, u_k)) = \sum_{\pi} (-1)^\pi \alpha(\pi(u_1), \ldots, \pi(u_k)) \beta(\pi(u_{k+1}, \ldots, u_{k+n})) = \det(\alpha) \beta(u_1, \ldots, u_k) \).

In particular, if \( \{e_1, \ldots, e_n\} \) is a basis of \( U \) and \( \{e_1, \ldots, e_n\} \) is the dual basis of \( V^* \), then \( \psi(e_1, \ldots, e_n) = (-1)^n \psi(1, \ldots, n) \).

Note: \( \text{Alt}^*(U) \otimes \text{Alt}^*(U) \cong \text{Alt}^*(U) \otimes \text{Alt}^*(U) \)

(\text{Cf: vector product } (u_1, v_1) \times (u_2, v_2) \mapsto (u_1v_2 - u_2v_1, v_1u_2 - v_2u_1))

Define \( d : \mathcal{L}^k(U) \to \mathcal{L}^{k+1}(U) \) such that:

(1) \( d \) is linear,

(2) \( d(\alpha \beta) = d(\alpha \beta) + (-1)^{\text{sign } \alpha} \alpha(d(\beta)) \) if \( \alpha \in \mathcal{L}^k(U) \),

(3) \( d(\alpha) = 0 \), e.g., curl, grad.

(\text{grad: } \nabla f(u_1, \ldots, u_n) = (f(u_1) \frac{\partial}{\partial u_1}, \ldots, f(u_n) \frac{\partial}{\partial u_n})\]

For each \( \alpha \), \( d \alpha \) is linear in \( u_1, \ldots, u_n \) for each \( x \), i.e., \( d \alpha \) is a \( \mathcal{L}^k(U) \to \mathcal{L}^{k+1}(U) \) and \( d \) is a differential operator.
Let $x^1, \ldots, x^n$ be the coordinate functions on $U \subset \mathbb{R}^n$, $x^i \in \mathcal{L}^0(U)$. Then, $dx^1$ is a constant element of $\mathcal{L}^1(\mathbb{R})$, in fact, the standard dual basis element. So, any element of $\mathcal{L}^1(U)$ can be written as $f^i dx^i$, where $f^i \in \mathcal{L}^0(U)$. And if $f \in \mathcal{L}^0(U)$, we have $df = \sum_i \partial f_i dx^i$, where $\partial f_i = \partial f / \partial x^i$.

If $i_1, \ldots, i_m$ is any sequence from $1, \ldots, n$, write $dx^{i_1} \wedge \ldots \wedge dx^{i_m}$. Any element $\alpha$ of $\mathcal{L}^m(U)$ is a sum $\sum_{\sigma} f_\sigma dx^{i_{\sigma_1}} \wedge \ldots \wedge dx^{i_{\sigma_m}}$ with $f_\sigma \in \mathcal{L}^0(U)$, where I runs over $\sigma$, and this expression is unique. So, we can define $d\alpha = \sum_{\sigma} \partial f_\sigma dx^{i_{\sigma_1}} \wedge \ldots \wedge dx^{i_{\sigma_m}}$.

We must check that this has the required properties. First note that $d(f dx^i) = df dx^i$ for all sequences $I$. Obviously, $d : \mathcal{L}^k \to \mathcal{L}^{k+1}$ is linear. Antiderivation: $d(f dx^i \wedge \ldots \wedge \ldots \wedge dx^j) = d(fg) dx^i \wedge \ldots \wedge dx^j = (g df + df) dx^i \wedge \ldots \wedge dx^j = (df \wedge dx^i \wedge \ldots \wedge dx^j + \ldots)$

$d(f dx^i \wedge \ldots \wedge \ldots \wedge dx^j) = d(f dx^i \wedge \ldots \wedge \ldots \wedge dx^j)$. So, enough to prove that $d f = 0$ for $f \in \mathcal{L}^0(U)$. But $df = \sum_i \partial f_i dx^i$, so $df\\big|_x = \sum_i \partial f_i dx^i$, and $df dx^i \wedge \ldots \wedge dx^j = 0$, as $D_{\partial f_i} f$ is symmetric in $i$ and $j$, and $dx^i \wedge \ldots \wedge dx^j$ is antisymmetric.

Examples: If we $\mathcal{L}^2(U)$, write $x^1 = f_1 dx^1 + \ldots + f_n dx^n$, then $dx^1 = \sum_i \partial f_i dx^i \wedge dx^i = \frac{1}{2} \Sigma f_{ij} dx^i \wedge dx^j$. Thus, $dx = dx^1 \wedge \ldots \wedge dx^n$. Similarly, if we $\mathcal{L}^{n-1}(U)$, we can write $x = \frac{1}{n!} f_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n}$. So, $dx = \frac{1}{n!} f_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n} = \frac{1}{n!} f_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n} = 0, f_{i_1 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n} = 0$.

Let $\Psi : U \to V$ be a smooth map, where $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, open. We shall define a homomorphism of graded rings, $\Psi^* : \mathcal{L}^S(U) \to \mathcal{L}^S(V)$ for all $S$. In, it is linear, and $\Psi^*(x_1 \wedge \ldots \wedge x_k) = \Psi^*(x_1) \wedge \ldots \wedge \Psi^*(x_k)$, with the additional propery: $d \Psi^* = \Psi^* \circ d$.

Definition of $\Psi^*$: An element $x \in \mathcal{L}^n(U)$ is a map $x : U \to \mathcal{L}^n(\mathbb{R}^n)$. We can write it: $x(y; v_1, \ldots, v_n) = x(y; v_1, \ldots, v_n)$ with $y \in V$ and $v_i \in \mathbb{R}^n$.

Define $(\Psi^* x)(y; v_1, \ldots, v_n) = x(y; \Psi(v_1), \ldots, \Psi(v_n))$. $x \in U, \Psi(\mathbb{R}) \subset \mathbb{R}^m, \mathbb{R} \subset \mathbb{R}^m)$

Trivially, $\Psi^*$ is a graded ring homomorphism.

We must prove that $\Psi^* \circ d - d \Psi^*$ : $\mathcal{L}^S(U) \to \mathcal{L}^S(U)$ is zero. But this map is an antiderivation, so $d(\Psi^* x - d \Psi^* x) = -d(\Psi^* x - d \Psi^* x)$. So, we have $\Psi^* x - d \Psi^* x$ are in the kernel. So, it is enough to prove $d \Psi^* (f) = \Psi^* df$ for $f \in \mathcal{L}^0(U)$. $(\Psi^* df)(y; v_1, \ldots, v_n) = df(\Psi(v_1), \ldots, \Psi(v_n)) = df(v_1, \ldots, v_n) = df(x(y; v_1, \ldots, v_n)). [x(y; v_1, \ldots, v_n)]$

We can now define $\mathcal{L}^S(X)$ where $X$ is a smooth manifold. An element $x$ of $\mathcal{L}^S(X)$ is a collection of elements $x_i \in \mathcal{L}^S(U_i)$ for each chart $U_i : U_i \to V_i \subset \mathbb{R}^n$, such that $(\Psi^* x_i)^* x^*_i = x_i$ in $U_i$. Define $d : \mathcal{L}^S(X) \to \mathcal{L}^{S+1}(X)$ by $(dv_i) = dv, e \in \mathcal{L}^{S+1}(U_i)$, each i.
Clearly, the $\mathbb{R}^n(x)$ form an anticommutative graded ring, and $d: \mathbb{R}^n(W) \to \mathbb{R}^n(W)$ is an antiderivation such that $d\cdot d = 0$.

**Change of variables in $\mathbb{R}^n$.**

Suppose $\varphi: U \to V$ is a diffeomorphism, $U, V, \text{open, } CR^n$. Suppose $F: V \to \mathbb{R}$ is smooth with compact support. Then, $\int F(y) dy\wedge F(y) \wedge \ldots \wedge dy^n$ is defined, and

$$\int F(y) dx\wedge F(y) \wedge \ldots \wedge dx^n = \int F(y) dy\wedge \ldots \wedge dy^n.$$ (If $\varphi$ has compact support in $U$).

Consider the element $\alpha = \int F dy\wedge \ldots \wedge dy^n \in \mathbb{R}^n(V)$. Then $\varphi^*\alpha = \int F \det(DF) dx\wedge \ldots \wedge dx^n \in \mathbb{R}^n(W)$.

**Proof:** Write $\varphi$ as $(x_1, \ldots, x_n) \rightarrow (\varphi_1(x), \ldots, \varphi_n(x))$. $DF(x)$ is the linear transformation with matrix $[D\varphi_i(x)]_{i,j}$.

$$\varphi^*(\alpha) = \int F(\varphi) d\varphi_1 dy\wedge \ldots \wedge d\varphi_n dy^n = \int (F \circ \varphi) d\varphi_1(y) \wedge \ldots \wedge d\varphi_n(y).$$

But, $\varphi^*F(y) = y^2 - \alpha$.

So, $d(\varphi^*F) = \sum_j (D\varphi_j)(DF) dx^j$.

So, $\varphi^*(\alpha) = \int \sum_j (D\varphi_j)(DF) dx^j dx^n = \int \det(DF) dx\wedge \ldots \wedge dx^n$.

And $\varphi^*F dx\wedge \ldots \wedge dy^n = \int F \det(DF) dx\wedge \ldots \wedge dx^n$.

**Corollary:** If $\alpha \in \mathbb{R}^n(U)$ has compact support, then $\int \alpha = \int \varphi^*\alpha$ if $\det DF(?) = 0$ everywhere. In particular, $\int \alpha = \int \varphi^*\alpha$ if $\varphi$ is orientation preserving.

Recall that $X$ is oriented if it has an oriented atlas, i.e., is covered by a set of charts $\{\varphi_i: U_i \to V_i, CR^n\}$ between which the transition functions are orientation preserving.

**Definition:** A smooth manifold with boundary is a set $X$ with an atlas $\varphi_i: U_i \to V_i, CR^n$, where $V_i$ is an open subset of $IR^n = \{(x_1, \ldots, x_n): x_n \geq 0\}$, the transition maps being diffeomorphisms as before.

We must distinguish between the different types of open sets $CR^n$.

Notice that if $\varphi: V_1 \to V_2$ is a diffeomorphism between open subsets of $IR^n$, then $\varphi$ induces a diffeomorphism $V_1 \cap IR^n \to V_2 \cap IR^n$, where $IR^n = \{(x_1, \ldots, x_n): x_n = 0\}$.

The points of $X$ which in some (and hence every) chart map to boundary points of $IR^n$ are called boundary points of $X$.

Clearly, they form a smooth manifold of dimension $n-1$ (without boundary) which can speak of an oriented manifold with boundary. The boundary of such an oriented manifold gives an orientation because an oriented atlas gives an oriented atlas for the boundary.

If $\varphi_i: V_1 \to V_2$ is a transition function, it takes $x_i = 0$ to $x_i = 0$. So, at boundary points $x_i$, $D\varphi_i(x_1) = 0\ldots 0$. But $a_{1i} > 0$ because $\varphi_i(x_i) \subset IR^n$, so the fact that $\det D\varphi_i(x_1) = a_{1i} \cdot \det(D\varphi_i(x_1))$ gives an orientation on the boundary.
Definition: Let $X$ be a smooth compact manifold with boundary $\partial X$. Suppose $X$ is oriented with oriented atlas $\{\Phi_i: U_i \rightarrow \mathbb{R}^n\}$. Define $\Phi$ for any $x \in \partial^m X$ by $\Phi(x) = \tilde{\phi}(\Phi_i(x))$, where $\Phi_i$ is a smooth partition of unity for the covering $\{U_i\}$. Let $\tilde{f}_i: X \rightarrow \mathbb{R}$ be smooth, $\partial^m$, $\text{supp}(\tilde{f}_i) \subset U_i$, $\sum \tilde{f}_i = 1$. And, where $\tilde{f}(\Phi_i(x)) \in \mathbb{R}^{n}(V_i)$ is the representative of $x$ in the chart $\Phi_i: U_i \rightarrow \mathbb{R}^n$.

We must check this is well-defined. Suppose $\{\tilde{f}_j, U_j \rightarrow \mathbb{R}^n\}$ is a compatible oriented atlas and $\tilde{f}_j$ is a partition of unity subordinate to it. We must show $\sum \tilde{f}_j(\Phi_i(x)) = \sum \tilde{f}_i(\Phi_j(x))$, $\Phi(U_i \cap U_j) = \sum \tilde{f}_j(\Phi_i(x))$, and $\text{RHS} = \sum \tilde{f}_i(\Phi_j(x))$. Now, $\tilde{f}_j \Phi_i \in \mathbb{R}^{n}(U_i)$ and has compact support in $U_i \cap U_j$. So, $(\tilde{f}_j \Phi_i) \in \mathbb{R}^{n}(U_i)$ has compact support, as does $(\tilde{f}_i \Phi_j) \in \mathbb{R}^{n}(U_j)$. By definition, $\tilde{f}_j = \sum \tilde{f}_i \Phi_i$, where $\beta = \tilde{f}_j \Phi_i$ and $\gamma$ is the transition map, $\gamma \equiv \Psi^{-1} \Phi_j$. So, $\sum \tilde{f}_j \Phi_i \beta_j = \sum \tilde{f}_i \beta_i$, because $\gamma$ is orientation preserving.

This definition would have been just as good without assuming $X$ compact, providing $\partial X$ has compact support. (We have proved the existence of partitions of unity only for compact $X$, but all we used was that the manifold could be covered by finitely many charts, and all we need is that $\text{supp}(\gamma)$ can be covered by finitely many charts).

Stokes' Theorem: If $X$ is a compact oriented manifold with boundary $\partial X$ and $\beta \in \mathbb{R}^{n}(X)$, then $\int_X \beta = \int_{\partial X} \partial \beta$.

Proof: Enough (writing $\beta = \sum \tilde{f}_j \Phi_i \beta_j$) to prove this when $\beta$ has support in some $U_i$.

So, enough to prove $\int_V \beta = \int_{\partial V} \partial \beta$ for an open subset $V \subset \mathbb{R}^n$ and $\beta \in \mathbb{R}^{n}(V)$ with compact support inside $V$. In fact, to show $\int_V \beta = \sum \tilde{f}_j \Phi_i \beta_j$ if $\beta$ has compact support in $V$.

But, if $\beta = \sum \tilde{f}_j \Phi_i \beta_j$, then $\beta = \sum \tilde{f}_j \Phi_i \beta_j = \sum \tilde{f}_i \Phi_j \beta_i$, and $\beta = \sum \tilde{f}_i \Phi_j \beta_i$; because $\sum \tilde{f}_i = 1$ and $\beta$ has compact support.

Now, $\int_V \partial \beta = \int_V \sum \tilde{f}_j \Phi_i \beta_j$, because $\sum \tilde{f}_j \Phi_i = 1$ on $\beta$ has compact support.

Let $R$ be an $n$-dimensional region $C_X$, $\beta \in \mathbb{R}^{n}(X)$. $\int_X \beta = \int_{\partial X} \partial \beta$. Suppose $X$ is an $n$-dimensional smooth manifold and $Y$ is a compact oriented $k$-dimensional manifold with boundary. Suppose $F: Y \rightarrow X$ is any smooth map. Then we can define $F^*: \mathbb{R}^{k}(Y) \rightarrow \mathbb{R}^{n}(X)$ for each $i$, so that $F^*$ is a ring homomorphism, and $dF^* = F^* d$. Then the general Stokes' Theorem is: $\int_Y F^*(\beta) = \int_{\partial Y} F^* \beta$ for any $\beta \in \mathbb{R}^{k-1}(Y)$.

Have $\alpha \in \mathbb{R}^{k}(X)$, $\Phi_i: U_i \rightarrow \mathbb{R}^n <\Rightarrow \{ \alpha \in \mathbb{R}^{n}(U) \}$. New definition of an element of $\mathbb{R}^{n}(X)$: it is a map which to each $\alpha$ associates an element of $\mathbb{R}^{n}(U)$, i.e., a map $X \rightarrow \mathbb{R}$ $\alpha^k : X \rightarrow \mathbb{R}^{n}(X)$. And, it is a smooth map. To define this, we must make $\mathbb{R}^{n}(X)$ into a smooth manifold. Chart $\Phi_i: U_i \rightarrow \mathbb{R}^n$ gives $\alpha \in \mathbb{R}^{n}(U)$, $\alpha_i \in \mathbb{R}^{n}(U)$ is $\alpha_i \in \mathbb{R}^{n}(U)$ also $\alpha^k : T_x X \rightarrow \mathbb{R}^n$, $\alpha_i \rightarrow v_i$. If we change charts to $\Phi_j: U_j \rightarrow \mathbb{R}^n$, let $\alpha = \Phi_j \Phi_i^{-1}$. Then $v_i$ changes to $DF_j(\Phi_i)(v_i) = v_j$.
Let $\text{Alt}^k(T_x X)$ be the disjoint union of $\text{Alt}^k(U_x X)$ for $X = \text{Alt}^k(T_x X)$. There is an obvious map $\pi: Y \to X$. For each chart $q_i: U_i \to V_i \subset \mathbb{R}^n$ for $X$, define a chart $q_i: \pi^{-1}(U_i) \to V_i \times \text{Alt}^k(\mathbb{R}^n) = \mathbb{R}^n \times \text{Alt}^k(\mathbb{R}^n)$, by $x \mapsto (x, q_i(x))$, where $q_i$ corresponds to $x$ under the isomorphism $T_x X \to \mathbb{R}^n$ given by the chart $q_i$.

Check that this makes $\text{Alt}^k(T_x X)$ into a smooth manifold.

It is then trivial to check that an element of $\mathbb{R}^k(x)$ in the first sense is exactly equivalent to a smooth map $\alpha: X \to \text{Alt}^k(T_x X)$ such that $\pi \circ \alpha = \text{id}$.

Return to the definition of $f^*\alpha: \mathbb{R}^k(x) \to \mathbb{R}^k(y)$.

First method: Represent $x \in \mathbb{R}^k(x)$ by $\alpha_i \in \mathbb{R}^k(U_i)$ for $q_i: U_i \to V_i \subset \mathbb{R}^n$, covering $X$.

Then, $\{f^*(\alpha_i)\}$ is an open covering of $Y$. Choose charts $\tilde{q}_m: \tilde{U}_m \to \tilde{V}_m \subset \mathbb{R}^k(y)$ covering $Y$ so that $f(\tilde{U}_m) \subset$ some $U_i$. Then define $f^*\alpha$ by giving it the representation $(\tilde{q}_m: \tilde{f}(\tilde{w}_m))$ in $\tilde{V}_m$.

Second method: Given $x \in \mathbb{R}^k(x)$, we have $\alpha(x) \in \text{Alt}^k(T_x X)$. But $f$ induces $Df(x): T_x X \to T_y Y$, so induces $\text{Alt}^k(T_x X) \to \text{Alt}^k(T_y Y)$.

Define $f^*(\alpha)(y) = \text{image of } x \circ f(x))$ under this map.

$\text{Alt}^{k}(T_x X)$ is a vector bundle. $\alpha \in \text{Alt}^{k}(T_x X) \mapsto \text{Alt}^{k}(T_y Y)$.

$\text{Alt}^{k}(T_x X) = \pi^* \frac{\partial}{\partial x} \times x$. $\pi^{-1}(x) = (T_x X)^* = T_x^* X$, the cotangent space at $x$.

Orientation: To give an orientation of a smooth $n$-dimensional manifold $X$ is equivalent to giving an element $w \in \mathbb{R}^n(x)$ which is non-zero at every $x \in X$.

Proof: Given $w$, say that a chart $q_i: U_i \to V_i \subset \mathbb{R}^n$ is oriented if the representative $w_i$ of $w$ is of the form $F(y)dy_1 \wedge \ldots \wedge dy_n$ with $F(y) > 0 \forall y \in V_i$. Clearly, if two charts are oriented in this sense then the transition between them is orientation preserving. So the oriented charts form an oriented atlas.

$(\Rightarrow)$ (for compact $X$). Choose a finite number of charts $q_i: U_i \to V_i \subset \mathbb{R}^n$ forming an oriented atlas. Choose a subordinate partition of unity $f_i: \mathbb{R} \to [0, 1]$ with support $C(U_i)$. Then consider the form $\tilde{w}_i \in \mathbb{R}^n(U_i)$, whose representative in $U_i$ is $dx_i \wedge \ldots \wedge dx_n$.

Let $w = \Sigma \tilde{w}_i \in \mathbb{R}^n(x)$. Check this that this is a well-defined nowhere-vanishing $n$-form.

$\mathbb{R}^n$ is not orientable. $\mathbb{R}^n \cong \mathbb{R}$. $\mathbb{R}^n = \mathbb{R}^n(S^{n-1}) \mapsto \Sigma (-1)^{\pi(i)} \xi_i \cdot dx_i \wedge \ldots \wedge dx_n$.

Suppose we $\mathbb{R}^n(\mathbb{R}^n)$. Then $\pi \circ w \in \mathbb{R}^n(S^{n-1})$, and $i^*(\pi \circ w) = \pi \circ w$ where $i: S^{n-1} \to S^n$, $x \mapsto -x$, as $\pi \circ i = \pi$. But as $w \in \mathbb{R}^n(S^{n-1})$ never vanishes, we can write $\pi \circ w = f w_i$ for some function $f$ on $S^n$. If $w$ never vanishes, then neither does $f$.

But $i^*(f w_i) = w_i$, i.e. $i^* f : i^* w_i = f w_i$, but $i^* w_i = - w_i$, so $i^* f = -f$, i.e. $f(1-x) = -f(x)$.

Essentially the same argument $\Rightarrow \mathbb{R}^n$ is orientable if $n$ is even. Consider a chart $q_i: U_i \to \mathbb{R}^n$ for $\mathbb{R}^n$. Then $q_i$ induces two charts $q_i: U_i \to \mathbb{R}^n$, $i = 1, 2$, for $\mathbb{R}^n$, where $U_1, U_2 = \pi^{-1}(U_i)$. But $i: U_1 \to U_2$. Now, $w_2$ is non-vanishing on $\mathbb{R}^n$. Suppose it is represented by $w_1, w_2$ in $\mathbb{R}^n(U_1), \mathbb{R}^n(U_2)$, respectively, then $i^* w_2 = w_2$, if $n$ is even. So we can take $i^* w_2 = w_2$ as representations of an element of $\mathbb{R}^n(\mathbb{R}^n)$.
A smooth manifold, dimension $n$. Then, $\alpha \in \Omega^k(X)$ is closed if $d\alpha = 0$, and exact if $\alpha = \text{d}\beta$ for some $\beta \in \Omega^{k-1}(X)$. $d \circ \text{d} = 0$, so exact $\subseteq$ closed.

**Definition:** $H^k(X) = \frac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}$, the $k$th de Rham cohomology of $X$.

If $\alpha$ is a closed $k$-form representing an element of $H^k(X)$, we can define an invariant $I^k_\alpha$ for $k$-cycles in $X$. A $k$-cycle in $X$ is a map $f: Y \to X$, where $Y$ is a $k$-dimensional compact oriented manifold without boundary.

Define $I^k_\alpha(Y,f) = \int_Y f^* \alpha$, with $f^* \alpha \in \Omega^k(Y)$.

$I^k_\alpha(Y_1,f_1) = I^k_\alpha(Y_2,f_2)$ if the cycles $(Y_1,f_1)$ and $(Y_2,f_2)$ are homologous, i.e., if $f_1$ and $f_2$ are homotopic maps $Y \to X$, where $Y_n = Y$, with reversed orientation, and any smooth map $Y \to X$ which restricts to $f_1$ and $f_2$ on $\partial Y$.

**Stokes' Theorem:** if $\alpha = \alpha_1 + d\beta$, then $I^k_\alpha = I^k_{\alpha_1}$, because $\int_Y f^* \alpha = \int_Y f^* \alpha_1 + \int_Y f^* d\beta$, but $\int_Y f^* d\beta = \int_{\partial Y} f^* \beta = 0$ as $Y$ has no boundary.

**Similarly:** $\text{Stokes} \Rightarrow I^k_\alpha(Y_1,f_1) = I^k_\alpha(Y_2,f_2)$ if the cycles are homologous.

**Degrees of Maps.**

**Theorem:** If $X$ is a compact connected oriented $n$-dimensional manifold without boundary, then $H^n(X) \cong \mathbb{R}$, by the map $\alpha \mapsto \int_X \alpha$ for $\alpha \in \Omega^n(X)$. Now, $\int_X \alpha + \int_X d\beta = \int_X \alpha$, by Stokes. Equivalently, if $\alpha \in \Omega^n(X)$, then $\alpha = \text{d}\beta$ for some $\beta$.

**Corollary:** If $X$, $Y$ are as in the theorem and $\varphi: X \to Y$ is smooth, then $\int_X \varphi^* \alpha = N \int_Y \alpha$ for every $\alpha \in \Omega^n(Y)$, where $N \in \mathbb{N}$, called the degree of $\varphi$.

Furthermore, if $y \in Y$ is such that $D\varphi(x)$ is an isomorphism whenever $\varphi(x) = y$, then $N = \sum_{f_0(y) = f_0(x)} f_x$, where $f_x = \text{det } f_x$, if $D\varphi(x)$ has det $\leq 0$ in local charts.

**Proof:** Take $y$ as in "Furthermore." Then, by the inverse function theorem, $f$ open neighborhood $V_y$ of $y$ such that $\varphi^{-1}(V_y) = \cup_i U_i$, where $\varphi: U_i \to V_y$ is a diffeomorphism $U_i \cong \varphi^{-1}(y)$. Now choose we $\varphi^*(\alpha)$ such that $\text{supp}(\varphi^*\alpha) \subseteq V_y$, and $\int_{V_y} \varphi^*\alpha = 1$. Then $\varphi^*\alpha$ vanishes outside $\varphi^{-1}(V_y)$. So $\int_X \varphi^*\alpha = \sum \int_{f_x} \varphi^*\alpha = \sum f_x = N \int_Y \alpha$, say. But by the theorem, if $\alpha \in \Omega^n(V_y)$, then $\alpha = \lambda \omega + \text{d}\beta$, where $\lambda = \frac{1}{\gamma} \int f_x \omega$ (as $\int_{V_y} f_x \omega = 0$) $\int_{f_x} \varphi^*\alpha = \lambda \int f_x \varphi^*\alpha = N \lambda \int f_x \omega = \lambda N \int \omega$. 

"Borromean rings."
Suppose we have $\Phi : X \to Y$, map between $n$-dimensional manifolds. Let $x \in \Phi^{-1}(y)$.  
$D\Phi(x) : T_x X \to T_y Y$, is invertible $\iff x \in \Phi^{-1}(y) \iff y$ is a regular value of $\Phi$.  
Sard's Theorem: Almost all $y \in Y$ are regular.  

Let $\Phi : \text{unit ball in } \mathbb{R}^m \to \mathbb{R}^n$. If $\det(D\Phi(x)) = 0$ and $r > 0$ then $x$ has a neighbourhood $U_x$ with vol $\Phi(U_x) < \varepsilon \cdot \text{vol}(U_x)$.  

Recall: $H^n(x) \cong \mathbb{R}$, if $X$ is compact, connected, oriented, dimension $n$.  
Let $H^n_{\text{cpt}}(X) = Z^n_{\text{cpt}}(X) \cap Z^m_{\text{cpt}}(X)$ with compact support.  
Let $\alpha = d\beta$, where $\beta$ has compact support.  
We shall prove $H^n_{\text{cpt}}(X) \cong \mathbb{R}$ via $\alpha = d\beta$, providing $X$ is oriented and covered by a finite number of open sets diffeomorphic to $\mathbb{R}^n$.  

Proof by induction on the number of balls covering $X$. We must prove that if $\alpha \in H^n_{\text{cpt}}(X)$ and $\int_X \alpha = 0$ then $\alpha = d\beta$ for some $\beta \in H^n_{\text{cpt}}(X)$.  

Write $X = X_1 \cup X_2$, where result is known for $X_1$ and $X_1 \cap X_2 \neq \emptyset$, so result is known for $X_2$. $X_1 \cap X_2 \neq \emptyset$ as $X$ is connected.  

Given $\alpha$ with $\int_X \alpha = 0$, write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1 \in H^n_{\text{cpt}}(X_1)$, where $\alpha_1 = f_1 \alpha$ with $f_1$ a partition of unity. Can assume $\int_{X_1} \alpha_1 = \int_{X_2} \alpha_2 = 0$ by replacing them by $\alpha_1 + \gamma$, $\alpha_2 - \gamma$, respectively, where $\gamma$ has compact support in a ball $\subset X_1 \cap X_2$.  

Then, $\alpha_2 = d\beta_2$, with $\beta_2 \in H^n_{\text{cpt}}(X_2)$. So $\alpha = d(\beta_1 + \beta_2)$.  

Case $X = \mathbb{R}^n$: Suppose $\int_X \alpha = 0$, where $\alpha \in H^n_{\text{cpt}}(\mathbb{R}^n)$. Suppose $n = 1$ and $\int_X \alpha \wedge \text{dt} = 0$. Define $\beta(x) = \int_0^x \alpha \wedge \text{dt}$. Then $d\beta = \alpha$. $\beta$ has compact support $\iff \int_{\mathbb{R}} \beta = 0$.  

Use induction on $n$. Consider $n = 2$.  
We have $H^n_{\text{cpt}}(\mathbb{R}^2), \alpha = \alpha(x,y) \text{d}x \text{d}y$. Define $\beta = \beta(y) \text{d}y$ by $\beta(y) = \int_0^y \alpha(x,y) \text{d}x$.  

If $\beta = 0$, then $\alpha = d\gamma$, where $\gamma = \gamma(x,y) \text{d}y$. $\gamma(x,y) = \int_0^x \alpha(x,y) \text{d}x$. Clearly $d\gamma = x \text{d}y$. $\gamma$ is compact $\iff \beta = 0$.  

If $\beta \neq 0$, consider $\alpha = \rho \beta$, where $\rho = \rho(x) \text{d}x$, with compact support and $\int \rho \text{d}x = 1$. Clearly, $\int_\mathbb{R} (\rho(x,y) - \rho(y)) \text{d}x \text{d}y = 0$. So $\alpha = \rho \beta + d\gamma$. But $\int \rho \text{d}x = 1$, so $\beta = d\gamma$, for some $\gamma$ with compact support. So $\alpha = \rho \beta + d\gamma = d(-\rho \gamma + \gamma)$.  

Application of degree: if $f$ is a nowhere-vanishing smooth tangent vector field on $S^{n-1}$ if $n$ is odd. Suppose $f : S^{n-1} \to \mathbb{R}^n$ such that $\langle x, f(x) \rangle = 0$ for all $x$. We can assume $\|x\| = 1$ for all $x$. Define $F_f : S^{n-1} \to \mathbb{R}^n$ for $x \in \mathbb{R}^n$, $F_f(x) = \langle x, f(x) \rangle$.  

$F_f = \text{id}$, has degree $1$. $F_f$ is $x \mapsto -x$, and has degree $-1$ if $n$ is odd. 
(Recall that $\omega = \Sigma (-1)^i x_i \text{d}x_i$ for $x \in S^{n-1}$, so $d\omega = 0$. Degree of $F_f = \frac{1}{S^{n-1}} \int_{S^{n-1}} F_f^* \omega$ where $\int_{S^{n-1}} \omega = 1$, so depends continuously on $t$. But degree takes integer values, so $F_f$ is constant. $x$.  

Poincare lemma: if $\text{d} \alpha = 0$ for some $\alpha \neq 0$, then $\alpha = \text{d} \beta$, some $\beta \in H^{n-1}(\mathbb{R}^n)$.  

Proposition: Suppose $U$ is open in $\mathbb{R}^n$ and $\Phi : \mathbb{R}^n \to U$ is smooth, where $U$ is open. Let $\Phi(x) = \Phi(x,u)$ for $x \in U$. Let $\beta \in \Omega^n(U)$ be closed ($\text{d} \beta = 0$). Then $\Phi^* \beta$ and $\Phi^* \beta$ are closed in $\Omega^n(U)$, and $\Phi^* \beta - \Phi^* \beta = \text{d} \gamma$, some $\gamma \in \Omega^{n-1}(U)$.  

From this, for the Poincaré lemma, take \( U = U (= \text{star-shaped region in } \mathbb{R}^n) \), and \( \Phi(t, x) = \Phi(t) x \), where \( \Phi : \mathbb{R} \to \mathbb{R}, \Phi(t) = \{ 1 \} \Sigma t \geq 1 \).

Then \( \Phi_1 = \text{id} \), so \( \Phi_1^* \beta = \beta \), and \( \Phi_0 \) is the constant map, so \( \Phi_0^* \beta = 0 \), as \( \forall t \geq 0 \).

So, proposition \( \Rightarrow \beta = d\gamma \).

Proof of proposition: Let \( \gamma \) be a closed form on \( \mathbb{R} \times U \), and \( \alpha_t = \gamma |_{\mathbb{R} \times U} \) \( \left[ i \gamma = i \gamma_t \right] \).

Then, \( \alpha_t - \alpha_0 = d \) (something), \( \alpha_t \in A(U) \).

\( \alpha = \beta + dt \wedge \gamma \), where \( \beta, \gamma \) involve no \( dt \)'s. \( \beta = \Sigma \beta_t (t, x) dx^i \).

Notice that \( \alpha_t = \beta_t \), so we want \( \beta_t - \beta_0 = d \) (something).

But \( d\gamma = d_0 \beta + (dt \wedge \frac{\partial}{\partial t} \beta) - dt \wedge d_0 \gamma \), where \( d_0 \) involves no \( dt \).

This is zero \( \Rightarrow \frac{\partial}{\partial t} \beta = d_0 \gamma \), i.e. \( \frac{\partial}{\partial t} \beta_t = d_0 \gamma_t \).

Integrate this: \( \beta_t - \beta_0 = d \gamma \), where \( \gamma = \frac{\partial}{\partial t} \gamma_t dt \).