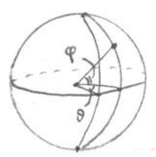
Differentiable Manifolds.

Assume smooth $\equiv$ differentiable $\equiv$ indefinitely often differentiable.

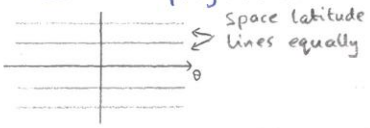
Example 1: Consider $S^2 = \{ x \in \mathbb{R}^3 : \|x\| = 1 \}$



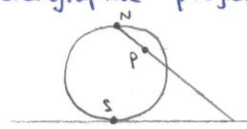
θ, longitude, $-\pi \le \theta < \pi$
φ, latitude, $-\frac{\pi}{2} \le \varphi \le \pi/2$

Conformal maps - preserve angles.

Mercator's projection:

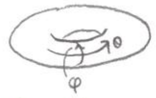


Stereographic projection:

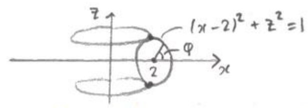


Definition: A smooth manifold is a set X together with a maximal atlas.
An atlas is a compatible collection of charts which cover X.
A chart is a pair (U, φ) consisting of a subset U of X and a map $\varphi : U \to \mathbb{R}^n$ which is injective, whose image is an open subset of $\mathbb{R}^n$.
Two charts $\varphi_i : U_i \to \mathbb{R}^{n_i}$, $i = 1, 2$, are compatible if:
(i) $\varphi_1(U_1 \cap U_2)$ is an open subset V_{12} of $\mathbb{R}^{n_1}$, $\varphi_2(U_1 \cap U_2)$ is an open subset V_{21} of $\mathbb{R}^{n_2}$
(ii) the bijections $\varphi_2 \circ \varphi_1^{-1} : V_{12} \to V_{21}$ and $\varphi_1 \circ \varphi_2^{-1} : V_{21} \to V_{12}$ are smooth (ie, C^∞)
"Charts cover X" means the sets U cover X.
An atlas is maximal means that any chart which is compatible with all the charts in the atlas is in the atlas.

Example 2:



Torus $\subset \mathbb{R}^3$. Or:



Surface of revolution: $(\sqrt{x^2+y^2} - 2)^2 + z^2 = 1$. Let $\frac{x}{\sqrt{x^2+y^2}} = \cos\theta$, $\frac{y}{\sqrt{x^2+y^2}} = \sin\theta$, $z = \sin\varphi$.
So, circle: $(2 + \cos\varphi, \sin\varphi)$, Surface: $((2+\cos\varphi)\cos\theta, (2+\cos\varphi)\sin\theta, \sin\varphi)$.
Torus $\leftrightarrow S^1 \times S^1$, $(\theta, \varphi) \leftrightarrow ((\cos\theta, \sin\theta), (\cos\varphi, \sin\varphi))$.
Torus is naturally in 1-1 correspondance with the subset of $\mathbb{R}^4$ consisting of all (x, y, z, t) with $x^2+y^2 = 1$, $z^2 + t^2 = 1$.

$X = $ Torus $\subset \mathbb{R}^3$. $X = \{ (x,y,z) \in \mathbb{R}^3 : (p-2)^2 + z^2 = 1$, where $p = +\sqrt{x^2+y^2} \}$ Find charts for X.
Define $\theta : X \to \mathbb{R}/2\pi\mathbb{Z}$ by $\cos\theta = \frac{x}{p}$, $\sin\theta = \frac{y}{p}$, $\varphi : X \to \mathbb{R}/2\pi\mathbb{Z}$ by $\cos\varphi = p - 2$, $\sin\varphi = z$.
One possible chart: take $U = $ all points of X with $p < 3$ and define $\Psi : U \to \mathbb{R}^2$ by $\Psi(x,y,z) = (\varphi\cos\theta, \varphi\sin\theta)$, where $\varphi \in [0, 2\pi)$.
Ψ is a 1-1 correspondance, $U \leftrightarrow \{ (\xi, \eta) \in \mathbb{R}^2 : 0 < \xi^2 + \eta^2 < (2\pi)^2 \}$.
Another chart: take $\tilde{U} = $ all points of X except those with $p = 1$.
Define $\tilde{\Psi} : \tilde{U} \to \mathbb{R}^2$ by $\tilde{\Psi}(x,y,z) = (\varphi\cos\theta, \varphi\sin\theta)$, where $\varphi \in (\pi, 3\pi)$.
This is a 1-1 correspondance, $\tilde{U} \leftrightarrow \{ (\xi, \eta) \in \mathbb{R}^2 : \pi^2 < \xi^2 + \eta^2 < (3\pi)^2 \}$.

$\Psi(U)$: 

, $\tilde{\Psi}(\tilde{U})$: 

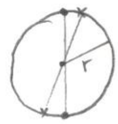
$\Psi(U \cap \tilde{U}) = $ punctured disc $-$ circle radius π
$\tilde{\Psi}(U \cap \tilde{U}) = $ annulus $-$ circle radius π.

We now have a 1-1 correspondance: $\Psi(U \cap \tilde{U}) \leftrightarrows \tilde{\Psi}(U \cap \tilde{U})$ given by:
$(x, y) \mapsto (x, y)$, if $\pi^2 < x^2+y^2 < (2\pi)^2$, $(x, y) \mapsto (\lambda x, \lambda y)$ if $0 < x^2+y^2 < \pi^2$, $\lambda = 2 + \sqrt{x^2+y^2}$
This is a smooth map. So is its inverse.

Remark: A smooth manifold has been described completely when we give a collection of compatible charts which cover it, because of:

Lemma: If $(U_\alpha, \varphi_\alpha)$ is a collection of compatible charts such that the U_α cover X, and (U', φ'), (U'', φ'') are charts which are compatible with all $(U_\alpha, \varphi_\alpha)$, then (U', φ'), (U'', φ'') are compatible with each other.

Example 3: Orthogonal group, $O_n = \{n\times n \text{ real matrices } A : A^TA = 1\}$ we are especially interested in the case $n=3$, and in $SO_n = \{A \in O_n : \det A = 1\}$.
$O_3 \subset 3\times 3$ real matrices $= \mathbb{R}^9$. O_3 = solution of 6 different equations.
We expect O_3 then to be 3-dimensional. Expect O_n to be of dimension $n^2 - \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$.
Elements of SO_3 are rotations about axes in $\mathbb{R}^3$ through angle θ, with $0 \le \theta \le \pi$. But, if $\theta = \pi$, the axis orientation is undefined.



Points of $SO_3 \leftrightarrow$ closed ball in $\mathbb{R}^3$ of radius π, with antipodal points on the boundary sphere representing the same element of SO_3.

First chart for SO_3: Take U = all rotations through π, all A with $\mathrm{Tr}(A) \neq -1$.
Define $\varphi : U \to \mathbb{R}^3$, $\varphi(A)$ = vector of length θ along the axis of rotation, where θ is the angle of rotation.
[Recall: $2\cos\theta + 1 = \mathrm{Tr}(A)$, as $A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ wrt some basis, and trace is invariant under change of basis.]
$\varphi(U)$ = open ball of radius π in $\mathbb{R}^3$.

Another chart for O_n: "Cayley parametrisation".
Let $\tilde{U} = \{A \in O_n : \det(A+I) \neq 0\}$. Then, $(A-I)(A+I)^{-1} = S$ is skew, since
$S^T = (A^T+I)^{-1}(A^T-I) = (A^{-1}+I)^{-1}(A^{-1}-I) = (A^{-1}(I+A))^{-1} A^{-1}(I-A) = (I+A)^{-1}(I-A) = -S$
Conversely, if S is skew, then $(S-I)(S+I)^{-1}$ is orthogonal.
Note that $\det(S+I) \neq 0$, as the eigenvalues of S are pure imaginary, as iS is Hermitian.
$\otimes$: $\tilde{U} \to \{n\times n \text{ real skew matrices}\} \cong \mathbb{R}^{\frac{1}{2}n(n-1)}$, $A \mapsto (A-I)(A+I)^{-1}$, is a chart.
If we identify (skew 3×3) $\leftrightarrow \mathbb{R}^3$ by $\otimes$, then the second chart is defined in the same region as the first, and $A \mapsto$ vector along axis of rotation, with length $\tan\frac{1}{2}\theta$.
So they are compatible.

Third chart for O_n: "exponential map".
If S is $n\times n$ real skew, then $e^S = I + S + \frac{1}{2!}S^2 + \dots$ is orthogonal.
For, $(e^S)^T = e^{S^T} = e^{-S} = (e^S)^{-1}$. Easy to check that $S \mapsto e^S$ is a bijection of skew matrices S such that $\|S\| < \pi$, and orthogonal matrices A such that $\|A - I\| < 2$.

Chart for SO_3: given by "Euler angles":

(θ, φ) = longitude, latitude of N'.
Let $A_\theta \in SO_3$ be rotation through θ about OZ.
$B_\varphi \in SO_3$ be rotation through φ about OY.

An element A of SO_3 has Euler angles (θ, φ, ψ), if $A = A_\theta B_\varphi A_\psi$.
This gives a 1-1 correspondance between a subset of $[0,2\pi]\times[0,\pi]\times[0,2\pi]$ and a subset of SO_3. It defines a chart of SO_3.

O_n is a group. Cayley parametrisation is a bijection $\mathcal{U} \to \{\text{skew matrices}\}$, where $\mathcal{U} = \{A \in O_n : \det(A+I) \neq 0\}$. Let $\mathcal{U}' = \{A \in O_n : \det(A-I) \neq 0\}$.
We have a bijection $\varphi' : \mathcal{U}' \to \{\text{skew matrices}\}$, by $\varphi'(A) = (A+I)(A-I)^{-1}$
$\varphi(\mathcal{U} \cap \mathcal{U}') = \{\text{invertible skew matrices}\}$ = open subset of all skew matrices.
Similarly, $\varphi'(\mathcal{U} \cap \mathcal{U}')$ = the same.
The bijection $\varphi' \cdot \varphi^{-1} : \varphi(\mathcal{U} \cap \mathcal{U}') \to \varphi'(\mathcal{U} \cap \mathcal{U}')$; $S \mapsto S^{-1}$ is smooth, with smooth inverse.
So these two charts are compatible.
Other charts: choose any $g \in O_n$, and define $\varphi_g : \mathcal{U}_g \to \{\text{skew matrices}\}$, where $\mathcal{U}_g = g\mathcal{U}$ and $\varphi_g(A) = \varphi(g^{-1}A)$. Because $g \in \mathcal{U}_g$, the sets $\{\mathcal{U}_g\}_{g \in O_n}$ cover O_n and are compatible, because $\varphi_{g_1}(\mathcal{U}_{g_1} \cap \mathcal{U}_{g_2}) \leftarrow \mathcal{U}_{g_1} \cap \mathcal{U}_{g_2} \to \varphi_{g_2}(\mathcal{U}_{g_1} \cap \mathcal{U}_{g_2})$, and $\varphi_{g_1}(\mathcal{U}_{g_1} \cap \mathcal{U}_{g_2}) = \{A : (g_1^{-1}A + I) \text{ and } (g_2^{-1}A + I) \text{ invertible}\}$ is open in $\varphi_{g_1}(\mathcal{U}_{g_1}) = \{\text{skew matrices}\}$.
$\varphi_{g_1}(\mathcal{U}_{g_1} \cap \mathcal{U}_{g_2}) = \{S : g_1^{-1}(\frac{S+I}{I-S}) + I \text{ and } g_2^{-1}(\frac{S+I}{I-S}) + I \text{ are invertible}\}$ - open set.
Smoothness of $\varphi_{g_1}(\mathcal{U}_{g_1} \cap \mathcal{U}_{g_2}) \to \varphi_{g_2}(\mathcal{U}_{g_1} \cap \mathcal{U}_{g_2})$. The map is a restriction of the composite: $S \mapsto \frac{S+I}{S-I} \mapsto g_1(\frac{S+I}{S-I}) \mapsto g_2 g_1^{-1}(\frac{S+I}{S-I}) = T \mapsto (T-I)(T+I)^{-1}$, of a sequence of smooth maps.

(Real) Projective Space.

The real projective plane $\mathbb{P}^2$ is the set of 1-dimensional vector subspaces of $\mathbb{R}^3$.
$\mathbb{P}^2$ is a smooth manifold. Let $\mathcal{U} \subset \mathbb{P}^2$ be all lines which meet the affine plane $x = 1$, ie, all lines which contain a vector of the form $(1, y, z)$. We have a 1-1 correspondance $\varphi : \mathcal{U} \to \mathbb{R}^2$, $\varphi(\text{line through } (1,y,z)) \mapsto (y,z)$.
$\mathbb{P}^2 \setminus \mathcal{U}$ = "points at ∞" from point of view of the plane $x = 1$.
Define $\mathcal{U}'$ = all lines which contain a point of the form $(x, 1, z)$. We have a 1-1 correspondance $\varphi' : \mathcal{U}' \to \mathbb{R}^2$, $\varphi(\text{line through } (x,1,z)) \mapsto (x,z)$. Similarly, define $\mathcal{U}''$ with $(x, y, 1)$ and $\varphi'' : \mathcal{U}'' \to \mathbb{R}^2$.
Clearly, $\mathcal{U}, \mathcal{U}', \mathcal{U}''$ cover $\mathbb{P}^2$. The charts are compatible.
$\varphi(\mathcal{U} \cap \mathcal{U}') = \{(y,z) \in \mathbb{R}^2 : y \neq 0\}$, $\varphi'(\mathcal{U} \cap \mathcal{U}') = \{(x,z) \in \mathbb{R}^2 : x \neq 0\}$, both open in $\mathbb{R}^2$.
The 1-1 correspondance $\varphi(\mathcal{U} \cap \mathcal{U}') \to \mathcal{U} \cap \mathcal{U}' \to \varphi'(\mathcal{U} \cap \mathcal{U}')$ takes:
$(y,z) \mapsto$ line through $(1,y,z)$ = line through $(y^{-1}, 1, y^{-1}z) \mapsto (y^{-1}, y^{-1}z)$. This is smooth.

Let X be a smooth manifold, with charts $\{\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n\}$. A subset W of X is called open if $\varphi_\alpha(W \cap \mathcal{U}_\alpha)$ is open in $\mathbb{R}^n$ for each chart. W is closed if $\varphi_\alpha(W \cap \mathcal{U}_\alpha)$ is a closed subset of $\varphi_\alpha(\mathcal{U}_\alpha)$ for each α. These "open" subsets give X the structure of a topological space, ie, (i) union of any family of open sets is open, (ii) intersection of any finite number of open sets is open. (iii) ϕ and X are open. We can now say "X is compact" or "X is connected."

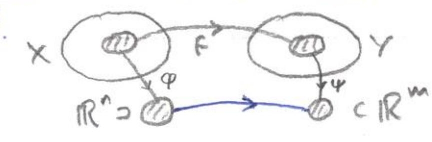
Two properties X may not have are:

(i) Hausdorffness: two points $x_1 \neq x_2$ are contained in two disjoint sets U_1, U_2.

[Example: $X = \mathbb{R}$, together with another point ω. Take two charts $(U_1 = \mathbb{R}) \xrightarrow{id} \mathbb{R}$, and $U_2 = \{\omega\} \cup (\mathbb{R} \setminus \{0\}) \xrightarrow{\varphi_2} \mathbb{R}$, where $\varphi_2(\omega) = 0$, $\varphi_2(\mathbb{R}\setminus\{0\}) = id$.]

(ii) Metrisability: $\exists$ metric on the set X such that a subset is open in the above sense (ie, for the atlas) iff it is open for the atlas.

If X, Y are two manifolds, then we say a map $f: X \to Y$ is $\left\{\begin{smallmatrix}\text{smooth}\\ \text{continuous}\end{smallmatrix}\right\}$ if for every chart $\varphi: U \to \mathbb{R}^n$ of X and $\psi: V \to \mathbb{R}^m$ of Y, the map $\psi \circ f \circ \varphi^{-1}: \varphi \circ f^{-1}(V) \to \mathbb{R}^m$ is $\left\{\begin{smallmatrix}\text{smooth}\\ \text{continuous}\end{smallmatrix}\right\}$.



We can thus speak about continuous or smooth $\mathbb{R}$-valued functions on X and continuous or smooth paths in X.

Exercise: A manifold is connected iff it is path-connected.

Proof of Compatibility Lemma: We must show (i) $\varphi'(U' \cap U'') \leftrightarrows \varphi''(U' \cap U'')$ are smooth, (ii) $\varphi'(U' \cap U'')$ is open in $\varphi'(U')$, and $\varphi''(U' \cap U'')$ is open in $\varphi''(U'')$

Now, $U' \cap U''$ is the union of $U' \cap U'' \cap U_\alpha$ for all α, so enough to show $\varphi'(W) = \varphi'(U' \cap U'' \cap U_\alpha)$ is open in $\varphi'(U')$, $\varphi''(W)$ is open in $\varphi''(U'')$, and that the bijections $\varphi'(W) \leftrightarrows \varphi''(W)$ are smooth.

Compatibility of U' and U_α and of U'' and U_α says that $\varphi_\alpha(U' \cap U_\alpha)$ is open in $\varphi_\alpha(U_\alpha)$ and $\varphi_\alpha(U'' \cap U_\alpha)$ is open in $\varphi_\alpha(U_\alpha)$.

So, $\varphi_\alpha(W) =$ (intersection of these) is open in $\varphi_\alpha(U_\alpha)$

But, we have smooth bijections $\varphi_\alpha(U_\alpha \cap U') \leftrightarrows \varphi'(U_\alpha \cap U')$. These take open sets to open sets, so $\varphi'(W)$ is open. Similarly, $\varphi''(W)$ is open, and we have smooth bijections $\varphi'(W) \rightleftarrows \varphi_\alpha(W) \rightleftarrows \varphi''(W)$, as we want.

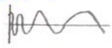
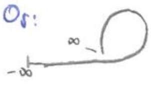
Suppose X is a smooth manifold and $Y \subset X$. Say Y is a submanifold of dimension m if for every point y of Y, $\exists$ chart $\varphi: U \to \mathbb{R}^n$ for X with $x \in U$ such that $\varphi(U \cap Y) = (\mathbb{R}^m \oplus \{0\}) \cap \varphi(U)$, $0 \in \mathbb{R}^{n-m}$.

Example: Let $Y = S^{n-1} \subset \mathbb{R}^n$. $Y = \{\xi \in \mathbb{R}^n : \|\xi\| = 1\}$. Consider $\begin{pmatrix}0\\ \vdots\\ 1\end{pmatrix} \in Y$. Take the chart for $\mathbb{R}^n$ given by $\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} \mapsto \begin{pmatrix}x_1\\ \vdots \\ x_{n-1}\\ x_n - \sqrt{1 - x_1^2 - \cdots - x_{n-1}^2}\end{pmatrix}$ defined in U, where $U = \left\{ \begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} : x_n > 0,\ x_1^2 + \cdots + x_{n-1}^2 < 1 \right\}$.

It takes $U \cap S^{n-1}$ to $(\mathbb{R}^{n-1} \oplus 0) \cap \varphi(U)$.

Example: To show O_n is a submanifold of $\mathbb{R}^{n^2}$ is a mess if done directly. Later, we shall prove a criterion for submanifolds.

Usually, we want only submanifolds which are either open or closed.

<u>Example</u>: We do not want the graph of $x \mapsto \sin 1/x$ on $0<x<1$, although the definition allows it. This is not closed: 
Or:  - This depicts a 1-1 smooth map $f: \mathbb{R} \to \mathbb{R}^2$ whose image is closed and whose derivative never vanishes. But the image is not a submanifold.

<u>Example</u>: dense winding on the torus, $S^1 \times S^1 = X$. Let $X = \{(u,v) \in \mathbb{C}^2 : |u| = |v| = 1\}$ 
Define $f: \mathbb{R} \to X$ by $f(t) = (e^{2\pi i t}, e^{2\pi i a t})$. If $a \in \mathbb{Q}$, say $a = p/q$ in lowest terms, then $f(\mathbb{R})$ is compact, in fact, a closed curve $X \cap \{u^p = v^q\}$, and is homeomorphic to a circle. If we embed X in $\mathbb{R}^3$ in the usual way, it is a "torus knot of type (p,q)". For example, $(2,3)$ gives the trefoil knot: 
Then, $f(\mathbb{R})$ is a closed submanifold of X.
But if $a \notin \mathbb{Q}$, then f is 1-1 and its image is dense in X.

Suppose $f: U \to \mathbb{R}^m$ is a continuously differentiable map, where U is open in $\mathbb{R}^n$. This means that for each $x \in U$, there is a linear map $Df(x): \mathbb{R}^n \to \mathbb{R}^m$, such that $f(x+h) = f(x) + Df(x).h + R(x,h)$, where $\frac{\|R(x,h)\|}{\|h\|} \to 0$ as $h \to 0$. "Continuously" differentiable means that, in addition, $x \mapsto Df(x)$ is a continuous map $\mathbb{R}^n \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$.

<u>Chain Rule</u>: If $U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}^k$, U open in $\mathbb{R}^m$, V open in $\mathbb{R}^n$, f, g continuously differentiable, then so is $g \circ f$, and $D(g \circ f)(x) = Dg(y) \circ Df(x)$, where $y = f(x)$.

In particular, if $U \to f(U) \subset \mathbb{R}^m$ is 1-1 with differentiable inverse, then $Df(x)$ is invertible for each $x \in U$ and so $n = m$.
Suppose U open $\subset \mathbb{R}^n$, $f: U \to \mathbb{R}^m$ continuously differentiable. $Df(x): \mathbb{R}^n \to \mathbb{R}^m$, linear. $\|Av\| \leq \|A\|.\|v\|$. Define $\|A\| = \sup_{\{v : \|v\| = 1\}} \|Av\|$. Then, $\|f(x) - f(y)\| \leq K\|x-y\|$, where $K = \sup_{z \in (x,y)} \|Df(z)\|$

<u>Inverse function theorem</u>: Let $f: U \to \mathbb{R}^n$ be continuously differentiable, U open $\subset \mathbb{R}^n$. Suppose $f(x) = y$, and $Df(x): \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Then, $\exists$ neighbourhood V of y in $\mathbb{R}^n$ and $g: V \to U$, continuously differentiable, with $f \circ g = id.: V \to V$ and $g \circ f = id.$ in $f^{-1}(V)$.
<u>Proof</u>: See handout I.

<u>Implicit function theorem</u>: Suppose $f: U \to \mathbb{R}^m$ (U open $\subset \mathbb{R}^n$) is continuously differentiable, and suppose $Df(x)$ has rank m, ie, is surjective, $\forall x \in U$. Then, $f^{-1}(y)$ is a smooth submanifold of U of dimension $n-m$.
<u>Proof</u>: We shall choose a linear map $p: \mathbb{R}^n \to \mathbb{R}^{n-m}$ such that $F: U \to \mathbb{R}^n$; $\xi \mapsto (f(\xi), p(\xi))$ has invertible $DF(x) = Df(x) \oplus p$. Choose $g: V \to \mathbb{R}^n$, where V is a neighbourhood of $(y, 0)$ in $\mathbb{R}^m \oplus \mathbb{R}^{n-m}$, such that $g \circ F = id$.
Then, $g^{-1}: g(V) \to \mathbb{R}^n$ is a chart for $\mathbb{R}^n$ near x, and takes $g(V) \cap f^{-1}(y)$ bijectively to $V \cap (\{y\} \times \mathbb{R}^{n-m})$. This is an open subset of $\{y\} \times \mathbb{R}^{n-m}$.
So it is a chart for $f^{-1}(y)$.

Example: Take U = all $n \times n$ invertible real matrices, and f: {all symmetric matrices}, with $f(A) = A^T A$. Then f is continuously differentiable.
$f(A+h) - f(A) = (A^T h + h^T A) + (h^T h)$. Clearly, $h^T h \to 0$ as $\|h\| \to 0$, and $h \mapsto A^T h + h^T A$ is linear. So, $Df(A).h = A^T h + h^T A \in$ {symmetric matrices}.
This is surjective, for if S is symmetric, take $h = \frac{1}{2}(A^T)^{-1} S$.
We have $Df(A).h = S$. Hence O_n is a submanifold of $\mathbb{R}^{n^2}$.

Lemma: If $A : \mathbb{R}^n \to \mathbb{R}^m$ is linear and of rank m, then $\exists$ linear $p : \mathbb{R}^n \to \mathbb{R}^{n-m}$ such that $A \oplus p : \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$.

Proof: We can choose p to be projection onto a coordinate subspace, ie $p\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \begin{pmatrix} p_1(\xi) \\ \vdots \\ p_{n-m}(\xi) \end{pmatrix}$ where $p_i(\xi) = \xi_{j(i)}$ for some $j(i)$.

Have: $\begin{array}{c} \\ m \\ \\ \end{array}\overset{n}{\begin{pmatrix} \boxed{A} \\ \boxed{P} \end{pmatrix}}$ Either, regard A as an $m \times n$ matrix with column rank m, hence row rank m, and add $n-m$ linearly independent rows.

Or, choose the linear maps $p_1, \ldots, p_{n-m} : \mathbb{R}^n \to \mathbb{R}$ successively so that p_i does not vanish identically on $\mathrm{Ker}(A \oplus p_1 \oplus \cdots \oplus p_{i-1})$

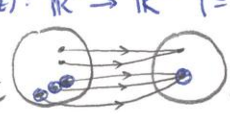
Inverse function theorem $\Rightarrow$ if $f : U \to V$ is a continuously differentiable bijection between open sets of $\mathbb{R}^n$, and $Df(x)$ is invertible $\forall x \in U$, then f^{-1} is continuously differentiable.

Definition: A diffeomorphism is a smooth map which is bijective with smooth inverse. (defined first for open subsets of $\mathbb{R}^n$, then for maps between smooth manifolds)

Theorem: $f \in \mathbb{C}[z]$, a polynomial, $f : \mathbb{C} \to \mathbb{C}$. f is surjective, say $f(z) = w$.

Proof 1: Consider the winding number of $f(z)$ around w, when z traverses a large circle $|z| = R$. Algebraic topology proof.

Proof 2: By inverse function theorem. Consider f as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$, and look at $Df(z) : \mathbb{R}^2 \to \mathbb{R}^2$ (= multiplication by $f'(z)$). $Df(z) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$
$\det Df(z) = |f'(z)|^2$.

$\mathbb{C}$  $\mathbb{C}$

- At most $n-1$ points where Df vanishes.

Inverse function theorem $\Rightarrow$ number of points in $f^{-1}(w)$ is locally constant as a function of w in the open set $\{w : w \neq f(\xi)$ with $f'(\xi) = 0\}$.
But, $\mathbb{C}$ − {finite number of points} is connected, so the number of points in $f^{-1}(w)$ is independent of w, so always $\neq 0$.

Orientability: Suppose $f : U \to V$ is a diffeomorphism where U, V are connected open subsets of $\mathbb{R}^n$. Then, $\det Df(x) \neq 0$ $\forall x \in U$. So it is either: >0 everywhere – say f is orientation preserving, or <0 everywhere – say f is orientation reversing.
More generally, say $f : U \to V$ is orientation preserving in $Df(x) > 0$ everywhere.
If X is a smooth manifold, say X is orientable if the set of all charts $\varphi : U \to \mathbb{R}^n$ can be divided into subsets C_1 and C_2 such that if $\varphi : U \to \mathbb{R}^n$, $\psi : V \to \mathbb{R}^n$ both belong to C_1 or both to C_2, then $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is orientation preserving.

<u>Definition</u>: An <u>orientation</u> of X is a choice of one of the classes C_1, C_2.

<u>Example</u>: Möbius band.  - one chart: "what you see" \ edge lines.
- another chart.

Tangent Vectors.

<u>Definition</u>: Let X be a smooth manifold. The <u>tangent space</u> T_xX to X at $x \in X$ is the set of all functions which assign to each chart (φ_u, U) with $x \in U$ a vector $\xi_u \in \mathbb{R}^n$ such that $\xi_u = D(\varphi_u \circ \varphi_v^{-1})(y)\xi_v$, where $y = \varphi_v(y)$, whenever (U, φ_u) and (V, φ_v) are two charts containing x.

Clearly, (i) T_xX is a vector space. $\dim(T_xX) = n = \dim X$.
(ii) an element is completely determined by giving ξ_u for one chart U containing x.
(iii) If $f: X \to Y$ is a smooth map between smooth manifolds, then f induces a linear map T_xX to $T_{f(x)}Y$ $\forall x \in X$. This is called $Df(x)$.
$Df(x)$ takes the family $\{\xi_u\}$ to the family $\{\eta_v\}$,
where $\eta_v = D(\varphi_v \circ f \circ \varphi_u^{-1})(w)\xi_u$, where $w = \varphi_u(x)$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cup & & \cup \\ U & & V \\ \downarrow \varphi_u & & \downarrow \varphi_v \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

<u>Alternatively</u>: X a smooth manifold. $\{\varphi_\alpha : U_\alpha \to \mathbb{R}\}_{\alpha \in \mathcal{T}}$ = collection of all charts for X. For $x \in X$, write $x_\alpha = \varphi_\alpha(x) \in \mathbb{R}^n$ if $x \in U_\alpha$. To define T_xX: an element of T_xX is a function $\xi : \{\alpha \in \mathcal{T} : x \in U_\alpha\} \to \mathbb{R}^n$, or a family $\{\xi_\alpha\}_{\alpha \in \mathcal{T}, x \in U_\alpha}$, with the property that $\xi_\beta = D(\varphi_\beta \circ \varphi_\alpha^{-1})(x_\alpha)\xi_\alpha$ -⊛, $\forall \alpha, \beta$ with $x \in U_\alpha, U_\beta$.

(i) T_xX is a vector space (because $D(\varphi_\beta \circ \varphi_\alpha^{-1})(x_\alpha)$ is a linear map).
(ii) For any α with $x \in U_\alpha$, the map $\xi \mapsto \xi_\alpha$ is an isomorphism $T_xX \to \mathbb{R}^n$.
[For, if we have ξ_α we can define ξ_β for any β with $x \in U_\beta$ by ⊛, and then ⊛ holds for all pairs β, γ with $x \in U_\beta \cap U_\gamma$, by chain rule.
$D(\varphi_\gamma \varphi_\beta^{-1})(x_\beta)\xi_\beta = D(\varphi_\gamma \varphi_\beta^{-1})(x_\beta) D(\varphi_\beta \varphi_\alpha^{-1})(x_\alpha)\xi_\alpha = D(\varphi_\gamma \varphi_\alpha^{-1})(x_\alpha)\xi_\alpha = \xi_\gamma$]
(iii) If X is a vector space, then T_xX is canonically isomorphic to X, by the map $\xi \in T_xX \mapsto \varphi_\alpha^{-1}\xi_\alpha \in X$, where $\varphi_\alpha : X \to \mathbb{R}^n$ is any linear isomorphism.
(This doesn't depend on α, because $D\varphi_\alpha(x) = \varphi_\alpha$ if φ_α is linear)
(iv) If $f: X \to Y$ is any smooth map, then f induces a linear map $Df(x) : T_xX \to T_{f(x)}Y$, for any $x \in X$. Let $\{\tilde{\varphi}_\alpha : \tilde{U}_\alpha \to \mathbb{R}^m\}_{\alpha \in \tilde{\mathcal{T}}}$ be the charts for Y.
Then define $(Df(x)\xi)_\alpha = D(\tilde{\varphi}_\alpha f \varphi_\beta^{-1})(x_\beta)\xi_\beta$. The chain rule shows that this is a well-defined tangent vector in $T_{f(x)}Y$.

$$\begin{array}{ccc} x \in X & \xrightarrow{f} & Y \ni f(x) \\ \downarrow \varphi_\beta & & \downarrow \tilde{\varphi}_\alpha \\ \mathbb{R}^n & & \mathbb{R}^m \end{array}$$

(v) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are smooth maps, then $Dg(y) \circ Df(x) = D(g \circ f)(x) : T_xX \to T_zZ$, with $y = f(x)$, $z = g(f(x))$. "<u>Chain rule</u>" - follows from chain rule for open subsets of $\mathbb{R}^n$, $\mathbb{R}^m$.

<u>Corollary</u>: If we have any rule which associates to each point x of each smooth manifold X a set $\mathcal{T}_xX$, and to each smooth $f: X \to Y$ a map $\mathcal{T}(f)(x) : \mathcal{T}_xX \to \mathcal{T}_{f(x)}Y$, such that:
(i) when X is a vector space, $\mathcal{T}_xX \cong X$, canonically.
(ii) $\mathcal{T}(g \circ f)(x) = \mathcal{T}(g)(y)\,\mathcal{T}(f)(x)$ when $X \xrightarrow{f} Y \xrightarrow{g} Z$; $x \mapsto y$.
(iii) If X' an open submanifold of X, then $Di(x) : \mathcal{T}_xX' \xrightarrow{\cong} \mathcal{T}_xX$, where $i: X' \to X$ is inclusion.
Then, $\mathcal{T}_xX \cong T_xX$, canonically.

Example: Suppose X is an n-dimensional submanifold of $\mathbb{R}^N$. Then, $T_x X$ can be identified with an n-dimensional vector subspace of $\mathbb{R}^N$, as follows: Choose a chart $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ with $x \in U_\alpha$. Let $\psi_\alpha : \varphi_\alpha(U_\alpha) \to X \hookrightarrow \mathbb{R}^N$ be φ_α^{-1} regarded as a map $\varphi_\alpha(U_\alpha) \to \mathbb{R}^N$. Then, image $(D\psi_\alpha(x_\alpha))$ is an n-dimensional subspace of $\mathbb{R}^N$ which does not depend on choice of chart. If $X \subset \mathbb{R}^N$ is "defined by equations", ie, we have $f : U \to \mathbb{R}^{N-n}$, where U is open in $\mathbb{R}^N$, f is smooth, and $Df(x)$ has rank $N-n$ $\forall x \in U$, and $X = f^{-1}(y)$, some $y \in \mathbb{R}^{N-n}$. Then, $T_x X =$ Kernel of $Df(x) : \mathbb{R}^N \to \mathbb{R}^{N-n}$.
More precisely, $\ker(Df(x)) = $ image $D\psi_\alpha(x_\alpha)$, with ψ_α as before.
For, $f \circ \psi_\alpha : \varphi_\alpha(U_\alpha) \to \mathbb{R}^N$ is constant, so $D(f\psi_\alpha) = Df \circ D\psi_\alpha = 0$, so $\mathrm{im}(D\psi_\alpha) \subset \ker(Df)$, and they have the same dimension, so are equal.

Example: $X = O_n \subset \mathbb{R}^{n^2}$. $X = f^{-1}(1)$, where $f : \mathbb{R}^2 \to \mathbb{R}^{\frac{1}{2}n(n-1)}$; $f(A) = A^T A$. $Df(A) = A^T h + h^T A$, so $\ker(Df(A)) = \{h : A^T h + h^T A = 0\} = \{h : A^T h \text{ is skew}\}$.
So, $T_A O_n = A\mathcal{S} = \mathcal{S}A \subset \mathbb{R}^2$, where $\mathcal{S} =$ all $n \times n$ skew matrices.
Alternatively, consider the parametrisation, (skew matrices) $\to O_n$, $S \mapsto (I+S)(I-S)^{-1}$
$\psi(S) = (S+I)(S-I)^{-1} = 1 + 2S + R(S)$, where $\frac{\|R(S)\|}{\|S\|} \to 0$ as $S \to 0$. So, $D\psi(0)$ is $S \mapsto 2S$.
So, $T_I O_n = $ image $D\psi(0) =$ all skew matrices.

$T_x X$ can be defined also:
(i) by means of curves through x.
Consider all smooth curves $\gamma : (-\varepsilon, \varepsilon) \to X$, such that $\gamma(0) = x$ (for any $\varepsilon > 0$).
Say two such curves are equivalent if they touch at x, ie, $\gamma_1 \sim \gamma_2$ if for any chart $\varphi : U \to \mathbb{R}^n$ in a neighbourhood of x we have $D(\varphi \circ \gamma_1)(0) = D(\varphi \circ \gamma_2)(0)$.
Let $\mathcal{T}_x X =$ set of equivalence classes.
The characterisation of $T_x X$ shows that $T_x X \cong \mathcal{T}_x X$, canonically.
(ii) in terms of derivatives.
Let $C^\infty(X) = \{$ all smooth maps $X \to \mathbb{R}\}$. This is a ring, and also a real vector space over the subring $\mathbb{R}$ of constant functions $X \to \mathbb{R}$. We can get the set X back from the ring $C^\infty(X)$, because $X \cong$ set of all ring homeomorphisms $C^\infty(X) \to \mathbb{R}$ via $x \mapsto \{f \mapsto f(x) = \varepsilon_x(f)\}$; $x \mapsto \varepsilon_x$. (Shall prove later that any homeomorphism $C^\infty(X) \to \mathbb{R}$ is ε_x for some x).
Given $x \in X$, let $\mathcal{D}_x X =$ all linear maps $\theta : C^\infty(X) \to \mathbb{R}$ which have the "Leibnitz property at x". Ie, $\theta(fg) = \theta(f)g(x)$ $f(x)\theta(g)$. (Such a θ is called a derivation).
[Notice that this is the same as saying that $f \mapsto f(x) + \varepsilon\theta(f)$ is a ring homeomorphism from $C^\infty(X)$ to $\mathbb{R}[\varepsilon]/(\varepsilon^2)$]
Clearly, $\mathcal{D}_x X$ is a vector space.

Theorem: $\mathcal{D}_x X$ is canonically isomorphic to $T_x X$.
Proof: First consider derivations of $C^\infty(\mathbb{R}^n)$ at $0 \in \mathbb{R}^n$. Shall prove that $\theta(f) = Df(0).\xi = \sum \xi_i \frac{\partial f}{\partial x_i}(0)$ for some $\xi \in \mathbb{R}^n$. ($Df = (D_1 f, \ldots, D_n f)$, $D_i f = \frac{\partial f}{\partial x_i}$). First note that $\theta(1)$, for $\theta(1) = \theta(1.1) = \theta(1).1 + 1.\theta(1) = 2\theta(1)$. So, $\theta(c) = 0$, c constant. But, any $f \in C^\infty(\mathbb{R}^n)$ can be expressed as $f = c + \sum x_i g_i$, where $x_i \in C^\infty(\mathbb{R}^n)$ is the ith coordinate function,

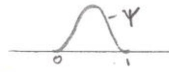
and $g_i \in C^\infty(\mathbb{R}^n)$ is such that $g_i(0) = D_i f(0)$. (For, $f(x) = f(0) + \int_0^1 \frac{d}{dt} f(tx)\, dt$. $\frac{d}{dt} f(tx) = \sum x_i D_i f(tx)$. So take $g_i(x) = \int_0^1 D_i f(tx)\, dt$. $g_i(0) = \int_0^1 D_i f(0)\, dt = D_i f(0)$)

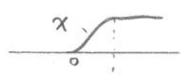
Now, $\theta(f) = \theta(c) + \sum(\theta(x_i) g_i(0) + 0 . \theta(g_i)) = \sum D_i f(0) \xi_i$, where $\xi_i = \theta(x_i)$.

Now let us prove that $\mathcal{D}_x X$ is an n-dimensional vector space for any n-dimensional smooth manifold X.

Lemma: $\exists$ smooth $\varphi : \mathbb{R}^n \to \mathbb{R}$ with $\varphi(x) = \begin{cases} 1 & \text{if } \|x\| \leq \varepsilon \\ 0 & \text{if } \|x\| \geq 2\varepsilon \end{cases}$ for any $\varepsilon > 0$.

Proof: First observe that e^{-1/x^2} on $[0, \infty)$ is C^∞ with all derivatives zero at 0.
So let $\psi(x) = e^{-1/x^2 - 1/(1-x)^2}$ if $x \in [0,1]$, and 0 otherwise. $\psi : \mathbb{R} \to \mathbb{R}$ is C^∞.
Let $\chi(x) = \int_0^x \psi(t)\, dt$ if $x \geq 0$, 0 otherwise.
Then, χ is C^∞ and constant, say c, for $x \geq 1$.





Finally, define $\varphi(x) = 1 - \frac{1}{c} \chi\left(\frac{\|x\| - 1}{\varepsilon}\right) : \mathbb{R}^n \to \mathbb{R}$. φ is as required.

Corollary: If X is any smooth manifold and $x \in X$ and U is a neighbourhood of x, we can find $\varphi : X \to \mathbb{R}$ such that $\operatorname{supp} \varphi \subset U$, and $\varphi = 1$ in a neighbourhood of x. (Recall: $\operatorname{supp} \varphi = \overline{\{y : \varphi(y) \neq 0\}}$)

Now suppose that $\theta : C^\infty(X) \to \mathbb{R}$ is a derivation at x. Choose φ such that $\varphi = \begin{cases} 1 & \text{near } x \\ 0 & \text{away from } x. \end{cases}$ Then, $\theta(\varphi) = 0$, because we can find $\tilde{\varphi}$ such that $\varphi \tilde{\varphi} = \tilde{\varphi}$, and then $\theta(\tilde{\varphi}) = \varphi(x) \theta(\tilde{\varphi}) + \theta(\varphi) \tilde{\varphi}(x) = \theta(\tilde{\varphi}) + \theta(\varphi)$. (Choose $\tilde{\varphi}$ to be 1 near x, and with $\operatorname{supp}(\tilde{\varphi}) =$ region where $\varphi = 1$). So, for any $f \in C^\infty(X)$ we have $\theta(f\varphi) = \theta(f)$.
This tells us that if $w : U \to \mathbb{R}^n$ is a chart near x then $\mathcal{D}_x X \cong \mathcal{D}_x U$.

For $C^\infty(X) \xrightarrow{\text{restriction}} C^\infty(U)$, $\theta : C^\infty(X) \to \mathbb{R}$, $\tilde{\theta} : C^\infty(U) \to \mathbb{R}$ - Given θ, define $\tilde{\theta}$ by $\tilde{\theta}(f) = \theta(f\varphi)$. $f \in C^\infty(U)$, $f\varphi$, extended by 0 outside U, $\in C^\infty(X)$.

But, by the chart, we have $\mathcal{D}_x U \cong \mathcal{D}_{w(x)}(w(U)) \cong \mathcal{D}_{w(x)}(\mathbb{R}^n) \cong \mathbb{R}^n$.

We now check trivially that $\mathcal{D}_x X$ has all the properties to be $T_x X$.
Main point: If $f : X \to Y$ is smooth, then f induces a ring homomorphism $f^* : C^\infty(Y) \to C^\infty(X)$, so map $\mathcal{D}_x X$ to $\mathcal{D}_{f(x)} Y$.

The disjoint union $\bigcup_{x \in X} T_x X$ is called the <u>tangent bundle</u>, written TX, of X.
It is a smooth manifold of dimension $2n$, where $n = \dim X$.
To give charts: Let $\varphi : U \to \mathbb{R}^n$ be a chart for X, then let $TU = \bigcup_{x \in U} T_x X$, and define $\tilde{\varphi} : TU \to \mathbb{R}^n \oplus \mathbb{R}^n$ by $\tilde{\varphi}(x, \xi) = (\varphi(x), \xi_u)$, where ξ_u is the representation of ξ in the chart. We have $\tilde{\varphi}(TU) = \varphi(U) \times \mathbb{R}^n$, which is open in $\mathbb{R}^{2n}$.
The transition between two charts corresponding to $\varphi_1 : U_1 \to \mathbb{R}^n$ and $\varphi_2 : U_2 \to \mathbb{R}^n$ is $(\varphi_2 \circ \varphi_1^{-1}) \times D(\varphi_2 \circ \varphi_1^{-1})$, ie, $(y, \eta) \mapsto (\varphi_2(\varphi_1^{-1}(y)), D(\varphi_2 \varphi_1^{-1})(y)\eta)$, which is smooth.

A <u>tangent vector field</u> on X is a smooth map $s : X \to TX$ such that $s(x) \in T_x X$ for all x.

Let $x \in X$, $x \in$ open $U \subset X$. If X is a compact smooth manifold, and $\{U_\alpha\}$ is a finite open covering of X by the domains of charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, then we can find smooth functions $f_\alpha : X \to \mathbb{R}$ such that: (i) $f_\alpha(x) \geq 0$ everywhere, (ii) $\operatorname{supp}(f_\alpha) \subset U_\alpha$, (iii) $\sum_\alpha f_\alpha(x) = 1 \ \forall x$.

$\{f_\alpha\}$ is called a partition of unity, subordinate to the covering $\{U_\alpha\}$.

Proof: For each x, choose $g_x : X \to \mathbb{R}$ smooth and ≥ 0 such that $g_x = 1$ near x, and $\operatorname{supp}(g_x) \subset$ some U_α. By compactness, $\exists\, x_1, \dots, x_n$ such that for every x, $g_{x_i}(x) \neq 0$ for some i. For each x_i, choose α_i such that $\operatorname{supp}(g_{x_i}) \subset U_{\alpha_i}$. Let $h_\alpha = \sum_{\alpha_i = \alpha} g_{x_i}$. Then, $h = \sum_\alpha h_\alpha$ is smooth and >0 everywhere. Take $f_\alpha = \frac{h_\alpha}{h}$.

Proposition: $X \cong$ set of non-zero ring homomorphisms $\theta : C^\infty(X) \to \mathbb{R}$.

Proof: See handout.

Theorem: Any smooth manifold of dimension n is diffeomorphic to a submanifold of $\mathbb{R}^{2n+1}$ - Whitney embedding theorem.

Proof: (Sketch for when X is compact)

- These are 1-d, but cannot embed in $\mathbb{R}^2$, but they are not manifolds.

First prove $X \cong$ submanifold of $\mathbb{R}^N$ for some N.

Cover X by finitely many charts, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, $\alpha = 1, \dots, k$. Then choose a partition of unity $f_\alpha : X \to \mathbb{R}$ with $\operatorname{supp}(f_\alpha) \subset U_\alpha$. Define $F : X \to \mathbb{R}^n \oplus \dots \oplus \mathbb{R}^n = \mathbb{R}^{nk}$ by $F(x) = (f_1(x)\varphi_1(x), \dots, f_k(x)\varphi_k(x))$, where we define $f_i(x)\varphi_i(x) = 0$ if $x \notin U_i$.

Then, F is smooth, and $F|_{(\text{region where } f_\alpha \neq 0)}$ is obviously 1-1.

But, points x for which $\{\alpha : f_\alpha(x) \neq 0\}$ is different obviously have different images. So F is 1-1.

We now need: Criterion: if $F : X \to \mathbb{R}^N$ is smooth, 1-1, and X is compact, and $DF(x)$ has rank n for each $x \in X$, then F is a diffeomorphism between X and a submanifold of $\mathbb{R}^N$.

In our case, the criterion is satisfied, because $DF(x) = (Df_1(x) . D\varphi_1(x), \dots)$, and at each point $D\varphi_i(x)$ has rank n and at least one $Df_i(x)$ is $\neq 0$.

Consider the criterion above. To get a good chart for $\mathbb{R}^N$ near $f(x)$, choose linear $h : \mathbb{R}^{N-n} \to \mathbb{R}^N$ such that $DF(x) \oplus h : T_x X \oplus \mathbb{R}^{N-n} \xrightarrow{\cong} \mathbb{R}^N$. Then apply inverse function theorem to $(f \cdot \psi) \oplus h$, where $\psi : V \to X$ is the inverse of a chart for X.

Consider: $X \hookrightarrow \mathbb{R}^N$ - if $N > 2n+1$

$\downarrow$ — orthogonal projection.

$\searrow V \subset \mathbb{R}^N$, V an $(N-1)$-dimensional linear subspace.

The space of all pairs of points of X joined by a line: dim $2n+1$ (Avoiding lines tangent to X).

Complex Manifolds (mostly one-dimensional)

Suppose U is an open subset in $\mathbb{C}^n$, and $f: U \to \mathbb{C}^m$ is a continuous map. Say f is analytic at $z \in U$ if $\exists$ $\mathbb{C}$-linear map $A: \mathbb{C}^n \to \mathbb{C}^m$ such that $f(z+h) = f(z) + Ah + R(z,h)$, where $\|R(z,h)\|/\|h\| \to 0$ as $h \to 0$. Ie, f is differentiable in the real sense, and the derivative, $Df(z)$, is $\mathbb{C}$-linear. Ie, $Df(z)$ satisfies the Cauchy-Riemann equations.

Example: $f = u + iv$. $Df = \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$. $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Need $Df.i = i.Df$.

Analytic maps are very tightly constrained. For example:
(i) once continuously differentiable $\Rightarrow$ derivatives of all orders exist.
(ii) if two analytic functions $f_1, f_2 : U \to \mathbb{C}^m$ agree in any open subset of U, they agree everywhere.
(iii) if $n = m = 1$, then either f = constant or f(open) = open.

Definition: X is a complex manifold of dimension n if it has a maximal atlas of compatible charts $\varphi: U \to \mathbb{C}^n$, where $\varphi(U)$ is open in $\mathbb{C}^n$, and "compatible" means that the transition maps $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ are analytic.

A complex manifold X is canonically oriented, for if $\Psi: U \to V$ is an analytic bijection, with U, V open in $\mathbb{C}^n$, then Ψ is automatically orientation preserving when regarded as a smooth map $U \to V$ between subsets of $\mathbb{R}^{2n}$.

Example: $\det \begin{pmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{pmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2 \geq 0$ (if Cauchy-Riemann equations hold).

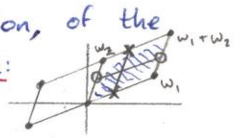
In general, when $Df(z)$, which is an $n \times n$ complex matrix, is regarded as a $2n \times 2n$ real matrix T, have $\det T = |\det Df(x)|^2$

Examples (i): Riemann sphere, $X = \mathbb{C} \cup \{\infty\}$. Charts: $\varphi: U = \mathbb{C} \xrightarrow{id.} \mathbb{C}$; $\tilde{\varphi}: \tilde{U} = (\mathbb{C} - \{0\}) \cup \{\infty\} \to \mathbb{C}$, $\tilde{\varphi}(z) = z^{-1}$.
$\varphi(U \cap \tilde{U}) = \tilde{\varphi}(U \cap \tilde{U}) = \mathbb{C} - \{0\}$, and the transition map is $z \mapsto z^{-1}$.
U is open in $\mathbb{C}$. Suppose $z \in U$ and $f: U - \{z\} \to \mathbb{C}$ is analytic. When can we extend f to an analytic map $f: U \to S = \mathbb{C} \cup \{\infty\}$? Precisely if f has at most a pole at z, ie, no essential singularity. Because f has a pole at z iff $w \mapsto \frac{1}{f(w-z)}$ has a removable singularity at $w = z$. But then, if we define $f(z) = \infty \in S$, then using the chart $S - \{0\} \to \mathbb{C}$, $\zeta \mapsto \zeta^{-1}$, $\infty \mapsto 0$, we see f is analytic.
Of course, if f has a removable singularity, we can extend f to an analytic map $f: U \to \mathbb{C} \subset S$. Notice that $z \mapsto e^{1/z}$ cannot be extended to $\mathbb{C} \to S$.
Suppose f is analytic, $U = \{z \in \mathbb{C} : |z| > R\} \to \mathbb{C}$. When does f extend to $f: U \cup \{\infty\} \to S$? Answer: when $|f(z)| \leq K|z|^k$, some k, K as $z \to \infty$.
We consider the chart for $U \cup \{\infty\}$ given by $\zeta \mapsto \frac{1}{\zeta}$. Then consider $\zeta \mapsto f(1/\zeta) : \{\zeta : 0 < |\zeta| < 1/R\} \to S$. When does this extend to $\zeta = 0$?
Precisely when $\zeta \mapsto f(1/\zeta)$ has at most a pole at $\zeta = 0$, ie, when $|f(1/\zeta)| \leq C|\zeta|^{-k}$ for some C, k.

Theorem: If $f: S \to S$ is analytic, then it is of the form $f(z) = \frac{p(z)}{q(z)}$, with p, q polynomials, ie, f is a "rational function."

Proof: First observe that $f^{-1}(\infty)$ consists of isolated points, for the zeroes of $1/f$ must be isolated. So, f has at most a finite number of poles, say at $z = a_1, \dots, a_k$. Near $z = a_1$, it can be expanded: $f(z) = \sum_{i=1}^{d_1} \frac{b_i}{(z-a_1)^i} + g_1(z)$, where g_1 is analytic near a_1. Doing this at each a_i, we find $f = r + g$, where r is rational and $\to 0$ as $z \to \infty$, and g is analytic $\forall z \in \mathbb{C}$. But $|f(z)| \leq K|z|^k$ and $|g(z)| \leq \tilde{K}|z|^k$ as $z \to \infty$. So g is a polynomial.

Examples (ii): Let $L \subset \mathbb{C}$ be a lattice, ie, a subgroup under addition, of the form $\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ where $\omega_1, \omega_2 \in \mathbb{C} - \{0\}$ and $\omega_1/\omega_2 \notin \mathbb{R}$. Example:



Let X = quotient group $\mathbb{C}/L$ = set of equivalence classes of $\mathbb{C}$ under the equivalence $z_1 \sim z_2$ if $z_1 - z_2 \in L$.

Atlas for X: Choose $\varepsilon > 0$ so that $L \cap \{z : |z| \leq 2\varepsilon\} = \{0\}$.

Let $V = \{z \in \mathbb{C} : |z| < \varepsilon\}$. Let $U = \pi(V) \subset X$, where $\pi : \mathbb{C} \to X$ is the obvious projection. Let $\varphi : U \to V \subset \mathbb{C}$ be π^{-1}. By construction, this is 1-1, so is our first chart for X. For any $w \in X$, let $U_w = w + U \subset X$ (X is a group) Define $\varphi_w : U_w \to \mathbb{C}$ by $\varphi_w(z) = \varphi(z - w)$.

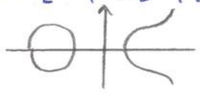
Clearly $\varphi_{w_1}(U_{w_1} \cap U_{w_2})$ and $\varphi_{w_2}(U_{w_1} \cap U_{w_2})$ are open in $V \subset \mathbb{C}$, and the transition map is just $\zeta \mapsto \zeta + w$, where $w \in \mathbb{C}$ such that $w \equiv w_1 - w_2 \mod L$.

- An atlas, so X is a complex manifold. X is compact because there is a continuous map $K \to X$ which is surjective, where $K = [0,1]\omega_1 + [0,1]\omega_2 \subset \mathbb{C}$ (ie, closed parallelogram). K is compact. X is also Hausdorff (check).

Consider $\mathbb{R}/\mathbb{Z}\lambda \cong$ circle $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, $t \mapsto \left(\cos \frac{2\pi t}{\lambda}, \sin \frac{2\pi t}{\lambda}\right)$

Let $\tilde{X}$ be the curve $w^2 = z^3 + az + b$ in $\mathbb{C}^2 \cup \{\infty\}$.

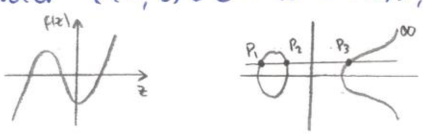
Over $\mathbb{R}$, 3 real roots:



- like cutting through a torus, with one a circle through ∞.

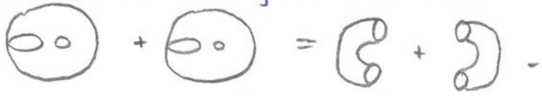
Note: A complex manifold of dimension 1 is called a Riemann surface.

Consider $\{(w, z) \in \mathbb{C}^2 : w^2 = f(z)\}$, with $f(z) = 4z^3 - az - b$, $a, b \in \mathbb{R}$.



$P_1 + P_2 + P_3 = 0$. How many points such that $2P = 0$?

Consider $f(z) \in \mathbb{C}$ - (negative real axis).



- ie, torus.

Consider $F(w, z) = w^2 - f(z)$. Y = Curve = $F^{-1}(0)$. $F : \mathbb{C}^2 \to \mathbb{C}$.

$DF(w, z) = (2w, f'(z)) \neq 0$ for $(w, z) \in F^{-1}(0)$, as long as f has distinct roots.

So, implicit function theorem $\Rightarrow F^{-1}(0)$ is a submanifold of $\mathbb{C}^2$.

Atlas for Y: Let $V = \mathbb{C}$, cut where $f(z) \in \mathbb{R}_- = \{x \in \mathbb{R} : x \leq 0\}$.

Then, $\{(w, z) \in Y : z \in V\} = \underset{(w = +\sqrt{f(z)})}{U_+} \cup \underset{(w = -\sqrt{f(z)})}{U_-}$, and $\varphi_\pm : U_\pm \to V \subset \mathbb{C}$ - two disjoint charts.

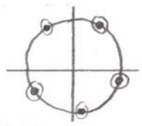
not literally, of course.

Let $W = \mathbb{C} - \{z : f(z) \in \mathbb{R}_+\}$. Then, $\{(w,z) \in Y : z \in W\} = \tilde{U}_+ \cup \tilde{U}_-$, where $\tilde{U}_+ = \{(w,z) : w = +\sqrt{f(z)}\}$, $\tilde{U}_- = \{(w,z) : w = -\sqrt{f(z)}\}$. Again, two disjoint charts: $\tilde{\varphi}_\pm : \tilde{U}_\pm \to \mathbb{C}$.

$$U_+ \cap \tilde{U}_+ \xrightarrow{\varphi} \mathbb{C} - \mathbb{R}, \qquad \xrightarrow{\tilde{\varphi}} \mathbb{C} - \mathbb{R}$$

We need three more charts to cover Y. Let z_1, z_2, z_3 be the zeroes of $f(z)$. We want a chart in the neighbourhood of $(0, z_i)$. Let U_1 be a small neighbourhood of $(0, z_1)$ in Y. Define $\varphi_1 : U_1 \to \mathbb{C}$ by $(w,z) \mapsto w$. This is injective if we can solve $f(z) = w^2$ uniquely as a function of w^2 for z near z_1. Similarly, for $i = 2, 3$.

Example: Fermat curve, $x^n + y^n = 1$, with $(x,y) \in \mathbb{C}^2$. $y = \sqrt[n]{1-x^n}$



Euler's formula: $V - E + F = 2$ for polyhedra.
Euler number: 2 for sphere, $\chi = 2 - 2g$ for a manifold of genus g.
For the Fermat curve: $n^2 - n$ edges, n faces, so $\chi = n(3-n)$, so $g = \frac{1}{2}(n-1)(n-2)$

Let $\hat{Y} = Y \cup \{\infty\}$. Chart near ∞ is given by: $\begin{cases} (w,z) \mapsto z/w \\ \infty \mapsto 0 \end{cases}$

Example: $\mathbb{P}^n_{\mathbb{C}}$ = complex projective space of dimension n = 1-dimensional complex vector subspaces of $\mathbb{C}^{n+1}$. Exactly like $\mathbb{P}^n$ over $\mathbb{R}$, but is a complex manifold.
$\mathbb{P}^2_{\mathbb{C}}$: points are: $\{(u,w,z) \in \mathbb{C}^3 - \{0\}\}/\sim$, where $(u,w,z) \sim (\lambda u, \lambda w, \lambda z)$ if $\lambda \neq 0$.
$\mathbb{C}^2 \subset \mathbb{P}^2_{\mathbb{C}}$, $(w,z) \mapsto (1, w, z)$. "line at infinity" is $u = 0$.
In $\mathbb{P}^2_{\mathbb{C}}$, we have the curve $\hat{Y}$: $uw^2 = 4z^3 - azu^2 - bu^3$. $\hat{Y} \cap \mathbb{C}^2 = Y$.
$\hat{Y} \cap$ (line at ∞) = points where $4z^3 = 0$, ie, $z = 0$, ie, $(u,w,z) = (0,1,0)$.

We shall now define an isomorphism of complex manifolds, $X = \mathbb{C}/L \to Y$, by using the Weierstrass elliptic function, $\wp : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$, defined by

$$\wp(z) = \sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) + \frac{1}{z^2} = \frac{1}{z^2} + Az^2 + Bz^4 + \dots, \text{ near } 0, \text{ where } A = 3 \sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \frac{1}{\lambda^4},\ B = 5 \sum_{\substack{\lambda \in L \\ \lambda \neq 0}} \frac{1}{\lambda^6}$$

Weierstrass $\wp$-function: associated to a lattice $L \subset \mathbb{C}$, $\wp = \wp_L$. This is a meromorphic function $\wp : \mathbb{C} - L \to \mathbb{C}$, with poles at the points of L. It is L-periodic, ie, $\wp(z + \lambda) = \wp(z)$ if $\lambda \in L$

$\wp$ is "doubly-periodic", eg, in the two directions shown.

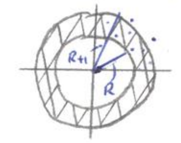
First notice that $\nexists f : \mathbb{C} \to \mathbb{C}$, analytic everywhere, which is L-periodic ($f \neq$ constant)
[This would contradict Liouville's Theorem, for f would be bounded on any one closed parallelogram (continuous on compact set), hence bounded everywhere, so constant]
We also cannot have an L-periodic function f with one simple pole in each parallelogram $\{0, w_1, w_2, w_1 + w_2\}$, as this would contradict Cauchy's integral formula.

[

For if we did, consider $\int_\Gamma f(z)\,dz$, $= 0$ by periodicity.
But, $\int f(z)\,dz = 2\pi i$(residue of f) $= 2\pi i A$, if $f(z) = \frac{A}{z-a} + B + C(z-a) + \dots$
So $A = 0$, hence f analytic, so constant]

Also, if the pole is on the lattice, we may translate to put it in the interior.
But, we can have, say, two simple poles or one double pole.

$\wp_L$ is characterised by the fact that it has a double pole at $z=0$ with principal part $1/z^2$, ie $f(z) = \frac{1}{z^2} + g(z)$ near $z=0$, with g analytic in a neighbourhood of $z=0$.
Consider $\sum_{z\in L} \frac{1}{(z-\lambda)^2}$. For given $z \in \mathbb{C}$, number of points of the form $z-\lambda$ ($\lambda \in L$) with $R \leq |z-\lambda| \leq R+1$ is $\leq KR$ for some constant K.

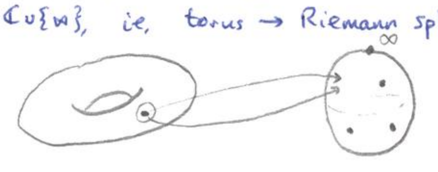


For such $z-\lambda$ we have $\left|\frac{1}{(z-\lambda)^2}\right| \leq \frac{1}{R^2}$. Now, $\sum_n \frac{n}{n^2}$ does not converge.
Compare with $\sum_{n\in\mathbb{Z}} \frac{1}{z-n} = \pi \cot \pi z = \frac{1}{z} + $ (analytic near 0).
This sum equals $\frac{1}{z} + \sum_{0\neq n\in\mathbb{Z}} \left(\frac{1}{z-n} + \frac{1}{n}\right)$. Note: $\frac{1}{z-n} + \frac{1}{z+n} = \frac{2z}{z^2-n^2} = O(n^2)$

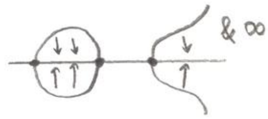
Define $\wp_L(z) = \frac{1}{z^2} + \sum_{\substack{\lambda\in L\\ \lambda\neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$. This does converge absolutely, because $\left|\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right| \leq \frac{A}{|z|^3} \leq \frac{A}{n^3}$ if $n \leq |\lambda| \leq n+1$, and $\sum_n \frac{n}{n^3} < \infty$.

The same argument shows that the series converges absolutely in any disc $|z-z_0| \leq \varepsilon$ which contains no lattice points. If the disc contains one lattice point λ, and we omit the term $\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}$ from the series, it will converge absolutely and uniformly in the disc. Hence, converges to an analytic function in the disc. Hence, $\wp_L$ is meromorphic, with second order poles at $z \in L$.

Clearly, $\wp_L(z)$ depends only on z modulo L, so it is really an analytic function $\wp_L : \mathbb{C}/L \to \mathbb{C} \cup \{\infty\}$, ie, torus $\to$ Riemann sphere.



All points on the Riemann sphere have two distinct points in their pre-image except ∞ and 3 otherpoints. See:



<u>Theorem</u>: $\wp_L(z)$ satisfies the equation: $\wp_L'(z)^2 = 4\wp_L(z)^3 - 60A\wp_L(z) - 140B$, where
$A = \sum_{0\neq\lambda\in L} \frac{1}{\lambda^4}$, $B = \sum_{0\neq\lambda\in L} \frac{1}{\lambda^6}$.
ie, point $(\wp(z), \wp'(z)) \in$ curve $Y = \{(z,w) : w^2 = 4z^3 - az - b.\}$
(<u>cf</u>: $(\cos z, -\sin z)$ parametrises $x^2+y^2=1$).

We shall show that $\wp_L$ induces an isomorphism from $X = \mathbb{C}/L$ to $Y = \mathbb{C} \cup \{\infty\}$.

<u>Proof</u>: Near $z=0$ we have $\wp(z) = \frac{1}{z^2} + g(z)$ with $g(z)$ analytic. But, $\wp(z) = \wp(-z)$, as L is a lattice, a group. So, $g(z) = g(-z)$. And, $g(0)=0$. So, $\wp(z) = \frac{1}{z^2} + \alpha z^2 + \beta z^4 + \dots$
So $\wp'(z) = -\frac{2}{z^3} + g'(z) = -\frac{2}{z^3} + 2\alpha z + 4\beta z^3 + \dots$ So, $\wp'(z)^2 = \frac{4}{z^6} - \frac{8\alpha}{z^2}$ + (analytic terms).
And, $4\wp(z)^3 = \frac{4}{z^6} + \frac{12\alpha}{z^2}$ + (analytic terms)
So, $\wp'(z)^2 - 4\wp(z)^3 = \frac{-20\alpha}{z^2}$ + (analytic terms) $= -20\alpha\wp(z)$ + (analytic terms).
So, $\wp'(z)^2 - 4\wp(z)^3 + 20\alpha\wp(z)$ is analytic near 0, hence analytic everywhere, hence constant.

We have $X = \mathbb{C}/L \to \hat{Y} = \{(u,v) \in \mathbb{C}^2 : u^2 = 4v^3 - av - b\} \cup \{\infty\}$; $z \mapsto (\wp'(z), \wp(z)) \in \mathbb{C}$ if $z = 0$,
$0 \mapsto \infty$.

$\wp(z) = \frac{1}{z^2} + \sum_{0\neq\lambda\in L} \left(\frac{1}{(z-\lambda)^2} + \frac{1}{\lambda^2}\right)$. We want to show that $\wp(z+\alpha) = \wp(z)$ for $\alpha \in L$.
$\wp(z+\alpha) - \wp(z) = \sum_{\lambda\in L} \left(\frac{1}{(z+\alpha-\lambda)^2} - \frac{1}{(z-\lambda)^2}\right) = \sum_{\lambda\in L} c_\lambda = -\sum_{\lambda\in L} c_{\lambda-\alpha} = -\sum_{\lambda\in L} c_\lambda$, so $=0$.

Theorem 1: $\mathbb{C}/L = X \to \hat{Y}$ is an isomorphism of complex manifolds.

(i) It is an analytic map, $z \mapsto (\wp'(z), \wp(z)) \mapsto \left\{ \begin{matrix} \wp(z) \in \mathbb{C} \\ \wp'(z) \in \mathbb{C} \end{matrix} \right\}$, under two kinds of chart.

To see what happens near $z = 0 \in \mathbb{C}/L$, consider $\hat{Y} \in \mathbb{P}^2_{\mathbb{C}}$.

$X \to \mathbb{P}^2_{\mathbb{C}}$; $z \mapsto (1, \wp'(z), \wp(z))$ if $z \neq 0$, $0 \mapsto (0, 1, 0)$.

Consider the chart for $\mathbb{P}^2$ given by $(p, q, r) \mapsto (p/q, r/q) \in \mathbb{C}^2$ when $q \neq 0$.

In this chart, our map is $z \mapsto \left(\frac{1}{\wp'(z)}, \frac{\wp(z)}{\wp'(z)}\right) \in \mathbb{C}^2$, $0 \mapsto (0,0)$.

Near $z = 0$, $\wp(z) = \frac{1}{z^2} + \cdots$, $\wp'(z) = \frac{-2}{z^3} + \cdots$, so $\left(\frac{1}{\wp'(z)}, \frac{\wp(z)}{\wp'(z)}\right) \sim \left(-\frac{1}{2}z^3 + O(z^4), -\frac{1}{2}z + O(z^2)\right)$

In the chart for $\hat{Y}$ near ∞, we have $z \mapsto \frac{\wp(z)}{\wp'(z)} = -\frac{1}{2}z + \cdots$

(ii) It is 1-1, onto, and its inverse is analytic.

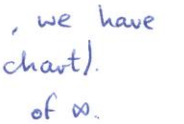
(ii) follows from:

Theorem 2: Suppose X and Y are two Riemann surfaces, both compact and Hausdorff, and suppose that $f: X \to Y$ is an analytic map. Then $\forall y \in Y$, the number of points in $f^{-1}(y)$ is finite, and this number of points is the same, except for finitely many points $y \in Y$. This number is called the degree of f. If the degree is 1, the map is an isomorphism.

Proof that Theorem 2 $\Rightarrow$ Theorem 1: X and $\hat{Y}$ are compact, Hausdorff, and X is connected. Enough to prove $\exists$ infinitely many points with $f^{-1}(y)$ having just one point. Clearly, only one point $\mapsto \infty$. By considering the chart near ∞ in $\mathbb{P}^2_{\mathbb{C}}$, we have $z \mapsto \frac{\wp(z)}{\wp'(z)} = -\frac{1}{2}z + \cdots$. This shows that $Df(0) \neq 0$ ($Df(0) = -\frac{1}{2}$ in this chart). So f is 1-1 by the inverse function theorem in a neighbourhood of ∞.

Proof of Theorem 2: Given $f: X \to Y$, suppose $f(z) = w$. Pick charts near z, w, say $\varphi, \tilde{\varphi}$. Then, $\tilde{\varphi} f \varphi^{-1}$ is an analytic function between open subsets of $\mathbb{C}$, and takes $\varphi(z)$ to $\tilde{\varphi}(w)$. As zeroes of analytic functions are isolated, $\tilde{\varphi} f \varphi^{-1}$ does not take any point near $\varphi(z)$ to $\tilde{\varphi}(w)$. So, the points in $f^{-1}(w)$ are isolated in X. But, X is compact and Hausdorff, so $f^{-1}(w)$ is finite. Now observe that for the same reason, $Df(z) \neq 0$ except for finitely many z. (Look in terms of charts: $Df(z) = 0 \iff D(\tilde{\varphi} f \varphi^{-1})(\varphi(z)) = 0$, and $D(\tilde{\varphi} f \varphi^{-1})$ has isolated zeroes). So, let $z_1, \ldots, z_k$ be the points where $Df(z) = 0$. Let $w_i := f(z_i)$, and $Y' = Y - \{w_1, \ldots, w_k\}$. We prove that the number of points in $f^{-1}(y)$ is locally constant as a function of y for $y \in Y'$. Hence it is constant in Y because removing $< \infty$ points cannot make a surface disconnected.

Number of points in $f^{-1}(y)$ is locally constant, by inverse function theorem, for if $y \in Y'$ and $f(z) = y$, then $Df(z) \neq 0$, so f gives a bijection $U_z \to V$ for some neighbourhoods U_z of z, V of y. Notice that because X is Hausdorff, we can choose disjoint open neighbourhoods U_z of the points z in $f^{-1}(y)$, and $f(X - \bigcup U_z) = f(\text{compact}) = $ compact, closed, and $\not\ni y$.



So any point y' near y has $f^{-1}(y) \subset \bigcup U_z$.

Finally, suppose the degree is 1. Then, $\exists$ unique $f^{-1}: Y' \to X$, and the inverse function theorem $\Rightarrow f^{-1}$ is analytic. But, in fact, $Y' = Y$, because if $Df(z) = 0$ for some z, then in terms of charts, $(\tilde{\varphi} f \varphi^{-1})(\varphi(z)) = 0$. So even in these charts, f is not locally 1-1.

Covering Spaces.

X, Y smooth manifolds of dimension n. Note: from now on, all manifolds are Hausdorff. A smooth map $p: X \to Y$ is a covering map if every $y \in Y$ has a neighbourhood V such that $f^{-1}(V) = \dot{\bigcup}_\alpha U_\alpha$, where f maps each U_α by a diffeomorphism onto V.

Examples: (0): The inclusion of an open subset of Y in Y is not a covering map.

(i) $p: \mathbb{C} \to \mathbb{C}^\times$, $p(z) = e^z$ is covering map, because if $|1-z| < 1$, then $z \mapsto \log z$ is an inverse map $V = \{z \in \mathbb{C} : |1-z| < 1\} \to \mathbb{C}$, and $p^{-1}(V) = \bigcup_\alpha \{\log(V) + 2\pi i\alpha\} = \dot{\bigcup}_\alpha U_\alpha$, $\alpha \in \mathbb{Z}$, and the U_i are disjoint. This deals with the property near $1 \in \mathbb{C}^\times$. For other points $y \in \mathbb{C}^\times$, consider $V_y = y^{-1}V$.

(ii) $p: \mathbb{C} \to \mathbb{C}/L$, natural map, where L is a lattice.

(iii) $X = \{(z,w) \in \mathbb{C}^2 : w^2 = 4z^3 - az - b,\ w \neq 0\}$, $Y = \mathbb{C} - \{\zeta : 4\zeta^3 - a\zeta - b = 0\}$. Then, $p: X \to Y$, $p(w,z) = z$ is a covering map.

(iv) $X = S^{n-1} \subset \mathbb{R}^n$, $Y = \mathbb{P}^{n-1}$. Define $p: X \to Y$ by $p(\xi) = \mathbb{R}\xi \subset \mathbb{R}^n$ (a 2-to-1 map).

Lemma: If $f: A \to B$ is a map of topological spaces and $A = A_1 \cup A_2$ where A_1 and A_2 are closed in A and $f|_{A_1}$, $f|_{A_2}$ are continuous, then f is continuous.

Theorem: Suppose $p: X \to Y$ is a covering map and $\gamma: [a,b] \to Y$ is a path. Suppose $p(x_0) = y_0 = \gamma(a)$. Then, $\exists$ unique lift $\tilde{\gamma}$ of γ starting at x_0, ie, unique $\tilde{\gamma}: [a,b] \to X$ such that $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(a) = x_0$.

Proof: Uniqueness: Suppose $\tilde{\gamma}$ and $\tilde{\tilde{\gamma}}$ are two such lifts. Suppose $\tilde{\gamma}(t) = \tilde{\tilde{\gamma}}(t)$ for some $t \in [a,b]$. Choose a neighbourhood V of $\gamma(t)$ as in definition of covering. Then $\tilde{\gamma}(s)$ and $\tilde{\tilde{\gamma}}(s)$ both belong to the same $U_\alpha \subset p^{-1}(V)$ $\forall$ s sufficiently close to t. But, $p: U_\alpha \to V$ is bijective and $p\tilde{\gamma} = p\tilde{\tilde{\gamma}}$, so $\tilde{\gamma}(s) = \tilde{\tilde{\gamma}}(s)$ $\forall$ s near t. Similarly, if $\tilde{\gamma}(t) \neq \tilde{\tilde{\gamma}}(t)$, then $\tilde{\gamma}(s) \neq \tilde{\tilde{\gamma}}(s)$ $\forall$ s sufficiently near t. So, as $[a,b]$ is connected, either $\tilde{\gamma}(t) = \tilde{\tilde{\gamma}}(t)$ $\forall$ $t \in [a,b]$, or $\tilde{\gamma}(t) \neq \tilde{\tilde{\gamma}}(t)$ $\forall$ $t \in [a,b]$. But we know $\tilde{\gamma}(t) = \tilde{\tilde{\gamma}}(t)$, some t.

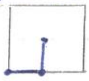
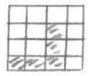
Existence: Because $[a,b]$ is compact we can successively bisect it, until it is the union of 2^k closed subintervals I_i such that each $\gamma(I_i) \subset$ some V_i, with $p^{-1}(V_i) = \dot{\bigcup}_\alpha U_{i\alpha}$. Take the intervals I_i in turn from the left. Suppose $\tilde{\gamma}$ has been defined on $I_1 \cup \dots \cup I_r$. Clearly, if $\gamma(I_{r+1}) \subset V$ and $p^{-1}V = \dot{\bigcup} U_\alpha$ and $\tilde{\gamma}(\text{end of } I_r) \in U_\alpha$, then $\tilde{\gamma}$ can be defined on I_{r+1} with $\tilde{\gamma}(I_{r+1}) \subset U_\alpha$. By the lemma, $\tilde{\gamma}$ is continuous on $I_1 \cup \dots \cup I_{r+1}$.

Definition: Paths $\gamma, \gamma^*: [a,b] \to Y$ are homotopic if $\exists$ a continuous map $F: [a,b] \times [0,1] \to Y$, such that $F(t,0) = \gamma(t)$, $F(t,1) = \gamma^*(t)$ for $t \in [a,b]$. Say γ, γ^* are homotopic rel. ends if $F(a,s) = \gamma(a) = \gamma^*(a)$, $F(b,s) = \gamma(b) = \gamma^*(b)$.

Definition: Y is simply connected if it is path-connected and for any $y_0, y_1 \in Y$, any two paths from y_0 to y_1 are homotopic rel ends.

If $p: X \to Y$ is a covering map and $F: [a,b] \times [0,1] \to Y$ and $p(x_0) = y_0 = F(a,0)$, then F has a unique lift $\tilde{F}$ such that $p \circ \tilde{F} = F$ and $F(a,0) = x_0$. In particular, if γ, γ^* are paths in Y which are homotopic rel ends, and $\tilde{\gamma}, \tilde{\gamma}^*$ are lifts both starting at x_0, then $\tilde{\gamma}, \tilde{\gamma}^*$ are homotopic rel. ends.

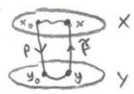
Sketch of proof: Uniqueness and "in particular" follow from the earlier lifting theorem.

Uniqueness:  Existence:  (?!!)

Theorem: $p: X \to Y$ as before. Suppose Z is a simply connected manifold and $f: Z \to Y$ is a smooth map. If, for some $z_0 \in Z$, we have $p(x_0) = f(z_0) = y_0 \in Y$, then $\exists$ unique smooth $\tilde{f}: Z \to X$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(z_0) = x_0$.

To show $\tilde{f}$ is analytic near $z \in Z$, choose a neighbourhood V of $f(z)$ in Y such that $V \subset$ domain of a chart and $p^{-1}(V) = \dot{\bigcup} U_\alpha$ as in definition of a covering map.
Clearly $\exists$ a unique analytic map $\varphi: V \to X$ such that $p \circ \varphi = id$. and $\varphi(f(z)) = \tilde{f}(z)$.
So $\varphi \circ f$ and $\tilde{f}$ are two lifts of $f|_{f^{-1}(V)}$ which agree at $f(z)$, so they agree in all of $f^{-1}(V)$. So $\tilde{f} = \varphi \circ f$ in $f^{-1}(V)$, and this is analytic.

Corollary: If $p: X \to Y$ is a covering map, X connected, Y simply connected, then p is an isomorphism.
Proof: Take $Y = Z$ in preceding, and $f: Y \to Y$ the identity. Get $\tilde{f}: Y \to X$ such that $p \circ \tilde{f} = id$.
And, $\tilde{f}$ is surjective, because any two points x_0, x can be joined by a path which is the unique lift of a path from $p(y_0)$ to $p(y)$.



Theorem: Let $X = \mathbb{C}/L$, $X' = \mathbb{C}/L'$. Then, $X \cong X'$ as complex manifolds iff $L' = \alpha L$ for some $\alpha \in \mathbb{C}$.
Proof: We have a covering map $p: \mathbb{C} \to X'$. $Z = \mathbb{C}$ is simply connected. Suppose we have an isomorphism $\varphi: X \to X'$. Then consider:
$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{\varphi}} & \mathbb{C} \\ \downarrow & & \\ \mathbb{C}/L & \xrightarrow{\varphi} & \mathbb{C}/L' \end{array}$$
By the preceding theorem, $\exists\, \tilde{\varphi}: \mathbb{C} \to \mathbb{C}$ such that $\tilde{\varphi}$ lifts $\mathbb{C} \to X \to X'$.
Consider the map $z \mapsto \tilde{\varphi}(z+\lambda) - \tilde{\varphi}(z)$ from $\mathbb{C} \to \mathbb{C}$, for some $\lambda \in L$. The images of $\tilde{\varphi}(z+\lambda)$ and $\tilde{\varphi}(z)$ in $\mathbb{C}/L'$ are the same, ie, $p \circ (\text{map})$ is constant.
So, $\tilde{\varphi}(z+\lambda) - \tilde{\varphi}(z)$ is independent of z. We can assume $\varphi(0) = 0$, as we have a group law on X and X'. So we can assume that $\tilde{\varphi}(0) = 0$.
So, $\tilde{\varphi}(z+\lambda) = \tilde{\varphi}(z) + \tilde{\varphi}(\lambda)$ if $z \in \mathbb{C}, \lambda \in L$. In particular, $\tilde{\varphi}|_L$ is a homomorphism $L \to L'$.
So $\tilde{\varphi}|_L$ is the restriction of an $\mathbb{R}$-linear map, $\mathbb{R}^2 \to \mathbb{R}^2$. So $|\tilde{\varphi}(\lambda)| \leq K|\lambda|$, some $K \in \mathbb{R}$.
But if $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$, let $M = \{z \in \mathbb{C} : z = \alpha w_1 + \beta w_2 \text{ with } 0 \leq \alpha, \beta \leq 1\}$
Clearly, $|\tilde{\varphi}(z)| \leq$ some C for $z \in M$. But any $z \in \mathbb{C}$ can be written as $z = z_0 + \lambda$, with $z_0 \in M$, $\lambda \in L$. So, $\tilde{\varphi}(z) = \tilde{\varphi}(z_0) + \tilde{\varphi}(\lambda)$, and $|\tilde{\varphi}(z)| \leq C + K|\lambda| \leq C' + K'|z|$, some $C', K' \in \mathbb{R}$.
But $\tilde{\varphi}$ is analytic $\mathbb{C} \to \mathbb{C}$, so $\tilde{\varphi}(z) = az + b$, for some $a, b \in \mathbb{C}$.
But $\tilde{\varphi}(0) = 0$, so $\tilde{\varphi}(z) = az$, so $aL \subset L'$.
But, $\varphi: \mathbb{C}/L \to \mathbb{C}/L'$ is bijective, so $\tilde{\varphi}(L) = L'$, so $aL = L'$.
Conversely, if $L' = aL$, then $z \mapsto az$ induces an isomorphism $\mathbb{C}/L \to \mathbb{C}/L'$.

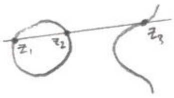
How do we classify lattices L up to the equivalence relation $L \sim aL$?
Clearly we may assume $L = \mathbb{Z} + \mathbb{Z}\tau$ (as $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \omega_1(\mathbb{Z} + \mathbb{Z}\,\omega_2/\omega_1)$), with $\operatorname{Im}\tau > 0$.
When is $\mathbb{Z} + \mathbb{Z}\tau = a(\mathbb{Z} + \mathbb{Z}\tau')$ for some $\tau, \tau' \in \{z : \operatorname{Im} z > 0\}$ and some a?
If so, $\tau = a(\alpha + \beta\tau')$, some $\alpha, \beta \in \mathbb{Z}$, and $1 = a(\gamma + \delta\tau')$, $\gamma, \delta \in \mathbb{Z}$. So, $\tau = \frac{\alpha + \beta\tau'}{\gamma + \delta\tau'}$.
Similarly, $\tau' = \frac{\alpha' + \beta'\tau}{\gamma' + \delta'\tau}$, with $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \in M_2(\mathbb{Z})$. So, $\det\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \pm 1$.
But, have $\det = +1$, else $z \mapsto \frac{\alpha + \beta z}{\gamma + \delta z}$ takes $\operatorname{Im} z > 0$ to $\operatorname{Im} z < 0$. $\left[\operatorname{Im}\left(\frac{\alpha + \beta z}{\gamma + \delta z}\right) = \frac{(\alpha\delta - \beta\gamma) \cdot \operatorname{Im} z}{|\gamma + \delta z|^2}\right]$.
So the condition is $\tau = g\tau'$ for some Möbius transformation $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$.

Recall that, given a lattice L, we defined $\wp_L : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ and an isomorphism $\mathbb{C}/L \to \hat{Y}$, where $\hat{Y} = $ (cubic curve with equation $y^2 = 4x^3 - ax - b$) $\cup \{\infty\}$. Let L be $\mathbb{Z} + \tau\mathbb{Z}$.
Say $y^2 = p(x)$. The roots of p are the points $\wp(\tfrac{1}{2})$, $\wp(\tau/2)$, $\wp(\tfrac{1}{2}\tau + \tfrac{1}{2})$, because $\wp(z) = \wp(z + \lambda)$ for any $\lambda \in L$, $\wp(z) = \wp(-z)$, $\wp(z) = \wp(\lambda - z)$. So, $\wp'(z) = -\wp'(\lambda - z)$. So, $\wp'(\lambda/2) = 0$ for any $\lambda \in L$ such that $\tfrac{1}{2}\lambda \notin L$.
Consider the cross-ratio of $(\wp(\tfrac{1}{2}), \wp(\tfrac{\tau}{2}), \wp(\tfrac{1}{2}\tau + \tfrac{1}{2}), \infty)$. Call it $g(\tau)$. We have a holomorphic map $\{z : \operatorname{Im} z > 0\} \to \mathbb{C} - \{0, 1\}$, by g. This map g is a covering map.

Corollary (Picard's Theorem): If $f : \mathbb{C} \to \mathbb{C} - \{0, 1\}$ is analytic, then f is constant.

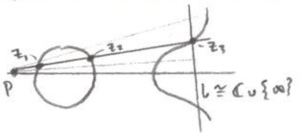
Suppose we had $\mathbb{C}/L \to \mathbb{C} \cup \{\infty\}$, degree 1. This would be an isomorphism, which is impossible, as $\mathbb{C}/L \cong$ torus, $\mathbb{C} \cup \{\infty\} \cong$ Riemann sphere, which are topologically different.
Any analytic map $\mathbb{C} \cup \{\infty\} \to \mathbb{C}/L$ is constant.

$$\begin{tikzcd} & \mathbb{C} \arrow[d, "p,\ \text{covering map}"] \\ \mathbb{C} \cup \{\infty\} \arrow[r] \arrow[ur, dashed, "\text{constant, by Liouville}"] & \mathbb{C}/L \end{tikzcd}$$



$\hat{Y} \cong \mathbb{C}/L$, group. Cubic curve Y parametrised by $z \mapsto (\wp(z), \wp'(z))$.
"Addition formula for elliptic functions": $z_1 + z_2 + z_3 \in L \iff \det\begin{pmatrix} 1 & 1 & 1 \\ \wp(z_1) & \wp(z_2) & \wp(z_3) \\ \wp'(z_1) & \wp'(z_2) & \wp'(z_3) \end{pmatrix} = 0$.

Take $P \notin$ curve, l a line in $\mathbb{P}^2_{\mathbb{C}}$.



Define $f : l \to \mathbb{C}/L$ by $f(Q) = z_1 + z_2 + z_3$, where z_1, z_2, z_3 are the parameters of the three points where the line PQ meets the curve $\hat{Y} = Y \cup \{\infty\}$. This is an analytic map, so is constant. A pencil - the set of lines through a point. Any two pencils have a line in common.
Take $L \subset \mathbb{C}$, $\wp_L : \mathbb{C}/L \to \mathbb{C} \cup \{\infty\}$ and consider cross-ratio of 4 points. $L \mapsto j(L) \in \mathbb{C}$ such that $j(L) = j(L') \iff L' = \alpha L$, some $\alpha \in \mathbb{C}$. $j(\tau) = j(\mathbb{Z} + \tau\mathbb{Z})$.
$j(\tau) = j(\tau') \iff \tau' = g\tau$, some $g \in SL_2(\mathbb{Z})$. $j : \{z : \operatorname{Im} z > 0\} \to \mathbb{C}$, analytic, surjective.
- Classical modular function. $j(\tau + 1) = j(\tau)$, $j(\tau) = \sum_{n \geq -1} a_n q^n$, $q = e^{2\pi i \tau}$. $j(\tau) = \frac{1}{q} + a_0 + a_1 q + \ldots$

Modular form of weight k: $f(L)$ such that $f(\alpha L) = \alpha^{-2k} f(L)$.

Example: $f(L) = \sum_{0 \neq \lambda \in L} \frac{1}{\lambda^{2k}}$ - Eisenstein Series.

Suppose X is a topological space, Y a set. Two maps, $f : U \to Y$, $g : V \to Y$, where U, V are neighbourhoods of some $x \in X$, have the same germ at x if $f|_W = g|_W$ for some neighbourhood W of contained in $U \cap V$.

Analytic Continuation.

Examples: (i) $\zeta : \{s \in \mathbb{C} : Re(s) > 1\}$, $\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$ – Analytic. But, ζ extends to an analytic function, $\zeta : \mathbb{C} - \{z : z \in \mathbb{Z}, z < 0\} \to \mathbb{C}$, with simple poles at some negative integers.

(ii) $f(q) = \sum p_n q^n$, where $p_n =$ number of partitions of n; analytic for $|q| < 1$. $\frac{1}{f(q)} = \prod_{n \geq 1} (1 - q^n)$. Dense zeroes on the unit circle, so this cannot be continued beyond $|q| = 1$. Letting $q = e^{2\pi i \tau}$, we get a modular function, $\prod_{0 \neq \lambda \in \mathbb{Z} + \tau\mathbb{Z}} \lambda$.

(iii) $-\log(1-z) = \sum z^k/k$ – becomes multivalued.

(iv) $\mathrm{dilog}(z) = \sum z^k/k^2 = \int_0^\zeta \mathrm{Log}(1-\zeta) \cdot \frac{d\zeta}{\zeta}$.

Construct a Riemann surface as follows: Take all pairs (U, f), U open $\subset \mathbb{C}$, $f : U \to \mathbb{C}$, analytic. Introduce an equivalence relation: $(z \in U, f) \sim (z' \in U', f')$ iff germ of f at z = germ of f' at z'.

Set of equivalence classes, X = all germs of maps $\mathbb{C} \to \mathbb{C}$. This is a manifold.

It has obvious charts, $\varphi_{U,f} : (U, f) \to U \subset \mathbb{C}$. Write U_f for (U, f). What is $\varphi_{U,f}(U_f \cap V_g)$?

It is $\{z \in U \cap V : f, g$ have the same germ at $z\}$, an open subset of U.

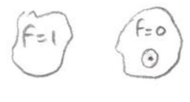
X is a Hausdorff 1-dimensional complex manifold. $\exists$ function $F : X \to \mathbb{C}$, $F|_{(U,f)} = f$.

Definition: A complete multivalued analytic function is a connected component of X.

Note: Inverse function to $\wp_L : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$. $\wp(z) = \zeta$. $dz = \left(\frac{d\zeta}{dz}\right)^{-1} d\zeta = p(\zeta)^{1/2} d\zeta$.
So $\int \frac{d\zeta}{p(\zeta)^{1/2}}$ is the inverse function to $\wp_L$.

Simple-connectedness:

$\mathbb{R}^3$ - (trefoil knot) is not simply connected.



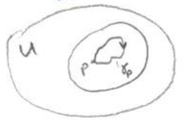
f is continuous, locally constant. When is f globally constant?

Consider $U = \mathbb{R}^3 - (z\text{-axis})$. $f : U \to \mathbb{R}^3$; $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 0 \right)^T$

For $\varphi : U \to \mathbb{R}$, $\gamma : [0,1] \to U$, suppose we can write $\int_\gamma \langle f(v), dv \rangle = \int_\gamma \langle \mathrm{grad}\, \varphi(v), dv \rangle$
$= \int_\gamma \frac{d}{dt} \varphi(v)\, dt = \varphi(\gamma(1)) - \varphi(\gamma(0)) = 0.$

$f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \mathrm{grad}\, \varphi \Rightarrow D_i f_j = D_j f_i \iff \mathrm{curl}\, f = 0.$

Locally, the converse is true: $\mathrm{curl}\, f = 0 \Rightarrow f = \mathrm{grad}\, \varphi$ in some neighbourhood of any point.



$\varphi(P) = \int_{\gamma_P} \langle f(v), dv \rangle$. $\int_{\tilde{\gamma}_P} - \int_{\gamma_P} = \int_{\text{surface } \Sigma \text{ with boundary } \tilde{\gamma}_P - \gamma_P} \langle \mathrm{curl}\, f, dS \rangle = 0.$

In the example, $f = \mathrm{grad}\, \varphi$, where $\varphi : \cos^{-1}\left(\frac{y}{\sqrt{x^2+y^2}}\right) = $ "θ" in cylindrical polar coordinates.

Whenever we have a vector-valued function $f : U \to \mathbb{R}^3$ such that $\mathrm{curl}\, f = 0$, we can define an invariant δ_f for closed curves γ in U by $\delta_f(\gamma) = \int_\gamma \langle f(v), dv \rangle$. If f_1, f_2 are two such functions, then $\delta_{f_1}(\gamma) = \delta_{f_2}(\gamma)$ for all γ iff $f_1 - f_2 = \mathrm{grad}\, \varphi$ for some $\varphi : U \to \mathbb{R}$.

We are led to introduce a group $H^1(U)$, the <u>first de Rham cohomology</u> of U, by: $H^1(U) = \{\text{smooth functions } f: U \to \mathbb{R}^3 \text{ such that } \operatorname{curl} f = 0\} \big/ \{\text{smooth } f: U \to \mathbb{R}^3 \text{ such that } f = d\varphi \text{ for some } \varphi: U \to \mathbb{R}\}$.

Note that curl.grad $= 0$, so $\{f : f = \operatorname{grad}\varphi\} \subset \{f : \operatorname{curl} f = 0\}$.

If $U = \mathbb{R}^3 - (z\text{-axis})$, then $H^1(U) = \mathbb{R}$, corresponding to the fact that the homotopy type of a closed curve is completely determined by just one invariant, $\delta_f =$ "winding number" for the particular f described.

$H^1(U) \cong \mathbb{R}$, generated by $\begin{pmatrix} \frac{y}{x^2+y^2} \\ \frac{-x}{x^2+y^2} \\ 0 \end{pmatrix} = f \iff$ If $\tilde{f}$ satisfies $\operatorname{curl}\tilde{f} = 0$, then $\tilde{f} = \lambda f + \operatorname{grad}\varphi$, some $\lambda \in \mathbb{R}$, some $\varphi: U \to \mathbb{R}$.

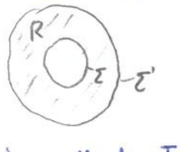
<u>Sketch proof</u>: Choose γ which winds once around the z-axis. Then, $\int_\gamma \langle f(v), dv\rangle = 2\pi$.
Let $\lambda = \frac{1}{2\pi}\int_\gamma \langle \tilde{f}(v), dv\rangle$. Then, $\int_\gamma \langle(\tilde{f} - \lambda f), dv\rangle = 0$. Now define
$\varphi(P) = \int_{\gamma_P} \langle(\tilde{f} - \lambda f), dv\rangle$, where γ_P is a path from some P_0 to P.

Similarly, $H^0(U) = \{\text{functions } \varphi: U \to \mathbb{R} \text{ such that } \operatorname{grad}\varphi = 0\}/0$
$= \{\text{Locally constant functions } \varphi: U \to \mathbb{R}\}$.
Ie, $\dim H^0(U) =$ number of connected components of U.

$H^2(U) = \{\text{functions } g: U \to \mathbb{R}^3 \text{ such that } \operatorname{div} g = 0\} \big/ \{\text{functions } g: U \to \mathbb{R}^3 \text{ such that } g = \operatorname{curl} f \text{ for some } f: U \to \mathbb{R}\}$

Notice that div. curl $= 0$, so $\{g : g = \operatorname{curl} f\} \subset \{g : \operatorname{div} g = 0\}$. Locally, the converse is true. Ie, if $\operatorname{div} g = 0$ then $g = \operatorname{curl} f$ for some f defined in a neighbourhood.

If one has g such that $\operatorname{div} g = 0$, we can define an invariant Φ_g for closed surfaces Σ in U by $\Phi_g(\Sigma) = \int_\Sigma \langle g, ds\rangle =$ "Flux of g through Σ"



$\partial R = \Sigma' - \Sigma$.
Then, $\int_{\Sigma'} g.ds - \int_\Sigma g.ds = \int_R (\operatorname{div} g)\, d(\text{vol.})$ - "<u>Green's Theorem</u>"
$= 0$ if $\operatorname{div} g = 0$.

Notice that $\Phi_{g_1}(\Sigma) = \Phi_{g_2}(\Sigma)$ for all closed Σ if $g_1 - g_2 = \operatorname{curl} f$, because $\int_\Sigma (g_1 - g_2).ds = \int_\Sigma (\operatorname{curl} f).ds = \int_{\partial\Sigma} f.ds = 0$ if Σ closed.

<u>Example</u>: $U = \mathbb{R}^3 - \{0\}$, $g: U \to \mathbb{R}^3$, field of point charge at 0. $g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x^2+y^2+z^2)^{-3/2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
Then, $\operatorname{div} g = 0$, but $g \neq \operatorname{curl} f$ for any $f: U \to \mathbb{R}^3$, because
$\int_\Sigma g.ds = 4\pi \times$(number of times Σ encloses the origin).

This all leads to: $\Omega^0(U) \xrightarrow{\text{grad}} \Omega^1(U) \xrightarrow{\text{curl}} \Omega^2(U) \xrightarrow{\text{div}} \Omega^3(U)$, where $\Omega^0(U) = \Omega^3(U) = \{\text{smooth } f: U \to \mathbb{R}\}$, and $\Omega^1(U) = \Omega^2(U) = \{\text{smooth } f: U \to \mathbb{R}^3\}$.

<u>Example</u>:
$$\left.\begin{aligned} &\int_{\text{curve}} \operatorname{grad}\varphi.\, ds = \varphi(\text{end}) - \varphi(\text{end}) = \int_{\partial(\text{curve})} \varphi \\ &\int_{\text{surface},\Sigma} (\operatorname{curl} v).\, ds = \int_{\partial\Sigma} v.\, d\underset{\sim}{s} \\ &\int_{\text{volume} V} (\operatorname{div} w)\, d(\text{vol}) = \int_{\partial V} w.\, ds \end{aligned}\right\} \text{"Stokes' Theorem"}$$

U open in $\mathbb{R}^n$. Define $\Omega^k(U)$ for $k=0,\dots,n$, by: $\Omega^0(U) = \{$smooth maps $U \to \mathbb{R}\}$, $\Omega^1(U) = \{$smooth maps $U \to \mathbb{R}^n\}$, $\begin{pmatrix} f_1 \\ f_n \end{pmatrix}$ is grad of something if $\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = 0$; $\Omega^2(U) = \{$smooth maps $U \to \mathbb{R}^{\binom{n}{2}}\}, \dots, \Omega^k(U) = \{$smooth maps $U \to \mathbb{R}^{\binom{n}{k}}\}, \dots, \Omega^n(U)$.
Have $\Omega^0 \xrightarrow{\text{grad}} \Omega^1 \xrightarrow{\text{curl}} \Omega^2 \to \dots \to \Omega^{n-1} \xrightarrow{\text{div}} \Omega^n$. In general, call each map d the <u>exterior derivative</u>.

Let V be a real vector space of dimension. Let $\mathrm{Alt}^k(V) =$ alternating multilinear maps $\alpha : \overbrace{V \times \dots \times V}^{k} \to \mathbb{R}$. Multilinear: linear in each variable separately, for example, $\alpha(\lambda_1 v_1 + \mu_1 v_1', v_2, \dots, v_n) = \lambda_1 \alpha(v_1, \dots, v_n) + \mu_1 \alpha(v_1', v_2, \dots, v_n)$. Alternating: if $\pi : \{1,\dots,k\} \to \{1,\dots,k\}$ is a permutation then $\alpha(v_{\pi(1)}, \dots, v_{\pi(n)}) = (-1)^\pi \alpha(v_1, \dots, v_n)$, where $(-1)^\pi = \mathrm{sign}\,\pi$ $= (-1)^{\#\text{crossings of } \pi} = \prod_{i>j} \frac{\pi(i) - \pi(j)}{i-j}$. $\mathrm{sign}(\pi_1 . \pi_2) = \mathrm{sign}\,\pi_1 . \mathrm{sign}\,\pi_2$.

By convention, $\mathrm{Alt}^0(V) = \mathbb{R}$; $\mathrm{Alt}^1(V) = V^*$, dual of V = linear maps $V \to \mathbb{R}$.
$\dim(\mathrm{Alt}^k(V)) = \binom{n}{k}$, because if $e_1, \dots, e_n$ is a basis for V then any $\alpha \in \mathrm{Alt}^k(V)$ is completely determined by giving $\alpha(e_{i_1}, \dots, e_{i_k})$ for all k-tuples $(i_1, \dots, i_k) \in \{1, \dots, n\}$, with $i_1 < \dots < i_k$. (If two i_j are equal then $\alpha(e_{i_1}, \dots) = 0$ by alternating property).
Conversely, if we give $\binom{n}{k}$ numbers $\alpha_{i_1 \dots i_k}$ $(i_1 < \dots < i_k)$ and define $\alpha_{i_1 \dots i_k} = 0$ if not all $i_1, \dots, i_k$ are distinct, and $= \mathrm{sign}(\pi)\, \alpha_{j_1 \dots j_k}$ where $j_1 < \dots j_k$ is the same set as $i_1, \dots, i_k$ and π is the permutation such that $\pi(i_r) = j_r$, then we can define an element $\alpha \in \mathrm{Alt}^k(V)$ by $\alpha(e_{i_1}, \dots, e_{i_k}) = \alpha_{i_1 \dots i_k}$ and extend linearly.

There is a multiplication, $\mathrm{Alt}^k(V) \times \mathrm{Alt}^m(V) \mapsto \mathrm{Alt}^{k+m}(V)$; $(\alpha, \beta) \mapsto \alpha \wedge \beta$, defined by $(\alpha \wedge \beta)(v_1, \dots, v_{k+m}) = \sum_{\substack{(k,m)\text{-shuffles} \\ \pi}} (-1)^\pi \alpha(v_{\pi(1)}, \dots, v_{\pi(k)}) \beta(v_{\pi(k+1)}, \dots, v_{\pi(k+m)})$.
A (k,m)-shuffle is a permutation $\pi : \{1, \dots, k+m\} \to \{1, \dots, k+m\}$ such that $\pi(1) < \dots < \pi(k)$ and $\pi(k+1) < \dots < \pi(k+m)$. There are $\binom{k+m}{k}$ such shuffles.
This multiplication is <u>bilinear</u>, <u>associative</u>, and <u>anticommutative</u>. (ie, $(-1)^{k+m} \beta \wedge \alpha = \alpha \wedge \beta$)
We have an <u>anticommutative graded ring</u>: $\mathrm{Alt}^0(V), \mathrm{Alt}^1(V), \mathrm{Alt}^2(V), \dots$

[Associativity: $(\alpha \wedge \beta \wedge \gamma)(v_1, \dots, v_{k+m+l}) = \sum_{\substack{(k,m,l)\text{-} \\ \text{shuffles}, \pi}} (-1)^\pi \alpha(v_{\pi(1)}, \dots, v_{\pi(k)}) \beta(v_{\pi(k+1)}, \dots, v_{\pi(k+m)}) \gamma(v_{\pi(k+m+1)}, \dots, v_{\pi(k+m+l)})$]

<u>Examples</u>: Suppose $\alpha, \beta \in \mathrm{Alt}^1(V) = V^*$. Then, $(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v) = \det \begin{pmatrix} \alpha(v) & \alpha(w) \\ \beta(v) & \beta(w) \end{pmatrix}$
If $\alpha_1, \dots, \alpha_k \in \mathrm{Alt}^1(V)$, then $(\alpha_1 \wedge \dots \wedge \alpha_k)(v_1, \dots, v_k) = \Sigma (-1)^\pi \alpha(v_{\pi(1)}) \dots \alpha(v_{\pi(k)}) = \det(\alpha_i(v_j))$.
In particular, if $\{e_1, \dots, e_n\}$ is a basis of V and $\{\alpha_1, \dots, \alpha_n\}$ is the dual basis of V^*, then $(\alpha_{i_1} \wedge \dots \wedge \alpha_{i_k})(e_{j_1}, \dots, e_{j_k}) = (-1)^\pi$, if π is the permutation $\pi(i_r) = j_r$, and $= 0$ if $\{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\}$. Thus, $\alpha_{i_1} \wedge \dots \wedge \alpha_{i_k}$ for $i_1 < \dots < i_k$ is a basis for $\mathrm{Alt}^k(V)$.

<u>Note</u>: $\mathrm{Alt}^1(\mathbb{R}^3) \cong \mathrm{Alt}^2(\mathbb{R}^3) \cong \mathbb{R}^3$. $\mathrm{Alt}^1 \times \mathrm{Alt}^1 \to \mathrm{Alt}^2$; $(v_1, v_2) \mapsto v_1 \times v_2$.
$\mathrm{Alt}^1 \times \mathrm{Alt}^2 \to \mathrm{Alt}^3 \cong \mathbb{R}$; $(v_1, v_2) \mapsto \langle v_1, v_2 \rangle$. (<u>cf</u>: "vector product" $(v_1 \times v_2) \times v_3 \neq v_1 \times (v_2 \times v_3)$)

Define $d : \Omega^k(U) \to \Omega^{k+1}(U)$ such that it is: (i) linear, (ii) an antiderivation (ie, $d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta)$ if $\alpha \in \Omega^k$), (iii) $d \circ d = 0$ (eg, curl.grad $= 0$), (iv) $d : \Omega^0(U) \to \Omega^1(U)$ is the usual "gradient", ie if $f : U \to \mathbb{R}$, then $Df(x) : \mathbb{R}^n \to \mathbb{R}$ is linear for each x, ie $df(x) = Df(x) \in \mathrm{Alt}^1(\mathbb{R}^n)$.

Let $x^1,\dots,x^n$ be the coordinate functions on $U\subset\mathbb{R}^n$, $x^i\in\Omega^0(U)$.
Then, dx^i is a constant element of $\mathrm{Alt}^1(\mathbb{R}^n)$, in fact, the standard dual basis element. So, any element of $\Omega^1(U)$ can be written as $\sum f_i\,dx^i$, where $f_i\in\Omega^0(U)$.
And if $f\in\Omega^0(U)$, we have $df=\sum_i (D_if)\,dx^i$, where $D_i=\partial/\partial x^i$

If $i_1,\dots,i_k$ is any sequence from $\{1,\dots,n\}$ write $dx^I=dx^{i_1}\wedge\dots\wedge dx^{i_k}$. Any element $\alpha\in\Omega^k(U)$ is a sum $\sum_I f_I(x)\,dx^I$ with $f_I\in\Omega^0(U)$, where I runs over $i_1<\dots<i_k$, and this expression is unique. So, we can define $d\alpha=\sum_I df_I\wedge dx^I$.
We must check that this has the required properties. First notice $d(f\,dx^I)=df\wedge dx^I$ for all sequences I. Obviously, $d:\Omega^k\to\Omega^{k+1}$ is linear.
Antiderivation: $d(f\,dx^I\wedge g\,dx^J) = d(fg\,dx^I\wedge dx^J) = d(fg)\wedge dx^I\wedge dx^J = (g\,df+f\,dg)\wedge dx^I\wedge dx^J$

$$= (df\wedge dx^I)\wedge(g\,dx^J) + f\,dg\wedge dx^I\wedge dx^J = d(f\,dx^I)\wedge(g\,dx^J) + (-1)^k (f\,dx^I)\wedge \underbrace{(dg\wedge dx^J)}_{=d(g\,dx^J)}$$

$d\circ d(f\,dx^I) = d(df\wedge dx^I) = (ddf)\wedge dx^I - df\wedge \overset{=0}{d(dx^I)}$. So, enough to prove that $ddf=0$ for $f\in\Omega^0(U)$. But, $df(x)=\sum_i (D_if)(x)\,dx^i$, so $ddf(x)=\sum_{j,i}(D_jD_if)(x)\,dx^j\wedge dx^i$ $=0$, as D_jD_if is symmetric in i and j, and $dx^j\wedge dx^i$ is antisymmetric.

Examples: If $\alpha\in\Omega^1(U)$, write $\alpha = f_1\,dx^1+\dots+f_n\,dx^n$, then $d\alpha=\sum df_i\wedge dx^i = \sum_{i,j}(D_jf_i)\,dx^j\wedge dx^i$
$= \sum_{i<j}(D_if_j-D_jf_i)\,dx^i\wedge dx^j$. Thus, $d\alpha$ = "curl" α.
Similarly, if $\alpha\in\Omega^{n-1}(U)$, we can write $\alpha=\sum_i(-1)^{i-1}f_i\,dx^1\wedge\dots\widehat{dx^i}\dots\wedge dx^n$.
So, $d\alpha = \sum_i (-1)^{i-1} df_i\wedge dx^1\wedge\dots\widehat{dx^i}\dots\wedge dx^n = \sum(-1)^{i-1}(D_if_i)\,dx^i\wedge dx^1\wedge\dots\widehat{dx^i}\dots\wedge dx^n$
$= (D_1f_1+\dots+D_nf_n).dx^1\wedge\dots\wedge dx^n$ = "div" α.

Let $\varphi: U\to V$ be a smooth map, where $U\subset\mathbb{R}^n$, $V\subset\mathbb{R}^m$, open. We shall define a homomorphism of graded rings, $\varphi^*:\Omega^k(V)\to\Omega^k(U)$ for all k. Ie, it is linear, and $\varphi^*(\alpha\wedge\beta)=\varphi^*(\alpha)\wedge\varphi^*(\beta)$, with the additional property: $d.\varphi^*=\varphi^*.d$.

Definition of φ^*: An element $\alpha\in\Omega^k(V)$ is a map $\alpha:V\to \mathrm{Alt}^k(\mathbb{R}^n)$. We can write it: $(y;v_1,\dots,v_k)\mapsto\alpha(y;v_1,\dots,v_k)$ with $y\in V$ and $v_i\in\mathbb{R}^n$.
Define $(\varphi^*\alpha)(x;v_1,\dots,v_k)=\alpha(\varphi(x);D\varphi(x)v_1,\dots,D\varphi(x)v_k)$. $(x\in U, \varphi(x)\in V, v_i\in\mathbb{R}^k, D\varphi(x)v_i\in\mathbb{R}^m)$

Trivially, φ is a graded ring homomorphism.
We must prove that $\varphi^*\circ d - d\circ\varphi^*:\Omega^k(V)\to\Omega^{k+1}(U)$ is zero. But this map is an antiderivation, so $d\circ(\varphi^*\circ d-d\circ\varphi^*) = -(\varphi^*\circ d - d\circ\varphi^*)\circ d$. So, $\alpha,\beta\in\ker(\varphi^*\circ d-d\circ\varphi^*)$ $\Rightarrow$ $d\alpha$, $\alpha\wedge\beta$ are in the kernel. So it is enough to prove $d\varphi^*(f)=\varphi^*df$ for $f\in\Omega^0(U)$.
$(\varphi^*df)(x;v) = df(\varphi(x);D\varphi(x)v) = Df[\varphi(x)]D\varphi(x)v = D(f\circ\varphi)(x)v = d(\varphi^*f)(x;v)$.
[Or, write as: $df=\sum\frac{\partial f}{\partial x^i}dx^i$. $\varphi^*df=\sum\frac{\partial f}{\partial x^i}(\varphi(x))\,\varphi^*(dx^i) = \sum\left(\frac{\partial f}{\partial x_i}\frac{\partial\varphi^i}{\partial x^j}\right)dx^j$]

We can now define $\Omega^k(X)$ where X is a smooth manifold. An element α of $\Omega^k(X)$ is a collection of elements $\alpha_i\in\Omega^k(V_i)$, one for each chart $\varphi_i:U_i\xrightarrow{\cong}V_i\subset\mathbb{R}^n$, such that $(\varphi_j\circ\varphi_i^{-1})^*\alpha_j=\alpha_i$ $\forall\, i,j$. Define $d:\Omega^k(X)\to\Omega^{k+1}(X)$ by $(d\alpha)_i=d\alpha_i\in\Omega^{k+1}(V_i)$, each i.

Clearly, the $\Omega^k(X)$ form an anticommutative graded ring, and $d: \Omega^k(X) \to \Omega^{k+1}(X)$ is an antiderivation such that $d \circ d = 0$.

Change of variables in $\mathbb{R}^n$.

Suppose $\varphi: U \to V$ is a diffeomorphism, U, V, open, $\subset \mathbb{R}^n$. Suppose $f: V \to \mathbb{R}$ is smooth with compact support. Then, $\int_V f(y)\, dy^1 \wedge \dots \wedge dy^n$ is defined, and
$\int_U f(\varphi(x)).|\det D\varphi(x)|.\, dx^1 \wedge \dots \wedge dx^n = \int_V f(y)\, dy^1 \wedge \dots \wedge dy^n$. ($f \circ \varphi$ has compact support $\subset U$).

Consider the element $\alpha = f\, dy^1 \wedge \dots \wedge dy^n \in \Omega^n(V)$. Then, $\varphi^*\alpha = (f \circ \varphi).\det(D\varphi)\, dx^1 \wedge \dots \wedge dx^n \in \Omega^n(U)$.

Proof: Write φ as $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_n(x) \end{pmatrix}$. $D\varphi(x)$ is the linear transformation with matrix $(D_j \varphi_i(x))$.

$\varphi^*(\alpha) = \varphi^*(f)\, \varphi^*(dy^1) \wedge \dots \wedge \varphi^*(dy^n) = (f \circ \varphi)\, d(\varphi^* y^1) \wedge \dots \wedge d(\varphi^* y^n)$. But, $\varphi^*(y^i) = y^i \circ \varphi = \varphi_i$.

So, $d(\varphi^* y^i) = d\varphi_i = \sum_j (D_j \varphi_i)\, dx^j$.

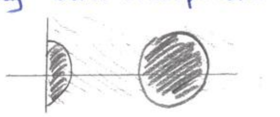
So, $\varphi^*(dy^1 \wedge \dots \wedge dy^n) = \sum_{j_1, \dots, j_n} (D_{j_1} \varphi_1) \dots (D_{j_n} \varphi_n)\, dx^{j_1} \dots dx^{j_n} = \det(D\varphi)\, dx^1 \wedge \dots \wedge dx^n$.

And $\varphi^*(f\, dy^1 \wedge \dots \wedge dy^n) = (f \circ \varphi) \det(D\varphi)\, dx^1 \wedge \dots \wedge dx^n$.

Corollary: If $\alpha \in \Omega^n(V)$ has compact support, then $\int_V \alpha = \left\{ \begin{smallmatrix} + \\ - \end{smallmatrix} \right\} \int_U \varphi^* \alpha$ if $\det D\varphi \left\{ \begin{smallmatrix} > \\ < \end{smallmatrix} \right\} 0$ everywhere.
In particular, $\int_V \alpha = \int_U \varphi^* \alpha$ if φ is orientation preserving.

Recall that X is oriented if it has an oriented atlas, ie, is covered by a set of charts $\{\varphi_i: U_i \to V_i \subset \mathbb{R}^n\}$ between which the transition functions are orientation preserving.

Definition: A smooth manifold with boundary is a set X with an atlas $\varphi_i: U_i \to V_i \subset \mathbb{R}^n_+$, where V_i is an open subset of $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) : x_1 \geq 0\}$, the transition maps being diffeomorphisms as before.



We must distinguish between these different types of open sets $\subset \mathbb{R}^n_+$.

Notice that if $\varphi: V_1 \to V_2$ is a diffeomorphism between open subsets of $\mathbb{R}^n_+$, then φ induces a diffeomorphism $V_1 \cap \mathbb{R}^{n-1} \to V_2 \cap \mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1} = \{(x_1, \dots, x_n) : x_1 = 0\}$.
The points of X which in some (and hence every) chart map to boundary points of $\mathbb{R}^n_+$ are called boundary points of X.
Clearly they form a smooth manifold of dimension $n-1$ (without boundary!) We can speak of an oriented manifold with boundary. The boundary of such acquires an orientation because an oriented atlas gives an oriented atlas for the boundary.

If $\Psi: V_1 \to V_2$ is a transition function, it takes $x_1 = 0$ to $x_1 = 0$. So, at boundary points, x, $D\Psi(x) = \left(\begin{array}{c|c} a_{11} & 0 \dots 0 \\ \hline * & (D\Psi_{bdry}) \end{array} \right)$. But $a_{11} > 0$ because $\Psi(V_1) \subset \mathbb{R}^n_+$, so the fact that $\det D\Psi(x) = a_{11}.\det(D\Psi_{bdry})$ gives an orientation on the boundary.

Definition: Let X be a smooth compact manifold with boundary ∂X. Suppose X is oriented with oriented atlas $\{\varphi_i : U_i \to V_i \subset \mathbb{R}^n_+\}$. Define $\int_X \alpha$ for any $\alpha \in \Omega^n(X)$ by $\int_X \alpha = \sum_i \int_{V_i} (f_i \alpha)_i$, where $\{f_i\}$ is a smooth partition of unity for the covering $\{U_i\}$. Ie, $f_i : X \to \mathbb{R}$ is smooth, ≥ 0, $\mathrm{supp}(f_i) \subset U_i$, $\sum f_i = 1$. And, where $(f_i\alpha)_i \in \Omega^n(V_i)$ is the representative of α in the chart $\varphi_i : U_i \to V_i$.

We must check this is well-defined. Suppose $\{\tilde{\varphi}_j : \tilde{U}_j \to \tilde{V}_j \subset \mathbb{R}^n_+\}$ is a compatible oriented atlas and $\{\tilde{f}_j\}$ is a partition of unity subordinate to it. We must show $\sum_i \int_{V_i} (f_i\alpha)_i = \sum_j \int_{\tilde{V}_j} (\tilde{f}_j \alpha)_j$. But, LHS $= \sum_i \int_{V_i} \sum_j (\tilde{f}_j f_i \alpha)_i$, and RHS $= \sum_{i,j} \int_{\tilde{V}_j} (\tilde{f}_j f_i \alpha)_j$.
Now, $\tilde{f}_j f_i \alpha \in \Omega^n(X)$ and has compact support in $U_i \cap \tilde{U}_j$. So, $(\tilde{f}_j f_i \alpha)_i \in \Omega^n(V_i)$ has compact support, as does $(\tilde{f}_j f_i \alpha)_j \in \Omega^n(\tilde{V}_j)$. By definition, $\beta_j = \Psi^* \beta_i$, where $\beta = \tilde{f}_j f_i \alpha$, and Ψ is the transition map, $\Psi = \varphi_i \cdot \tilde{\varphi}_j^{-1}$.
So, $\int_{\tilde{V}_j} \beta_j = \int_{V_i} \Psi^* \beta_i$, because Ψ is orientation preserving.

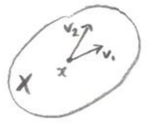
This definition would have been just as good without assuming X compact, providing α has compact support. (We proved the existence of partitions of unity only for compact X, but all we used was that the manifold could be covered by finitely many charts, and all we need is that $\mathrm{supp}^+(\alpha)$ can be covered by finitely many charts).

Stokes' Theorem: If X is a compact oriented manifold with boundary ∂X and $\beta \in \Omega^{n-1}(X)$, then $\int_X d\beta = \int_{\partial X} \beta$.

Proof: Enough (writing $\beta = \sum f_i \beta$) to prove this when β has support in some U_i.
So, enough to prove $\int_V d\beta = \int_{V \cap \mathbb{R}^{n-1}} \beta$ for an open subset $V \subset \mathbb{R}^n$, and $\beta \in \Omega^n(V)$ with compact support inside V. In fact, to show $\int_{\mathbb{R}^n_+} d\beta = \int_{\mathbb{R}^{n-1}} \beta$ if β has compact support $\subset \mathbb{R}^n_+$. But, if $\beta = \sum (-1)^{i-1} g_i \, dx^1 \wedge \dots \widehat{dx^i} \dots dx^n$, then $d\beta = (D_1 g_1 + \dots + D_n g_n)\, dx^1 \wedge \dots \wedge dx^n$. So we must show $\int_{\mathbb{R}^n_+} D_i g_i = \int_{\mathbb{R}^{n-1}} g_1 \, dx^1 \wedge \dots \wedge dx^n$ if $i = 1$, and $= 0$ if $i > 1$, because $\int \frac{\partial g_i}{\partial x^i} dx^i = 0$ as g_i has compact support.
Now, $\int_0^\infty D_1 g_1 \, dx^1 = -g_1(0, x^2, \dots, x^n)$. So $\int_X d\beta = -\int_{\partial X} \beta$, and we should have being $\mathbb{R}^n_-$, not $\mathbb{R}^n_+$. But it's close...

Let R be a k-dimensional region $\subset X$, $\beta \in \Omega^{k-1}(X)$. $\int_R d\beta = \int_{\partial R} \beta$. Suppose X is an n-dimensional smooth manifold and Y is a compact oriented k-dimensional manifold with boundary. Suppose $f : Y \to X$ is any smooth map. Then we can define $f^* : \Omega^i(X) \to \Omega^i(Y)$ for each i, so that it is a ring homomorphism, and $d.f^* = f^*.d$.
Then, the general Stokes' Theorem is: $\int_Y f^*(d\beta) = \int_{\partial Y} f^* \beta$ for any $\beta \in \Omega^{k-1}(X)$.

Have $\alpha \in \Omega^k(X)$, $\varphi_i : U_i \xrightarrow{\cong} V_i \subset \mathbb{R}^n$ $\iff$ $\{\alpha_i \in \Omega^k(V_i)$ for each chart$\}$.



New definition of an element of $\Omega^k(X)$: it is a map which to each $x \in X$ associates an element of $\mathrm{Alt}^k(T_x X)$, ie, a map $X \to \bigcup_{x \in X} \mathrm{Alt}^k(T_x X)$.
And, it is a smooth map. To define this, we must make $\bigcup_x \mathrm{Alt}^k(T_x X)$ into a smooth manifold. Chart $\varphi_i : U_i \to \mathbb{R}^n$ gives $\alpha_i \in \Omega^k(V_i)$, $\alpha_i(x) \in \mathrm{Alt}^k(\mathbb{R}^n)$ also gives $T_x X \xrightarrow{\cong} \mathbb{R}^n$, $v \mapsto v_i$. If we change charts to $\varphi_j : U_j \to V_j \subset \mathbb{R}^n$, let $\Psi = \varphi_j \varphi_i^{-1}$. Then v_i changes to $D\Psi(\varphi_i(x)) v_i = v_j$.

Let $\mathrm{Alt}^k(TX)$ be the disjoint union of $\mathrm{Alt}^k(T_xX)$ $\forall x \in X$. There is an obvious map $\pi: Y \to X$ taking $\mathrm{Alt}^k(T_xX)$ to x. For each chart $\varphi_i: U_i \to V_i \subset \mathbb{R}^n$ for X, define a chart $\tilde{\varphi}_i: \pi^{-1}(U_i) \to V_i \times \mathrm{Alt}^k(\mathbb{R}^n) \subset \mathbb{R}^n \times \mathrm{Alt}^k(\mathbb{R}^n) = \mathbb{R}^{n+\binom{n}{k}}$, by $\alpha \mapsto (x, \alpha_i)$, where α_i corresponds to α under the isomorphism $T_xX \to \mathbb{R}^n$ given by the chart φ_i.
Check that this makes $\mathrm{Alt}^k(TX)$ into a smooth manifold.
It is then trivial to check that an element of $\Omega^k(X)$ in the first sense is exactly equivalent to a smooth map $\alpha: X \to \mathrm{Alt}^k(TX)$ such that $\pi \circ \alpha = id$.

Return to the definition of $f^*: \Omega^j(X) \to \Omega^j(Y)$.
First method: Represent $\alpha \in \Omega^j(X)$ by $\alpha_i \in \Omega^j(V_i)$ for $\varphi_i: U_i \to V_i \subset \mathbb{R}^n$, covering X.
Then, $\{f^{-1}(U)\}$ is an open covering of Y. Choose charts $\{\tilde{\varphi}_m: \tilde{U}_m \to \tilde{V}_m \subset \mathbb{R}^k\}$ covering Y so that $f(\tilde{U}_m) \subset$ some U_i. Then define $f^*\alpha$ by giving it the representation $(\varphi_i \cdot f \cdot \tilde{\varphi}_m)^* \alpha_i$ in $\tilde{V}_m$.
Second method: Given $\alpha \in \Omega^j(X)$, we have $\alpha(x) \in \mathrm{Alt}^j(T_xX)$. But f induces $Df(y): T_yY \to T_{f(y)}X$ $\forall y \in Y$, so induces $\mathrm{Alt}^j(T_{f(y)}X) \to \mathrm{Alt}^j(T_yY)$.
Define $(f^*\alpha)(y)$ = image of $\alpha(f(y))$ under this map.

$\mathrm{Alt}^k(TX)$ is a vector bundle. $X \xleftarrow{\pi} \mathrm{Alt}^k(TX) \longleftrightarrow \mathrm{Alt}^k(T_xX)$. id. $\subseteq X \underset{\pi}{\rightleftarrows} TX$
$\mathrm{Alt}^1(TX) = T^*X \xrightarrow{\pi} X$. $\pi^{-1}(x) = (T_xX)^* = T_x^*X$, the cotangent space at x.

Orientation: To give an orientation of a smooth n-dimensional manifold X is equivalent to giving an element $\omega \in \Omega^n(X)$ which is non-zero at every $x \in X$.
Proof: $(\Leftarrow)$ Given ω, say that a chart $\varphi_i: U_i \to V_i \subset \mathbb{R}^n$ is oriented if the representative ω_i of ω is of the form $f(y)\,dy^1 \wedge \dots \wedge dy^n$ with $f(y) > 0$ $\forall y \in V_i$. Clearly, if two charts are oriented in this sense then the transition between them is orientation preserving. So the oriented charts form an oriented atlas.
$(\Rightarrow)$ (for compact X). Choose a finite number of charts $\varphi_i: U_i \to V_i \subset \mathbb{R}^n$ forming an oriented atlas. Choose a subordinate partition of unity $f_i: X \to \mathbb{R}$ with $\mathrm{supp}(f_i) \subset U_i$. Then consider the form $\tilde{\omega}_i \in \Omega^n(U_i)$, whose representation in V_i is $dx^1 \wedge \dots \wedge dx^n$. Let $\omega = \sum f_i \tilde{\omega}_i \in \Omega^n(X)$. Check that this is a well-defined nowhere-vanishing n-form.

$\mathbb{P}^2$ is not orientable. $\mathbb{R}^3 \supset S^2 \xrightarrow{\pi} \mathbb{P}^2$. S^{n-1}: $\Omega^{n-1}(S^{n-1}) \ni \omega_0 = \sum (-1)^{i-1} x_i\, dx^1 \wedge \dots \widehat{dx^i} \dots \wedge dx^n$.
Suppose $\omega \in \Omega^2(\mathbb{P}^2)$. Then $\pi^*\omega \in \Omega^2(S^2)$, and $i^*(\pi^*\omega) = \pi^*\omega$, where $i: S^2 \to S^2$, $x \mapsto -x$, as $\pi \circ i = \pi$. But as $\omega_0 \in \Omega^2(S^2)$ never vanishes, we can write $\pi^*\omega = f\omega_0$ for some function f on S^2. If ω never vanishes, then neither does f.
But $i^*(f\omega_0) = \omega_0$, ie, $i^*f \cdot i^*\omega_0 = f\omega_0$, but $i^*\omega_0 = -\omega_0$, so $i^*f = -f$, ie $f(-x) = -f(x)$.

Essentially the same argument $\Rightarrow$ $\mathbb{P}^{n-1}$ is orientable if n is even. Consider a chart $\varphi: U \to \mathbb{R}^{n-1}$ for $\mathbb{P}^{n-1}$. Then φ induces two charts: $\varphi_i: U_i \to \mathbb{R}^{n-1}$, $i = 1, 2$, for S^{n-1}, where $U_1 \dot{\cup} U_2 = \pi^{-1}(U)$. But $i: U_1 \xrightarrow{\cong} U_2$. Now, ω_0 is non-vanishing on S^{n-1}. Suppose it is represented by ω_1, ω_2 in $\Omega^{n-1}(U_1)$, $\Omega^{n-1}(U_2)$, respectively, then $i^*\omega_1 = \omega_2$, if n is even. So we can take $i^*\omega_1 = \omega_2$ as representations of an element of $\Omega^{n-1}(\mathbb{P}^{n-1})$.

X a smooth manifold, dimension n. Then, $\alpha \in \Omega^k(X)$ is closed if $d\alpha = 0$, and exact if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(X)$. $d \circ d = 0$, so $\{\text{exact}\} \subset \{\text{closed}\}$.

Definition: $H^k(X) = \dfrac{\text{closed } k\text{-forms}}{\text{exact } k\text{-forms}}$, the k^{th} de Rham cohomology of X.

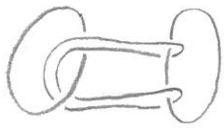
If α is a closed k-form representing an element of $H^k(X)$, we can define an invariant I_α for k-cycles in X. (A k-cycle in X is a map $f: Y \to X$, where Y is a k-dimensional compact oriented manifold without boundary).

Define $I_\alpha(Y, f) = \int_Y f^*\alpha$, $f^*\alpha \in \Omega^k(Y)$.
$I_\alpha(Y_1, f_1) = I_\alpha(Y_2, f_2)$ if the cycles (Y_1, f_1) and (Y_2, f_2) are homologous, ie, if $\exists$ a $(k+1)$-dimensional compact oriented manifold with boundary ∂Y, such that $\partial Y = \tilde{Y}_1 \dot\cup Y_2$, where $\tilde{Y}_1 = Y_1$ with reversed orientation, and a smooth map $Y \to X$ which restricts to f_1 and f_2 on ∂Y.

Stokes' Theorem $\Rightarrow$ if $\alpha_1 = \alpha_2 + d\beta$, then $I_{\alpha_1} = I_{\alpha_2}$, because $\int_Y f^*\alpha_1 = \int_Y f^*\alpha_2 + \int_Y f^* d\beta$, but $\int_Y f^* d\beta = \int_Y d(f^*\beta) = 0$ as Y has no boundary.
Similarly, Stokes $\Rightarrow$ $I_\alpha(Y_1, f_1) = I_\alpha(Y_2, f_2)$ if the cycles are homologous.

$$v(x) = \frac{1}{2\pi} \int_\gamma \frac{(x - \gamma(t)) \wedge \dot\gamma(t)}{\|x - \gamma(t)\|^3} \, dt.$$

$$\frac{1}{2\pi} \int_{\gamma, \tilde\gamma} \frac{[\tilde\gamma(s) - \gamma(t), \dot\gamma(t), \dot{\tilde\gamma}(s)]}{\|x - \gamma(t)\|^3} \, dt \, ds.$$



– "Borromean rings".

Degrees of Maps.

Theorem: If X is a compact connected oriented n-dimensional manifold without boundary, then $H^n(X) \cong \mathbb{R}$, by the map $\alpha \mapsto \int_X \alpha$ for $\alpha \in \Omega^n(X)$. Now, $\int_X (\alpha + d\beta) = \int_X \alpha + \int_X d\beta = \int_X \alpha$, by Stokes. Equivalently, if $\alpha \in \Omega^n(X)$, then $\alpha = d\beta$ for some $\beta \iff \int_X \alpha = 0$.

Corollary: If X, Y are as X in the theorem and $\varphi: X \to Y$ is smooth, then $\int_X \varphi^*\alpha = N \int_Y \alpha$ for every $\alpha \in \Omega^n(Y)$, where $N \in \mathbb{N}$, called the degree of φ.
Furthermore, if $y \in Y$ is such that $D\varphi(x)$ is an isomorphism whenever $\varphi(x) = y$, then $N = \sum_{\{x : \varphi(x) = y\}} \varepsilon_x$, where $\varepsilon_x = \{{+1 \atop -1}\}$, if $D\varphi(x)$ has det $\{{> 0 \atop < 0}\}$ in local charts.

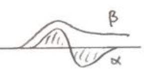
Proof: Take y as in "Furthermore". Then, by the inverse function theorem, $\exists$ open neighbourhood V_y of y such that $\varphi^{-1}(V_y) = \dot\bigcup_{x \in \varphi^{-1}(y)} U_x$, where $\varphi: U_x \to V_y$ is a diffeomorphism $\forall x \in \varphi^{-1}(y)$. Now choose $w \in \Omega^n(Y)$ such that $\text{supp}(w) \subset V_y$, and $\int_Y w = 1$. Then $\varphi^* w$ vanishes outside $\varphi^{-1}(V_y)$. So, $\int_X \varphi^* w = \sum_{x \in \varphi^{-1}(y)} \int_{U_x} \varphi^* w = \sum_{x \in X} \varepsilon_x = N \int_Y w$, say.
But, by the theorem, if $\alpha \in \Omega^n(Y)$, then $\alpha = \lambda w + d\beta$, where $\lambda = \int_Y \alpha$ (as $\int_Y (\alpha - \lambda w) = 0$)
So, $\int_X \varphi^* \alpha = \lambda \int_X \varphi^* w = \lambda N = N \int_Y \alpha$.

Suppose we have $\varphi: X \to Y$, map between n-dimensional manifolds. Let $x \in \varphi^{-1}(Y)$.
$D\varphi(x): T_x X \to T_y Y$, is invertible $\forall x \in f^{-1}(y) \Leftrightarrow$ "y is a regular value of f".
Sard's Theorem: Almost all $y \in Y$ are regular.

Let φ: (unit ball in $\mathbb{R}^n$) $\to \mathbb{R}^n$. If $\det(D\varphi(x)) = 0$ and $\varepsilon > 0$ then x has a neighbourhood U_x with $\mathrm{vol}(\varphi(U_x)) < \varepsilon . \mathrm{vol}(U_x)$.

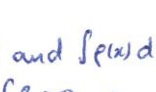
Recall: $H^n(X) \cong \mathbb{R}$ if X is compact, connected, oriented, dimension n. $\alpha \mapsto \int_X \alpha$.
Let $H^n_{cpt}(X) = \{n\text{-forms on } X \text{ with compact support}\} / \{\alpha : \alpha = d\beta, \text{ where } \beta \text{ has compact support}\}$.
We shall prove $H^n_{cpt}(X) \cong \mathbb{R}$ via $\alpha \mapsto \int_X \alpha$, providing X is oriented and covered by a finite number of open sets diffeomorphic to $\mathbb{R}^n$.

Proof: by induction on the number of balls covering X. We must prove that if $\alpha \in \Omega^n_{cpt}(X)$ and $\int_X \alpha = 0$ then $\alpha = d\beta$ for some $\beta \in \Omega^{n-1}_{cpt}(X)$.
Write $X = X_1 \cup X_2$, where result is known for X_1 and $X_2 \cong \mathbb{R}^n$, so result is known for X_2. $X_1 \cap X_2 \neq \emptyset$ as X is connected.
Given α with $\int \alpha = 0$, write $\alpha = \alpha_1 + \alpha_2$ with $\alpha_i \in \Omega^n_{cpt}(X_i)$, where $\alpha_i = f_i \alpha$ with f_i a partition of unity. Can assume $\int_{X_1} \alpha_1 = \int_{X_2} \alpha_2 = 0$ by replacing them by $\alpha_1 + \gamma$, $\alpha_2 - \gamma$ respectively, where γ has support in a ball $\subset X_1 \cap X_2$.
Then, $\alpha_i = d\beta_i$, with $\beta_i \in \Omega^{n-1}_{cpt}(X_i)$. So $\alpha = d(\beta_1 + \beta_2)$.
Case $X = \mathbb{R}^n$: Suppose $\int_{\mathbb{R}^n} \alpha = 0$, where $\alpha \in \Omega^n_{cpt}(\mathbb{R}^n)$. Suppose $n = 1$ and $\int_{\mathbb{R}} \alpha(t)\,dt = 0$.
Define $\beta(x) = \int_{-\infty}^{x} \alpha(t)\,dt$. Then, $d\beta = \alpha$. β has compact support $\Leftrightarrow \int_{\mathbb{R}} \alpha = 0$.



Use induction on n. Consider $n = 2$.
$\alpha \in \Omega^2_{cpt}(\mathbb{R}^2)$, $\alpha = \alpha(x,y)\,dx\,dy$. Define $\beta = \beta(y)\,dy$ by $\beta(y) = \int_{\mathbb{R}} \alpha(x,y)\,dx$.
If $\beta = 0$, then $\alpha = d\gamma$, where $\gamma = \gamma(x,y)\,dy$. $\gamma(x,y) = \int_{-\infty}^{x} \alpha(t,y)\,dt$.
Clearly $d\gamma = \alpha$. $\mathrm{Supp}(\gamma)$ is compact $\Leftrightarrow \beta = 0$.
If $\beta \neq 0$, consider $\alpha - \rho \wedge \beta$, where $\rho = \rho(x)\,dx$, with compact support and $\int \rho(x)\,dx = 1$.
Clearly, $\int_{\mathbb{R}} (\alpha(x,y) - \rho(x)\beta(y))\,dx\,dy = 0$. So, $\alpha = \rho \wedge \beta + d\gamma$. But $\int \alpha = \int \beta = 0$, so $\beta = df$, for some f with compact support. So $\alpha = \rho \wedge df + d\gamma = d(-f\rho + \gamma)$.

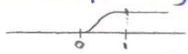
Application of degree: $\nexists$ a nowhere-vanishing smooth tangent vector field on S^{n-1} if n is odd. Suppose $\exists\, v: S^{n-1} \to \mathbb{R}^n$ such that $\langle x, v(x) \rangle = 0 \;\forall x$. We can assume $\|v(x)\| = 1 \;\forall x$. Define $\varphi_t : S^{n-1} \to S^{n-1}$ for $t \in \mathbb{R}$, $\varphi_t(x) = (\cos t)x + (\sin t)v(x)$.



$\varphi_0 = id.$, has degree 1. φ_π is $x \mapsto -x$, and has degree -1 if n is odd.
(Recall that $\omega = \sum (-1)^{i-1} x^i\, dx^1 \wedge .. \widehat{dx^i} .. \wedge dx^n \in \Omega^{n-1}(S^{n-1})$ has $\int \omega \neq 0$).
$\mathrm{Degree}(\varphi_t) = \int_{S^{n-1}} \varphi_t^*(\omega)$ where $\int_{S^{n-1}} \omega = 1$, so depends continuously on t. But degree takes integer values, so is constant. ※

Poincaré lemma: If $d\alpha = 0$ for some $\alpha \in \Omega^k(\mathbb{R}^n)$, $k > 0$, then $\alpha = d\beta$, some $\beta \in \Omega^{k-1}(\mathbb{R}^n)$.

Proposition: Suppose U is open in $\mathbb{R}^n$ and $\varphi : \mathbb{R} \times U \to V$ is smooth, where V is open in $\mathbb{R}^m$. Let $\varphi_t(x) = \varphi(x,t)$ for $x \in U$. Let $\beta \in \Omega^k(V)$ be closed ($k > 0$). Then $\varphi_0^*\beta$ and $\varphi_1^*\beta$ are closed in $\Omega^k(U)$, and $\varphi_1^*\beta - \varphi_0^*\beta = d\gamma$, some $\gamma \in \Omega^{k-1}(U)$.

From this, for the Poincaré lemma, take $U = V$ (= star-shaped region in $\mathbb{R}^n$),
and $\varphi(t,x) = \rho(t)\,x$, where $\rho : \mathbb{R} \to \mathbb{R}$, $\rho(t) = \begin{cases} 0 & \text{if } t \le 0 \\ 1 & \text{if } t \ge 1 \end{cases}$ 
Then $\varphi_1 = id_.$, so $\varphi_1^*\beta = \beta$, and φ_0 is the constant map, so $\varphi_0^*\beta = 0$, as $k > 0$.
So, proposition $\Rightarrow \beta = d\gamma$.

Proof of proposition: Let α be a closed form on $\mathbb{R}\times U$, and $\alpha_t = i_t^*\alpha = \alpha|_{\{t\}\times U}$ $[i_t(x) = i(t,x)]$.

Then, $\alpha_1 - \alpha_0 = d$ (something), $\alpha_t \in \Omega(U)$.

$\alpha = \beta + dt \wedge \gamma$, where β, γ involve no dt's. $\beta = \sum \beta_I(t,x)\,dx^I$.

Notice that $\alpha_t = \beta_t$, so we want $\beta_1 - \beta_0 = d$ (something)

But $d\alpha = d_0\beta + (dt \wedge \frac{\partial}{\partial t}\beta) - dt \wedge d_0\gamma$, where d_0 involves no dt.

This is zero $\Rightarrow \frac{\partial}{\partial t}\beta = d_0\gamma$, ie $\frac{\partial}{\partial t}\beta_t = d(\gamma_t)$.

Integrate this: $\beta_1 - \beta_0 = d\zeta$, where $\zeta = \int_0^1 \gamma_t\, dt$.